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The Riesz-Kantorovich formula for lexicographically ordered spaces

Master Thesis

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Contents

Introduction

If E and F are partially ordered vector spaces, then one can consider regular linear maps from E to F, i.e. linear maps which can be written as the difference of two positive linear maps. If the space E is directed, then the space $L^r(E,F)$ of all regular linear operators becomes a partially ordered vector space itself. We will mainly concern ourselves with the questions when the space $L^r(E,F)$ is itself a Riesz space and how, even if it is not a Riesz space, its lattice operations work. The so-called *Riesz-Kantorovich theorem* gives sufficient conditions for which $L^r(E,F)$ is a Riesz space and it also specifies the lattice operations by means of the *Riesz-Kantorovich formula*: if $S,T \in L^r(E,F)$ and $x \in E$ with $x \geq 0$ then it holds that

$$(S \vee T)(x) = \sup\{S(y) + T(x-y) : 0 \leq y \leq x\}.$$

It is still an open problem whether if the supremum of two regular operators exists in $L^r(E,F)$, it automatically satisfies the Riesz-Kantorovich formula. We will discuss several new results on this subject, including our main theorem, which is discussed in Chapter 4.

In Chapter 1 we will discuss some general theory on partially ordered vector spaces, (pre-)Riesz spaces and even some basic results on ordinal numbers.
In Chapter 2 we will state two conjectures and give counter-examples to them. The conjectures are concerned with the following situation: if E,F are Riesz spaces, but F is not Dedekind complete, then in general $L^r(E,F)$ is only a pre-Riesz space and not a Riesz space. As with all pre-Riesz spaces it is of interest to find a more explicit description of its Riesz completion. Naively it might seem logical that this Riesz completion might have something to do with the space $L^r(E,F^\delta)$, where F^δ is the Dedekind completion of F, or, if F is a normed Riesz space, with $L^r(E,\overline{F})$, where $\overline{F}$ is the norm completion of F. However, we will show that, in general, there is no clear connection between the two.
After that in Chapter 3, we will formulate and prove the Riesz-Kantorovich theorem. Subsequently we prove a theorem which combines the result of Chapter 2 with the Riesz-Kantorovich theorem.
Finally in Chapter 4 we will state and prove our main theorem. This theorem states that if the image space is an arbitrary product of the real line, equipped with the lexicographic ordering, we only need very minor conditions on the domain space for the Riesz-Kantorovich formula to hold. We will present two versions of this theorem, one using ordinal numbers and one only considering the finite-dimensional case.

The new results in this thesis are the major part of Chapter 2 as well as Theorem 3.4 and of course our main theorem in its two versions Theorem 4.5 and Theorem 4.6.

1. General theory

1.1. **Partially ordered vector spaces and Riesz spaces.**

We will start by giving some of the most important definitions and results, which we will need in the rest of the thesis.

Definition 1.1. *A vector space V over the real numbers with a partial order $\leq$ is called a* ***(partially) ordered vector space*** *if for all $x, y, z \in V$ and all $\lambda \in \mathbb{R}_{\geq 0}$ we have:*

- *If $x \leq y$ then also $x + z \leq y + z$.*
- *If $x \leq y$ then also $\lambda x \leq \lambda y$.*

Vectors $v \in V$ which satisfy $v \geq 0$ are called ***positive****. The set of all positive vectors is usually denoted by V_+, so we have $V_+ = \{v \in V : v \geq 0\}$, and is called the* ***positive cone*** *of V. The positive cone is said to be* ***generating*** *if for all $x \in V$ there exist $y, z \in V_+$ such that $x = y - z$ or, equivalently, if for any $x \in V$ there exists $y \in V_+$ with $x \leq y$. If the positive cone is generating then we call the partially ordered vector space* ***directed****.*

The positive cone of a partially ordered vector space V is always convex and satisfies the following three properties.

(1) $V_+ + V_+ \subset V_+$
(2) $\lambda V_+ \subset V_+$ for all $\lambda \in \mathbb{R}_{\geq 0}$
(3) $V_+ \cap (-V_+) = \{0\}$

Any convex subset of a vector space V which satisfies the properties (1), (2) and (3) above is called a *convex cone* in V. Instead of defining partially ordered vector spaces as a vector space with a suitable partial order one can also define it in terms of a convex cone: if L is a convex cone in a real vector space V then we define the order $\leq$ on V by saying that $x \leq y$ if and only if $y - x \in L$. Then $(V, \leq)$ is a partially ordered vector space in the sense of the previous definition. This means that a partially ordered vector space is in fact completely characterized by its positive cone.

From now on we always write V instead of $(V, \leq)$ when considering a partially ordered vector space. Furthermore, if V is a partially ordered vector space and U is a linear subspace of V then we will view U as a partially ordered vector space with the ordering inherited from V.

We will now give some other basic definitions:

Definition 1.2. *Let V be a partially ordered vector space. If $a, b \in V$ then we define the* ***order interval*** *$[a, b] := \{x \in V : x \geq a, x \leq b\}$. A subset $B \subset V$ is said to be* ***bounded from above*** *if there exists $u \in V$ such that $u \geq b$ for all $b \in B$. Similarly B is* ***bounded from below*** *if there exists $l \in V$ such that $l \leq b$ for all $b \in B$. The elements u and l in the previous definitions are called an* ***upper bound of*** *B and a* ***lower bound of*** *B, respectively. B is called* ***order-bounded*** *if it is bounded from above and below or, equivalently, if it is contained in an order-interval. We will denote for a subset B which is bounded from above, the set of all upper bounds of B with $B^u = \{x \in V : x \geq b \text{ for all } b \in B\}$.*

Now we will define suprema and infima in a general partially ordered vector space, which will lead to the definition of a Riesz space.

Definition 1.3. *A subset B of a partially ordered vector space V is said to have a* ***least upper bound*** *or* ***supremum*** *if there exists an element $s \in V$ such that s is an upper bound for B and such that for all $x \in V$ for which it holds that x is an upper bound of B we must have that $s \leq x$. This element is necessarily unique if it exists and we will denote it with $\sup B$. Similarly, B is said to have a* ***greatest lower bound*** *or* ***infimum*** *if there exists an element $r \in V$ which is a lower bound of B and such that for all $y \in V$ for which it holds that y is a lower bound we must have that $y \leq r$. This r is again unique if it exists and we will denote it with $\inf B$. A partially ordered vector space for which $\sup B$ and $\inf B$ exist for all order-bounded sets B is said to be* ***Dedekind complete****. We will use the following notation for the suprema and infima of sets with only two elements*

$$x \vee y = \sup\{x, y\} \text{ and } x \wedge y = \inf\{x, y\}$$

for $x, y \in V$.

Note that in general $\sup B$ and $\inf B$ need not to be elements of B. If they are, they are sometimes referred to as the *maximum* and *minimum* of B respectively. Now we are ready to define a Riesz space.

Definition 1.4. *A partially ordered vector space V is said to be a* ***Riesz space*** *or a* ***vector lattice*** *if for all $x, y \in V$ we have that $x \vee y$ and $x \wedge y$ exist in V.*

If V is a Riesz space and $x \in V$ then we can define the following vectors

$$x^+ := x \vee 0, x^- := (-x) \vee 0, |x| := x \vee (-x),$$

called the *positive part, negative part* and *absolute value* of x respectively. We will also use this terminology if V is not a Riesz space, whenever the suprema in these definitions exist anyway. The positive part, negative part and absolute value satisfy the following important identities

$$x = x^+ - x^- \text{ and } |x| = x^+ + x^-.$$

Note that this implies that every Riesz space is automatically directed. Using the above notation we obtain the following list of basic results, which we will use later on without reference.

Lemma 1.5. *Let V be a partially ordered vector space. We have for $u, v, w \in V$ and $\lambda \in \mathbb{R}_{\geq 0}$, whenever, in (2)-(4), one of the two sides of the equality sign exists*

(1) $u = 0$ *if and only if* $u, -u \in V_+$;
(2) $(u + w) \vee (v + w) = (u \vee v) + w; (u + w) \wedge (v + w) = (u \wedge v) + w$;
(3) $(\lambda v) \vee (\lambda w) = \lambda(v \vee w)$;
(4) $((u \vee v) \vee w) = (u \vee (v \vee w))$;
(5) $u^+, u^- \in V_+; u^+ \wedge u^- = 0; |u| \in V_+; u^+ \vee u^- = |u|$;
(6) $|u| \wedge |v| = 0$ *if and only if* $|u - v| = |u + v|$.

Proof See [8, Theorem 1.4]. ∎

Now for a slightly more interesting result, we obtain the following well-known proposition.

Proposition 1.6. *Let V be a partially ordered vector space. Suppose that x^+ exists for all $x \in V$. Then V is a Riesz space.*

Proof Let $x, y \in V$. Then $(x-y)^+$ and $(x-y)^- = (y-x)^+$ exist by assumption and thus $|x-y| = (x-y)^+ + (x-y)^-$ exists. We claim that $x \vee y = \frac{1}{2}\big(x+y+|x-y|\big)$. We have

$$\frac{1}{2}\big(x+y+|x-y|\big) \geq \frac{1}{2}(x+y+x-y) = x$$

and

$$\frac{1}{2}\big(x+y+|x-y|\big) \geq \frac{1}{2}(x+y+y-x) = y.$$

Now let $z \in V$ with $z \geq x$ and $z \geq y$. Then we have

$$\begin{aligned} z - \frac{1}{2}(x+y) &= \left(\frac{1}{2}z - \frac{1}{2}x\right) + \frac{1}{2}z - \frac{1}{2}y \\ &\geq 0 + \frac{1}{2}z - \frac{1}{2}y \geq \frac{1}{2}x - \frac{1}{2}y = \frac{1}{2}(x-y) \end{aligned}$$

and

$$\begin{aligned} z - \frac{1}{2}(x+y) &= \left(\frac{1}{2}z - \frac{1}{2}y\right) + \frac{1}{2}z - \frac{1}{2}x \\ &\geq 0 + \frac{1}{2}z - \frac{1}{2}x \geq \frac{1}{2}y - \frac{1}{2}x = \frac{1}{2}(y-x). \end{aligned}$$

So we must have that

$$z - \frac{1}{2}(x+y) \geq \frac{1}{2}|x-y|$$

and therefore

$$z \geq \frac{1}{2}\big(x+y+|x-y|\big).$$

Hence $x \vee y$ exists and is equal to $\frac{1}{2}\big(x+y+|x-y|\big)$. Therefore V is a Riesz space. ∎

An important definition is that of an Archimedean space. It reads as follows.

Definition 1.7. *A partially ordered vector space V is said to be* ***Archimedean*** *if for elements $u, v \in V$ with $nv \leq u$ for all $n \in \mathbb{Z}_{>0}$ we must have that $v \leq 0$. If V is a Riesz space then this is equivalent to the statement that if for elements $u, v \in V_+$ satisfying $0 \leq nv \leq u$ for all $n \in \mathbb{Z}_{>0}$ we must have that $v = 0$.*

A Riesz space is not necessarily Dedekind complete, but it is always contained in a Riesz space that is, as can be seen from the following result.

Proposition 1.8. *Let V be an Archimedean Riesz space. Then there exists a Dedekind complete Riesz space W such that $V \subset W$ and such that for any Dedekind complete Riesz space U with $V \subset U \subset W$ we must have that $U = W$. This space W is unique up to isomorphism and is usually denoted by V^δ and is called the* ***Dedekind completion*** *of V.*

Proof See [9, Theorem 32.5].

A slightly weaker version of being a Riesz space is having the so-called Riesz-decomposition property. It is defined as follows.

Definition 1.9. *A partially ordered vector space V is said to have the* ***Riesz decomposition property*** *if for any $x_1, x_2, y \in V_+$ with $y \leq x_1 + x_2$ there exist $y_1, y_2 \in V$ with $0 \leq y_1 \leq x_1, 0 \leq y_2 \leq x_2$ and $y_1 + y_2 = y$.*

We obtain the following basic result, which can be found in [8].

Proposition 1.10. *Every Riesz space has the Riesz decomposition property.*

Proof Let V be a Riesz space and let $x_1, x_2, y \in V_+$ with $0 \leq y \leq x_1 + x_2$. Define $y_1 = y \wedge x_1$ and $y_2 = y - y \wedge x_1$. Then we immediately obtain that $0 \leq y_1 \leq x_1$ and $y_1 + y_2 = y$. Since $y_1 \leq y$ we obtain that $y_2 \geq 0$. Finally we have that

$$y_2 = y - y \wedge x_1 = y + (-y \vee -x_1) = 0 \vee (y - x_1) \leq 0 \vee x_2 = x_2.$$

So this gives $0 \leq y_2 \leq x_2$. ∎

Now we are ready to give a few examples.

Example 1.11. On the vector space $\mathbb{R}^n$ there are two often-used partial orders which turn $\mathbb{R}^n$ into a partially ordered vector space. The first one is the **componentwise ordering** and is defined by $(x_1, ..., x_n) \leq (y_1, ..., y_n)$ if and only if $x_1 \leq y_1, ..., x_n \leq y_n$. The other one is called the **lexicographic ordering** and is defined by $(x_1, ..., x_n) \leq (y_1, ..., y_n)$ if and only if $(x_1, ..., x_n) = (y_1, ..., y_n)$ or there exists $j \in \{1, ..., n-1\}$ such that $x_i = y_i$ for all $i \in \{1, ..., j\}$ and $x_{j+1} < y_{j+1}$. One can easily check that in both cases $\mathbb{R}^n$ becomes a Riesz space. For the componentwise ordering it is also Archimedean and even Dedekind complete. However, $\mathbb{R}^n$ with the lexicographic ordering is neither, unless of course $n = 1$ in which case the lexicographic and the componentwise ordering coincide. It is not Archimedean for $n > 1$ since $(1, 0, ..., 0), (0, ..., 0, 1) \in \mathbb{R}^n_+$ and $0 \leq m(0, ..., 0, 1) = (0, ..., 0, m) \leq (1, 0, ..., 0)$ for all $m \in \mathbb{Z}_{>0}$, while $(0, ..., 0, 1) \neq 0$. Also the set $\{(0, ..., 0, \lambda) : \lambda \in \mathbb{R}\}$ is bounded from above by $(1, 0, ..., 0)$ and from below by $(-1, 0, ..., 0)$, but it clearly does not have a supremum.

Example 1.12. If (X, τ) is any topological space then the space $C(X)$ of all continuous functions from X to $\mathbb{R}$ is a Riesz space if we say that $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in X$. This follows from the fact that if $f, g \in C(X)$ then the functions $x \mapsto \max\{f(x), g(x)\}$ and $x \mapsto \min\{f(x), g(x)\}$ are also continuous. The space $C(X)$ is Archimedean, but in general it is not Dedekind complete. For example, if $X = [0, 1]$ with the Euclidean topology and for $n \in \mathbb{Z}_{>1}$ we define f_n to be 0 on $[0, \frac{1}{2}]$, linearly increasing on $[\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]$ and 1 on $[\frac{1}{2} + \frac{1}{n}, 1]$, then each f_n is continuous, but the set $\{f_n : n \in \mathbb{Z}_{>1}\}$ has no supremum, as the only function from X to $\mathbb{R}$ which potentially could be a supremum is the function which is 0 on $[0, \frac{1}{2}]$ and 1 on $(\frac{1}{2}, 1]$, which is not a continuous function.

Example 1.13. Let (S, Σ, μ) be a measure space and let $1 \leq p < \infty$. Let $\sim$ denote the equivalence relation on $M(S)$, the measurable functions from (S, Σ) to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel-σ-algebra on $\mathbb{R}$, given by $f \sim g$ if and only if $f = g$ μ-almost everywhere. Now let $L^p = \{f \in M(S) : \int |f|^p d\mu < \infty\}/\sim$. We define the ordering $\leq$ on L^p by $f \leq g$ if and only if $f \leq g$ almost everywhere. Now if $f \in L^p$ then we let f^+ be given by $f^+(x) = \begin{cases} f(x) \text{ if } f(x) \geq 0 \\ 0 \text{ if } f(x) < 0 \end{cases}$. Then f^+ is clearly measurable and we have that

$$\int |f^+|^p d\mu \leq \int |f|^p d\mu < \infty,$$

since $|f^+|(x) \leq |f|(x)$ for all $x \in S$. Hence $f^+ \in L^p$. Clearly $f^+ \geq f$, $f^+ \geq 0$ and f^+ is the smallest function which satisfies $f^+ \geq f$ and $f^+ \geq 0$. Hence f^+ is indeed the positive part of f. So by Proposition 1.6 we get that L^p is a Riesz space.

It is also Archimedean. In particular we get that the space

$$\ell^p = \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{R} \text{ for all } n \in \mathbb{N}, \sum_{n=1}^{\infty} |a_n|^p < \infty\}$$

is an Archimedean Riesz space.
Also for $p = \infty$ we let

$$L^\infty = \{f \in M(S) : \text{there exists } B \in \mathbb{R}_{\geq 0} \text{ such that } |f| \leq B \text{ a.e.}\}.$$

Again this space is an Archimedean Riesz space. In particular the space

$$\ell^\infty = \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{R} \text{ for all } n \in \mathbb{N}, \sup_{n\in\mathbb{N}} |a_n| < \infty\}$$

is an Archimedean Riesz space.

Example 1.14. Finally we give an example of a space which is not a Riesz space. Let $C_1[0,1]$ denote the set of functions from $[0,1]$ to $\mathbb{R}$ which are continuously differentiable and we say that $f \leq g$ if and only $f(x) \leq g(x)$ for all $x \in [0,1]$. Then the functions $x \mapsto x$ and $x \mapsto 1-x$ do not have a supremum. Intuitively this is because the pointwise maximum of the two functions is not differentiable, but a slightly more rigorous argument is needed to fully prove it. It does, however, have the Riesz decomposition property: if $0 \leq f, g_1, g_2$ and $f \leq g_1+g_2$ then for $i \in \{1,2\}$ we define the function

$$f_i : [0,1] \to \mathbb{R}, x \mapsto \begin{cases} \frac{g_i(x)f(x)}{g_1(x)+g_2(x)} & \text{if } g_1(x)+g_2(x) \neq 0 \\ 0 \text{ else.} \end{cases}$$

One can check that these f_1 and f_2 have the desired properties.

1.2. **Linear operators between partially ordered vector spaces.**
Now we will discuss several spaces of operators between two partially ordered vector spaces. We start with the following definition.

Definition 1.15. *Let V and W be two partially ordered vector spaces. A linear map $f : V \to W$ is called* ***positive*** *if $f(x) \in W_+$ for all $x \in V_+$. The map f is called* ***order-bounded*** *if f carries order-bounded subsets of V into order-bounded subsets of W. Finally, f is called* ***regular*** *if it can be written as the difference of two positive linear maps.*

If in the setting of Definition 1.15 the space V is directed then we can define the ordering $\leq$ on the space of linear operators from V to W, which we denote by $L(V,W)$, by saying that $f \leq g$ if and only if $g - f$ is positive. Then $L(V,W)$ with this ordering becomes a partially ordered vector space itself. Similarly, this ordering turns the space of all order-bounded operators and the space of all regular operators into partially ordered vector spaces, which we will denote by $L^\sim(V,W)$ and $L^r(V,W)$ respectively. We always have the following sequence of inclusions

$$L^r(V,W) \subset L^\sim(V,W) \subset L(V,W).$$

These inclusions hold because any positive operator is clearly order-bounded and therefore any regular operator is order-bounded as well. Furthermore if $L^\sim(V,W)$ is directed, in particular if it is a Riesz space, then we have $L^r(V,W) = L^\sim(V,W)$. Finally we also have the following definition.

Definition 1.16. *A map $f \in L(V,W)$ is called* ***bipositive*** *if for all $x \in V$ we have $x \in V_+$ if and only if $f(x) \in W_+$.*

Note that a bipositive map is always positive. Bipositive maps are also automatically injective: let $f : V \to W$ be bipositive and suppose $f(x) = 0$ for some $x \in V$. Then $f(x) \geq 0$ so $x \geq 0$. Then also $f(-x) = -f(x) \geq 0$ and thus $-x \geq 0$.

Hence we obtain that $x = 0$, so f is injective.

1.3. **Strong order units and ideals.**
Next we will define what a strong order unit is.

Definition 1.17. *Let V be a partially ordered vector space. An element $e \in V_+$ is a **strong order unit** if for all $x \in V$ there exists a $\lambda \in \mathbb{R}_{>0}$ such that $x \leq \lambda e$.*

If e is a strong order unit then for all $x \in V$ we have $x \leq \lambda e$ for some $\lambda \in \mathbb{R}_{>0}$ and since $\lambda e \geq 0$ we obtain that then V is automatically directed. Note that if e is a strong order unit, then so are λe for all $\lambda \in \mathbb{R}_{>0}$ and $x + e$ for all $x \in V_+$. We also have the following definition:

Definition 1.18. *If X is a vector space over $\mathbb{R}$ and A is a subset of X, then a vector $a \in A$ is called an **internal point** of A if for each $x \in X$ there exists a $\lambda_0 \in \mathbb{R}_{>0}$ such that $a + \lambda x \in A$ for each $-\lambda_0 \leq \lambda \leq \lambda_0$.*

The following lemma shows the connection between strong order units and internal points, which we will need later on.

Lemma 1.19. *A vector $e \in V_+$ is a strong order unit if and only if it is an internal point of V_+.*

Proof See [1, Lemma 2.1]. ∎

Definition 1.20. *Let V be a partially ordered vector space and let $x \in V_+$. Then we define the **ideal generated by the element** x to be the linear subspace $\bigcup_{n=1}^{\infty} [-nx, nx]$ and we denote this subspace by V_x.*

Although more general ideals exist, they are not of use for us and hence we will only consider ideals generated by a specific element. Note that if $x \in V$ and $y \in V_x$ with $y \geq 0$ then for all $z \in V$ with $0 \leq z \leq y$ we have $z \in V_x$. Furthermore x is a strong order unit in V_x, which is an important fact we will need later on.

1.4. **Pre-Riesz spaces.**
Now we will introduce the concept of a pre-Riesz space. For this we need some terminology.

Definition 1.21. *We call a linear subspace W of a partially ordered vector space V **order-dense in** V if for all $v \in V$ we have $v = \inf\{w \in W : w \geq v\}$, where the infimum is taken in V.*

Definition 1.22. *We say that a subspace W of a Riesz space V **generates** V if for every $v \in V$ there exist $a_1, ..., a_m, b_1, ..., b_n \in W$ such that $v = \vee_{i=1}^{m} a_i - \vee_{i=1}^{m} b_i$.*

Now we can define the concept of a pre-Riesz space.

Definition 1.23. *Let V be a partially ordered vector space. Then V is called a **pre-Riesz space** if one of the following equivalent conditions hold.*

(1) For every $x, y, z \in V$ the inclusion $\{x+y, x+z\}^u \subset \{y, z\}^u$ implies $x \in V_+$.
(2) There exists a Riesz space W and a bipositive linear map $\iota : V \to W$ such that $\iota(V)$ is order-dense in W.
(3) There exists a Riesz space W and a bipositive linear map $\iota : V \to W$ such that $\iota(V)$ is order-dense in W and generates W as a Riesz space.

*All spaces W as in (3) are isomorphic as Riesz spaces. A pair (W, ι) as in (3) is called, since it is unique up to isomorphism, **the Riesz completion of** X.*

The equivalence of these definitions can be found in [6, Corollaries 4.9-11, Theorems 3.5, 3.7, 4.13]. We have the following observations.

Proposition 1.24. *(1) Every pre-Riesz space is directed.*
(2) Every directed Archimedean partially ordered vector space is a pre-Riesz space.
(3) Every Riesz space is a pre-Riesz space.

Proofs of these results can be found in [6]. However, the construction of the Riesz completion of a pre-Riesz space is far from explicit. It is often an interesting question whether we can explicitly describe it. We have the following important theorem (see [6]).

Theorem 1.25. *Let E, F be Riesz spaces, such that F is Archimedean. Then $L^r(E, F)$ is directed and Archimedean. In particular, $L^r(E, F)$ is a pre-Riesz space.*

Proof For $T \in L^r(E, F)$ we have since T is regular that we can find positive T_1, T_2 such that $T = T_1 - T_2$ and therefore $L^r(E, F)$ is directed.
Let $S, T \in L^r(E, F)$ and suppose that $nS \leq T$ for all $n \in \mathbb{Z}_{>0}$. Suppose that $S \not\leq 0$. Since E is directed we can let $x \in E_+$ with $S(x) \not\leq 0$. Then $nS(x) \leq T(x)$ for all $n \in \mathbb{Z}_{>0}$ since $x \in E_+$. Since F is Archimedean this implies that $S(x) \leq 0$ which gives a contradiction. So $S \leq 0$. Hence $L^r(E, F)$ is Archimedean. By Proposition 1.24 this means that $L^r(E, F)$ is a pre-Riesz space. ∎

In Chapter 2 we will consider the question if we can determine the Riesz completion of the space of regular operators.

Now we will introduce the concept of disjointness.

Definition 1.26. *Let V be a partially ordered vector space. The elements $v, w \in V$ are called **disjoint**, which we denote by $v \perp w$, if $\{v + w, -v - w\}^u = \{v - w, -v + w\}^u$. If V is a Riesz space then $v \perp w$ is equivalent to the equality $|v| \wedge |w| = 0$.*

We have the following useful result which shows under what circumstances disjointness is preserved in subspaces. It will be used later on.

Proposition 1.27. *Let V and W be partially ordered vector spaces and $x, y \in W$. If W is a subspace of V then $x \perp y$ in V implies $x \perp y$ in W. Furthermore, if W is an order dense subspace of V, then $x \perp y$ in V if and only if $x \perp y$ in W.*

Proof See [5, proposition 2.1]. ∎

1.5. **Normed Riesz spaces.**

To conclude our discussion of partially ordered vector spaces we consider the concept of normed Riesz spaces. We start with the following definition.

Definition 1.28. *Let V be a Riesz space and $\|\cdot\|$ be a norm on V. Then $\|\cdot\|$ is called a **Riesz norm** if for $x, y \in V$ with $|x| \leq |y|$ we have $\|x\| \leq \|y\|$. The pair $(V, \|\cdot\|)$ is called a **normed Riesz space**. If V is complete with respect to the norm $\|\cdot\|$ then the pair $(V, \|\cdot\|)$ is called a **Banach lattice**.*

We will first discuss a few examples.

Example 1.29. Let $1 \leq p \leq \infty$ and consider the space ℓ^p from Example 1.13. For $p < \infty$ we define the norm $\|(a_n)_{n\in\mathbb{N}}\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}}$ and for $p = \infty$ we define the norm $\|(a_n)_{n\in\mathbb{N}}\|_\infty := \sup_{n\in\mathbb{N}} |a_n|$ for $(a_n)_{n\in\mathbb{N}} \in \ell^p$. Then the space $\left(\ell^p, \|\cdot\|_p\right)$ is a Banach lattice.

Example 1.30. If X is a compact and Hausdorff topological space then the space $(C(X), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the usual supremum-norm on $C(X)$, is a Banach lattice.

Example 1.31. The space $\mathbb{R}^n$ with the *lexicographic ordering* and the usual Euclidean norm $\|\cdot\|$ is not a normed Riesz space. To see this note that $|(0,2)| = (0,2) \leq (1,0) = |(1,0)|$, while $\|(0,2)\| = 2 > 1 = \|(0,1)\|$.

We first have the following elementary result.

Proposition 1.32. *Let $(V, \|\cdot\|)$ be a normed Riesz space. Then we have:*

(1) The maps $(x,y) \mapsto x \vee y, (x,y) \mapsto x \wedge y, x \mapsto x^+, x \mapsto x^-$ and $x \mapsto |x|$ are all continuous as maps either from $V \times V$ to V or from V to V.

(2) The positive cone V_+ is closed with respect to the topology induced by $\|\cdot\|$.

Proof See [11, proposition 5.2]. ∎

Now we obtain a more interesting result (see [2]).

Proposition 1.33. *Let $(V, \|\cdot\|)$ be a Banach lattice and let $(W, \|\cdot\|')$ be a normed Riesz space. Let $T \in L^r(V,W)$. Then T is norm-bounded, i.e. $\|T\| < \infty$.*

Proof Since the difference of two norm-bounded operators is norm-bounded we restrict ourselves to the case where $T \geq 0$. We will argue by contradiction. Suppose that T is not norm-bounded. Then we can find a sequence $(x_n)_{n \in \mathbb{N}}$ in V such that $\|x_n\| = 1$ and $\|Tx_n'\| \geq n^3$ for all $n \in \mathbb{N}$. Since V is a Banach lattice and both $|x| \leq \left|(|x|)\right|$ and $\left|(|x|)\right| \leq |x|$ (since $\left|(|x|)\right| = |x|$) for all $x \in V$ we get that $\||x_n|\| = \|x_n\| = 1$ for all $n \in \mathbb{N}$. Since T is positive and $|x_n| \geq x_n \geq -|x_n|$ we obtain that $T|x_n| \geq Tx_n \geq -T|x_n|$ and in particular $T|x_n| \geq |Tx_n|$ for all $n \in \mathbb{N}$. Hence $\|T|x_n|\|' \geq \|Tx_n\|' \geq n^3$ since W is a normed Riesz space. Now we let $y = \sum_{n=1}^{\infty} \frac{|x_n|}{n^2}$, which is a well-defined element of V since the series converges absolutely and V is norm complete. For $m \in \mathbb{N}$ we see that $y \geq \frac{|x_m|}{m^2} \geq 0$, since by Proposition 1.32 the positive cone of V is closed, so we obtain that $Ty \geq \frac{T|x_m|}{m^2} \geq 0$. Hence we get $\|Ty\|' \geq \left\|\frac{T|x_m|}{m^2}\right\|' \geq m$ for all $m \in \mathbb{N}$. This gives a contradiction since naturally $\|Ty\|' < \infty$ and thus $\|Ty\|' \geq m$ cannot hold for all $m \in \mathbb{N}$. Hence we must have that T is norm-bounded. ∎

1.6. **Ordinal numbers.**

Although this subject has (almost) nothing to do with the concept of partially ordered vector spaces, we will introduce the concept of ordinal numbers and show a few basic results, since one version of our main theorem is formulated in terms of ordinal numbers and uses transfinite induction.

The concept of an ordinal number originates in set theory and that is where it is used mostly. We do not assume that the reader has any knowledge of axiomatic set theory, but we also do not need that: all concepts can be defined and discussed using 'intuitive' set theory. For a more detailed discussion and proofs of the results, we refer to [7]. We start with the following definition.

Definition 1.34. *A set S is called **transitive** if for all $x \in S$ it holds that $x \subset S$. An **ordinal number** is a transitive set such that if we say that $x \leq y$ if and only if $x = y$ or $x \in y$ that then S is well-ordered by $\leq$.*

The most basic example of an ordinal number is the empty set. We have the following result.

Proposition 1.35. *If α is a non-empty ordinal number and $\beta \in \alpha$ then β is also an ordinal number. The class of all ordinal numbers is itself well-ordered by $\in$ (although it is not a set). In particular, any two ordinal numbers α and β with $\alpha \neq \beta$ satisfy either $\alpha \in \beta$ or $\beta \in \alpha$.*

Instead of $\beta \in \alpha$ we also often write $\beta < \alpha$. $\emptyset$ is the smallest ordinal number, followed by $\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$. $\emptyset$ is often identified with the natural number 0, $\{\emptyset\}$ with 1 etc. In this way we can view each natural number as an ordinal number.
The ordinal numbers are an important class of well-ordered sets. In fact we get that every well-ordered set is isomorphic to a unique ordinal number in the following sense.

Proposition 1.36. *Let $(V, \preceq)$ be a well-ordered set. Then there exists a unique ordinal number α and a bijective map $f : V \to \alpha$ such that $f(x) < f(y)$ if and only if $x \preceq y$ for every $x, y \in V$.*

We also use the following terminology.

Definition 1.37. *Let α be an ordinal number. The **immediate successor of** α, notation $\alpha + 1$, is the unique ordinal number $\min\{\beta : \alpha \in \beta\}$ and it satisfies $\alpha + 1 = \alpha \cup \{\alpha\}$. The ordinal α is called a **successor ordinal** if $\alpha = \beta + 1$ for some ordinal β and α is called a **limit ordinal** if it is not 0 and it is no successor ordinal.*

The set of all natural numbers is the first limit ordinal. If α is a limit ordinal then one has the following identity:

$$\alpha = \bigcup_{\beta \in \alpha} \beta.$$

Using this terminology we can say what we mean by transfinite induction.

Theorem 1.38. ***(Transfinite induction)*** *Let A be a class of ordinal numbers. Suppose A has the following three properties:*

(1) $0 \in A$.
(2) If $\alpha \in A$ then also $\alpha + 1 \in A$.
(3) If α is a limit ordinal and $\beta \in A$ for all $\beta \in \alpha$ then also $\alpha \in A$.

Then A is the class of all ordinal numbers.

This allows us to prove statements in the following way: first we prove that for some minimal ordinal number the property holds; then we assume the property holds for some ordinal α and we prove that this implies it holds for $\alpha + 1$ and finally we assume the property holds for all $\beta \in \alpha$ where α is a limit ordinal that this implies that the property holds for α. Since natural numbers are also ordinal numbers this allows us to include 'normal' induction in transfinite induction. To conclude we give an example of a partially ordered vector space which uses ordinal numbers.

Example 1.39. Let $\alpha \neq 0$ be an ordinal number. Let $M = \prod_{1 \leq j \leq \alpha} \mathbb{R}$. If $x \in M$ and $\beta < \alpha$ then we let $x^{(\beta)}$ denote the β^{th} coordinate of x and $x^{(1),\ldots,(\beta)}$ the vector consisting of the first β coordinates of x. Then we can give M the **lexicographic ordering** by saying that $x \leq y$ if and only if either $x = y$ or there exists some $\beta \leq \alpha$ such that for all $\gamma < \beta$ we have $x^{(\gamma)} = y^{(\gamma)}$ and $x^{(\beta)} < y^{(\beta)}$. Just as for the lexicographic ordering on $\mathbb{R}^n$ we see that M becomes a Riesz space as it is totally ordered. If $\alpha > 1$ then M not Archimedean and not Dedekind complete. It also has the following useful property. We give M the product topology. Let $K \subset M$ be non-empty and compact. We claim that K has an order-maximum and show this by transfinite induction. For $\alpha = 1$ the result is well-known. Now let $\alpha \geq 1$

and assume each non-empty compact set in $\prod_{1\leq j\leq\alpha}\mathbb{R}$ has an order-maximum. Let $K\subset M$ be compact. Let π_α be the projection from M onto its first α coordinates. Then we know that π_α is continuous. Hence $\pi_\alpha(K)$ is compact in $\prod_{1\leq j\leq\alpha}\mathbb{R}$ and thus has an order-maximum, say x. Then $\{y^{(\alpha+1)} : y\in K, \pi_\alpha(y)=x\}$ is a compact and non-empty subset of $\mathbb{R}$ and thus has an order-maximum y. Then $(x,y)\in K$ and if $z\in K$ then $\pi_\alpha z\leq x$ and if $\pi_\alpha z = x$ then $z^{(\alpha+1)}\leq y$. Hence $z\leq (x,y)$. So K has an order-maximum. Finally let α be a limit ordinal and assume that for each $\beta<\alpha$ we have that every non-empty compact subset of $\prod_{1\leq j\leq\beta}\mathbb{R}$ has an order-maximum. Let $K\subset M$ be compact. Let $\beta<\alpha$ and let π_β be the projection from M on the first β coordinates. As before we can let $x_\beta\in\pi_\beta(K)$ be the order-maximum. If $\gamma<\beta$ then clearly the first γ coordinates of x_β must be x_γ since we use the lexicographic ordering. Hence we can let $x\in K$ be defined by $x^{(\beta)} = x_\beta^{(\beta)}$ and by construction we indeed have $x\in K$. Now if $z\in K$ with $z\neq x$ then we let $\beta<\alpha$ be minimal such that $z^{(\beta)}\neq x^{(\beta)}$ and thus we must have that $\pi_\beta z < x_\beta = \pi_\beta x$. Hence also $z<x$. So indeed x is an order-maximum of K. So we get that for all $\alpha\geq 1$ and all compact K in $\prod_{1\leq j\leq\alpha}\mathbb{R}$ we must have that K has an order-maximum.

2. The Riesz completion of the space $L^r(E,F)$

Let E and F be Archimedean Riesz spaces. In Theorem 1.25 we saw that $L^r(E,F)$ is always a pre-Riesz space. As with all pre-Riesz spaces it is interesting to wonder if we can find an explicit expression for its Riesz completion. The following theorem, which is part of the theorem we will discuss and prove in Chapter 3, might give us a tool in that.

Theorem 2.1. *If F is Dedekind complete, then $L^r(E,F)$ is a Riesz space and it is even Dedekind complete.*

Since any Riesz space F has a Dedekind completion, which we will denote by F^δ, it is a natural question to wonder whether the Riesz completion of $L^r(E,F)$ might have something to do with the Riesz space $L^r(E,F^\delta)$, since there is clearly a linear order-preserving injection from $L^r(E,F)$ to $L^r(E,F^\delta)$. It might be too much to ask that the Riesz completion of $L^r(E,F)$ is isomorphic to $L^r(E,F^\delta)$, but at least it would help if $L^r(E,F)$ would be an order dense subset of $L^r(E,F^\delta)$, since then the Riesz completion of $L^r(E,F)$ would be isomorphic to the Riesz subspace of $L^r(E,F^\delta)$ generated by $L^r(E,F)$.

Conjecture 2.2. *The space $L^r(E,F)$ lies order dense in the space $L^r(E,F^\delta)$, where F^δ is the Dedekind completion of F.*

In [3], Deckers studies this question for the simplest cases where $E=\mathbb{R}^n$ or $E=c_{00}$, the space of sequences which are eventually constant and he showed that in these cases, this conjecture is true. In the following part we will investigate some more difficult examples and discuss if this result remains true.

We let $1\leq p\leq\infty$ and let $E=\ell^p$ be the space of all p-summable real sequences with the componentwise ordering. Let $q\in[1,\infty]$ satisfy the usual relation $\frac{1}{p}+\frac{1}{q}=1$. Let F be any Archimedean Riesz space, which has a Riesz norm, i.e. a norm $\|\cdot\|$ which satisfies

$$\|f\|\leq\|g\| \text{ if } |f|\leq|g|.$$

For now, we assume that this norm is complete, i.e. we assume that F is a Banach lattice. We define the following space, which will turn out to be very important in the upcoming part,

$$\begin{gathered} X(F)=\{f:\mathbb{N}\to F|\sum_{n=1}^{\infty}\alpha_n f(n) \text{ converges for all } (\alpha_n)\in\ell^p \\ \text{and there exist } g\geq 0, h\geq 0 \text{ with } f=g-h \\ \text{such that } \sum_{n=1}^{\infty}\alpha_n g(n), \sum_{n=1}^{\infty}\alpha_n h(n) \text{ converge for all } (\alpha_n)\in\ell^p\}. \end{gathered}$$

Note that if $f\geq 0$ satisfies that $\sum_{n=1}^{\infty}\alpha_n f(n)$ converges for all $(\alpha_n)\in\ell^p$ that then $f\in X(F)$, since we can write $f=f-0$ and both sequences have the desired property. At first sight it is not clear what F-valued sequences are elements of $X(F)$. Luckily, by Hölder's inequality we at least get the following inclusion.

Lemma 2.3. *We have*

$$\{f : \mathbb{N} \to F | \sum_{n=1}^{\infty} \|f(n)\|^q < \infty\} \subset X(F)$$

if $p > 1$ *and*

$$\{f : \mathbb{N} \to F | f \textit{ is norm-bounded}\} \subset X(F)$$

if $p = 1$.

Proof If $f : \mathbb{N} \to F$ satisfies $\sum_{n=1}^{\infty} \|f(n)\|^q < \infty$ for $p > 1$ or if f is norm-bounded for $p = 1$, then that means that $\|(f(n))\|_q < \infty$, so then by Hölder's inequality we have for $(\alpha_n) \in \ell^p$ that

$$\sum_{n=1}^{\infty} \|\alpha_n f(n)\| \leq \|(\alpha_n)\|_p \|(f(n))\|_q < \infty.$$

So since F is Banach, we get that absolute convergence implies convergence. Since $|f^+(n)| \leq |f(n)|$, $|f^-(n)| \leq |f(n)|$ for all $n \in \mathbb{N}$ we get since $\|\cdot\|$ is a Riesz norm that

$$\sum_{n=1}^{\infty} \|f^+(n)\|^q \leq \sum_{n=1}^{\infty} \|f(n)\|^q < \infty$$

if $p > 1$ and

$$\sup_{n \in \mathbb{N}} \|f^+(n)\| \leq \sup_{n \in \mathbb{N}} \|f(n)\| < \infty$$

if $p = 1$ and a similar result holds for f^-. Hence we get, using the same argument as before, that $\sum_{n=1}^{\infty} \alpha_n f^+(n)$ and $\sum_{n=1}^{\infty} \alpha_n f^-(n)$ converge for all $(\alpha_n) \in \ell^p$. Since $f = f^+ - f^-$ we get that $f \in X(F)$. ∎

Note that if $p > 1$ then $X(F)$ need not be equal to the set of F-valued sequences of which the series of q-th power of norms converges. The simplest counterexample is to look at $F = \ell^p$, which is a Banach lattice, and look at $f(n) = e_n$ for $n \in \mathbb{N}$, where e_n is the sequence which has a 1 at the n-th coordinate and is 0 elsewhere. Then for all $(\alpha_n) \in \ell^p$ we see that

$$\sum_{n=1}^{\infty} \alpha_n f(n) = (\alpha_n), \quad \sum_{n=1}^{\infty} \alpha_n f^+(n) = (\alpha_n), \quad \sum_{n=1}^{\infty} \alpha_n f^-(n) = (0).$$

Hence $f \in X(F)$, but we clearly have

$$\sum_{n=1}^{\infty} \|f(n)\|^q = \sum_{n=1}^{\infty} 1 = \infty.$$

Now we get the following result, which shows the relation between $X(F)$ and $L^r(E, F)$, where we still assume that $E = \ell^p$ and that F is a Banach lattice.

Theorem 2.4. *The map*

$$j : X(F) \to L^r(E, F), f \mapsto ((\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n f(n))$$

is a bipositive isomorphism.

Proof First we show that j is well-defined. Let $f \in X(F)$ and $g, h \geq 0$ with $f = g - h$ such that for all $(\alpha_n) \in \ell^p$ we have that $\sum_{n=1}^{\infty} \alpha_n g(n)$ and $\sum_{n=1}^{\infty} \alpha_n h(n)$ converge. Clearly we see that $(\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n g(n)$ is well-defined and linear. Since $g \geq 0$ we get if $(\alpha_n) \geq 0$ that $\sum_{n=1}^{\infty} \alpha_n g(n) \geq 0$, since in Banach lattices the positive cone is closed. So $(\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n g(n)$ is a positive operator from E

to F. Similarly $(\alpha_n) \mapsto \sum_{n=1}^{\infty} \alpha_n h(n)$ is a positive operator from E to F. Since $f = g - h$ we see for $(\alpha_n) \in \ell^p$ that

$$\sum_{n=1}^{\infty} \alpha_n f(n) = \sum_{n=1}^{\infty} \alpha_n g(n) - \sum_{n=1}^{\infty} \alpha_n h(n)$$

So $j(f)$ is the difference of two positive operators and is therefore a regular operator and we see that $j(f) = j(g) - j(h)$. So j is well-defined.

The linearity of j is clear. We have already seen that if $f \geq 0$ that then $j(f) \geq 0$. Now suppose that for some $f \in X(F)$ we have that $j(f) \geq 0$. For $m \in \mathbb{N}$ we can look at $(\alpha_n) = e_m$. Since $e_m \geq 0$ and $j(f) \geq 0$ we get that $j(f)(e_m) = f(m) \geq 0$. So $f \geq 0$. Hence we see that j is bipositive. As remarked below Definition 1.16, we obtain that j is injective.

Finally we let $T : E \to F$ be a positive operator. Let for $i \in \mathbb{N}$ $x_i = T(e_i)$ and since $T \geq 0$ we get $x_i \geq 0$. Since positive operators on Banach lattices are continuous (see Proposition 1.33), we get for $(\alpha_n) \in \ell^p$ that

$$T((\alpha_n)) = T(\sum_{n=1}^{\infty} \alpha_n e_n) = \sum_{n=1}^{\infty} \alpha_n T(e_n) = \sum_{n=1}^{\infty} \alpha_n x_n.$$

Since $x_n \geq 0$ and $x_n = x_n - 0$ for all $n \in \mathbb{N}$, we see that $(x_n)_{n\in\mathbb{N}} \in X(F)$. Furthermore we see that $j((x_n)_{n\in\mathbb{N}}) = T$. Now if $T = T_1 - T_2$ with $T_1, T_2 \geq 0$, then as shown above we can find $f_1, f_2 \in X(F)$ with $T_1 = j(f_1)$ and $T_2 = j(f_2)$. So then by linearity we must have that $T = j(f_1 - f_2)$. So T is surjective.

Hence j is a bipositive isomorphism. ■

Clearly this representation of $L^r(E, F)$ as the sequence space $X(F)$ is much easier to grasp than the operator space $L^r(E, F)$. One application is immediately an important one: it turns out that for this choice of E and F, $L^r(E, F)$ is already a Riesz space!

Theorem 2.5. *The space $X(F)$ as described before is a Riesz space.*

Proof We use Proposition 1.6 to obtain that for $X(F)$ to be a Riesz space, it is sufficient to check that for each $f \in X(F)$ we have that $f^+ := f \vee 0 \in X(F)$ exists. Let $f \in X(F)$. Now if we can show that the F valued sequence $f^+ := (f(n)^+)_{n\in\mathbb{N}}$ is again an element of $X(F)$ then we are done, since this element clearly is the positive part of f since the ordering on $X(F)$ is pointwise.
Since $f \in X(F)$ we can let $g, h \in X(F)$ with $g \geq 0, h \geq 0$ and $f = g - h$. We see that $0 \leq f^+ \leq g$. Let $(\alpha_n) \in (\ell^p)^+$ then we see that for $k \leq m \in \mathbb{N}$ we have that

$$0 \leq \sum_{n=k}^{m} \alpha_n f(n)^+ \leq \sum_{n=k}^{m} \alpha_n g(n).$$

Now since the norm on F is a Riesz norm, we get that for $k \leq m \in \mathbb{N}$ we have that

$$\left\| \sum_{n=k}^{m} \alpha_n f(n)^+ \right\| \leq \left\| \sum_{n=k}^{m} \alpha_n g(n) \right\|.$$

Since the series $\sum\limits_{n=1}^{\infty} \alpha_n g(n)$ converges, we get that the righthandside of the previous equation converges to 0 as $k, m \to \infty$ and therefore the lefthandside must converge

to 0 as well. So $(\sum_{n=1}^{N} \alpha_n f(n)^+)_{N\in\mathbb{N}}$ is a Cauchy sequence in F. Since F is a Banach lattice, it is complete, so we get that $\sum_{n=1}^{\infty} \alpha_n f(n)^+$ converges.
Now if $(\alpha_n) \in \ell^p$, then we write $(\alpha_n) = (\alpha_n^+) - (\alpha_n^-)$. Since $(\alpha_n^+), (\alpha_n^-) \in (\ell^p)^+$, we get that $\sum_{n=1}^{\infty} \alpha_n^+ f(n)^+$ and $\sum_{n=1}^{\infty} \alpha_n^- f(n)^+$ converge. Hence we obtain that

$$\sum_{n=1}^{\infty} \alpha_n f(n)^+ = \sum_{n=1}^{\infty} \alpha_n^+ f(n)^+ - \sum_{n=1}^{\infty} \alpha_n^- f(n)^+$$

converges. So since $f^+ \geq 0$ we see that $f^+ \in X(F)$. So $X(F)$ is a Riesz space. ■

An immediate corollary is the following.

Corollary 2.6. *If $E = \ell^p$ and F is a Banach lattice, then $L^r(E, F)$ is a Riesz space.*

Proof Since $X(F)$ is a Riesz space and j is a bipositive isomorphism from $X(F)$ to $L^r(E, F)$ we get that $X(F)$ is Riesz if and only if $L^r(E, F)$ is Riesz. ■

Now we separate the cases where $p = 1$ and where $p > 1$. We start with $p = 1$ and obtain the following lemma:

Lemma 2.7. *Suppose that $p = 1$. Then we have that*

$$\{f : \mathbb{N} \to F | f \text{ is norm-bounded}\} = X(F).$$

Proof The inclusion $\{f : \mathbb{N} \to F | f \text{ is norm-bounded}\} \subset X(F)$ follows from Lemma 2.3. Now let $f \in X(F)$. Let $T = j(f)$. Since T is regular, it is continuous. For $j \in \mathbb{N}$ we see for $(\alpha_n) = e_j \in \ell^p$ that

$$T((\alpha_n)) = \sum_{n=1}^{\infty} \alpha_n f(n) = f(j).$$

Since T is continuous, it is bounded, so we obtain that

$$\|f(j)\| = \|T((\alpha_n))\| \leq \|T\| \|(\alpha_n)\| = \|T\|.$$

So we see that f is norm-bounded. So $X(F) \subset \{f : \mathbb{N} \to F | f \text{ is norm-bounded}\}$. Therefore we get $\{f : \mathbb{N} \to F | f \text{ is norm-bounded}\} = X(F)$. ■

Now we will return to our main question: is $L^r(E, F)$ order dense in $L^r(E, F^\delta)$? It is well-known that F^δ is again a Banach lattice and we let $\|\cdot\|$ denote its norm, which is an extension of the norm on F. It will turn out that in the case where $p = 1$, the answer to our question is indeed yes, which is the statement of the following theorem.

Theorem 2.8. *If $p = 1$, then $L^r(E, F)$ is order dense in $L^r(E, F^\delta)$.*

Proof Let $f \in X(F^\delta)$. We define

$$\rho : F^\delta \to [0, \infty), y \mapsto \inf\{\|x\| : x \in F, x \geq y\}.$$

By, for instance, [4, Theorem 3.43] we have that ρ defines a norm on F^δ and that $\rho(y) \geq \|y\|$ for all $y \in F^\delta$. Hence the identity map from $(F, \|\cdot\|)$ to (F, ρ) is bounded and therefore continuous. Since F^δ is complete with respect to $\|\cdot\|$ and $(F^\delta)^+$ is $\|\cdot\|$-closed, $(F^\delta)^+$ is complete with respect to $\|\cdot\|$ as well. Now with [4, Theorem 3.46]

we get that F^δ is complete with respect to ρ. So then by the Banach isomorphism theorem the inverse of the identity is bounded as well and thus we can conclude that $\|\cdot\|$ and ρ are equivalent norms. Hence we can let $C > 0$ such that $\rho(y) \leq C\|y\|$ for all $y \in F^\delta$.
Now for $n \in \mathbb{N}$ we get that $\rho(f(n)) \leq C\|f(n)\|$ and by definition of ρ this means that we can find $h_n \in F$ with $\|h_n\| \leq C\|f(n)\| + 1$ and $h_n \geq f(n)$. Since f is norm-bounded we let $M \geq 0$ such that $\|f(n)\| \leq M$ for all $n \in \mathbb{N}$. Then for $n \in \mathbb{N}$ we have that

$$\|h_n\| \leq C\|f(n)\| + 1 \leq CM + 1.$$

So $(h_n)_{n\in\mathbb{N}}$ is norm-bounded. So $(h_n)_{n\in\mathbb{N}} \in X(F)$ by Lemma 2.7. Furthermore we see that $f \leq (h_n)_{n\in\mathbb{N}}$. So $\{h \in X(F) : h \geq f\} \neq \emptyset$. Now by Theorem 2.1 we have that $X(F^\delta) \cong L^r(E, F^\delta)$ is Dedekind complete. Since $\{h \in X(F) : h \geq f\}$ is non-empty and bounded from below by f, its infimum exists. Suppose that $g \in X(F^\delta)$ satisfies $g \leq h$ for all $h \in X(F)$ with $h \geq f$. Suppose that $g \not\leq f$. Then we let $n \in \mathbb{N}$ with $g(n) \not\leq f(n)$. Since F is order dense in F^δ we can let $h(n) \in F$ such that $f(n) \leq h(n)$ and $g(n) \not\leq h(n)$, since $f(n) = \inf\{x \in F : x \geq f(n)\}$. Now for each $m \in \mathbb{N}$ with $m \neq n$ we saw that we can find $h(m) \in F$ with $h(m) \geq f(m)$ such that $h = (h(m))_{m\in\mathbb{N}}$ is norm-bounded and thus $h \in X(F)$. Then $h \geq f$, but $h \not\geq g$, since $h(n) \not\geq g(n)$. So that gives a contradiction. So $X(F)$ is order dense in $X(F^\delta)$ and therefore $L^r(E, F)$ is order dense in $L^r(E, F^\delta)$. ∎

Note that it took quite some effort to prove that for each $f \in X(F^\delta)$ there is an element in $X(F)$ majorizing f. We will see that precisely this is what might go wrong in other cases, see Counterexamples 2.9 and 2.10.

Now we look at the case where $p > 1$. Interestingly enough, it will turn out that in this case $L^r(E, F)$ is not necessarily order dense in $L^r(E, F^\delta)$ and we will show this using a few counterexamples.

Counterexample 2.9. In this counterexample we assume that $1 < p < \infty$. Consider $F = P_2[0, 1]$, the set of polynomials of degree at most 2 on the interval $[0, 1]$ with the pointwise ordering. For $n \in \mathbb{N}$ we let $f_n \in C[0, 1]$ be a piecewise linear function, with values between 0 and 1, which is 1 at three points, one of which is in the interval $(0, \frac{1}{3})$, one in $(\frac{1}{3}, \frac{2}{3})$ and one in $(\frac{2}{3}, 1)$ such that $f_n(\frac{1}{3}) = f_n(\frac{2}{3}) = 0$. We also need that if for $n \in \mathbb{N}$ and $x \in [0, 1]$ we have $f_n(x) > 0$ that then $f_m(x) = 0$ for all $m \in \mathbb{N}$ with $m \neq n$. Then for each $x \in [0, 1]$ we have that $f_n(x) \neq 0$ for at most one $n \in \mathbb{N}$ so we see that for each $(\alpha_n) \in \ell^p$ we have that $\sum_{n=1}^{\infty} \alpha_n f_n$ converges to a continuous function: the only points x where it possibly would not be continuous is where for each $\delta > 0$ we have that there are infinitely many n such that f_n is non-zero on $[x, x + \delta)$ or $(x - \delta, x]$. Without loss of generality we assume it is on $[x, x + \delta)$. In such a case we must clearly have that $f_n(x) = 0$ for all $n \in \mathbb{N}$, since otherwise we can find $\delta > 0$ such that $f_n > 0$ on $[x, x+\delta)$ and by the disjointness of the supports of the functions $(f_m)_m$ we get that only $f_n \neq 0$ on $[x, x+\delta)$. However, for each $\varepsilon > 0$ we can find $N \in \mathbb{N}$ with $|\alpha_n| < \varepsilon$ for all $n \geq N$, since $\lim_{n\to\infty} \alpha_n = 0$ and if we chose $\delta > 0$ small enough such that there are only f_n with $n \geq N$ non-zero on $[x, x + \delta)$, then clearly $|\sum_{n=1}^{\infty} \alpha_n f_n(y)| < \varepsilon$ for all $y \in [x, x + \delta)$. So then $\sum_{n=1}^{\infty} \alpha_n f_n$ is continuous in x and therefore it is continuous on $[0, 1]$. So since $(f_n) \geq 0$ we see that $(f_n) \in X(C[0, 1])$. Let (p_n) be a sequence in F such that $p_n \geq f_n$ for all $n \in \mathbb{N}$. Since non-constant polynomials of degree at most two either have precisely one local minimum on $[0, 1]$ or they have two local minima at 0 and at 1 and since for each $n \in \mathbb{N}$ we have that f_n equals 1 on three different points, we get that $p_n(\frac{1}{3}) \geq 1$ or $p_n(\frac{2}{3}) \geq 1$. Since $p > 1$ we have that $(\alpha_n) = (\frac{1}{n}) \in \ell^p$ and we have

$$\sum_{n=1}^{\infty} \frac{1}{n} p_n(\frac{1}{3}) + \sum_{n=1}^{\infty} \frac{1}{n} p_n(\frac{2}{3}) = \sum_{n=1}^{\infty} \frac{1}{n}(p_n(\frac{1}{3}) + p_n(\frac{2}{3})) \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

So we see that $\sum\limits_{n=1}^{\infty} \frac{1}{n} p_n$ does not converge, since it is infinite at $x = \frac{1}{3}$ or at $x = \frac{2}{3}$. Hence there is no sequence $(p_n) \in X(F)$ with $(p_n) \geq (f_n)$. So in particular we have that

$$(f_n) \neq \inf\{(p_n) \in X(F) : (p_n) \geq (f_n)\},$$

since the righthandside does not exist, since the infimum of the empty set does not exist. So $X(F)$ is not order dense in $X(C[0,1])$. Since F is order dense in $C[0,1]$, we see that $C[0,1] \subset F^\delta$ and thus $X(F)$ is in particular not order dense in $X(F^\delta)$. Therefore $L^r(E,F)$ is not order dense in $L^r(E,F^\delta)$ by Theorem 2.4. ■

Now since $P_2[0,1]$ is not a Riesz space, it is not too surprising that our conjecture is not true in this case. However, we can find another counterexample in which F is indeed a Banach lattice:

Counterexample 2.10. Let $1 < p \leq \infty$. Consider $F = C[0,1]$, which is a Banach lattice. We have already seen that F is not Dedekind complete, and it is well-known that F^δ contains all piecewise continuous bounded functions on $[0,1]$. For each $n \in \mathbb{N}$ let $f_n \in F^\delta$ be continuous and strictly increasing on $[0, \frac{1}{2})$ such that $f_n(0) = 0$ and $\lim\limits_{x \uparrow \frac{1}{2}} f_n(x) = 1$, such that $f_n(x) = 0$ for $x \geq \frac{1}{2}$ and such that for each $x \in [0, \frac{1}{2})$ we have that $(f_n(x))_{n \in \mathbb{N}}$ is q-summable with $\|f_n(x)\|_q \leq 2$. Then by Hölder's inequality we have for all $x \in [0, \frac{1}{2})$ and all $(\alpha_n) \in \ell^p$ that

$$\sum_{n=1}^{\infty} |\alpha_n||f_n(x)| \leq \|(\alpha_n)\|_p \|(f_n(x))_{n \in \mathbb{N}}\|_q \leq 2\|(\alpha_n)\|_p.$$

So $\sum\limits_{n=1}^{\infty} \alpha_n f_n$ converges pointwise (and even uniformly) and the limit is a bounded piecewise continuous function and therefore an element of F^δ. Since $f \geq 0$ we get that $f \in X(F^\delta)$. Now for each $n \in \mathbb{N}$ and each $g \in F$ with $g \geq f_n$ we must have that $g(\frac{1}{2}) \geq 1$, since g is continuous and $\lim\limits_{x \uparrow \frac{1}{2}} f_n(x) = 1$. Hence if $g = (g_n)_{n \in \mathbb{N}}$ is a majorizing sequence in F of f, then by taking $(\alpha_n) = (\frac{1}{n}) \in \ell^p$ we get that

$$\sum_{n=1}^{\infty} \alpha_n g_n(\frac{1}{2}) \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

So there is no sequence $g \in X(F)$ with $g \geq f$. So just as in the previous counterexample we get that $X(F)$ is not order-dense in $X(F^\delta)$. Hence by Theorem 2.4 we obtain that $L^r(E,F)$ is not order-dense in $L^r(E,F^\delta)$. ■

Now we have shown that in general $L^r(E,F)$ need not be an order dense subspace of $L^r(E,F^\delta)$ and thus that we cannot say anything about the Riesz completion of $L^r(E,F)$ in terms of $L^r(E,F^\delta)$. **So Conjecture 2.2 is not true in general**.

Now one might wonder what happens if F is a normed space with Riesz norm, which is not norm-complete. It might be possible that $L^r(E,F)$ is related to $L^r(E,\overline{F})$, where $\overline{F}$ is the norm completion of F, which is known to be a Banach lattice. Therefore we can formulate our second conjecture.

Conjecture 2.11. *If F is a normed Riesz space and $\overline{F}$ is its norm completion, then $L^r(E,F)$ is order dense in $L^r(E,\overline{F})$.*

In the remainder of the chapter we again let $1 \leq p \leq \infty$ and let $E = \ell^p$. Let F be a normed Riesz space, which is not necessarily norm-complete. We again want to establish a bipositive isomorphism from $X(F)$, which we can also define for this F, and $L^r(E,F)$. If we look at the proof of Theorem 2.4, then we see that the only time we used that F is a Banach lattice is to show that F^+ is norm-closed. Now let (x_n) be a sequence in F^+ which converges to some $x \in F$. Then (x_n) is also a sequence in $\overline{F}^+$ which converges to x. Since $\overline{F}$ is a Banach lattice, we get $x \in \overline{F}^+$. Since $x \in F$ we get $x \in F^+$, so also in this case we have that F^+ is closed. Thus we again have that there is a bipositive isomorphism from $X(F)$ to $L^r(E,F)$. Note that we cannot lift Lemma 2.3 to this case, since if F is not norm-complete, then there are always sequences which converge absolutely, but do not converge, so we cannot conclude that the q-summable F-valued sequences are contained in $X(F)$ any more. However, we do get the following result.

Theorem 2.12. *If F is a normed Riesz space, then $L^r(E,F)$ is a Riesz space.*

Proof Let $f \in X(F)$. Then $f \in X(\overline{F})$, since $X(F) \subset X(\overline{F})$. Hence we get since $X(\overline{F})$ is Riesz that $f^+ \in X(\overline{F})$, so for all $(\alpha_n) \in \ell^p$ we get that $\sum\limits_{n=1}^{\infty} \alpha_n f(n)^+$ converges in $\overline{F}$. Since lattice operations are continuous we see that for $(\alpha_n) \in (\ell^p)^+$ we have that

$$\sum_{n=1}^{\infty} \alpha_n f(n)^+ = \left(\sum_{n=1}^{\infty} \alpha_n f(n)\right)^+$$

and that for $(\alpha_n) \in -(\ell^p)^+$ we have that

$$\sum_{n=1}^{\infty} \alpha_n f(n)^+ = -\left(\sum_{n=1}^{\infty} |\alpha_n| f(n)\right)^+ .$$

Hence for $(\alpha_n) \in \ell^p$ we have that

$$\sum_{n=1}^{\infty} \alpha_n f(n)^+ = \left(\sum_{n=1}^{\infty} \alpha_n^+ f(n)\right)^+ - \left(\sum_{n=1}^{\infty} \alpha_n^- f(n)\right)^+ .$$

Since $(\alpha_n^+) \in \ell^p$ and $(\alpha_n^-) \in \ell^p$ and since $f \in X(F)$ we get that $\sum\limits_{n=1}^{\infty} \alpha_n^+ f(n)$ and $\sum\limits_{n=1}^{\infty} \alpha_n^- f(n)$ converge in F. Since F is Riesz, their positive parts are also in F and thus we get that $\sum_{n=1}^{\infty} \alpha_n f(n)^+ \in F$. Hence $f^+ \in X(F)$. So $X(F)$ is a Riesz space and therefore $L^r(E,F)$ is a Riesz space as well. ∎

Unfortunately we can still find a counterexample for our conjecture, which is given by the following example.

Counterexample 2.13. Let $F = C[0,1]$, but now we give it the 1-norm instead of the supremum norm, i.e. for $f \in F$ we let $\|f\|_1 = \int |f| d\lambda$ where λ is the Lebesgue measure on $[0,1]$. Then it is well-known that F is norm dense in the Banach lattice $\overline{F} = L_1[0,1]$, which has the ordering that $f \leq g$ if and only if $f(x) \leq g(x)$ for almost every $x \in [0,1]$. Let $f \in \overline{F}$ be any unbounded function. Since $[0,1]$ is compact, we get that every $g \in F$ is bounded, so there is no $g \in F$ with $g \geq f$. Now if we let (f_n) be a sequence in $\overline{F}$ given by $f_1 = f, f_n = 0$ for $n \geq 2$, then for each $(\alpha_n) \in \ell^p$ we get that

$$\sum_{n=1}^{\infty} \alpha_n f_n = \alpha_1 f, \quad \sum_{n=1}^{\infty} \alpha_n f_n^+ = \alpha_1 f^+, \quad \sum_{n=1}^{\infty} \alpha_n f_n^- = \alpha_1 f^-.$$

So $(f_n) \in X(\overline{F})$. Since there are no $g \in F$ with $g \geq f$ there are no $h \in X(F)$ with $h \geq (f_n)$. So using the same argument as before, $X(F)$ is not order dense in $X(\overline{F})$. ∎

So we conclude that Conjecture 2.11 is also not true in general.

3. The Riesz-Kantorovich formula

In this chapter we will look at the main subject of this thesis: the Riesz-Kantorovich formula. It is stated as follows (see [10, Theorem 1.3.2]).

Theorem 3.1. ***(The Riesz-Kantorovich formula)*** *Let E, F be Riesz spaces. Assume that F is Dedekind complete. Then $L^r(E, F)$ is a Riesz space and for $S, T \in L^r(E, F)$ and $x \in E_+$ we have the so-called* ***Riesz-Kantorovich formula***

$$(S \vee T)(x) = \sup\{S(y) + T(z) : 0 \leq y \leq x, 0 \leq z \leq x, y + z = x\}.$$

Our main object of interest is the following long-standing open problem.

If for some partially ordered vector spaces E, F and regular operators $S, T : E \to F$ the supremum $S \vee T$ exists in the space $L^r(E, F)$, does it necessarily satisfy the Riesz-Kantorovich formula?

As before, it is easier to only consider $T \in L^r(E, F)$ for which T^+ exists. For $x \in E_+$ the Riesz-Kantorovich formula then states that

$$T^+(x) = \sup\{T(y) : 0 \leq y \leq x\}.$$

Similar to the above, if we have Riesz-Kantorovich for all positive parts of operators for which the positive part exists, we have it for all pairs of operators for which the supremum exists.

First we will give a proof of Theorem 3.1, but we will formulate it slightly differently, so that it might give an insight how to prove the general Riesz-Kantorovich formula for regular operators between two Riesz spaces.

Theorem 3.2. *Let E, F be Riesz spaces. Let $T \in L^r(E, F)$. Then T has a positive part in $L^r(E, F)$ if for every $x \in E_+$ it holds that $\sup\{T(u) : 0 \leq u \leq x\}$ exists in F, which, in particular, is always the case if F is Dedekind complete, and furthermore the Riesz-Kantorovich formula holds.*

Of course the existence of $\sup\{T(u) : 0 \leq u \leq x\}$ for $x \in E_+$ is necessary for the Riesz-Kantorovich formula to hold. If one can prove the reverse statement of this theorem, namely that if the positive part of some operator $T \in L^r(E, F)$ exists, that then $\sup\{T(u) : 0 \leq u \leq x\}$ exists for $x \in E_+$, then the Riesz-Kantorovich formula always holds in $L^r(E, F)$. We will now prove the theorem.

Proof Let $T \in L^r(E, F)$ such that $\sup\{T(u) : 0 \leq u \leq x\}$ exists for all $x \in E_+$. Define for $x \in E^+$ $S(x) = \sup\{T(u) : 0 \leq u \leq x\}$. We will first show linearity of S. Let $x, y \in E_+$ and $\lambda > 0$. Then we have that

$$S(\lambda x) = \sup\{T(u) : 0 \leq u \leq \lambda x\} = \sup\{T(u) : 0 \leq \frac{u}{\lambda} \leq x\}$$
$$= \sup\{T(\lambda v) : 0 \leq v \leq x\} = \lambda \sup\{T(v) : 0 \leq v \leq x\} = \lambda S(x),$$

where we used the linearity of T.
Let $0 \leq u \leq x + y$. Since E is a Riesz space, it has the Riesz decomposition property, so we can let $u_1 \geq 0, u_2 \geq 0$ with $u_1 + u_2 = u$ and $u_1 \leq x, u_2 \leq y$. Then we get that

$$T(u) = T(u_1) + T(u_2) \leq S(x) + S(y).$$

Hence we also obtain that

$$S(x + y) \leq S(x) + S(y).$$

We also have if $0 \leq u_1 \leq x$ and $0 \leq u_2 \leq y$ then $0 \leq u_1 + u_2 \leq x + y$ and thus that

$$S(x + y) \geq T(u_1 + u_2) = T(u_1) + T(u_2).$$

Therefore we must also have that

$$S(x+y) \geq S(x) + T(u_2).$$

and thus we obtain that

$$S(x+y) \geq S(x) + S(y).$$

Hence it holds that $S(x+y) = S(x) + S(y)$. Finally we also have that

$$S(0) = \sup\{T(u) : 0 \leq u \leq 0\} = T(0) = 0.$$

So S is linear on E_+. Furthermore we see if $x \geq 0$ that then

$$S(x) = \sup\{T(u) : 0 \leq u \leq x\} \geq T(0) = 0,$$

since $0 \leq 0 \leq x$.

Now we extend S to E by defining $S(e) = S(e^+) - S(e^-)$ for $e \in E$. Then if $e \geq 0$ we have $e^- = 0$ and $e^+ = e$ so indeed this definition gives an extension of S. Now we will show linearity of S. Let $e, x, y \in E$ and $\lambda \in \mathbb{R}$. Then we have if $\lambda \geq 0$ that

$$\begin{aligned} S(\lambda e) &= S((\lambda e)^+) - S((\lambda e)^-) = S(\lambda e^+) - S(\lambda e^-) \\ &= \lambda S(e^+) - \lambda S(e^-) = \lambda S(e) \end{aligned}$$

and if $\lambda < 0$ that

$$\begin{aligned} S(\lambda e) &= S((\lambda e)^+) - S((\lambda e)^-) = S(|\lambda| e^-) - S(|\lambda| e^+) \\ &= |\lambda| S(e^-) - |\lambda| S(e^+) = |\lambda|(-S(e)) = \lambda S(e). \end{aligned}$$

Now if $z \geq 0$ then we have that

$$S(e^+ + z) - S(e^- + z) = S(e^+) + S(z) - S(e^-) - S(z) = S(e).$$

So if we can write $e = f - g$ with $f \geq 0$ and $g \geq 0$ then we can let $z \geq 0$ with $f = e^+ + z$ and $g = e^- + z$ and then it holds that

$$S(f) - S(g) = S(e^+ + z) - S(e^- + z) = S(e).$$

Hence we obtain since $x + y = x^+ + y^+ - (x^- + y^-)$ and $x^+ + y^+ \geq 0, x^- + y^- \geq 0$ that

$$S(x+y) = S(x^+ + y^+) - S(x^- + y^-) = S(x^+) + S(y^+) - S(x^-) - S(y^-) = S(x) + S(y).$$

Hence we see that S is linear. Furthermore we saw that $S(e) \geq 0$ for $e \in E_+$ so we must have that $S \geq 0$. In particular $S \in L^r(E, F)$. Furthermore for $e \in E_+$ we have that

$$S(e) = \sup\{T(u) : 0 \leq u \leq x\} \geq T(x).$$

Hence we also obtain $S \geq T$.

Now if $R \in L^r(E, F)$ also satisfies $R \geq 0$ and $R \geq T$, then for $e \in E_+$ and $0 \leq u \leq x$ we must have that

$$R(x) \geq R(u) \geq T(u),$$

where the first inequality follows since $R \geq 0$ and the second since $R \geq T$. Hence we must have that $R(x) \geq S(x)$ and thus $R \geq S$.
Thus we obtain that indeed $S = \sup\{0, T\}$ and thus $S = T^+$. So T^+ exists in $L^r(E, F)$ and satisfies the Riesz-Kantorovich formula. ■

Remark 3.3. The condition in the previous theorem that E should be a Riesz space can be slightly relaxed to only requiring that E is directed and has the Riesz decomposition property.

Our first new result is immediately linked to the contents of Chapter 2.

Theorem 3.4. *Let E, F be Riesz spaces. We do not assume F to be Dedekind complete and let F^δ be its Dedekind completion. If $L^r(E,F)$ is order-dense in $L^r(E,F^\delta)$ and for $S,T \in L^r(E,F)$ the supremum $S \vee T$ exists in $L^r(E,F)$, then it satisfies the Riesz-Kantorovich formula.*

Proof We only consider $T \in L^r(E,F)$ for which T^+ exists in $L^r(E,F)$. First we will show that T^+ is also the positive part of T in the Riesz space $L^r(E,F^\delta)$. Let S be the positive part of T in $L^r(E,F^\delta)$. Then $S \leq T^+$ since T^+ still satisfies $T^+ \geq T$ and $T^+ \geq 0$. Let R be the negative part of T in $L^r(E,F^\delta)$ and T^- the negative part of T in $L^r(E,F)$ (which exists since it equals $T^+ - T$). We let $U = T^+ - S$. Then $U \geq 0$. Then we have that

$$T = T^+ - T^- = S - R = T^+ - U - R.$$

Hence we obtain that

$$T^- = U + R.$$

Let $V \in L^r(E,F)$ such that $V \geq T^+ + T^-, V \geq -T^+ - T^-$. Then it also holds that

$$V \geq T^+ + T^- \geq T^+ \geq T^+ - T^-$$

and

$$V \geq T^+ + T^- \geq -T^+ + T^-.$$

So $\{T^+ + T^-, -T^+ - T^-\}^u \subset \{T^+ - T^-, -T^+ - T^-\}^u$.
Now let $W \in L^r(E,F)$ such that $W \geq T^+ - T^- = T, W \geq -T^+ + T^- = -T$. Then it also holds that

$$W \geq T \vee (-T) = |T| = T^+ + T^- \geq -T^+ - T^-.$$

So we obtain $\{T^+ + T^-, -T^+ - T^-\}^u \supset \{T^+ - T^-, -T^+ - T^-\}^u$ and thus that

$$\{T^+ + T^-, -T^+ - T^-\}^u = \{T^+ - T^-, -T^+ - T^-\}^u.$$

So $T^+ \perp T^-$ in $L^r(E,F)$ by definition. By Proposition 1.27 we get that $T^+ \perp T^-$ in $L^r(E,F^\delta)$ since $L^r(E,F)$ is order-dense in $L^r(E,F^\delta)$ by assumption. Now in $L^r(E,F^\delta)$ we have that

$$|T^+ - T^-| = |S + U - (U + R)| = |S - R| = |T|$$

and

$$|T^+ + T^-| = |S + U + R + U| = ||T| + 2U| = |T| + 2U,$$

since $|T| + 2U \geq 0$. However, since T^+ and T^- are disjoint in the Riesz space $L^r(E,F^\delta)$, we saw that this means that $|T^+| \wedge |T^-| = 0$ and thus $|T^+ - T^-| = |T^+ + T^-|$. Hence we must have that

$$|T| = |T| + 2U.$$

which implies $U = 0$. Hence $S = T^+$ and thus T^+ is the positive part of T in $L^r(E,F^\delta)$.

Now let $x \in E_+$. Since the Riesz-Kantorovich formula holds in $L^r(E,F^\delta)$ we see that

$$T^+(x) = \sup\{T(u) : 0 \leq u \leq x\},$$

where the supremum is taken in F^δ. Hence $T^+(x) \geq T(u)$ for all $u \in E$ with $0 \leq u \leq x$ and if $y \in F$ satisfies $y \geq T(u)$ for all $0 \leq u \leq x$ then in F^δ we should have that $y \geq T^+(x)$, but since $y, T^+(x) \in F$ it follows that $y \geq T^+(x)$ in F. Hence $T^+(x) = \sup\{T(u) : 0 \leq u \leq x\}$ in F. ∎

Corollary 3.5. *If $E = \ell^1$ and F any Banach lattice, then for all $S, T \in L^r(E, F)$ for which $S \vee T$ exists, the Riesz-Kantorovich formula holds.*

Proof This follows immediately from Theorem 3.4 and Theorem 2.8. ∎

To show the non-triviality of taking suprema in the space of regular operators, we consider the following example.

Example 3.6. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is called **Lipschitz continuous** if there exists a constant $L \geq 0$ such that for all $x, y \in \mathbb{R}$ we have $|f(x) - f(y)| \leq L|x - y|$. The smallest such constant is called the Lipschitz constant of f and is denoted by $\mathrm{Lip}(f)$. Then one can show that the space of all Lipschitz continuous functions is a Riesz space under the order it inherits from $C(\mathbb{R})$. However, for this example we consider a new ordering on the space $E = \{f \in C([0, 4]) : f \text{ is Lipschitz continuous}\}$, namely we say that $g \succeq 0$ if and only if $g \geq 0$ and $\mathrm{Lip}(g) \leq g(\frac{3}{2})$. Then E becomes a partially ordered vector space and we will show it is directed. Let $g \in E$. Let $c \in \mathbb{R}$ such that $c - g(\frac{3}{2}) \geq \mathrm{Lip}(g)$ and $c \geq 0$. Let $f = c$. Then f is Lipschitz continuous and $\mathrm{Lip}(f) = 0 \leq c$. So $f \succeq 0$. Furthermore since f is constant we get $\mathrm{Lip}(f - g) = \mathrm{Lip}(-g) = \mathrm{Lip}(g) \leq c - g(\frac{3}{2}) = (f - g)(\frac{3}{2})$. So $f - g \succeq 0$ and thus $f \succeq g$. Hence E is directed.
We let $F = C([0, 4])$. For $h \in C([0, 4])$ we consider the following map

$$T_h : E \to F, g \mapsto (x \mapsto \int_0^x h(t)g(t)dt).$$

It is clear that $T_h = T_{h^+} - T_{h^-}$ and that $T_{h^+} \geq 0, T_{h^-} \geq 0$. In particular $T_h \in L^r(E, F)$. We define the following function

$$f : x \mapsto \begin{cases} 1 \text{ if } x \in [0, 1] \\ -2x + 3 \text{ if } 1 \leq x \leq 2 \\ -1 \text{ if } 2 \leq x \leq 4. \end{cases}$$

Then indeed $f \in C[0, 4]$. It would make sense that $T_{f^+} = (T_f)^+$ but we will show that that cannot be the case assuming that the Riesz-Kantorovich formula holds. We let $g = 1$. Then g is constant so $\mathrm{Lip}(g) = 0 \leq 1 = g(\frac{3}{2})$, so $g \succeq 0$. Let $0 \preceq u \preceq g$. Since g is constant we have $\mathrm{Lip}(g - u) = \mathrm{Lip}(-u) = \mathrm{Lip}(u)$. Since also $0 \leq u \leq g$ we get $0 \leq u(\frac{3}{2}) \leq 1$. Suppose $1 \geq u(\frac{3}{2}) \geq \frac{1}{2}$. Then since $u \preceq g$ we get that

$$\mathrm{Lip}(u) = \mathrm{Lip}(g - u) \leq (g - u)(\frac{3}{2}) = 1 - u(\frac{3}{2}) \leq \frac{1}{2}.$$

Suppose that $u(\frac{3}{2}) \leq \frac{1}{2}$. Then since $0 \preceq u$ we get $\mathrm{Lip}(u) \leq \frac{1}{2}$. So in each case we have $\mathrm{Lip}(u) \leq \frac{1}{2}$. Now some estimations give that we must have that

$$T_f u(2) \leq \frac{3}{2}$$

for every u with $0 \preceq u \preceq g$ and that the maximum of $T_f u(2)$ over such functions u is attained for a piecewise linear function u satisfying $u(\frac{3}{2}) = \frac{3}{4}$, which is as large as allowed on $[0, \frac{3}{2}]$ and as small as allowed on $[\frac{3}{2}, 4]$. Since $T_f u$ is decreasing on $[\frac{3}{2}, 4]$ since $u \geq 0$ we get that $T_f u(2 + \varepsilon) \leq T_f u(2) \leq \frac{13}{12}$ for all u with $0 \preceq u \preceq g$ and all $0 \leq \varepsilon \leq 2$. So if $(T_f)^+$ would exist and would satisfy the Riesz-Kantorovich formula then we should have, since $(T_f)^+ g$ is continuous, that

$$(T_f)^+ g(2 + \varepsilon) \leq \frac{13}{12}$$

for all $0 \leq \varepsilon \leq 2$. However, it holds that

$$(T_{f^+})g(2+\varepsilon) = \int_0^1 1dx + \int_1^{\frac{3}{2}} (-2x+3)dx = \frac{5}{4} > \frac{13}{12}$$

for all $0 \leq \varepsilon \leq 2$ and that we have equality for $\varepsilon = 0$. So T_{f^+} cannot be equal to $(T_f)^+$ if the latter exists and satisfies the Riesz-Kantorovich formula. However, it is unknown if this is a counter-example for the Riesz-Kantorovich formula or that $(T_f)^+$ does not exist or that it exists, but is simply not equal to T_{f^+}. In any case, it may be interesting to investigate this example in any further research.

4. Main theorem

To start we will formulate the Riesz-Kantorovich formula for the supremum of finitely many regular operators:
If $S_1, ..., S_n \in L^r(E,F)$ and $\bigvee_{i=1}^n S_i$ exists in $L^r(E,F)$ and $x \in E_+$ then the Riesz-Kantorovich formula is given by

$$\bigvee_{i=1}^n S_i(x) = \sup\{\sum_{i=1}^n S(x_i) : 0 \leq x_i \text{ for all } i \in \{1, ..., n\} \text{ and } \sum_{i=1}^n x_i = x\}$$

In this chapter we will discuss our main theorem. In their article [1], Aliprantis et al. show the following theorem.

Theorem 4.1. *Let L be an ordered vector space with an order unit and a Hausdorff linear topology such that all order intervals are compact. If for some linear continuous functionals $f_1, ..., f_m \in L^\sim(L, \mathbb{R})$ the supremum $g = f_1 \vee ... \vee f_m$ exists in $L^\sim(L, \mathbb{R})$ then g satisfies the Riesz-Kantorovich formula.*

In other words, they show the Riesz-Kantorovich formula for linear functionals, where the domain space has a strong order unit and where we assume some continuity properties. Of course $\mathbb{R}$ is a Dedekind complete Riesz space, so in that sense it is a 'good space'. However, the domain space need not be a Riesz space; it does not even need to have the Riesz decomposition property as can be seen from the following example, which is based on [1, Example 2.3].

Example 4.2. Consider the vector space $\mathbb{R}^3$ with the following 'ice cream' cone:

$$\begin{aligned} C &= \{(x,y,z) \in \mathbb{R}^3 : z \geq 0, z^2 \geq 4(x^2+y^2)\} \\ &= \{\lambda(x,y,2) : \lambda \geq 0, x^2+y^2 \leq 1\}. \end{aligned}$$

Since C is a convex cone, it induces a partial order on $\mathbb{R}^3$, which we will denote by $\leq_C$, by $x \leq_C y$ if and only if $y - x \in C$. We equip the space with the Euclidean topology. Then we obtain the following properties:

(1) C is a closed cone.
(2) The space $(\mathbb{R}^3, \leq_C)$ has a strong order unit; for example $e = (0,0,2)$ is a strong order unit. This can be seen as follows: let $u = (x,y,z) \in \mathbb{R}^3$. Choose $\alpha > 0$ large enough such that $\lambda = \frac{-z+2\alpha}{2}$ satisfies $\lambda > \sqrt{x^2+y^2}$. Then we see that

$$-u + \alpha e = (-x,-y,-z) + \alpha(0,0,2) = \lambda\left(\frac{-x}{\lambda}, \frac{-y}{\lambda}, 2\right) \in C$$

and thus $u \leq_C \alpha e$.
(3) The order intervals of $(\mathbb{R}^3, \leq_C)$ are closed bounded subsets of $\mathbb{R}^3$ and thus by Heine-Borel theorem they are compact.
(4) $(\mathbb{R}^3, \leq_C)$ does not have the Riesz decomposition property and is thus in particular is not a Riesz space. To show this we first show the following: if $u = \lambda(x,y,2) \in C$ and $x^2+y^2 = 1$ then $v \in \mathbb{R}^3$ satisfies $0 \leq_C u \leq_C v$ if and only if there exists $0 \leq \mu \leq 1$ such that $v = \mu u$. The 'if' part is clear, so we only show the 'only if' part. So assume we have $0 \leq_C \lambda(a,b,2) \leq_C (x,y,2)$ with $x^2+y^2 = 1$. Then $0 \leq \lambda \leq 1$ and if either $\lambda = 0$ or $\lambda = 1$ then the result follows immediately. Hence we assume $0 < \lambda < 1$. Since $(x,y,2) - \lambda(a,b,2) \in C$ we get that there exists (α, β) with $\alpha^2 + \beta^2 \leq 1$ and some $\mu \geq 0$ such that $(x,y,2) - \lambda(a,b,2) = \mu(\alpha,\beta,2)$. Hence we obtain that $(x,y) = \lambda(a,b) + \mu(\alpha,\beta)$. Since $x^2+y^2 = 1$, the point (x,y) is an extreme point of the unit disk in $\mathbb{R}^2$ and thus we must have that $(x,y) = (a,b)(= (\alpha,\beta))$. Hence $\lambda(x,y,2) = \lambda(a,b,2)$.

Now we show that $(\mathbb{R}^3, \leq_C)$ does not have the Riesz decomposition property by considering the vectors $(1,0,2),(-1,0,2),(0,1,2) \in C$. We begin by observing that

$$0 \leq_C (0,1,2) \leq_C (0,0,4) = (1,0,2) + (-1,0,2).$$

If $(\mathbb{R}^3, \leq_C)$ would have the Riesz decomposition property then we would be able to find $u, v \in \mathbb{R}^3$ with $0 \leq_C u \leq_C (1,0,2)$, $0 \leq_C v \leq_C (-1,0,2)$ and $u + v = (0,1,2)$. Now by what we showed we have $0 \leq \lambda, \mu \leq 1$ with $u = \lambda(1,0,2)$ and $v = \mu(-1,0,2)$. However, then we would get that

$$(0,1,2) = u + v = \lambda(1,0,2) + \mu(-1,0,2) = (\lambda + \mu, 0, 2(\lambda + \mu)),$$

which is impossible. Hence $(\mathbb{R}^3, C)$ indeed does not have the Riesz decomposition property and is thus also not a Riesz space.

In particular this means that $(\mathbb{R}^3, C)$ is a space we can apply Theorem 4.1 and our upcoming main theorem to, but not the standard results on the Riesz-Kantorovich formula.

Before we go on we introduce some notation.

Notation 4.3. *Let L be an ordered vector space. We use the short-hand notation $L^\sim := L^\sim(L, \mathbb{R})$. For any integer $m > 0$ and $x \in L_+$ we define the following non-empty convex sets*

$$\mathcal{A}_x^m = \{(y_1, ..., y_m) \in L_+^m : \sum_{i=1}^m y_i \leq x\}$$

and

$$\mathcal{F}_x^m = \{(y_1, ..., y_m) \in L_+^m : \sum_{i=1}^m y_i = x\}.$$

If $x \in \prod_{j \in J} \mathbb{R}$ for some index set J then we write $x^{(j)}$ for the j-th component of x and similarly if $f : L \to \prod_{j \in J} \mathbb{R}$ is any map then we write $f^{(j)}$ for the map $f^{(j)} : L \to \mathbb{R}, x \mapsto (f(x))^{(j)}$. Similarly, if α is an ordinal, then for $x \in \prod_{1 \leq \beta \leq \alpha} \mathbb{R}$ and $\gamma < \alpha$ we mean by $x^{(1),...,(\gamma)}$ the vector in $\prod_{1 \leq \beta \leq \gamma} \mathbb{R}$ which corresponds to the first γ coordinates of x and if $f : L \to \prod_{1 \leq \beta \leq \alpha} \mathbb{R}$ is any map and $\gamma < \alpha$ then we mean by $f^{(1),...,(\gamma)}$ the map $f^{(1),...,(\gamma)} : L \to \prod_{1 \leq \beta \leq \gamma} \mathbb{R}, x \mapsto (f(x))^{(1),...,(\gamma)}$.

We have to prove one more lemma before we can state our main theorem.

Lemma 4.4. *Let L be a directed partially ordered vector space. Let $\alpha \geq 1$ be an ordinal number and let $M = \prod_{1 \leq\ j \leq \alpha} \mathbb{R}$ with the lexicographic ordering. Let $f_1, ..., f_m \in L^\sim(L, M)$ such that $f_i^{(j)}$ is regular for all $i \in \{1, ..., m\}$ and all $1 \leq j \leq \alpha$ and such that $g = f_1 \vee ... \vee f_m$ exists in $L^\sim(L, M)$. Let $\beta < \alpha$. Then $f_1^{(1),...,(\beta)} \vee ... \vee f_m^{(1),...,(\beta)}$ exists and equals $g^{(1),...,(\beta)}$.*

Proof Suppose this is not the case. Then we can let $h : L \to \prod_{1 \leq j \leq \beta} \mathbb{R}$ be order-bounded and $x \in L_+$ such that $h \geq f_i^{(1),...,(\beta)}$ for all $1 \leq i \leq m$ and $h(x) < g^{(1),...,(\beta)}(x)$. By regularity we can write

$$f_i^{(j)} = g_i^{(j)} - h_i^{(j)}$$

where $g_i^{(j)} \geq 0, h_i^{(j)} \geq 0$ for $1 \leq i \leq m$ and $1 \leq j \leq \alpha$. We define

$$h' : L \to M, y \mapsto (h^{(1)}(y), ..., h^{(\beta)}(y), \sum_{i=1}^m g_i^{(\beta+1)}(y), ..., \sum_{i=1}^m g_i^{(\alpha)}(y)).$$

Then clearly $h' \in L^{\sim}(L,M)$ and we see that $h' \geq f_i$ for all $i \in \{1,...,m\}$ since we use the lexicographic ordering on M and $h \geq f_i^{(1),...,(\beta)}$ for all $1 \leq i \leq m$. Furthermore we see that

$$h'(x) = (h^{(1)}(x),...,h^{(\beta)}(x), \sum_{i=1}^{m} g_i^{(\beta+1)}(x),..., \sum_{i=1}^{m} g_i^{(\alpha)}(x)) < g(x)$$

since $h(x) < g^{(1),...,(\beta)}(x)$ and we use the lexicographic ordering on M. Hence $h' \not\geq g$ and that gives a contradiction with the definition of g. ∎

Now we are ready to state our main theorem, which is an improved version of Theorem 4.1. Since we can imagine that not every reader appreciates ordinal numbers, we present this theorem in two versions: the first version is for the spaces $\mathbb{R}^n$ and uses only 'ordinary' induction, while the second one is for arbitrary products of copies of $\mathbb{R}$ and uses transfinite induction. Other than that, the differences in the proofs are very minor.

Theorem 4.5. ***(Main theorem)*** *Let L be an ordered vector space with an order unit e and a Hausdorff linear topology such that all order intervals are compact. Let $M = \mathbb{R}^n$ with the lexicographic ordering. Assume we have $f_1, f_2, ..., f_m$ continuous, linear maps from L to M such that $f_i^{(j)}$ is regular for all $i \in \{1,...,m\}$ and all $j \in \{1,...,n\}$ and such that the supremum $g = \vee_{i=1}^m f_i$ exists in $L^{\sim}(L,M)$. Then g satisfies the Riesz-Kantorovich formula.*

Proof The proof will be done by induction on n. The case $n = 1$ follows from Theorem 4.1. So we let $n \geq 1$ and assume the theorem holds for image spaces $\mathbb{R}^1, ..., \mathbb{R}^{n-1}$. The induction step will consist of proving the following three statements:

(i) Assume we have $f_1, f_2, ..., f_m$ continuous, linear maps from L to M such that $f_i^{(j)}$ is regular for all $i \in \{1,...,m\}$ and all $j \in \{1,...,n\}$ and such that the supremum $(\vee_{i=1}^m f_i)^+$ exists in $L^{\sim}(L,M)$. Then there exists some $(x_1^*, x_2^*, ..., x_m^*) \in \mathcal{A}_e^m$ satisfying
$$\sum_{i=1}^{m} f_i(x_i^*) \geq \sum_{i=1}^{m} f_i(x_i)$$
for each $(x_1, x_2, ..., x_m) \in \mathcal{A}_e^m$, and then
$$(\vee_{i=1}^m f_i)^+ (e) = \sum_{i=1}^{m} f_i(x_i^*).$$

(ii) Assume we have $f_1, f_2, ..., f_m$ continuous, linear maps from L to M such that $f_i^{(j)}$ is regular for all $i \in \{1,...,m\}$ and all $j \in \{1,...,n\}$ and such that the supremum $\vee_{i=1}^m f_i$ exists in $L^{\sim}(L,M)$. Then there exists some $(x_1^*, x_2^*, ..., x_m^*) \in \mathcal{F}_e^m$ satisfying
$$\sum_{i=1}^{m} f_i(x_i^*) \geq \sum_{i=1}^{m} f_i(x_i)$$
for each $(x_1, x_2, ..., x_m) \in \mathcal{F}_e^m$, and then
$$(\vee_{i=1}^m f_i)(e) = \sum_{i=1}^{m} f_i(x_i^*).$$

(iii) Assume we have $f_1, f_2, ..., f_m$ continuous, linear maps from L to M such that $f_i^{(j)}$ is regular for all $i \in \{1,...,m\}$ and all $j \in \{1,...,n\}$ and such that the supremum $\vee_{i=1}^m f_i$ exists in $L^{\sim}(L,M)$. Then the Riesz-Kantorovich formula holds.

Proof of (i) The first statement in (i) follows from the fact that if $K \subset \mathbb{R}^n$ is compact (and since all f_i are continuous we get that $K = \{\sum\limits_{i=1}^{m} f_i(x_i) : (x_1, ..., x_m) \in \mathcal{A}_e^m\}$ is compact) then K contains a maximum with respect to the lexicographic ordering on $\mathbb{R}^n$.

Write $g = (f_1 \vee ... \vee f_m)^+$. We let $(x_1^*, ..., x_m^*) \in \mathcal{A}_e^m$ such that

$$\sum_{i=1}^{m} f_i(x_i^*) \geq \sum_{i=1}^{m} f_i(x_i)$$

for all $(x_1, ..., x_m) \in \mathcal{A}_e^m$.
Now we have

$$g(e) = g(\sum_{i=1}^{m} x_i^*) = \sum_{i=1}^{m} g(x_i^*) \geq \sum_{i=1}^{m} f_i(x_i^*).$$

So we need to show that the other inequality holds.

First we assume that $f_i^{(j)} \geq 0$, $f_i^{(n)} \geq 0$ and $f_i^{(n)} \geq f_i^{(j)}$ for all $i \in \{1, ..., m\}$ and all $j \in \{1, ..., n-1\}$. Then without loss of generality we assume that $\sum_{i=1}^m x_i^* = e$, since all maps are positive. Suppose that $\sum_{i=1}^m f_i(x_i^*)^{(n)} = 0$. Then also $\sum_{i=1}^m f_i(x_i^*)^{(j)} = 0$ for all $j \in \{1, ..., n-1\}$, since $f_i^{(n)} \geq f_i^{(j)}$ for all $i \in \{1, ..., m\}$. So since $f_i^{(j)} \geq 0, f_i^{(n)} \geq 0$ this is only possible if $f_i = 0$ for all $i \in \{1, ..., m\}$ by definition of $(x_1^*, ..., x_m^*)$. In that case the result holds. So we assume that $\sum_{i=1}^m f_i(x_i^*)^{(n)} > 0$.
We know from Lemma 4.4 that $f_1^{(1),...,(n-1)} \vee ... \vee f_m^{(1),...,(n-1)} = (f_1 \vee ... \vee f_m)^{(1),...,(n-1)}$, since we use the lexicographic ordering on $\mathbb{R}^n$. Hence by the induction hypothesis we get if we let $h = f_1^{(1),...,(n-1)} \vee .... \vee f_m^{(1),...,(n-1)}$ then h is positive, linear and satisfies for $x \in L^+$

$$h(x) = \max\{\sum_{i=1}^{m} f_i(x_i)^{(1),...,(n-1)} : (x_1, ..., x_m) \in \mathcal{F}_x^m\},$$

where the usual supremum is now a maximum by the continuity of the functionals $f_i^{(1),...,(n-1)}$ and the compactness of order intervals. Then we have

$$h(e) = h(\sum_{i=1}^{m} x_i^*) = \sum_{i=1}^{m} f_i(x_i^*)^{(1),...,(n-1)},$$

by our choice of $x_1^*, ..., x_m^*$. We define

$$Y = \left\{(y_1, ..., y_m) \in L_+^m : h(\sum_{i=1}^{m} y_i) = \sum_{i=1}^{m} f_i(y_i)^{(1),...,(n-1)}, \sum_{i=1}^{m} f_i(y_i)^{(n)} > \sum_{i=1}^{m} f_i(x_i^*)^{(n)}\right\}.$$

This set is clearly convex and non-empty (it contains $t(x_1^*, ..., x_m^*)$ for all $t > 1$). We also let

$$Z = \{(z_1, ..., z_m) \in L^m : \sum_{i=1}^{m} z_i \leq e\}.$$

Also this set Z is non-empty and convex. We will show that Y and Z are disjoint. Suppose that $Y \cap Z \neq \emptyset$ and let $(y_1, ..., y_m) \in Y \cap Z$. Since in the definition of h we only have maxima, we can let $z_1, ..., z_m \in L_+$ such that $\sum_{i=1}^m z_i = e - \sum_{i=1}^m y_i \geq 0$ and $h(\sum_{i=1}^m z_i) = \sum_{i=1}^m f_i(z_i)^{(1),...,(n-1)}$. Then $\sum_{i=1}^m (y_i + z_i) = e$, $y_i + z_i \geq 0$ for all $i \in \{1,...,m\}$ and we have

$$\sum_{i=1}^{m} f_i(y_i+z_i)^{(1),...,(n-1)} = \sum_{i=1}^{m} f_i(y_i)^{(1),...,(n-1)} + \sum_{i=1}^{m} f_i(z_i)^{(1),...,(n-1)}$$
$$= h(\sum_{i=1}^{m} y_i) + h(e - \sum_{i=1}^{m} y_i) = h(e) = \sum_{i=1}^{m} f_i(x_i^*)^{(1),...,(n-1)}$$

and by positivity of $f_i^{(n)}$ for $i \in \{1,...,m\}$ we also have

$$\sum_{i=1}^{m} f_i(y_i+z_i)^{(n)} \geq \sum_{i=1}^{m} f_i(y_i)^{(n)} > \sum_{i=1}^{m} f_i(x_i^*)^{(n)}.$$

So $\sum_{i=1}^{m} f_i(y_i+z_i) > \sum_{i=1}^{m} f_i(x_i^*)$ which gives a contradiction with the definition of $(x_1^*,...,x_m^*)$. So we must have that $Y \cap Z = \emptyset$.

This set Z has an internal point in L^m by [1, Lemma 2.1]. By the Separation Theorem [2, Theorem 5.61] there exists a non-zero linear functional $(h_1^{(n)},...,h_m^{(n)}) \in (L^*)^m$ that separates Z and Y. That is

$$\text{(1)} \qquad \sum_{i=1}^{m} h_i^{(n)}(y_i) \geq \sum_{i=1}^{m} h_i^{(n)}(z_i) \text{ for all } (y_1,...,y_m) \in Y \text{ and } (z_1,...,z_m) \in Z.$$

Since $(x_1^*,...,x_m^*) \in Z$ it follows from (1) that

$$\text{(2)} \qquad \sum_{i=1}^{m} h_i^{(n)}(y_i) \geq \sum_{i=1}^{m} h_i^{(n)}(x_i^*) \text{ for all } (y_1,...,y_m) \in Y.$$

Furthermore, it follows from (1) that

$$\text{(3)} \qquad \sum_{i=1}^{m} h_i^{(n)}(x_i^*) = \lim_{\alpha \downarrow 1} \sum_{i=1}^{m} h_i^{(n)}(\alpha x_i^*) \geq \sum_{i=1}^{m} h_i^{(n)}(z_i) \text{ for all } (z_1,...,z_m) \in Z.$$

Next, we show that $h_1^{(n)} = ... = h_m^{(n)} := h^{(n)}$. Suppose, by way of contradiction, that there exists some $z \in L$ such that $h_1^{(n)}(z) > h_2^{(n)}(z)$. Then for some $\alpha > 1$ we have

$$h_1^{(n)}(x_1^*+z) + h_2^{(n)}(x_2^*-z) + \sum_{i=3}^{m} h_i^{(n)}(x_i^*) > \sum_{i=1}^{m} h_i^{(n)}(\alpha x_i^*).$$

However, $(\alpha x_1^*,...,\alpha x_m^*) \in Y$ since $\alpha > 1$ and $(x_1^*+z, x_2^*-z, x_3^*,...,x_m^*) \in Z$, which contradicts (1). By the symmetry of the situation we see that $h_1^{(n)} = h_2^{(n)} = \cdots = h_m^{(n)} := h^{(n)}$.

Now we show that $h^{(n)} \geq 0$. Let $x \in L^+$. Note that $(e - \alpha x, 0, 0, ...) \in Z$ for all $\alpha > 0$. Therefore, from (3) we get $\sum_{i=1}^{m} h^{(n)}(x_i^*) \geq h^{(n)}(e) - \alpha h^{(n)}(x)$ or

$$h^{(n)}(x) \geq \frac{h^{(n)}(e) - \sum_{i=1}^{m} h^{(n)}(x_i^*)}{\alpha}$$

for each $\alpha > 0$. Letting $\alpha \to \infty$ yields $h^{(n)}(x) \geq 0$. Hence $h^{(n)} \geq 0$ and thus $h^{(n)} \in L^{\sim}$.

Furthermore, since $h^{(n)} \neq 0$ it must be the case that $h^{(n)}(e) > 0$ since e is an order unit. Since $(e, 0, ..., 0) \in Z$ it follows from (3) that $\sum_{i=1}^{m} h^{(n)}(x_i^*) > 0$. So we can define

$$\delta^{(n)} = \frac{\sum_{i=1}^{m} f_i(x_i^*)^{(n)}}{\sum_{i=1}^{m} h^{(n)}(x_i^*)} \in \mathbb{R}_{>0}.$$

We claim that $\delta^{(n)} h^{(n)}(x) \geq f_i^{(n)}(x)$ for $i \in \{1,...,m\}$ and $x \in L^+$ such that $h(x) = f_i(x)^{(1),...,(n-1)}$. To see this, fix $i \in \{1,...,m\}$ and $x \in L_+$ such that

$h(x) = f_i(x)^{(1),...,(n-1)}$. If $f_i(x)^{(n)} \leq 0$ then $\delta^{(n)} h^{(n)}(x) \geq 0 \geq f_i(x)^{(n)}$ is trivially true. Assume, therefore, that $f_i(x)^{(n)} > 0$ and let

$$\gamma^{(n)} = \frac{\sum_{i=1}^m f_i(x_i^*)^{(n)}}{f_i(x)^{(n)}} \in \mathbb{R}_{>0}.$$

Now we have for $\alpha > 1$

$$f_i(\alpha\gamma^{(n)} x)^{(n)} = \alpha \frac{\sum_{i=1}^m f_i(x_i^*)^{(n)}}{f_i(x)^{(n)}} f_i(x)^{(n)} = \alpha \sum_{i=1}^m f_i(x_i^*)^{(n)} > \sum_{i=1}^m f_i(x_i^*)^{(n)},$$

since we assumed that $\sum_{i=1}^m f_i(x^*)^{(n)} > 0$. Hence by linearity of h and $f_i^{(1),...,(n-1)}$ we get that $(0, ..., 0, \alpha\gamma^{(n)} x, 0, ..., 0) \in Y$, where $\alpha\gamma^{(n)} x$ takes the i-th position of this vector. Hence with equation (2) we get

$$\begin{aligned} \delta^{(n)} \gamma^{(n)} h^{(n)}(x) &= \delta^{(n)} \lim_{\alpha \downarrow 1} h^{(n)}(\alpha\gamma^{(n)} x) \\ &\geq \delta^{(n)} \sum_{i=1}^m h^{(n)}(x_i^*) = \sum_{i=1}^m f_i(x_i^*)^{(n)} = \gamma^{(n)} f_i(x)^{(n)}. \end{aligned}$$

Hence $\delta^{(n)} h^{(n)}(x) \geq f_i(x)^{(n)}$ for all $x \in L_+$ with $h(x) = f_i(x)^{(1),...,(n-1)}$. Thus, due to the lexicographic ordering, for all $x \geq 0$ we have

$$f_i(x) \leq (h(x), \delta^{(n)} h^{(n)}(x)).$$

Since this holds for all $i \in \{1, ..., m\}$, since $h \geq 0$ and $\delta^{(n)} h^{(n)} \geq 0$, we get that

$$g(x) \leq (h(x), \delta^{(n)} h^{(n)}(x))$$

for all $x \geq 0$. In particular we get

$$g(e) \leq (h(e), \delta^{(n)} h^{(n)}(e)) = \sum_{i=1}^m f_i(x_i^*)$$

since $e = \sum_{i=1}^m x_i^*$. Hence we obtain that

$$g(e) = \sum_{i=1}^m f_i(x_i^*).$$

Now we do not assume that $f_i^{(j)} \geq 0, f_i^{(n)} \geq 0$ or $f_i^{(n)} \geq f_i^{(j)}$ for $i \in \{1, ..., m\}$ and $j \in \{1, ..., n-1\}$. Since the $f_i^{(j)}$ are all regular for $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$ we can let $h_1, ..., h_n, g_1^{(1)}, ..., g_m^{(1)}, ..., g_1^{(n)}, ..., g_m^{(n)}$ be positive linear functionals such that for $i \in \{1, ...m\}$ we have:

$$f_i = (g_i^{(1)} - h_1, ..., g_i^{(n)} - h_n)$$

Due to compactness of $\mathcal{A}_e^{m+1}$ we can let $(y_1^*, ..., y_{m+1}^*) \in \mathcal{A}_e^m$ such that

$$\begin{aligned} &\sum_{i=1}^m (g_i^{(1)}(y_i^*), ..., g_i^{(n-1)}(y_i^*), g_i^{(n)}(y_i^*) + \sum_{j=1}^{n-1} \sum_{k=1}^m g_k^{(j)}(y_i^*) + \sum_{j=1}^{n-1} h_j(y_i^*)) \\ &\qquad + (h_1, ..., h_{n-1}, \sum_{j=1}^{n-1} \sum_{k=1}^m g_k^{(j)} + \sum_{j=1}^n h_j)(y_{m+1}) \\ &\geq \sum_{i=1}^m (g_i^{(1)}(x_i), ..., g_i^{(n-1)}(x_i), g_i^{(n)}(x_i) + \sum_{j=1}^{n-1} \sum_{k=1}^m g_k^{(j)}(x_i) + \sum_{j=1}^{n-1} h_j(x_i)) \\ &\qquad + (h_1, ..., h_{n-1}, \sum_{j=1}^{n-1} \sum_{k=1}^m g_k^{(j)} + \sum_{j=1}^n h_j)(x_{m+1}) \end{aligned}$$

for each $(x_1, x_2, ..., x_{m+1}) \in \mathcal{A}_e^{m+1}$. By positivity we can assume without loss of generality that $\sum_{j=1}^{m+1} y_j^* = e$ so that $y_{m+1}^* = e - \sum_{j=1}^{m} y_j^*$. Then we obtain that the set of functions $\{(g_i^{(1)}, ..., g_i^{(n-1)}, g_i^{(n)} + \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)} + \sum_{j=1}^{n-1} h_j) : i \in \{1, ..., m\}\} \cup \{(h_1, ..., h_{n-1}, \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)} + \sum_{j=1}^{n} h_j)\}$ satisfies the assumptions we made in the first half of this proof. Hence we get by our previous results

$$
\begin{aligned}
g(e) &= \left(\bigvee_{i=1}^{m}(g_i^{(1)} - h_1, ..., g_i^{(n)} - h_n)\right)^+ \\
&= \Bigg(\Bigg(\bigvee_{i=1}^{m}\Bigg((g_i^{(1)}, ..., g_i^{(n-1)}, g_i^{(n)} + \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)} + \sum_{j=1}^{n-1} h_j) \\
&\quad - (h_1, ..., h_{n-1}, \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)} + \sum_{j=1}^{n} h_j)\Bigg)\Bigg) \\
&\quad \vee \left((h_1, ..., h_{n-1}, \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)} + \sum_{j=1}^{n} h_j) - (h_1, ..., h_{n-1}, \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)} + \sum_{j=1}^{n} h_j)\right)\Bigg)(e) \\
&= \left(\bigvee_{i=1}^{m}(g_i^{(1)}, ..., g_i^{(n-1)}, g_i^{(n)} + \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)} + \sum_{j=1}^{n-1} h_j)\right)(e) \\
&\quad - \left(h_1, ..., h_{n-1}, \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)} + \sum_{j=1}^{n} h_j\right)(e) \\
&= \sum_{i=1}^{m}(g_i^{(1)}(y_i^*), ..., g_i^{(n-1)}(y_i^*), g_i^{(n)}(y_i^*) + \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)}(y_i^*) + \sum_{j=1}^{n-1} h_j(y_i^*)) \\
&\quad + \left(h_1(e - \sum_{i=1}^{m} y_i^*), ..., h_{n-1}(e - \sum_{i=1}^{m} y_i^*), \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)}(e - \sum_{i=1}^{m} y_i^*) + \sum_{j=1}^{n} h_j(e - \sum_{i=1}^{m} y_i^*)\right) \\
&\quad - (h_1(e), ..., h_{n-1}(e), \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)}(e) + \sum_{j=1}^{n} h_j(e)) \\
&= \sum_{i=1}^{m}(g_i^{(1)}(y_i^*), ..., g_i^{(n-1)}(y_i^*), g_i^{(n)}(y_i^*) + \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)}(y_i^*) + \sum_{j=1}^{n-1} h_j(y_i^*)) \\
&\quad - \sum_{i=1}^{m}(h_1(y_i^*), ..., h_{n-1}(y_i^*), \sum_{j=1}^{n-1}\sum_{k=1}^{m} g_k^{(j)}(y_i^*) + \sum_{j=1}^{n} h_j(y_i^*)) \\
&= \sum_{i=1}^{m}(g_i^{(1)}(y_i^*) - h_1(y_i^*), ..., g_i^{(n)}(y_i^*) - h_2(y_i^*)) \\
&= \sum_{i=1}^{m}(f_i^{(1)}(y_i^*), ..., f_i^{(n)}(y_i^*)) \leq \sum_{i=1}^{m} f_i(x_i^*)
\end{aligned}
$$

by our choice of $x_1^*, ..., x_m^*$. Since in the very first part of the proof we saw that $g(e) \geq \sum_{i=1}^{m} f_i(x_i^*)$ we have $g(e) = \sum_{i=1}^{m} f_i(x_i^*)$. That finishes the proof of part (i). ■

Proof of (ii) The first statement follows for the same reason as in part (i). So let $(x_1^*, ..., x_m^*) \in \mathcal{F}_e^m$ such that $\sum\limits_{i=1}^{m} f_i(x_i^*) \geq \sum\limits_{i=1}^{m} f_i(x_i)$ for each $(x_1, x_2, ..., x_m) \in \mathcal{F}_e^m$. First note since $(x_1^*, ..., x_m^*) \in \mathcal{F}_e^m$ that $x_1^* = e - \sum\limits_{i=2}^{m} x_i^*$. Let $g = \vee_{i=1}^m f_i$ and start by observing that $g - f_1 = [\vee_{i=2}^m (f_i - f_1)]^+$ and

$$\sum_{i=2}^{m}(f_i - f_1)(x_i^*) = \sum_{i=1}^{m} f_i(x_i^*) - f_1(e)$$

$$\geq \sum_{i=2}^{m} f_i(x_i) + f_1\left(e - \sum_{i=2}^{m} x_i\right) - f_1(e) = \sum_{i=2}^{m}(f_i - f_1)(x_i)$$

for any $(x_2, ..., x_m) \in \mathcal{A}_e^{m-1}$ since if $(x_2, ..., x_m) \in \mathcal{A}_e^{m-1}$ then $(e - \sum\limits_{i=2}^{m} x_i, x_2, ..., x_m) \in \mathcal{F}_e^m$. Therefore by part (i) we have

$$(g - f_1)(e) = \sum_{i=2}^{m}(f_i - f_1)(x_i^*).$$

In particular it holds that

$$g(e) = f_1\left(e - \sum_{i=2}^{m} x_i^*\right) + \sum_{i=2}^{m} f_i(x_i^*) = \sum_{i=1}^{m} f_i(x_i^*),$$

which is the desired formula. ■

Proof of (iii) Now fix an arbitrary $x \in L^+$ and then select an order unit $e \in L$ such that $x \leq e$.
For each $0 < \alpha < 1$ let $x_\alpha = \alpha x + (1 - \alpha)e$. Clearly, each x_α is an order unit and $0 \leq x_\alpha \leq e$ holds for each $0 < \alpha < 1$. We consider the index set $(0, 1)$ directed by the increasing order relation $\succeq$, i.e. $\alpha \succeq \beta$ in $(0, 1)$ if and only if $\alpha \geq \beta$. Clearly $x_\alpha \to x$. Since $\mathcal{F}_e^m$ is compact, the continuous function $\mathcal{F}_e^m \to M, (x_1, ..., x_m) \mapsto \sum_{i=1}^m f_i(x_i)$ attains its maximum. By part (ii), we know that g satisfies the Riesz-Kantorovich formula for each order unit. Therefore, by our assumption on the space M and the compactness of $\mathcal{F}_{e_\alpha}^m$, for each $0 < \alpha < 1$ there exists some $(z_1^\alpha, ..., z_m^\alpha) \in L_+^m$ such that $\sum_{i=1}^m z_i^\alpha = x_\alpha$ and

$$\alpha g(x) + (1 - \alpha) g(e) = g(x_\alpha) = \sum_{i=1}^{m} f_i(z_i^\alpha). \tag{4}$$

Since $z_i^\alpha \in [0, e]$ for each $\alpha \in (0, 1)$ and each $i \in \{1, ..., m\}$ and the order interval $[0, e]$ is compact, there exists a subnet of $\{(z_1^\alpha, ..., z_m^\alpha)\}$ (which without loss of generality we also denote by $\{(z_1^\alpha, ..., z_m^\alpha)\}$) such that $z_i^\alpha \to z_i$ for some $z_i \in [0, e]$. From $x_\alpha - z_i^\alpha \in [0, e]$ for each α and the closedness of $[0, e]$, we see that $0 \leq x - z_i \leq e$. So $0 \leq z_i \leq x$ and $\sum_{i=1}^m z_i = x$. Finally, letting $\alpha \to 1$ in equation (4) it follows from the continuity of each f_i that

$$g(x) = \sum_{i=1}^{m} f_i(z_i).$$

Since we already know that $g(x) \geq \sum_{i=1}^m f_i(y_i)$ for all $(y_1, ..., y_m) \in \mathcal{F}_x^m$ we get that $\sup\{\sum_{i=1}^m f_i(y_i) : (y_1, ..., y_m) \in \mathcal{F}_x^m\} = g(x)$. So the proof is finished. ■

Theorem 4.6. ***(Main theorem, ordinal version)*** *Let L be an ordered vector space with an order unit e and a Hausdorff linear topology such that all order intervals are compact. Let α be an ordinal number with $\alpha \geq 1$ and let $M = \prod_{1\leq j\leq\alpha}\mathbb{R}$ with the lexicographic ordering and the product topology. Assume we have $f_1, f_2, ..., f_m$ continuous, linear maps from L to M such that $f_i^{(j)}$ is regular for all $i \in \{1, ..., m\}$ and all ordinals j with $1 \leq j \leq \alpha$ and such that the supremum $g = \vee_{i=1}^m f_i$ exists in $L^\sim(L, M)$. Then g satisfies the Riesz-Kantorovich formula.*

Proof The proof will be done by transfinite induction on α. The case $\alpha = 1$ follows from Theorem 4.1. So we let $\alpha \geq 1$ and assume the theorem holds for image spaces $\mathbb{R}^1, ..., \prod_{1\leq j\leq\alpha}\mathbb{R}$. Let $M = \prod_{1\leq j\leq\alpha}\mathbb{R}$. The induction step for successor ordinals will consist of proving the following three statements:

(i) Assume we have $f_1, f_2, ..., f_m$ continuous, linear maps from L to M such that $f_i^{(j)}$ is regular for all $i \in \{1, ..., m\}$ and all ordinals j with $1 \leq j \leq \alpha$ and such that the supremum $(\vee_{i=1}^m f_i)^+$ exists in $L^\sim(L, M)$. Then there exists some $(x_1^*, x_2^*, ..., x_m^*) \in \mathcal{A}_e^m$ satisfying

$$\sum_{i=1}^m f_i(x_i^*) \geq \sum_{i=1}^m f_i(x_i)$$

for each $(x_1, x_2, ..., x_m) \in \mathcal{A}_e^m$, and then

$$\left(\vee_{i=1}^m f_i\right)^+(e) = \sum_{i=1}^m f_i(x_i^*).$$

(ii) Assume we have $f_1, f_2, ..., f_m$ continuous, linear maps from L to M such that $f_i^{(j)}$ is regular for all $i \in \{1, ..., m\}$ and all ordinals j with $1 \leq j \leq \alpha$ and such that the supremum $\vee_{i=1}^m f_i$ exists in $L^\sim(L, M)$. Then there exists some $(x_1^*, x_2^*, ..., x_m^*) \in \mathcal{F}_e^m$ satisfying

$$\sum_{i=1}^m f_i(x_i^*) \geq \sum_{i=1}^m f_i(x_i)$$

for each $(x_1, x_2, ..., x_m) \in \mathcal{F}_e^m$, and then

$$\left(\vee_{i=1}^m f_i\right)(e) = \sum_{i=1}^m f_i(x_i^*).$$

(iii) Assume we have $f_1, f_2, ..., f_m$ continuous, linear maps from L to M such that $f_i^{(j)}$ is regular for all $i \in \{1, ..., m\}$ and all ordinals j with $1 \leq j \leq \alpha$ and such that the supremum $\vee_{i=1}^m f_i$ exists in $L^\sim(L, M)$. Then the Riesz-Kantorovich formula holds.

Proof of (i) The first statement in (i) follows from the fact that if $K \subset \prod_{1\leq j\leq\alpha}\mathbb{R}$ is compact (and since all f_i are continuous we get that $K = \{\sum_{i=1}^m f_i(x_i) : (x_1, ..., x_m) \in \mathcal{A}_e^m\}$ is compact) then K contains a maximum with respect to the lexicographic ordering on $\prod_{1\leq j\leq\alpha}\mathbb{R}$.

Write $g = (f_1 \vee ... \vee f_m)^+$. We let $(x_1^*, ..., x_m^*) \in \mathcal{A}_e^m$ such that

$$\sum_{i=1}^m f_i(x_i^*) \geq \sum_{i=1}^m f_i(x_i)$$

for all $(x_1, ..., x_m) \in \mathcal{A}_e^m$.
Now we have

$$g(e) = g(\sum_{i=1}^{m} x_i^*) = \sum_{i=1}^{m} g(x_i^*) \geq \sum_{i=1}^{m} f_i(x_i^*).$$

So we need to show that the other inequality holds.

First we assume that $f_i^{(\alpha+1)} \geq 0$ for all $i \in \{1, ..., m\}$. We know from Lemma 4.4 that $f_1^{(1),...,(\alpha)} \vee ... \vee f_m^{(1),...,(\alpha)} = (f_1 \vee ... \vee f_m)^{(1),...,(\alpha)}$, since we use the lexicographic ordering on $\prod_{1\leq j\leq \alpha} \mathbb{R}$. Hence by the induction hypothesis we get if we let $h = \left(f_1^{(1),...,(\alpha)} \vee .... \vee f_m^{(1),...,(\alpha)}\right)^+$ then h is positive, linear and satisfies for $x \in L^+$

$$h(x) = \max\{\sum_{i=1}^{m} f_i(x_i)^{(1),...,(\alpha)} : (x_1, ..., x_m) \in \mathcal{A}_x^m\},$$

where the usual supremum is now a maximum by the continuity of the functionals $f_i^{(1),...,(\alpha)}$ and the compactness of order intervals. Then we have that

$$h(e) = \sum_{i=1}^{m} f_i(x_i^*)^{(1),...,(\alpha)} = h(\sum_{i=1}^{m} x_i^*)$$

by our choice of $x_1^*, ..., x_m^*$.

Suppose that $\sum_{i=1}^m f_i(x_i^*)^{(\alpha+1)} = 0$. Let $j \in \{1, ..., m\}$ and $y \geq 0$ and suppose we have $h(y) = f_j(y)^{(1),...,(\alpha)}$ and $f_j(y)^{(\alpha+1)} > 0$. Since in the definition of h we only have maxima, we can let $z_1, ..., z_m \in L_+$ such that $\sum_{i=1}^m z_i \leq e - y$ and $h(e - y) = \sum_{i=1}^m f_i(z_i)^{(1),...,(\alpha)}$. Then $y + \sum_{i=1}^m z_i \leq e$, $y + z_j \geq 0, z_i \geq 0$ for all $i \in \{1, ..., m\}$ and we have

$$\sum_{i=1, i\neq j}^{m} f_i(z_i)^{(1),...,(\alpha)} + f_j(y + z_j)^{(1),...,(\alpha)} = f_j(y)^{(1),...,(\alpha)} + \sum_{i=1}^{m} f_i(z_i)^{(1),...,(\alpha)}$$

$$= h(y) + h(e - y) = h(e) = \sum_{i=1}^{m} f_i(x_i^*)^{(1),...,(\alpha)}$$

and by positivity of $f_i^{(\alpha+1)}$ for $i \in \{1, ..., m\}$ we also have

$$\sum_{i=1, i\neq j}^{m} f_i(z_i)^{(\alpha+1)} + f_j(y + z_j)^{(\alpha+1)} \geq f_j(y)^{(\alpha+1)} > 0 = \sum_{i=1}^{m} f_i(x_i^*)^{(\alpha+1)}.$$

So $\sum_{i=1,i\neq j}^m f_i(z_i) + f_j(y + z_j) > \sum_{i=1}^m f_i(x_i^*)$, which gives a contradiction with the definition of $(x_1^*, ..., x_m^*)$. So for all $j \in \{1, ..., m\}$ and $y \in L_+$ with $h(y) = f_j(y)^{(1),...,(\alpha)}$ we have $f_j(y)^{(\alpha+1)} = 0$. Clearly we also have for all $j \in \{1, ..., m\}$ and $y \in L_+$ that $h(y) \geq f_j(y)^{(1),...,(\alpha)}$ by our definition of h. Hence we see that for all $j \in \{1, ..., m\}$ and all $y \in L_+$ we have

$$f_j(y) \leq (h(y), 0)$$

and thus

$$f_j \leq (h, 0).$$

This implies that $g \leq (h, 0)$. In particular we see that

$$g(e) \leq (h(e), 0) = (\sum_{i=1}^{m} f_i(x_i^*)^{(1),...,(\alpha)}, \sum_{i=1}^{m} f_i(x_i^*)^{(\alpha+1)}) = \sum_{i=1}^{m} f_i(x_i^*).$$

Therefore we obtain

$$g(e) = \sum_{i=1}^{m} f_i(x_i^*).$$

So it remains to consider the case where $\sum_{i=1}^m f_i(x_i^*)^{(\alpha+1)} > 0$. We define

$$Y = \{(y_1, ..., y_m) \in L_+^m : h(\sum_{i=1}^m y_i) = \sum_{i=1}^m f_i(y_i)^{(1),...,(\alpha)},$$

$$\sum_{i=1}^m f_i(y_i)^{(\alpha+1)} > \sum_{i=1}^m f_i(x_i^*)^{(\alpha+1)}\}.$$

This set is clearly convex and non-empty (it contains $t(x_1^*, ..., x_m^*)$ for all $t > 1$). We also let

$$Z = \{(z_1, ..., z_m) \in L^m : \sum_{i=1}^m z_i \leq e\}.$$

Also this set Z is non-empty and convex. We will show that Y and Z are disjoint. Suppose that $Y \cap Z \neq \emptyset$ and let $(y_1, ..., y_m) \in Y \cap Z$. Since in the definition of h we only have maxima and $e - \sum_{i=1}^m y_i \geq 0$, we can let $z_1, ..., z_m \in L_+$ such that $\sum_{i=1}^m z_i \leq e - \sum_{i=1}^m y_i$ and $h(e - \sum_{i=1}^m y_i) = \sum_{i=1}^m f_i(z_i)^{(1),...,(\alpha)}$. Then $\sum_{i=1}^m (y_i + z_i) \leq e$, $y_i + z_i \geq 0$ for all $i \in \{1, ..., m\}$ and we have

$$\sum_{i=1}^m f_i(y_i + z_i)^{(1),...,(\alpha)} = \sum_{i=1}^m f_i(y_i)^{(1),...,(\alpha)} + \sum_{i=1}^m f_i(z_i)^{(1),...,(\alpha)}$$

$$= h(\sum_{i=1}^m y_i) + h(e - \sum_{i=1}^m y_i) = h(e) = \sum_{i=1}^m f_i(x_i^*)^{(1),...,(\alpha)}$$

and by positivity of $f_i^{(\alpha+1)}$ for $i \in \{1, ..., m\}$ we also have

$$\sum_{i=1}^m f_i(y_i + z_i)^{(\alpha+1)} \geq \sum_{i=1}^m f_i(y_i)^{(\alpha+1)} > \sum_{i=1}^m f_i(x_i^*)^{(\alpha+1)}.$$

So $\sum_{i=1}^m f_i(y_i + z_i) > \sum_{i=1}^m f_i(x_i^*)$, which gives a contradiction with the definition of $(x_1^*, ..., x_m^*)$. So we must have that $Y \cap Z = \emptyset$.

This set Z has an internal point in L^m by [1, Lemma 2.1]. By the Separation Theorem [2, Theorem 5.61] there exists a non-zero linear functional $(h_1^{(\alpha+1)}, ..., h_m^{(\alpha+1)}) \in (L^*)^m$ that separates Z and Y. That is

$$(5) \quad \sum_{i=1}^m h_i^{(\alpha+1)}(y_i) \geq \sum_{i=1}^m h_i^{(\alpha+1)}(z_i) \text{ for all } (y_1, ..., y_m) \in Y \text{ and } (z_1, ..., z_m) \in Z.$$

Since $(x_1^*, ..., x_m^*) \in Z$ it follows from (5) that

$$(6) \qquad \sum_{i=1}^m h_i^{(\alpha+1)}(y_i) \geq \sum_{i=1}^m h_i^{(\alpha+1)}(x_i^*) \text{ for all } (y_1, ..., y_m) \in Y.$$

Furthermore, it follows from (5) that

$$(7) \quad \sum_{i=1}^m h_i^{(\alpha+1)}(x_i^*) = \lim_{\beta \downarrow 1} \sum_{i=1}^m h_i^{(\alpha+1)}(\beta x_i^*) \geq \sum_{i=1}^m h_i^{(\alpha+1)}(z_i) \text{ for all } (z_1, ..., z_m) \in Z.$$

Next, we show that $h_1^{(\alpha+1)} = ... = h_m^{(\alpha+1)} := h^{(\alpha+1)}$. Suppose, by way of contradiction, that there exists some $z \in L$ such that $h_1^{(\alpha+1)}(z) > h_2^{(\alpha+1)}(z)$. Then for some

$\beta > 1$ we have

$$h_1^{(\alpha+1)}(x_1^*+z) + h_2^{(\alpha+1)}(x_2^*-z) + \sum_{i=3}^{m} h_i^{(\alpha+1)}(x_i^*) > \sum_{i=1}^{m} h_i^{(\alpha+1)}(\beta x_i^*).$$

However, $(\beta x_1^*, ..., \beta x_m^*) \in Y$ since $\beta > 1$ and $(x_1^*+z, x_2^*-z, x_3^*, ..., x_m^*) \in Z$, which contradicts (5). By the symmetry of the situation we see that $h_1^{(\alpha+1)} = h_2^{(\alpha+1)} = \cdots = h_m^{(\alpha+1)} := h^{(\alpha+1)}$.
Now we show that $h^{(\alpha+1)} \geq 0$. Let $x \in L^+$. Note that $(e - \beta x, 0, 0, ...) \in Z$ for all $\beta > 0$. Therefore, from (7) we get $\sum_{i=1}^m h^{(\alpha+1)}(x_i^*) \geq h^{(\alpha+1)}(e) - \beta h^{(\alpha+1)}(x)$ or

$$h^{(\alpha+1)}(x) \geq \frac{h^{(\alpha+1)}(e) - \sum_{i=1}^m h^{(\alpha+1)}(x_i^*)}{\beta}$$

for each $\beta > 0$. Letting $\beta \to \infty$ yields $h^{(\alpha+1)}(x) \geq 0$. Hence $h^{(\alpha+1)} \geq 0$ and thus $h^{(\alpha+1)} \in L^\sim$.
Now since $(e, 0, ..., 0) \in Z$ and $\sum\limits_{i=1}^{m} x_i^* \leq e$ we get from equation (7) that

$$h^{(\alpha+1)}(\sum_{i=1}^{m} x_i^*) \leq h^{(\alpha+1)}(e) \leq h^{(\alpha+1)}(\sum_{i=1}^{m} x_i^*)$$

and thus we obtain

$$h^{(\alpha+1)}(\sum_{i=1}^{m} x_i^*) = h^{(\alpha+1)}(e).$$

Furthermore, since $h^{(\alpha+1)} \neq 0$ it must be the case that $h^{(\alpha+1)}(e) > 0$ since e is an order unit. Since $(e, 0, ..., 0) \in Z$ it follows from (7) that $\sum_{i=1}^m h^{(\alpha+1)}(x_i^*) > 0$. So we can define

$$\delta^{(\alpha+1)} = \frac{\sum_{i=1}^m f_i(x_i^*)^{(\alpha+1)}}{\sum_{i=1}^m h^{(\alpha+1)}(x_i^*)} \in \mathbb{R}_{>0}.$$

We claim that $\delta^{(\alpha+1)} h^{(\alpha+1)}(x) \geq f_i^{(\alpha+1)}(x)$ for $i \in \{1, ..., m\}$ and $x \in L^+$ such that $h(x) = f_i(x)^{(1),...,(\alpha)}$. To see this, fix $i \in \{1, ..., m\}$ and $x \in L_+$ such that $h(x) = f_i(x)^{(1),...,(\alpha)}$. If $f_i(x)^{(\alpha+1)} \leq 0$ then $\delta^{(\alpha+1)} h^{(\alpha+1)}(x) \geq 0 \geq f_i(x)^{(\alpha+1)}$ is trivially true. Assume, therefore, that $f_i(x)^{(\alpha+1)} > 0$ and let

$$\gamma^{(\alpha+1)} = \frac{\sum_{i=1}^m f_i(x_i^*)^{(\alpha+1)}}{f_i(x)^{(\alpha+1)}} \in \mathbb{R}_{>0}.$$

Now we have for $\beta > 1$

$$f_i(\beta\gamma^{(\alpha+1)} x)^{(\alpha+1)} = \beta \frac{\sum_{i=1}^m f_i(x_i^*)^{(\alpha+1)}}{f_i(x)^{(\alpha+1)}} f_i(x)^{(\alpha+1)}$$

$$= \beta \sum_{i=1}^{m} f_i(x_i^*)^{(\alpha+1)} > \sum_{i=1}^{m} f_i(x_i^*)^{(\alpha+1)},$$

since we assumed that $\sum_{i=1}^m f_i(x^*)^{(\alpha+1)} > 0$. Hence by linearity of h and $f_i^{(1),...,(\alpha)}$ we get that $(0, ..., 0, \beta\gamma^{(\alpha+1)} x, 0, ..., 0) \in Y$, where $\beta\gamma^{(\alpha+1)} x$ takes the i-th position of this vector. Hence with equation (6) we get

$$\delta^{(\alpha+1)} \gamma^{(\alpha+1)} h^{(\alpha+1)}(x) = \delta^{(\alpha+1)} \lim_{\beta \downarrow 1} h^{(\alpha+1)}(\beta\gamma^{(\alpha+1)} x)$$

$$\geq \delta^{(\alpha+1)} \sum_{i=1}^{m} h^{(\alpha+1)}(x_i^*) = \sum_{i=1}^{m} f_i(x_i^*)^{(\alpha+1)} = \gamma^{(\alpha+1)} f_i(x)^{(\alpha+1)}.$$

Hence $\delta^{(\alpha+1)}h^{(\alpha+1)}(x) \geq f_i(x)^{(\alpha+1)}$ for all $x \in L_+$ with $h(x) = f_i(x)^{(1),\ldots,(\alpha)}$. Thus, due to the lexicographic order, for all $x \geq 0$ we have

$$f_i(x) \leq (h(x), \delta^{(\alpha+1)}h^{(\alpha+1)}(x)).$$

Since this holds for all $i \in \{1,...,m\}$ we get that

$$g(x) \leq (h(x), \delta^{(\alpha+1)}h^{(\alpha+1)}(x))$$

for all $x \geq 0$. In particular we get

$$g(e) \leq (h(e), \delta^{(\alpha+1)}h^{(\alpha+1)}(e)) = \sum_{i=1}^{m} f_i(x_i^*)$$

since $h(e) = h(\sum_{i=1}^m x_i^*)$ and $h^{(\alpha+1)}(e) = h^{(\alpha+1)}(\sum_{i=1}^m x_i^*)$. Hence we obtain

$$g(e) = \sum_{i=1}^{m} f_i(x_i^*).$$

Now we do not assume that $f_i^{(\alpha+1)} \geq 0$ for $i \in \{1,...,m\}$. Since the $f_i^{(\alpha+1)}$ are all regular for $i \in \{1,...,m\}$ we can let g_i, h be positive linear functionals such that for $i \in \{1,...m\}$ we have:

$$f_i^{(\alpha+1)} = g_i - h.$$

Due to compactness of $\mathcal{A}_e^{m+1}$ we can let $(y_1^*,...,y_{m+1}^*) \in \mathcal{A}_e^m$ such that

$$\sum_{i=1}^{m}(f_i^{(1)}(y_i^*),...,f_i^{(\alpha)}(y_i^*), g_i(y_i^*)) + (0,...,0,h)(y_{m+1}^*)$$
$$\geq \sum_{i=1}^{m}(f_i^{(1)}(x_i),...,f_i^{(\alpha)}(x_i), g_i(x_i)) + (0,...,0,h)(x_{m+1})$$

for each $(x_1, x_2, ..., x_{m+1}) \in \mathcal{A}_e^{m+1}$. By positivity of $(0,...,0,h)$ we can assume without loss of generality that $\sum\limits_{j=1}^{m+1} y_j^* = e$ so that $y_{m+1}^* = e - \sum\limits_{j=1}^{m} y_j^*$. Then we obtain that the set of functions $\{(f_i^{(1)},...,f_i^{(\alpha)}, g_i) : i \in \{1,...,m\}\} \cup \{(0,...,0,h)\}$ satisfies the assumptions we made in the first half of this proof. Hence we get by our previous results

$$g(e) = \left(\bigvee_{i=1}^{m}(f_i^{(1)},...,f_i^{(\alpha)}, g_i^{(\alpha+1)} - h)\right)^+(e)$$
$$= \left(\left(\bigvee_{i=1}^{m}\left((f_i^{(1)},...,f_i^{(\alpha)}, g_i) - (0,...,0,h)\right)\right) \vee ((0,...,0,h) - (0,...,0,h))\right)(e)$$
$$= \left(\bigvee_{i=1}^{m}(f_i^{(1)},...,f_i^{(\alpha)}, g_i) \vee (0,...,h)\right)(e) - (0,...,0,h)\,(e)$$
$$= \sum_{i=1}^{m}(f_i^{(1)}(y_i^*),...,f_i^{(\alpha)}(y_i^*), g_i(y_i^*)) + (0,...,0,h(e - \sum_{i=1}^{m} y_i^*)) - (0,...,0,h(e))$$
$$= \sum_{i=1}^{m}(f_i^{(1)}(y_i^*),...,f_i^{(\alpha)}(y_i^*), g_i(y_i^*) - h(y_i^*)) + (0,...,0,h(e) - h(e))$$
$$= \sum_{i=1}^{m}(f_i^{(1)}(y_i^*),...,f_i^{(\alpha)}(y_i^*), f_i^{(\alpha+1)}(y_i^*))$$
$$= \sum_{i=1}^{m} f_i(y_i^*) \leq \sum_{i=1}^{m} f_i(x_i^*)$$

by our choice of $x_1^*, ..., x_m^*$. Since in the very first part of the proof we saw that $g(e) \geq \sum_{i=1}^m f_i(x_i^*)$ we must have $g(e) = \sum_{i=1}^m f_i(x_i^*)$ and that finishes the proof of part (i). ■

Proof of (ii) The first statement follows for the same reason as in part (i). So let $(x_1^*, ..., x_m^*) \in \mathcal{F}_e^m$ such that $\sum\limits_{i=1}^m f_i(x_i^*) \geq \sum\limits_{i=1}^m f_i(x_i)$ for each $(x_1, x_2, ..., x_m) \in \mathcal{F}_e^m$. First note since $(x_1^*, ..., x_m^*) \in \mathcal{F}_e^m$ that $x_1^* = e - \sum\limits_{i=2}^m x_i^*$. Let $g = \vee_{i=1}^m f_i$ and start by observing that $g - f_1 = [\vee_{i=2}^m (f_i - f_1)]^+$ and

$$\sum_{i=2}^m (f_i - f_1)(x_i^*) = \sum_{i=1}^m f_i(x_i^*) - f_1(e)$$

$$\geq \sum_{i=2}^m f_i(x_i) + f_1\left(e - \sum_{i=2}^m x_i\right) - f_1(e) = \sum_{i=2}^m (f_i - f_1)(x_i)$$

for any $(x_2, ..., x_m) \in \mathcal{A}_e^{m-1}$ since if $(x_2, ..., x_m) \in \mathcal{A}_e^{m-1}$ then $(e - \sum\limits_{i=2}^m x_i, x_2, ..., x_m) \in \mathcal{F}_e^m$. Therefore by part (i) we have that

$$(g - f_1)(e) = \sum_{i=2}^m (f_i - f_1)(x_i^*).$$

In particular it holds that

$$g(e) = f_1\left(e - \sum_{i=2}^m x_i^*\right) + \sum_{i=2}^m f_i(x_i^*) = \sum_{i=1}^m f_i(x_i^*),$$

which is the desired formula. ■

Proof of (iii) Now fix an arbitrary $x \in L^+$ and then select an order unit $e \in L$ such that $x \leq e$.
For each $0 < \beta < 1$ let $x_\beta = \beta x + (1 - \beta)e$. Clearly, each x_β is an order unit and $0 \leq x_\beta \leq e$ holds for each $0 < \beta < 1$. We consider the index set $(0, 1)$ directed by the increasing order relation $\succeq$, i.e. $\beta \succeq \gamma$ in $(0, 1)$ if and only if $\beta \geq \gamma$. Clearly $x_\beta \to x$. Since $\mathcal{F}_e^m$ is compact, the continuous function $\mathcal{F}_e^m \to M, (x_1, ..., x_m) \mapsto \sum_{i=1}^m f_i(x_i)$ attains its maximum. By part (ii), we know that g satisfies the Riesz-Kantorovich formula for each order unit. Therefore, by our assumption on the space M and the compactness of $\mathcal{F}_{e_\beta}^m$, for each $0 < \beta < 1$ there exists some $(z_1^\beta, ..., z_m^\beta) \in L_+^m$ such that $\sum_{i=1}^m z_i^\beta = x_\beta$ and

$$\beta g(x) + (1 - \beta)g(e) = g(x_\beta) = \sum_{i=1}^m f_i(z_i^\beta). \tag{8}$$

Since $z_i^\beta \in [0, e]$ for each $\beta \in (0, 1)$ and each $i \in \{1, ..., m\}$ and the order interval $[0, e]$ is compact, there exists a subnet of $\{(z_1^\beta, ..., z_m^\beta)\}$ (which without loss of generality we also denote by $\{(z_1^\beta, ..., z_m^\beta)\}$) such that $z_i^\beta \to z_i$ for some $z_i \in [0, e]$. From $x_\beta - z_i^\beta \in [0, e]$ for each β and the closedness of $[0, e]$, we see that $0 \leq x - z_i \leq e$. So $0 \leq z_i \leq x$ and $\sum_{i=1}^m z_i = x$. Finally, letting $\beta \to 1$ in equation (8) it follows from the continuity of each f_i that

$$g(x) = \sum_{i=1}^m f_i(z_i).$$

Since we already know that $g(x) \geq \sum_{i=1}^m f_i(y_i)$ for all $(y_1, ..., y_m) \in \mathcal{F}_x^m$ we get that $\sup\{\sum_{i=1}^m f_i(y_i) : (y_1, ..., y_m) \in \mathcal{F}_x^m\} = g(x)$. So the proof of this induction step is finished. ■

Finally let α be a limit ordinal and assume that the theorem holds for all image spaces $\prod_{1 \leq j \leq \beta}$ for $\beta < \alpha$. Let $M = \prod_{1 \leq j \leq \alpha} \mathbb{R}$ and let $f_1, f_2, ..., f_m$ be continuous, linear maps from L to M such that $f_i^{(j)}$ is regular for all $i \in \{1, ..., m\}$ and all ordinals j with $1 \leq j \leq \alpha$ and such that the supremum $g := \bigvee_{i=1}^m f_i$ exists in $L^\sim(L, M)$. Let $x \in L_+$. Let $\beta < \alpha$. Since we know from Lemma 4.4 that $\bigvee_{i=1}^m (f_i^{(1),...,(\beta)}) = f_1^{(1),...,(\beta)} \vee ... \vee f_m^{(1),...,(\beta)}$ we can use the induction hypothesis to obtain that

$$g(x)^{(1),...,(\beta)} = \sup\{\sum_{i=1}^m f_i(x_i)^{(1),...,(\beta)} : (x_1, ..., x_m) \in \mathcal{F}_x^m\}.$$

Since we use the lexicographic ordering on M we know that $\sup\{\sum_{i=1}^m f_i(x_i) : (x_1, ..., x_m) \in \mathcal{F}_x^m\}$ exists in M and that we have

$$\sup\{\sum_{i=1}^m f_i(x_i) : (x_1, ..., x_m) \in \mathcal{F}_x^m\}^{(1),...,(\beta)}$$

$$= \sup\{\sum_{i=1}^m f_i(x_i)^{(1),...,(\beta)} : (x_1, ..., x_m) \in \mathcal{F}_x^m\} = g(x)^{(1),...,(\beta)}.$$

Since this holds for all $\beta < \alpha$ we immediately obtain

$$g(x) = \sup\{\sum_{i=1}^m f_i(x_i) : (x_1, ..., x_m) \in \mathcal{F}_x^m\}$$

and the proof is finished. ■

Remark 4.7. One might wonder why we used the lexicographic ordering and not the componentwise ordering. The reason is as follows: throughout the proof we use that in the lexicographic ordering each compact set has an order-maximum. This is not true for the componentwise ordering as can be seen from the following example. Let $f : \mathbb{R} \to \mathbb{R}^2, \lambda \mapsto \lambda(-1, 1)$ and $g : \mathbb{R} \to \mathbb{R}^2, \lambda \mapsto 0$. Then it holds that $\{f(1-y) + g(y) : 0 \leq y \leq 1\} = \{\lambda(-1, 1) : 0 \leq \lambda \leq 1\}$, which does not have an order maximum, since its order supremum equals $(0, 1)$ which is not an element of this set. Since sets of the form $\{f(1-y) + g(y) : 0 \leq y \leq 1\}$ are precisely the sets for which we need an order maximum to exist, there is no hope to prove a similar statement for the componentwise ordering in the same way we did for the lexicographic ordering. Note, however, that [1, Lemma 3.1] and [1, Lemma 3.2], which form the base for steps (i) and (ii) in the proof above, can easily be proven for the componentwise ordering.

The following corollary might give a more general solution to the Riesz-Kantorovich problem than before. It is concerned with the situation of Theorem 4.6 but then for spaces without a strong order unit.

Corollary 4.8. *Let L be a partially ordered vector spaces with a Hausdorff topology for which the order-intervals are compact. Let $\alpha \geq 1$ be an ordinal number and let $M = \prod_{1 \leq j \leq \alpha} \mathbb{R}$ equipped with the lexicographic ordering. Then the following statements are equivalent.*

(1) *There is a pair $f, g \in L^r(L, M)$ of continuous maps such that $f^{(j)}$ and $g^{(j)}$ are regular for all $1 \leq j \leq \alpha$, such that $f \vee g$ exists in $L^{\sim}(L, M)$, but does not satisfy the Riesz-Kantorovich formula for all $x \in L_+$.*

(2) *There is a pair $f, g \in L^r(L, M)$ of continuous maps and a point $x \in L_+$ such that $f^{(j)}$ and $g^{(j)}$ are regular for all $1 \leq j \leq \alpha$, such that $f \vee g$ exists in $L^{\sim}(L, M)$, but the restriction $(f \vee g)|_{L_x}$ is not the supremum of $f|_{L_x}$ and $g|_{L_x}$ in $L^{\sim}(L_x, V)$.*

Proof Assume that (1) holds and let $f, g \in L^r(L, M)$ be a pair of continuous maps such that $f^{(j)}$ and $g^{(j)}$ are regular for all $1 \leq j \leq \alpha$, such that $f \vee g$ exists in $L^{\sim}(L, M)$, but does not satisfy the Riesz-Kantorovich formula. Then we can let $x \in L_+$ such that

$$f \vee g(x) > \sup\{f(y) + g(x - y) : 0 \leq y \leq x\}.$$

Consider the space L_x. As stated before we get that x is a strong order unit for L_x. We will argue by contradiction. Assume that $f|_{L_x} \vee g|_{L_x}$ exists in $L^{\sim}(L_x, M)$ and that

$$(f \vee g)|_{L_x} = f|_{L_x} \vee g|_{L_x}.$$

Then by Theorem 4.6, which we can apply since $f|_{L_x}$ and $g|_{L_x}$ satisfy the conditions of Theorem 4.6 and the space L_x has a strong order unit, we obtain that

$$\begin{aligned}(f|_{L_x} \vee g|_{L_x})(x) &= \sup\{f|_{L_x}(y) + g|_{L_x}(x - y) : y \in L_x, 0 \leq y \leq x\} \\ &= \sup\{f|_{L_x}(y) + g|_{L_x}(x - y) : y \in L, 0 \leq y \leq x\} \\ &< (f \vee g)(x) = (f|_{L_x} \vee g|_{L_x})(x),\end{aligned}$$

where we used the definition of L_x and the assumptions. So we get a contradiction. Therefore (2) holds and we obtain that (1) $\Rightarrow$ (2).

Conversely, suppose that (2) is true. Let $f, g \in L^r(L, M)$ be a pair of continuous maps and let $x \in L_+$ such that $f^{(j)}$ and $g^{(j)}$ are regular for all $1 \leq j \leq \alpha$, such that $f \vee g$ exists in $L^{\sim}(L, M)$, but the restriction $(f \vee g)|_{L_x}$ is not the supremum of $f|_{L_x}$ and $g|_{L_x}$ in $L^{\sim}(L_x, V)$. Again we argue by contradiction. Assume that $f \vee g$ satisfies the Riesz-Kantorovich formula for all $z \in L_+$. Let $h \in L^{\sim}(L_x, M)$ such that $h \geq f|_{L_x}$ and $h \geq g|_{L_x}$. Let $z \in L_x^+$ and $0 \leq y \leq z$. Then we obtain since $y, z - y \in L_x$ that

$$h(z) = h(y) + h(z - y) \geq f|_{L_x}(y) + g|_{L_x}(z - y).$$

Therefore we must have that

$$\begin{aligned}h(z) &\geq \sup\{f|_{L_x}(y) + g|_{L_x}(z - y) : 0 \leq y \leq z\} \\ &= \sup\{f(y) + g(z - y) : 0 \leq y \leq z\} = (f \vee g)(z) = (f \vee g)|_{L_x}(z)\end{aligned}$$

since $f \vee g$ satisfies the Riesz-Kantorovich formula. Therefore we must have that $h \geq (f \vee g)|_{L_x}$. Furthermore we obtain for $z \in L_+$ that

$$(f \vee g)|_{L_x}(z) = (f \vee g)(z) \geq f(z) = f|_{L_x}(z)$$

and similarly $(f \vee g)|_{L_x}(z) \geq g|_{L_x}(z)$. Hence $(f \vee g)|_{L_x} \geq f|_{L_x} \vee g|_{L_x}$. Combining these two facts gives that $(f \vee g)|_{L_x} = f|_{L_x} \vee g|_{L_x}$ which gives a contradiction to our assumption. Therefore (1) must hold and we obtain (2) $\Rightarrow$ (1). ∎

Discussion

In this thesis we discussed some new results on the space $L^r(E,F)$ of regular operators from E to F, where E and F are partially ordered vector spaces. We showed that, if E and F are Riesz spaces, in general the space $L^r(E,F)$ need not be an order-dense subspace of $L^r(E,F^\delta)$, but if it is, that then its lattice operations must be given by the Riesz-Kantorovich formula. Furthermore we showed that if the space E has a strong order unit and F is an arbitrary product of $\mathbb{R}$, equipped with the lexicographic ordering, that, under a few continuity conditions, the Riesz-Kantorovich formula holds. We also showed, in this situation, how to deal with the absence of a strong order unit in E.

There are several possibilities for future research, most of them dealing with the Riesz-Kantorovich formula, which in general is still an open problem. One could for instance investigate the case when E and F are Riesz spaces, but $L^r(E,F)$ is not an order-dense subset of $L^r(E,F^\delta)$. For example, one can try to prove the reverse statement of Theorem 3.2. It may also be interesting to look more into Example 3.6, since this might give a counterexample to the Riesz-Kantorovich theorem and even if it turns out to be no counterexample, it is still interesting to investigate it further.
Since the componentwise ordering on products of $\mathbb{R}$ is used more often than the lexicographic ordering, it is of course of much interest to prove a statement similar to Theorem 4.6 or even Theorem 4.5 for the componentwise ordering. As stated before it is not difficult to show an equivalent for [1, Lemma 3.1] and [1, Lemma 3.2] for this case, but the problem lies in the fact that a compact set no longer needs to have an order-maximum, so the proof of [1, Theorem 3.3] cannot be adapted easily.
Finally it might be interesting to find an explicit condition for when the space $L^r(E,F)$ is an order-dense subset of $L^r(E,F^\delta)$, since then we have a more explicit class of spaces we know we are allowed to apply the Riesz-Kantorovich formula to.

References

[1] C.D. Aliprantis, R. Tourky, N.C. Yannelis, The Riesz-Kantorovich formula and general equilibrium theory, *Journal of Mathematical Economics* 34 (2000), 55-76.

[2] C.D. Aliprantis and K.C. Border, *Infinite dimensional analysis: A hitchhikers guide*, Springer-Verlag, New York, 1999, third edition.

[3] J. Deckers, *Rieszcompleteringen van ruimten van operatoren*, Bachelor thesis, Leiden University, 2015.

[4] O. van Gaans, *Seminorms on ordered vector spaces*, Ph. D. thesis, University of Nijmegen, 1999.

[5] O. van Gaans and A. Kalauch, Disjointness in partially ordered vector spaces, *Positivity* 10 (2006), 573-589.

[6] M. van Haandel, *Completions in Riesz space theory*, Ph. D. thesis, University of Nijmegen, 1993.

[7] T. Jech, *Set theory: the third millennium*, Springer, 2006.

[8] E. De Jonge, A.C.M. Van Rooij, *Introduction to Riesz spaces*, Mathematical Centre, Amsterdam, 1977.

[9] W.A.J. Luxemburg and A.C. Zaanen, *Riesz spaces Volume I*, North-Holland Publishing Company, Amsterdam-London, 1971

[10] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, 1991.

[11] H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, 1974.