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Computing Pareto Fronts of Implicitly Defined Objective Functions

Master thesis

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Contents

Abstract

We propose and discuss by means of examples different methods of finding an Efficient Set and a Pareto Frontier in multi-objective optimization problems. Among those methods are both the probabilistic and the deterministic search e.g. 'the gradient method', Karlin theorem, Karush-Kuhn-Tucker condition. We get the Efficient Set in the explicit form and then we find the Pareto Frontier in parametric or explicit form. The key part of the thesis is a method for finding the Efficient Set and the Pareto Frontier for the implicitly defined objective functions. We consider first the general case and later focus on the special case of two equations of the form: $f_1(x_1,...,x_n)-\phi_1(y_1,y_2)=0, \quad f_2(x_1,...,x_n)-\phi_2(y_1,y_2)=0$ that define two implicitly defined objective functions $y_1=y_1(x_1,...,x_n)$, $y_2=y_2(x_1,...,x_n)$. Next we study the Pareto optimization problem to minimize y_1 and y_2. The solution of this problem is divided into two parts. In the first part we solve the problem of finding the Pareto Efficient Set for functions that are given explicitly. In the second part we discuss and propose methods for finding the Pareto Frontier and Efficient Set for implicitly given objective functions. The results of this work were presented at 28th European Conference on Operational Research "EURO 2016" in Poznań, Poland on July 3-6, 2016, Emmerich M., Sklyar M. "Computing Pareto Fronts of Implicitly Defined Functions", Conference Handbook p. 20.

1 Introduction

We consider a vector valued function:
$f(x)=(f_1(x_1,...x_n),f_2(x_1,...x_n),...,f_m(x_1,...x_n))$.

Definition: The point $\bar{x}$ is called **Pareto optimal**, if there do not exist any point x such that:
$f_i(x)\leq f_i(\bar{x})$ for all $i=1,...,m$ and for some $i_0\in\{1,..,m\}$: $f_{i_0}(x)<f_{i_0}(\bar{x})$.

- *The set of all Pareto optimal points is called* ***the Efficient Set*** *and denoted by* E_f.
- *The set* $\{(f_1(\bar{x}),...,f_m(\bar{x})), \quad \bar{x}\in E_f\}$ *is called* ***the Pareto Frontier*** *and denoted by* F_f.

The problem of multi-criteria optimization (minimization) for the function $f(x)$ *consists in determining of both sets:* E_f *and* F_f.

The setting of the multi-criteria optimization problem is coming from the Italian economist V. Pareto (1848-1923). In order to honour him the optimal points are called Pareto Optimal and multi-criteria optimization is often referred to as Pareto Optimization. The main idea of the Pareto Optimality, which was used by him in his work in economy is, that with the availability of couple of criteria for optimal points, one cannot improve one criteria, without worsen others. Especially in the control of complicated technological objects, in finance and supply chain management, there are many criteria and they are disparate with each other. For example this is the case in the problem of time transition of technological system from one state to another combined with the consumption of various types of resources. Nevertheless, the Pareto approach is a useful tool in formulating the economic policy. The Pareto optimal

solutions of continuous problems typically form an infinite set. The subsequent selection of the proper optimal Pareto points is made by the experts in the field, while non-Pareto optimal points are discarded a-priori.
Generally, finding the minimal elements by Pareto one can treat as finding the minimal element of a partially ordered set.
Let the system criteria consist of two functions f_1, f_2. The two-dimensional codomain can be equipped with a partial order as follows: vector function $f(x) = (f_1(x_1, ..., x_n), f_2(x_1, ..., x_n)) \leq f(y) = (f_1(y_1, ..., y_n), f_2(y_1, ..., y_n))$ means that $f(x) \leq f(y)$ if $f_1(x) \leq f_1(y)$ and $f_2(x) \leq f_2(y)$. A Pareto optimal solution is an element $\hat{x}$ of the domain of f such that $f(\hat{x})$ is a minimal element of the range of f. The Pareto Frontier or Pareto set is the set of parameterizations (allocations) that are all Pareto efficient. That is, it is the image under f of the Efficient Set. In the case of two objective functions $f_1(x_1, ..., x_n)$ and $f_2(x_1, ..., x_n)$, the Pareto Frontier is represented as a curve or a couple of curves in-plane (f_1, f_2), when the f_i are sufficiently smooth. In other words one cannot change the arguments from the Efficient Set in such way that both functions f_1 and f_2 decrease their values.
When finding the optimal points by Pareto one may use deterministic, and probability methods of Bayesian type, Kriging optimization, set-oriented Newton methods see eg. Deutz, Emmerich, Schutze, etc. [1]-[3]. In the deterministic approach, an important application has also The Karlin Theorem [4]-[5].

Karlin Theorem [4]-[5, p. 51] *Let $f : \mathbb{R}^n \to \mathbb{R}$, will be continuously differentiable in $x^* \in \mathbb{R}^n$ and convex functions. For x^* to be in the Efficient Set it is necessary and sufficient that there exist non-negative numbers $\lambda_1, \lambda_2, ..., \lambda_m$ with $\sum_{i=1}^m \lambda_i = 1$ such that the point x^* is a minimum of the function $\sum_{i=1}^m \lambda_i f_i(x)$; this is, $\sum_{i=1}^m \lambda_i f_i(x^*) \leq \sum_{i=1}^m \lambda_i f_i(x)$ for all $x \in \mathbb{R}^n$.*

More generally when considering the problem with constrains $g_i(x) \leq 0$, $i = 1, ..., k$, the solutions must satisfy the additional restrictions $g_i(x) \leq 0$, one can use the Karush-Kuhn-Tucker conditions [6]-[7].

Karush-Kuhn-Tucker conditions [6]-[7] *Let the following condition be satisfied: $\nabla g_i(x)$ for all $i = 1, ..., k$, are linearly independent and $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable in $x^* \in \mathbb{R}$. If point x^* is an optimal point by Pareto, then there exist vectors: $\lambda \in \mathbb{R}^k$ and $\mu \in \mathbb{R}^m$ such that $\sum_{i=1}^m \lambda_i \cdot \nabla f_i(x^*) + \sum_{j=1}^k \mu_j \cdot \nabla g_j(x^*) = 0$, $\mu_j \cdot g_j(x^*) = 0$, for $j = 1, ..., k$, and $\lambda > 0$, $\mu \geq 0$.*

In this work we discuss different approaches to multi-criteria optimization. The main result of this work is given in Section 3, where we consider the problem of multi-criteria optimization for implicitly defined functions. We first show that gradient-based methods introduced in the first part of the thesis for explicitly defined functions can be used to solve problems for implicitly defined objective functions. Then, we introduce a class of implicitly given functions of so-called separable type. For this class we show that the Efficient Set for the vector function of the arguments is also the Efficient Set for the vector of implicit functions. This allows to obtain the Pareto Frontier for these implicit functions.

When solving Pareto problems for implicitly defined functions, it is necessary to introduce first methods for solving multicriteria-optimization problems for certain explicitly defined functions.

Hence the first part of this work is devoted to different approaches of solving the Pareto problem for explicitly defined functions.

2 Method based on the proportionality of gradients

Lemma. *Let function $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, let vector $l \in \mathbb{R}^n$ is such that $\langle \nabla f, l \rangle < 0$. Then $\exists \epsilon > 0$ such that for $0 < t < \epsilon : f(x_0 + tl) < f(x_0)$.*

Proof. From Taylor's form we have $f(x_0 + tl) = f(x_0) + \langle \nabla f, tl \rangle + \bar{o}(t \cdot |l|) = f(x_0) + t\langle \nabla f, l \rangle + \bar{o}(t \cdot |l|)$. Using our conditions we know: $\langle \nabla f, l \rangle < 0$, $t > 0$, then there exists $\epsilon > 0$ such that for $0 < t < \epsilon$; $t\langle \nabla f, l \rangle + \bar{o}(t \cdot |l|) < 0$. From which directly follows: $f(x_0 + tl) < 0$.

If $0 > \langle \nabla f, l \rangle = |\nabla f| \cdot |l| \cos \alpha$, where $0 < \alpha < 2\pi$ is an angle between them, means that should hold: $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$.

Theorem. *Let two objective functions $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable and $\nabla f_1(x) \neq 0 \quad \nabla f_1(x) \neq 0$ for all $x \in \mathbb{R}^n$. Then for $\hat{x} \in \mathbb{R}^n$ to belong to the Efficient Set it is necessary that $\nabla f_1(\hat{x}) = -\lambda \nabla f_2(\hat{x})$ where $\lambda \geq 0$.*

Proof. The condition $\nabla f_1 = -\lambda \nabla f_2$, $\lambda > 0$, means that an angle between gradients ∇f_1 and ∇f_2 (similarly the angle between $-\nabla f_1$ and $-\nabla f_2$) is equal to π.[1] We will prove it by contradiction. Let the angle between $-\nabla f_1$ and $-\nabla f_2$, which we denote as α will be smaller than π, then in the direction $l = -\frac{\nabla f_1}{2|\nabla f_1|} - \frac{\nabla f_2}{2|\nabla f_2|}$ functions f_1 and f_2 are decreasing. Indeed $\langle \nabla f_1, l \rangle = \langle \nabla f_1, -\frac{\nabla f_1}{2|\nabla f_1|} - \frac{\nabla f_2}{2|\nabla f_2|} \rangle = -\frac{1}{2}\langle -\nabla f_1, -\frac{\nabla f_1}{|\nabla f_1|} \rangle - \frac{1}{2}\langle -\nabla f_1, -\frac{\nabla f_2}{|\nabla f_2|} \rangle = -\frac{1}{2} - \frac{1}{2}\cos \alpha < 0$, since $0 < \alpha < \pi$, because by the assumptions an angle α is smaller than π. Therefore using Lemma from above the function f_1 in the direction l is decreasing. Analogously $\langle \nabla f_2, l \rangle < 0$, which means that function f_2 in the direction l is decreasing as well. Then there exist a direction in which both functions are decreasing. In this case a point x_0 cannot be a Pareto efficient point, since in this case there exist arbitrarily close points x in which both functions f_1 and f_2 take values smaller than in point x_0.

Note. The proportionality condition on the gradient that is presented above in the Theorem is sufficient in case when the two continuously differentiable objective functions $f_1 : \mathbb{R}^n \to \mathbb{R}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}$ are convex. It follows directly from Karlin Theorem. Indeed in case of two functions f_1 and f_2 the condition of reaching minimum of function $\lambda_1 f_1(x_0) + \lambda_2 f_2(x_0)$, $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 = 1$, means that $\lambda_1 \nabla f_1 + \lambda_2 \nabla f_2 = 0$ in point x_0. The condition that $\lambda_1 + \lambda_2 = 1$, means that at least one of $\lambda_1, \quad \lambda_2$ is not equal to zero. Let for example $\lambda_1 \neq 0$, then $\nabla f_1 = \frac{-\lambda_2}{\lambda_1} \nabla f_2$. Let we denote $\lambda = -\frac{\lambda_2}{\lambda_1}$, then $\nabla f_1 = -\lambda \nabla f_2$, where $\lambda > 0$ since if $\lambda = 0$ then $\nabla f_1 = 0$ which is contradictory to our assumptions.

A method to construct the Pareto Frontier could now be based on the following principle, where we are able to note that $\langle \nabla f_1, \nabla f_2 \rangle = |\nabla f_1| \cdot |\nabla f_2| \cdot \cos \alpha = |\nabla f_1| \cdot |\nabla f_2| \cdot \cos \pi = -|\nabla f_1| \cdot |\nabla f_2|$, where α is an angle between $\nabla f_1(x_1, x_2, ..., x_n)$ and $\nabla f_2(x_1, x_2, ..., x_n)$. In order to build this level curves (on which the equation from above will be satisfied) we normalize those vectors and consider:

$$H_1 = \frac{\nabla f_1(x_1, x_2, ..., x_n)}{|\nabla f_1(x_1, x_2, ..., x_n)|},$$

[1]Means that vectors of anti-gradients and similarly gradients have opposite directions.

$$H_2 = \frac{\nabla f_2(x_1, x_2, ..., x_n)}{|\nabla f_2(x_1, x_2, ..., x_n)|}.$$

Then level curves that we are looking for can be described with the equation:

$$\langle H_1, H_2 \rangle = \left\langle \frac{\nabla f_1(x_1, x_2, ..., x_n)}{|\nabla f_1(x_1, x_2, ..., x_n)|}, \frac{\nabla f_2(x_1, x_2, ..., x_n)}{|\nabla f_2(x_1, x_2, ..., x_n)|} \right\rangle = \frac{|\nabla f_1| \cdot |\nabla f_2| \cdot \cos\alpha}{|\nabla f_1| \cdot |\nabla f_2|} = \cos\pi$$
$$= -1.$$

Based on this equation different methods can be used to find points that (approximately) satisfy the equation:

- Probabilistic method: Sample uniformly points in the search space and evaluate the equation.

- Deterministic method: Solve the equation analytically or by using symbolic equation solver (e.g. Mathematica)

Let us proceed with examples that demonstrate these methods. The first example will demonstrate the stochastic method. Moreover we will demonstrate the deterministic method by a number of examples. The examples increase in their difficulty. They not only serve to demonstrate the method, but some of them are also of interest i.e. the Generalized Schaffer Problems [8, p.922-936] and the Dent problem [9], [10, pp.198-199], because they are test problems used to assess the performance of multi-criteria optimization heuristics in the literature. The exact solution of them is important as a reference solution.

2.1 First example: probabilistic approach

Let us consider the simplest example where level curves of the functions $f_1(x_1, x_2)$, $f_2(x_1, x_2)$ are circles with the centres in $(1, 1)$ and $(-1, -1)$:

$$\begin{cases} f_1(x_1, x_2) = \sqrt{(x_1 - 1)^2 + (x_2 - 1)^2}, \\ f_2(x_1, x_2) = \sqrt{(x_1 + 1)^2 + (x_2 + 1)^2}. \end{cases}$$

In order to make the equations look cleaner by abuse of notation we will denote $f_1(x_1, x_2) = f_1$ and $f_2(x_1, x_2) = f_2$.

We proceed with finding the gradients of each function:

$$\nabla f_1 = \left(\frac{x_1 - 1}{\sqrt{(x_1 - 1)^2 + (x_2 - 1)^2}}, \frac{x_2 - 1}{\sqrt{(x_1 - 1)^2 + (x_2 - 1)^2}} \right),$$
$$\nabla f_2 = \left(\frac{x_1 + 1}{\sqrt{(x_1 + 1)^2 + (x_2 + 1)^2}}, \frac{x_2 + 1}{\sqrt{(x_1 + 1)^2 + (x_2 + 1)^2}} \right).$$

The scalar product of the gradients is given by:

$$\langle \nabla f_1, \nabla f_2 \rangle =$$
$$\frac{x_1^2 - 1}{\sqrt{(x_1 - 1)^2 + (x_2 - 1)^2}\sqrt{(x_1 + 1)^2 + (x_2 + 1)^2}} + \frac{x_2^2 - 1}{\sqrt{(x_1 - 1)^2 + (x_2 - 1)^2}\sqrt{(x_1 + 1)^2 + (x_2 + 1)^2}} =$$
$$\frac{x_1^2 + x_2^2 - 2}{\sqrt{(x_1 - 1)^2 + (x_2 - 1)^2}\sqrt{(x_1 + 1)^2 + (x_2 + 1)^2}}.$$

In this case:

$$|\nabla f_1| = 1,$$
$$|\nabla f_2| = 1.$$

On the other hand:

$$\langle \nabla f_1, \nabla f_2 \rangle = |\nabla f_1| \cdot |\nabla f_2| \cdot \cos\alpha.$$

Here α is an angle between those two gradient vectors. Hence

$$\cos\alpha = \frac{x_1^2 + x_2^2 - 2}{\sqrt{(x_1-1)^2 + (x_2-1)^2}\sqrt{(x_1+1)^2 + (x_2+1)^2}}, \ \cos\alpha \in [-1, 1].$$

The anti-gradient is a vector which points to the direction of steepest descent. In the effective point (Pareto in 2D), anti-gradients have opposite direction, since in those point one cannot decrease the values of both functions simultaneously and an angle between them will be equal to π i.e $\cos\alpha = -1$. Therefore, in the probabilistic method, we write a program to find points in a uniform random sample across the function's domain that will satisfy that condition (for example $(\nabla f_1, \nabla f_2) \leq -0,99$) to find the Efficient Set, from which we can calculate approximately the Pareto Front.
In Figure 1 we provide an example of a Wolfram Mathematica script that implements this method:

```
f1 = Sqrt[(x1 - 1)^2 + (x2 - 1)^2]; f2 = Sqrt[(x1 + 1)^2 + (x2 + 1)^2];
g1 = Grad[f1, {x1, x2}];
g2 = Grad[f2, {x1, x2}];
gg = g1.g2; d = 2; e = 0; Do[{Subscript[z,
   i] = {Subscript[a1, i] = -d + 2 d RandomReal[],
    Subscript[a2, i] = -d + 2 d RandomReal[]},
  If[(gg /. x1 -> Subscript[a1, i] /.
     x2 -> Subscript[a2, i]) <= -0.999, {Print[i, "   ",
    Subscript[f1, i] =
     f1 /. x1 -> Subscript[a1, i] /. x2 -> Subscript[a2, i], "     ",
     Subscript[f2, i] =
     f2 /. x1 -> Subscript[a1, i] /. x2 -> Subscript[a2, i]],
    e = e + 1, q = e, Subscript[n, e] = Subscript[z, i],
    Subscript[m, e] = {Subscript[f1, i], Subscript[f2, i]},
    Subscript[k, e] = Subscript[f1, i],
    Subscript[b, e] = Subscript[f2, i]}]}, {i, 10000}]

Do[{If[((Subscript[b, j] - Subscript[b, i]) <=
      0) && (Subscript[k, j] - Subscript[k, i]) <= 0, {Subscript[k,
     i] = Subscript[k, j], Subscript[b, i] = Subscript[b, j],
    Subscript[n, i] = Subscript[n, j]}],
  Subscript[v1, i] = {Subscript[k, i], Subscript[b, i]}}, {j, 1,
  q}, {i, 1, q}]
aa = ListPlot[Table[Subscript[v1, i], {i, 1, q}],
  PlotStyle -> PointSize[0.01]]
```

Figure 1: Wolfram Mathematica code

We get the Efficient Set:

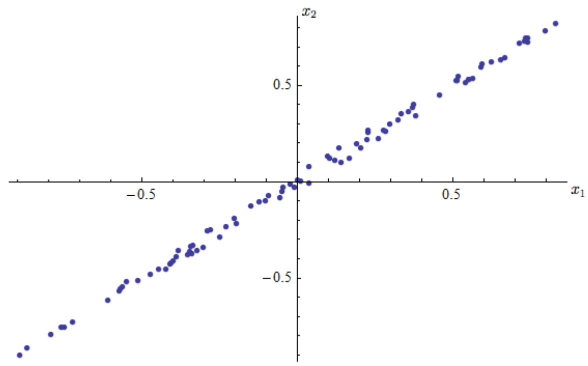

Figure 2: The Efficient Set

and the Pareto Frontier:

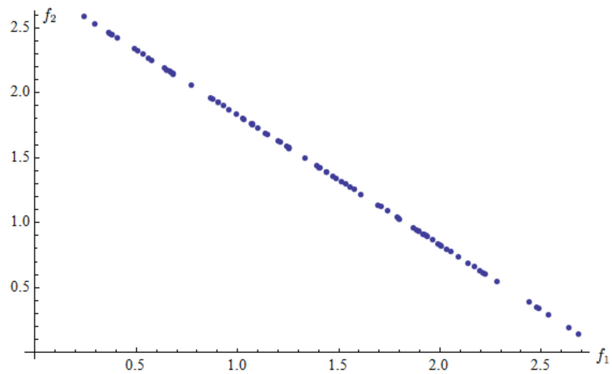

Figure 3: The Pareto Frontier

Thereby it is visible that Efficient Set is $x_1, x_2 \in [-1, 1]$ and $x_1 = x_2$, when $|x_1| = |x_2| \geq 1$ then $(\triangledown f_1, \triangledown f_2) = 1$. The Pareto Frontier is a line segment defined by $f_1, f_2 \geq 0$ and $f_1 + f_2 = 2\sqrt{2}$.

2.2 Analytic approach: first example

This method is much more precise and will determine the exact Pareto Frontier line. This method is similar to the previous one but here we have to find analytically solutions that satisfy $\cos\alpha = -1$, where α is the angle between the gradients.
Moreover previous example was easy since $|f_1| = 1$ and $|f_2| = 1$. Nevertheless one can notice that if we denote:

$$H_1 = \frac{\triangledown f_1}{|\triangledown f_1|},$$

$$H_2 = \frac{\triangledown f_2}{|\triangledown f_2|}.$$

Then

$$\langle H_1, H_2 \rangle = \frac{\triangledown f_1, \triangledown f_2}{|\triangledown f_1| \cdot |\triangledown f_2|} = \cos\alpha = -1.$$

Let us proceed with a less trivial example. Let us consider $f_1(x_1, x_2) = f_1$ and $f_2(x_1, x_2) = f_2$:

$$\begin{cases} f_1 = \sqrt{\frac{(x_1-1)^2}{2} + (x_2-1)^2}, \\ f_2 = \sqrt{(x_1+1)^2 + \frac{(x_2+1)^2}{2}}. \end{cases}$$

We get

$$\nabla f_1 = \left(\frac{x_1 - 1}{2\sqrt{\frac{(x_1-1)^2}{2} + (x_2-1)^2}}, \frac{x_2 - 1}{\sqrt{\frac{(x_1-1)^2}{2} + (x_2-1)^2}} \right),$$

$$\nabla f_2 = \left(\frac{x_1 + 1}{\sqrt{(x_1+1)^2 + \frac{(x_2+1)^2}{2}}}, \frac{x_2 + 1}{2\sqrt{(x_1+1)^2 + \frac{(x_2+1)^2}{2}}} \right).$$

Hence

$$H_1 = \left(\frac{x_1 - 1}{\sqrt{-2x_1 + x_1^2 - 8x_2 + 4x_2^2 + 5}}, \frac{2(x_2 - 1)}{\sqrt{-2x_1 + x_1^2 - 8x_2 + 4x_2^2 + 5}} \right),$$

$$H_2 = \left(\frac{2(x_1 + 1)}{\sqrt{8x_1 + 4x_1^2 + 2x_2 + x_2^2 + 5}}, \frac{x_2 + 1}{\sqrt{8x_1 + 4x_1^2 + 2x_2 + x_2^2 + 5}} \right),$$

$$\langle H_1, H_2 \rangle = \frac{2(x_1^2 - 1) + 2(x_2^2 - 1)}{\sqrt{-2x_1 + x_1^2 - 8x_2 + 4x_2^2 + 5}\sqrt{8x_1 + 4x_1^2 + 2x_2 + x_2^2 + 5}},$$

$\langle H_1, H_2 \rangle$ should be equal to -1. Therefore we get the equation

$$\frac{2(x_1^2 + x_2^2 - 2) + \sqrt{-2x_1 + x_1^2 - 8x_2 + 4x_2^2 + 5}\sqrt{8x_1 + 4x_1^2 + 2x_2 + x_2^2 + 5}}{\sqrt{-2x_1 + x_1^2 - 8x_2 + 4x_2^2 + 5}\sqrt{8x_1 + 4x_1^2 + 2x_2 + x_2^2 + 5}} = 0.$$

But one cannot divide by zero so only the numerator can be equal to zero i.e.

$$2(x_1^2 + x_2^2 - 2) + \sqrt{-2x_1 + x_1^2 - 8x_2 + 4x_2^2 + 5}\sqrt{8x_1 + 4x_1^2 + 2x_2 + x_2^2 + 5} = 0, \tag{1}$$

hence

$$4(x_1^2 + x_2^2 - 2)^2 = (-2x_1 + x_1^2 - 8x_2 + 4x_2^2 + 5)(8x_1 + 4x_1^2 + 2x_2 + x_2^2 + 5).$$

Expanding this equation we get:

$$9 + 30x_1 + 25x_1^2 - 30x_2 - 68x_1x_2 - 30x_1^2x_2 + 25x_2^2 + 30x_1x_2^2 + 9x_1^2x_2^2 = 0,$$

additionally simplifying it:

$$(-3 + 5x_2 + x_1(-5 + 3x_2))^2 = 0.$$

The equation from above gives us $x_1 = \frac{3-5x_2}{-5+3x_2}$, since one cannot divide by zero the following condition has to be satisfied: $x_2 \neq \frac{5}{3}$. Let us notice that in the equation (1):

$$\sqrt{-2x_1 + x_1^2 - 8x_2 + 4x_2^2 + 5}\sqrt{8x_1 + 4x_1^2 + 2x_2 + x_2^2 + 5} \geq 0$$

therefore $2(x_1^2+x_2^2-2)$ should be less than zero. Then substituting $\frac{3-5x_2}{-5+3x_2}$ gives us:

$$\left(\frac{3-5x_2}{-5+3x_2}\right)^2 + x_2^2 - 2 < 0,$$

solving it with Mathematica solver we get that $-1 < x_2 < 1$. Additionally when looking at the values under the roots in the equation (1) the values are positive since:

$$\sqrt{-2x_1+x_1^2-8x_2+4x_2^2+5} = \sqrt{\frac{(1+x_2)^2(41-30x_2+9x_2^2)}{(5-3x_2)^2}}$$

and

$$\sqrt{8x_1+4x_1^2+2x_2+x_2^2+5} = \sqrt{\frac{(1+x2)^2(41-30x2+9x2^2)}{(5-3x2)^2}}.$$

Hence $x_2 \in (-1,1)$, gives us the Efficient Set:

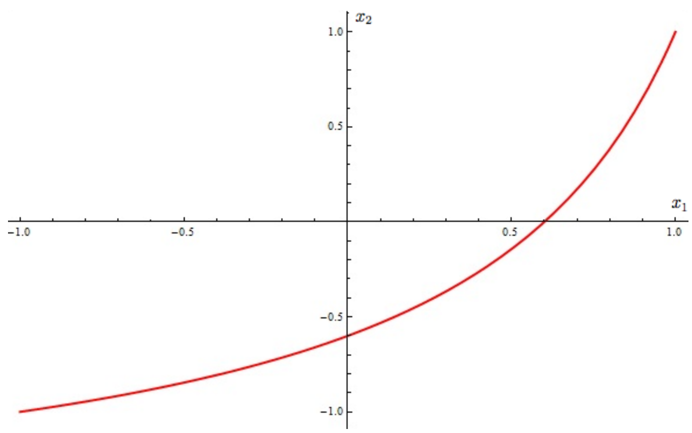

Figure 4: The Efficient Set

In the previous example, level curves were the circles with the centres belonging to the interval $[-1,1]$. In this example level curves are ellipses. Let us consider for example the following system:

$$\begin{cases} \frac{(x_1-1)^2}{2} + (x_2-1)^2 = 1.98338, \\ (x1+1)^2 + \frac{(x2+1)^2}{2} = 0.82410, \end{cases}$$

then $(x_1,x_2) = (-0.53191, 0.1)$ is the point, where level curves touch. The functions f_1, f_2 are equal here $f_1 = \sqrt{1.98338} = 1.40833$ and $f_2 = \sqrt{0.82410} = 0.90780$. In Figure 5 we show the situation of touching level curves.

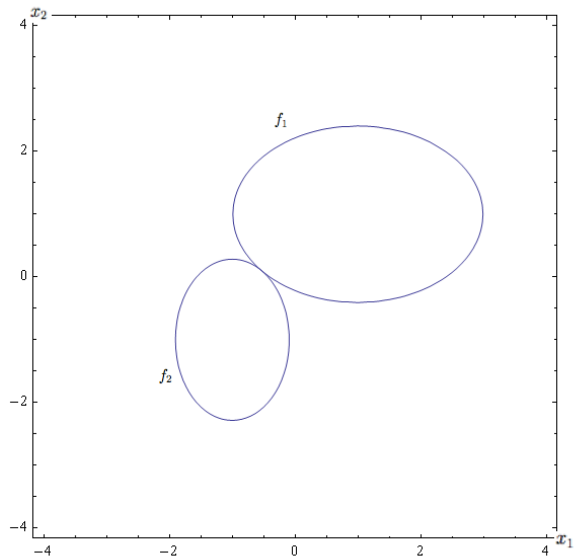

Figure 5: Picture of touching level curves of $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$

In the Figure 6 the upper line is a straight line and the bottom line is the Pareto Frontier, for the problem described above:

$$\begin{cases} f_1 = \sqrt{(-1+x_2)^2 + 1/2(-1+(3-5x_2)/(-5+3x_2))^2} = \sqrt{3}\frac{1-x_2}{5-3x_2}\sqrt{19-10x_2+3x_2^2}, \\ f_2 = \sqrt{\frac{1}{2}(1+x_2)^2 + (1+(3-5x_2)/(-5+3x_2))^2} = \sqrt{\frac{3}{2}}\frac{1+x_2}{5-3x_2}\sqrt{11-10x_2+3x_2^2}. \end{cases}$$

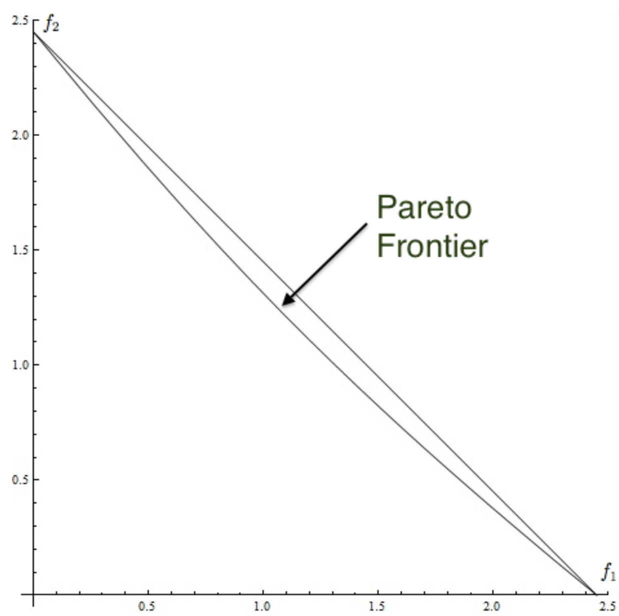

Figure 6: The Pareto Frontier. The straight line satisfies the condition $f_1 + f_2 = \sqrt{6}$.

2.3 Analytic approach: second example

Let us consider the following example, which is based on the previous one. This example is a bit harder but level curves of the functions are ellipses as well. It is done in order to show a different method of the solution.

$$\begin{cases} f_1(x_1,x_2) = \sqrt[4]{(x_1-1)^4+(x_2-1)^4}, \\ f_2(x_1,x_2) = \sqrt{(x_1+1)^2+\frac{(x_2+1)^2}{2}}. \end{cases}$$

Analogously, as in the previous example we will denote $f_1(x_1,x_2) = f_1$ and $f_2(x_1,x_2) = f_2$. We will use the our method a bit differently than in the previous case (without the normalization of the gradients). This is caused by the fact that with the normalization of them the example is pretty hard to solve. In the points from the Efficient Set level curves of f_1 and f_2 touch and the gradients have different directions, hence we can use the following property: $\triangledown f_1 = -\lambda \triangledown f_2$ for $\lambda > 0$, $\lambda \in \mathbb{R}$.
We have:

$$\triangledown f_1 = \left(\frac{(-1+x_1)^3}{((-1+x_1)^4+(-1+x_2)^4)^{3/4}}, \frac{(-1+x_2)^3}{((-1+x_1)^4+(-1+x_2)^4)^{3/4}} \right),$$

$$\triangledown f_2 = \left(\frac{1+x_1}{\sqrt{(1+x_1)^2+\frac{1}{2}(1+x_2)^2}}, \frac{1+x_2}{2\sqrt{(1+x_1)^2+\frac{1}{2}(1+x_2)^2}} \right).$$

Therefore one must have the two conditions:

$$\frac{(-1+x_1)^3}{((-1+x_1)^4+(-1+x_2)^4)^{3/4}} = -\lambda \frac{1+x_1}{\sqrt{(1+x_1)^2+\frac{1}{2}(1+x_2)^2}}, \tag{2}$$

$$\frac{(-1+x_2)^3}{((-1+x_1)^4+(-1+x_2)^4)^{3/4}} = -\lambda \frac{1+x_2}{2\sqrt{(1+x_1)^2+\frac{1}{2}(1+x_2)^2}}, \tag{3}$$

for $\lambda > 0$ it is not so hard to conclude that $-1 < x_1 < 1$ and $-1 < x_2 < 1$, otherwise one side of equation and/or will have different sign than the other. Because in case when:

- $x_1 \le -1$ in the equation (2): $\frac{(-1+x_1)^3}{((-1+x_1)^4+(-1+x_2)^4)^{3/4}} < 0$ and $-\lambda \frac{1+x_1}{\sqrt{(1+x_1)^2+\frac{1}{2}(1+x_2)^2}} \ge 0$,
- $x_1 \ge 1$ in the equation (2): $\frac{(-1+x_1)^3}{((-1+x_1)^4+(-1+x_2)^4)^{3/4}} \ge 0$ and $-\lambda \frac{1+x_1}{\sqrt{(1+x_1)^2+\frac{1}{2}(1+x_2)^2}} < 0$,
- $x_2 \le -1$ in the equation (3): $\frac{(-1+x_2)^3}{((-1+x_1)^4+(-1+x_2)^4)^{3/4}} < 0$ and $-\lambda \frac{1+x_2}{2\sqrt{(1+x_1)^2+\frac{1}{2}(1+x_2)^2}} \ge 0$,
- $x_2 \ge 1$ in the equation (3): $\frac{(-1+x_2)^3}{((-1+x_1)^4+(-1+x_2)^4)^{3/4}} \ge 0$ and $-\lambda \frac{1+x_2}{2\sqrt{(1+x_1)^2+\frac{1}{2}(1+x_2)^2}} < 0$

which would bring us to contradictions. Hence $0 < f_1 < 2\sqrt{2} \approx 2.8284$ and $0 < f_2 < \sqrt{6} \approx 2.4495$.
From the equations (2) by (3) and we get by division the following:

$$\frac{(-1+x_1)^3}{(-1+x_2)^3} = \frac{2(1+x_1)}{1+x_2}, \tag{4}$$

which we solve for the parameter x_2.

In order to do so we have to approximate the solution using one of the well known methods which is either the Taylor approximation or using an interpolation polynomial. Either way the answer will be similar.
Solving the cubic equation (4) with the fixed x_2 gives us two complex solutions and one real solution:

$$x_1 = 1 + ((6^{2/3}(-1+x_2)^3(1+x_2) + 6^{1/3}(-9+9x_2+18x_2^2-18x_2^3-9x_2^4+9x_2^5+\sqrt{3}\sqrt{-(-1+x_2)^6(1+x_2)^3(-29-21x_2-6x_2^2+2x_2^3)})^{2/3}))/(3(-9+9x_2+18x_2^2-18x_2^3-9x_2^4+9x_2^5+\sqrt{3}\sqrt{-(-1+x_2)^6(1+x_2)^3(-29-21x_2-6x_2^2+2x_2^3)})^{1/3}(1+x_2)).$$

One can approximate the solution by resolving it by Taylor series. On the other hand one can build the Lagrange interpolation polynomial [2] in, for example, points:
$x_2 = (0, 0.15, 0.3, 0.45, 0.6, 0.75, 0.9)$. Finding the roots of the equation we get

x_2	0	0.15	0.3	0.45	0.6	0.75	0.9
x_1	-0.179509	-0.0165239	0.152708	0.327187	0.506091	0.688739	0.874572

Using this information we can build the Lagrange interpolation polynomial which has the form:

$x_1 = -0.179509 + 1.0632x_2^2 + 0.165008x_2^2 - 0.0643367x_2^3 + 0.0180231x_2^4 - 0.00106339x_2^5 - 0.00278178x_2^6.$

Let us notice that we chose to pick seven points in order to get the sixth-degree polynomial, which is already quite accurate. This gives us an approximation of the Efficient Set which is depicted in Figure 7:

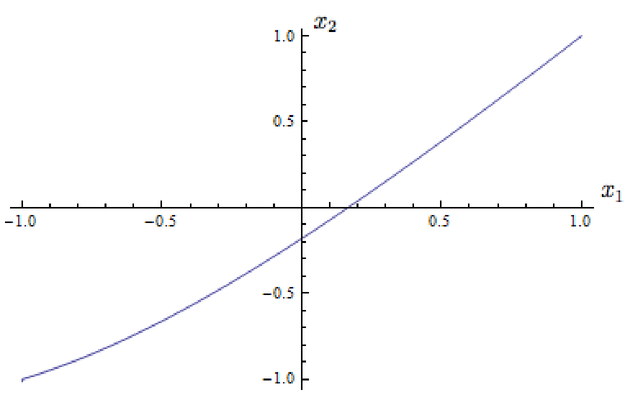

Figure 7: An approximation of the Efficient Set by therms of Lagrange interpolation as described in the main text.

On the other hand we can consider this problem from a different perspective. Solving the equation (4):

$$\frac{(-1+x_1)^3}{(-1+x_2)^3} = \frac{2(1+x_1)}{1+x_2},$$

[2]The Lagrange interpolation polynomial is the unique polynomial of lowest degree that goes through the requested points.

let us denote

$$y = \frac{x_1 - 1}{x_2 - 1}.$$

So

$$y^3 = \frac{2(1 + x_1)}{1 + x_2} = \frac{2((x_2 - 1)y + 2)}{x_2 + 1}$$

$$y^3 = 2\frac{x_2 - 1}{x_2 + 1}y + \frac{4}{x_2 + 1} =$$

$$2\frac{x_2 + 1 - 2}{x_2 + 1}y + \frac{4}{x_2 + 1} =$$

$$2(1 - \frac{2}{x_2 + 1})y + \frac{4}{x_2 + 1},$$

hence with $a = \frac{2}{x_2+1}$, one has to solve for $y \in \mathbb{R}$:

$$y^3 = 2(1 - a)y + 2a.$$

Note that x_2 has restrictions: $-1 < x_2 < 1$ precisely established. Hence $a > 1$. Thus there is only one sign change in the coefficients of the monomials in y in the equation. Hence it has one real positive root $y_1 = y(a)$ by Discrete's Rule of Signs. Since there are no sign change in the polynomial with y replaced by $-y,$ (i.e. $-y^3 - 2(a - 1)y - 2a$). The cubic equation $y^3 = 2(1-a)y+2a$ has no negative roots. So it has a unique positive root y_1 and two complex (conjugative) roots. Thus we obtain:

$$x_1 = y_1(a) \cdot (x_2 - 1) + 1 = y_1\left(\frac{2}{x_2 + 1}\right) \cdot (x_2 - 1) + 1.$$

The Efficient Set can be found using following way. We assign the values of x_2, we find a then we can find the only positive root of the equation $y^3 = 2(1 - a)y + 2a$ and finally we find x_1 from $x_1 = y_1\left(\frac{2}{x_2+1}\right) \cdot (x_2 - 1) + 1.$

Let us now find the function f_1 in dependence of f_2. We can do this with the usage of the sixth-degree Lagrange interpolation polynomial. In order to do this let us give seven random values for the point x_2 which would be $(-1, -0.4, -0.1, 0.3, 0.5, 0.7, 1)$ and find the values of f_1, f_2 in those points. Those values are

x_2	-1	-0.4	-0.1	0.3	0.5	0.7	1
f_1	2.37164	1.77716	1.43013	0.93230	0.67230	0.40673	5.68627
f_2	0.01022	0.60144	0.95786	1.47436	1.74556	2.02323	2.44893

hence the interpolation polynomial is:

$$f_2 = 2.382336764776 - 1.047576180478f_1 + 0.1038896808395f_1^2 - 0.07648397121814f_1^3$$
$$+0.03505132435256f_1^4 - 0.008799351949231296f_1^5 + 0.000919102458746087f_1^6.$$

We get an approximation of the Pareto Frontier:

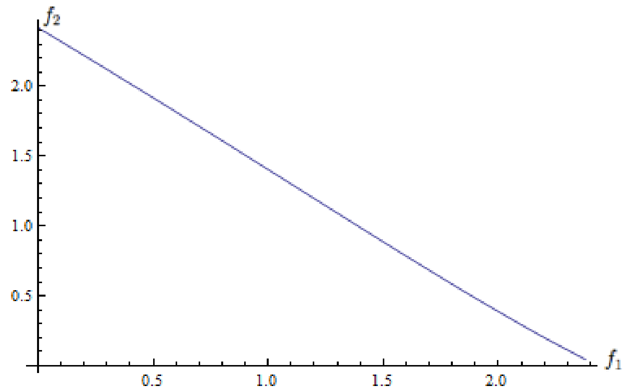

Figure 8: An approximation of the Pareto Frontier

As an example, for:

$$\begin{cases} f_1 = \sqrt[4]{(x_1-1)^4 + (x_2-1)^4} = 1, \\ f_2 = \sqrt{(x_1+1)^2 + \frac{(x_2+1)^2}{2}} = 1.4. \end{cases}$$

We get the contour plot

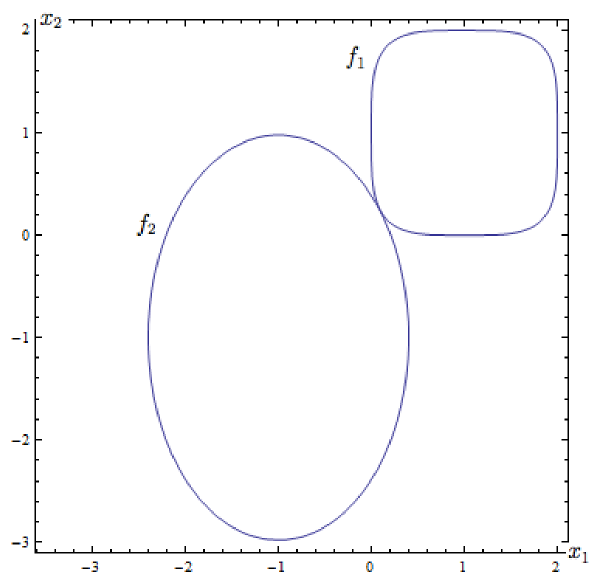

Figure 9: The contour plot, where points of the Efficient Set are $x_1 \approx 0.09$ and $x_2 \approx 0.25$.

2.4 Analytic approach: third example

Let us consider an example where f_1, f_2 are the functions of three variables:

$$\begin{cases} f_1 = x_1^2 + x_2^2 + x_3^2, \\ f_2 = (x_1-1)^2 + (x_2-1)^2 + 2(x_3-1)^2. \end{cases}$$

It is worth to notice that level curves of the first function is a sphere with centre in $(0,0,0)$ and for the second function is an ellipsoid with the centre in $(1,1,1)$. Gradients of f_1 and f_2 are:

$$\nabla f_1 = (2x_1, 2x_2, 2x_3),$$
$$\nabla f_2 = (2(x_1-1), 2(x_2-1), 4(x_3-1))$$

and

$$|\nabla f_1| = \sqrt{4x_1^2+4x_2^2+4x_3^2} = 2\sqrt{x_1^2+x_2^2+x_3^2},$$

$$|\nabla f_2| = \sqrt{4(x_1-1)^2+4(x_2-1)^2+16(x_3-1)^2} = 2\sqrt{(x_1-1)^2+(x_2-1)^2+4(x_3-1)^2} =$$
$$= 2\sqrt{6-2x_1+x_1^2-2x_2+x_2^2-8x_3+4x_3^2}.$$

Therefore normalized gradients of f_1 and f_2 are:

$$\frac{\nabla f_1}{|\nabla f_1|} = \left(\frac{x_1}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_2}{\sqrt{x_1^2+x_2^2+x_3^2}}, \frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}}\right),$$

$$\frac{\nabla f_2}{|\nabla f_2|} = \left(\frac{x_1-1}{\sqrt{6-2x_1+x_1^2-2x_2+x_2^2-8x_3+4x_3^2}}, \frac{x_2-1}{\sqrt{6-2x_1+x_1^2-2x_2+x_2^2-8x_3+4x_3^2}}, \frac{2(x_3-1)}{\sqrt{6-2x_1+x_1^2-2x_2+x_2^2-8x_3+4x_3^2}}\right).$$

Then in order to get the Pareto Frontier functions will have to satisfy:

$$\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \frac{\nabla f_2}{|\nabla f_2|} \right\rangle = \frac{x_1^2-x_1+x_2^2-x_2+2x_3^2-2x_3}{\sqrt{6-2x_1+x_1^2-2x_2+x_2^2-8x_3+4x_3^2}\sqrt{x_1^2+x_2^2+x_3^2}} = -1.$$

Hence
$x_1^2-x_1+x_2^2-x_2+2x_3^2-2x_3+\sqrt{6-2x_1+x_1^2-2x_2+x_2^2-8x_3+4x_3^2}\sqrt{x_1^2+x_2^2+x_3^2}$,
this should be equal to zero. Solving it by x_1 one will get that

$$x_1 = \frac{x_2+2x_3-x_3^2-\sqrt{-(x_2(-2+x_3)+x_3)^2(6-4x_3+x_3^2)}}{5-4x_3+x_3^2}$$

or

$$x_1 = \frac{x_2+2x_3-x_3^2+\sqrt{-(x_2(-2+x_3)+x_3)^2(6-4x_3+x_3^2)}}{5-4x_3+x_3^2},$$

but let us notice that in both cases the value under the root, can be negative as well as zero (the squared part is always positive, the other part has $\triangle = 16-24=-8$ and since there is a minus ahead of all the value, the value under the root can be or negative or zero).
Therefore we have to solve $x_2(-2+x_3)+x_3=0$, hence $x_2 = \frac{x_3}{2-x_3}$ and $x_1 = \frac{x_3}{2-x_3}$.
It is not hard to show in this example that the domain is $x_1, x_2, x_3 \in [0,1]$.
The Efficient Set is shown on a Figure 10.

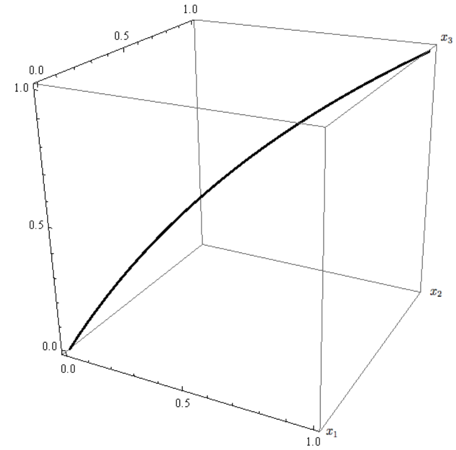

Figure 10: The Efficient Set

In the points from the Efficient Set, the functions have the parametric form:

$$\begin{cases} f_1 = \frac{2x_3^2}{(2-x_3)^2+x_3^2} = \frac{x_3^2(6-4x_3+x_3^2)}{(-2+x_3)^2,} \\ f_2 = 2(\frac{x_3}{(2-x_3)-1})^2 + (x_3-1)^2 = 2\frac{(-1+x_3)^2(8-4x_3+x_3^2)}{(-2+x_3)^2}. \end{cases}$$

Which gives us the graph of the Pareto Frontier:

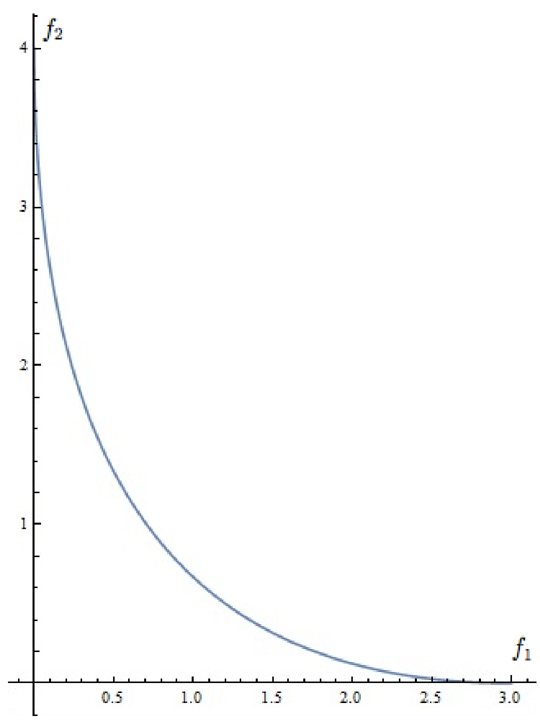

Figure 11: The Pareto Frontier

2.5 Analytic approach: fourth example

Let us now consider the example of f_1, f_2 each depending on four variables. This example will be more general than the previous one, level curves for f_1 will be an ellipsoid and for f_2 be a sphere but in the four dimensional space:

$$
\begin{cases} f_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ f_2 = (x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 + 2(x_4 - 1)^2. \end{cases} \tag{5}
$$

Let us find the gradients of those functions in order to use the gradient method as we did in the previous examples:

$$
\triangledown f_1 = (2x_1, 2x_2, 2x_3, 2x_4),
$$

$$
\triangledown f_2 = (2(x_1 - 1), 2(x_2 - 1), 2(x_3 - 1), 4(x_4 - 1)).
$$

As well

$$
|\triangledown f_1| = \sqrt{4x_1^2, 4x_2^2, 4x_3^2, 4x_4^2} = 2\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}
$$

and

$$
\begin{aligned}
&|\triangledown f_2| = \sqrt{4(x_1 - 1)^2 + 4(x_2 - 1)^2 + 4(x_3 - 1)^2 + 16(x_4 - 1)^2} = \\
&= 2\sqrt{(x_1 - 1)^2 + (x_2 - 1)^2 + (x_3 - 1)^2 + 4(x_4 - 1)^2} = \\
&= 2\sqrt{x_1^2 - 2x_1 + 1 + x_2^2 - 2x_2 + 1 + x_3^2 - 2x_3 + 1 + 4x_4^2 - 8x_4 + 4} = \\
&= 2\sqrt{7 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2 - 8x_4 + 4x_4^2},
\end{aligned}
$$

therefore

$$
\begin{aligned}
&\frac{\triangledown f_1}{|\triangledown f_1|} = \\
&\left(\frac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}}, \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}}, \frac{x_4}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}} \right),
\end{aligned}
$$

$$
\frac{\triangledown f_2}{|\triangledown f_2|} = (a, b, c, d)\,,
$$

where

$$
\begin{aligned}
a &= \frac{x_1 - 1}{\sqrt{7 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2 - 8x_4 + 4x_4^2}}, \\
b &= \frac{x_2 - 1}{\sqrt{7 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2 - 8x_4 + 4x_4^2}}, \\
c &= \frac{x_3 - 1}{\sqrt{7 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2 - 8x_4 + 4x_4^2}}, \\
d &= \frac{2(x_4 - 1)}{\sqrt{7 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2 - 8x_4 + 4x_4^2}}.
\end{aligned}
$$

Then the scalar product is:

$$\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \frac{\nabla f_2}{|\nabla f_2|} \right\rangle = \frac{x_1^2 - x_1 + x_2^2 - x_2 + x_3^2 - x_3 + 2x_4^2 - 2x_4}{\sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}\sqrt{7 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2 - 8x_4 + 4x_4^2}}$$

$$= -1.$$

From that we obtain that the term

$$x_1^2 - x_1 + x_2^2 - x_2 + x_3^2 - x_3 + 2x_4^2 - 2x_4 + ...$$

$$... + \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}\sqrt{7 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2 - 8x_4 + 4x_4^2}$$

should be equal to zero, in this case calculations are pretty hard and time consuming, they are contained in the Wolfram Mathematica program, but for the purposes of this work it is preferred to give an answer. Moreover let us underline one important note that in this example we use analogical principal as in Example 3, that the value of the expression under the root can be either negative or zero.

Hence we get the Efficient Set: $x_1 = x_2 = x_3 = \frac{x_4}{2-x_4}$, $0 < x_1, x_2, x_3, x_4 < 1.$

The Efficient Set is not possible to draw since it would be in the fourth dimension.

The parametric representation of the Pareto Frontier has the form:

$$\begin{cases} f_1 = \frac{x_4^2(7-4x_4+x_4^2)}{(-2+4x_4)^2}, \\ f_2 = \frac{2(-1+x_4)^2(10-4x_4+x_4^2)}{(-2+4x_4)^2}. \end{cases}$$

The graph looks as:

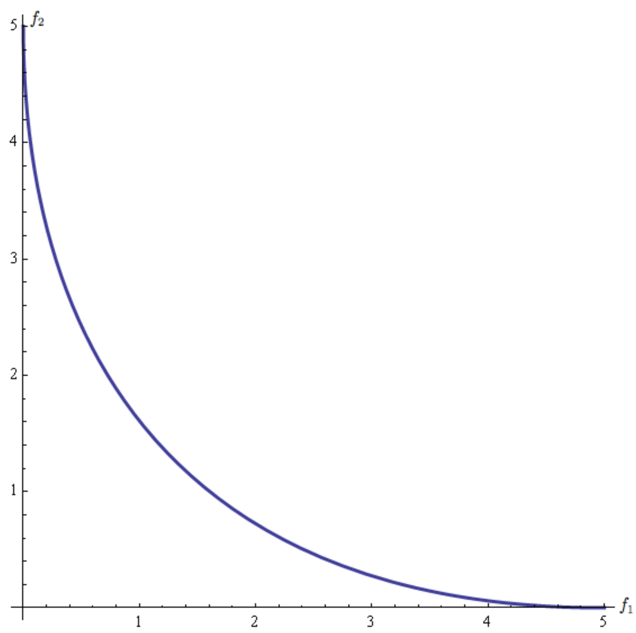

Figure 12: The Pareto Frontier

On the other hand if we change the problem a bit we can get an interesting outcome. Let us consider the case when:

$$\begin{cases} F_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ F_2 = -(x_1 - 1)^2 - (x_2 - 1)^2 - (x_3 - 1)^2 - 2(x_4 - 1)^2. \end{cases} \tag{6}$$

Therefore, $F_1(x_1, x_2, x_3, x_4) = f_1(x_1, x_2, x_3, x_4)$ and $F_2(x_1, x_2, x_3, x_4) = -f_1(x_1, x_2, x_3, x_4)$. Then the problem can be viewed as finding a Pareto Frontier in the case where one function symbolizes wins and the other one costs that we have to bear with those wins. But considering what we proved in old case, we can conclude that points $x_1, x_2, x_3, x_4 > 1$ or $x_1, x_2, x_3, x_4 < 0$ are not the points from the Efficient Set for the case (5). Therefore in case (6), they actually are points from the Efficient Set since for those points $1 = \langle \frac{\nabla f_1}{|\nabla f_1|}, \frac{\nabla f_2}{|\nabla f_2|} \rangle = -\langle \frac{\nabla F_1}{|\nabla F_1|}, \frac{\nabla F_2}{|\nabla F_2|} \rangle$, hence $\langle \frac{\nabla F_1}{|\nabla F_1|}, \frac{\nabla F_2}{|\nabla F_2|} \rangle = -1$, for those points. Having the Efficient Set $x_1 = x_2 = x_3 = \frac{x_4}{2 - x_4}$ and where $x_4 < 0$ or $1 < x_4 \neq 2$ (the right hand side of the inequality can be easily proved). We get the parametric representation of the Pareto Frontier:

$$\begin{cases} F_1 = \frac{x_4^2(11 - 8x_4 + 2x_4^2)}{(-2 + x_4)^2}, \\ F_2 = -\frac{2(-1 + x_4)^2(10 - 4x_4 + x_4^2)}{(-2 + x_4)^2}. \end{cases}$$

This gives us the following picture:

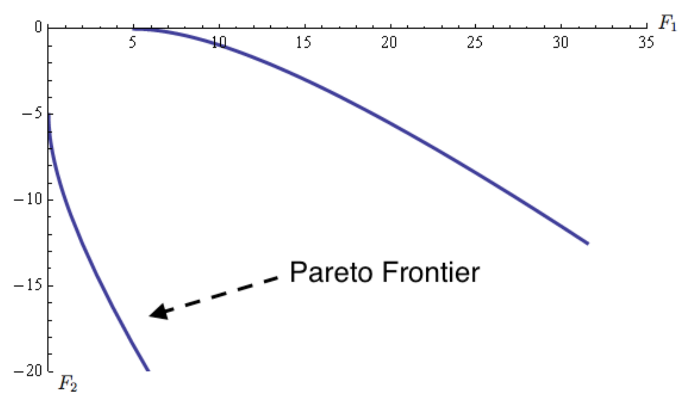

Figure 13: Candidate Set

On the Figure 13 one can see the Candidate Set: bottom line shows the case where $x_4 < 0$, the upper shows the case where $1 < x_4 < 1.5$. Due to the definition of Pareto Frontier only the bottom line is a Pareto Frontier.

2.6 Generalized Schaffer Problems

Let us next look at the Generalized Schaffer Problem. Originally it comes from the Schaffer Problem [8], where one has to optimize by Pareto objective functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$. The Generalized Schaffer Problems was proposed and analyzed by Michael T.M. Emmerich and Andre H. Deutz in [9, p.922-936]. Here, we provide an alternative way to compute the exact solutions of this family of test problems. Let $f_1(x_1, x_2, ..., x_n) = f_1$ and $f_2(x_1, x_2, ..., x_n) = f_2$ the optimization problem is:

$$\begin{cases} f_1 = \frac{1}{n^\alpha}(\sum_{i=1}^n x_i^2)^\alpha \to \min, \\ f_2 = \frac{1}{n^\alpha}(\sum_{i=1}^n (1 - x_i)^2)^\alpha \to \min, \end{cases}$$

$x_i \in \mathbb{R}_+$ and parameter $\alpha \in \mathbb{R}_+$. The parameter α controls the convexity/concavity and the resolvability of the Pareto Front.
To solve this let us start with the simpler version to see some coincidence. Let us consider the example where $\alpha = 2$ and $n = 3$:

$$\begin{cases} f_1 = (\frac{1}{3})^2(x_1^2 + x_2^2 + x_3^2)^2, \\ f_2 = (\frac{1}{3})^2((1-x_1)^2 + (1-x_2)^2 + (1-x_3)^2)^2, \end{cases}$$

then

$$\nabla f_1 = \left(\frac{4}{9}x_1(x_1^2 + x_2^2 + x_3^2), \frac{4}{9}x_2(x_1^2 + x_2^2 + x_3^2), \frac{4}{9}x_3(x_1^2 + x_2^2 + x_3^2)\right),$$

$$\nabla f_2 = (a, b, c),$$

$$a = -\frac{4}{9}(1-x_1)((1-x_1)^2 + (1-x_2)^2 + (1-x_3)^2),$$

$$b = -\frac{4}{9}(1-x_2)((1-x_1)^2 + (1-x_2)^2 + (1-x_3)^2),$$

$$c = -\frac{4}{9}(1-x_3)((1-x_1)^2 + (1-x_2)^2 + (1-x_3)^2).$$

But because the sufficient condition is that $\nabla f_1 = -\lambda \nabla f_2$, where $\lambda > 0$. Then

$$\frac{\frac{4}{9}x_1(x_1^2 + x_2^2 + x_3^2)}{-\frac{4}{9}(1-x_1)((1-x_1)^2 + (1-x_2)^2 + (1-x_3)^2)} = -\lambda < 0,$$

hence

$$\frac{x_1}{x_1 - 1} < 0$$

so $0 < x_1 < 1$, analogically $0 < x_2 < 1$ and $0 < x_3 < 1$.

$$\frac{\nabla f_1}{|\nabla f_1|} = \left(\frac{x_1}{\sqrt{(x_1^2 + x_2^2 + x_3^2)}}, \frac{x_2}{\sqrt{(x_1^2 + x_2^2 + x_3^2)}}, \frac{x_3}{\sqrt{(x_1^2 + x_2^2 + x_3^2)}}\right),$$

$$\frac{\nabla f_2}{|\nabla f_2|} = (k, l, m),$$

where

$$k = \frac{(x_1 - 1)}{\sqrt{(3 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2)}},$$

$$l = \frac{(x_2 - 1)}{\sqrt{(3 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2)}},$$

$$m = \frac{(x_3 - 1)}{\sqrt{(3 - 2x_1 + x_1^2 - 2x_2 + x_2^2 - 2x_3 + x_3^2)}},$$

and

$$\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \frac{\nabla f_2}{|\nabla f_2|} \right\rangle = \frac{(x_1-1)x_1+(x_2-1)x_2+(x_3-1)x_3}{\sqrt{(x_1^2+x_2^2+x_3^2)}\sqrt{(3-2x_1+x_1^2-2x_2+x_2^2-2x_3+x_3^2)}}.$$

This expression should equals -1, so

$$(x_1-1)x_1+(x_2-1)x_2+(x_3-1)x_3 = \sqrt{(x_1^2+x_2^2+x_3^2)}\sqrt{(3-2x_1+x_1^2-2x_2+x_2^2-2x_3+x_3^2)}$$

raising it to the second power we get

$$((x_1-1)x_1+(x_2-1)x_2+(x_3-1)x_3)^2-(x_1^2+x_2^2+x_3^2)(3-2x_1+x_1^2-2x_2+x_2^2-2x_3+x_3^2) = 0.$$

From which one obtains

$$x_1^2+x_2^2+x_3^2 = x_2x_3+x_1(x_2+x_3)$$

hence

$$x_1 = \frac{1}{2}\left(x_2+x_3 \pm \sqrt{3}\sqrt{-x_2^2+2x_2x_3-x_3^2}\right) = \frac{1}{2}\left(x_2+x_3 \pm \sqrt{3}\sqrt{-(x_2-x_3)^2}\right).$$

Therefore, there is a real solution x_1 if and only if $x_2 = x_3$. In that case $x_1 = \frac{1}{2}(x_2+x_3) = x_2 = x_3$. From the second equation, one can conclude that it can be zero only in case $x_1 = x_2 = x_3$. So the second equation does not provide additional solutions. The Efficient Set is depicted below

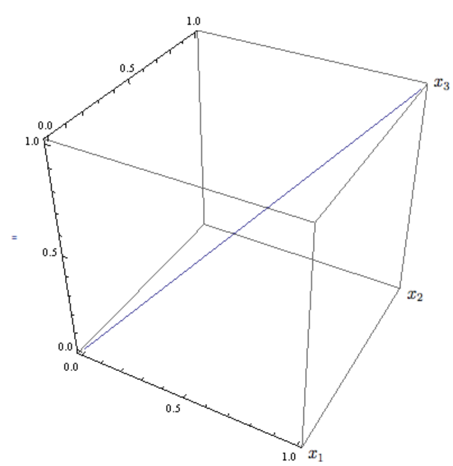

Figure 14: The Efficient Set for the Generalized Schaffer Problem with $\alpha = 2$, $n = 3$ and $0 < x < 1$.

and the Pareto Frontier is given by $(f_1, f_2) = (x^4, (1-x)^4)$, $x \in [0, 1]$ It is depicted in Figure 15.

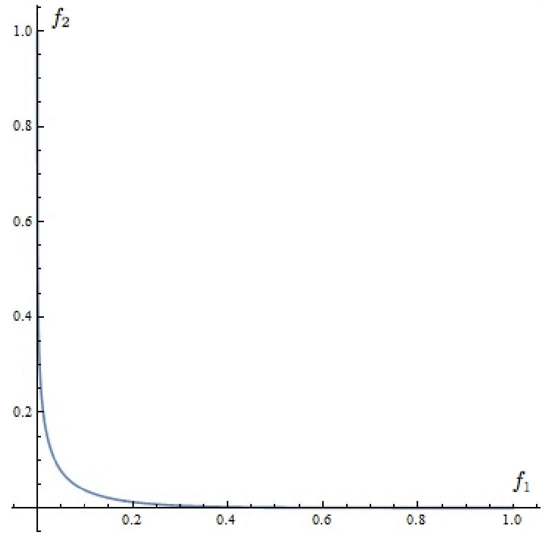

Figure 15: The Pareto Frontier for the Generalized Schaffer Problem with $n = 3$, $\alpha = 2$.

Let us look consider the Generalized Schaffer Problem in case of n parameters: We can use the proportionality of gradients theorem [p.5];

$$\nabla f_1 = -\lambda \nabla f_2, \quad \lambda > 0,$$

$$\nabla f_1 = \left(x_1 \left(\frac{2\alpha}{n^\alpha} \sum_{i=1}^{n} x_i^2 \right)^{\alpha-1}, ..., x_n \left(\frac{2\alpha}{n^\alpha} \sum_{i=1}^{n} x_i^2 \right)^{\alpha-1} \right)$$

$$\nabla f_2 = \left((1-x_1) \left(\frac{-2\alpha}{n^\alpha} \sum_{i=1}^{n} (1-x_i)^2 \right)^{\alpha-1}, ..., (1-x_n) \left(\frac{-2\alpha}{n^\alpha} \sum_{i=1}^{n} (1-x_i) \right)^{\alpha-1} \right),$$

hence

$$\left(\frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} (1-x_i)^2} \right)^{\alpha-1} \frac{x_1}{x_1 - 1} = ... = \left(\frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} (1-x_i)^2} \right)^{\alpha-1} \frac{x_n}{x_n - 1} = -\lambda < 0,$$

from which we obtain

$$\frac{x_1}{x_1 - 1} = \frac{x_2}{x_2 - 1} = ... = \frac{x_n}{x_n - 1} < 0,$$

therefore $0 < x_k < 1$, for $k = 1, ..., n$. As well for all $l, m = 1, ..., n$:

$$\frac{x_l}{x_l - 1} = \frac{x_m}{x_m - 1}$$

we get

$$x_l x_m - x_m = x_l x_m - x_l$$

so

$$x_l = x_m.$$

Concluding the Efficient Set for Generalized Schaffer problem in case of $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$ is $x_1 = x_2 = ... = x_n$, $x_i \in (0, 1)$ for $i = 1, ..., n$ and correspondingly parametric form of Pareto Frontier:

$$\begin{cases} f_1(x_1, ..., x_n) = \frac{1}{n^\alpha}(nx^2)^\alpha = x^{2\alpha}, \\ f_2(x_1, ..., x_n) = \frac{1}{n^\alpha}(n(1-x)^2)^\alpha = (1-x)^{2\alpha}, \end{cases}$$

where $x = x_1 = ... = x_n \in (0,1)$. The explicit form of Pareto Frontier is

$$f_2 = (1 - f_1^{\frac{1}{2\alpha}})^{2\alpha}, \quad f_1 \in (0,1).$$

2.7 The Dent Problem

The Dent Problem has been proposed as an example of a parametric multi-criteria optimization problem, that has a gap in the Pareto frontier. It was first proposed in the doctoral thesis of Katrin Witting [10]. The problem has also been used in subsequent studies on the performance of numerical multi-criteria optimization algorithms [11, pp.198-199]. So far, only numerically computed Pareto fronts were reported and with the new method i.e. gradient method, we will be able to compute analytically a symbolic expression for the Pareto front.

The Dent Problem is given by:

$$\begin{cases} f_1(x_1,x_2) = \frac{1}{2}(\sqrt{1+(x_1+x_2)^2} + \sqrt{1+(x_1-x_2)^2} + x_1 - x_2) + \lambda \cdot e^{-(x_1-x_2)^2}, \\ f_2(x_1,x_2) = \frac{1}{2}(\sqrt{1+(x_1+x_2)^2} + \sqrt{1+(x_1-x_2)^2} - x_1 + x_2) + \lambda \cdot e^{-(x_1-x_2)^2}. \end{cases}$$

To minimize f_1 and f_2. To deal with the problem one should start with the critical value $\lambda = 0.85$, and then analyse what happens with the Pareto Frontier when changing values of λ. So consider first:

$$\begin{cases} f_1 = \frac{1}{2}(\sqrt{1+(x_1+x_2)^2} + \sqrt{1+(x_1-x_2)^2} + x_1 - x_2) + 0,85 \cdot e^{-(x_1-x_2)^2}, \\ f_2 = \frac{1}{2}(\sqrt{1+(x_1+x_2)^2} + \sqrt{1+(x_1-x_2)^2} - x_1 + x_2) + 0,85 \cdot e^{-(x_1-x_2)^2}. \end{cases}$$

Then

$$\begin{aligned} \triangledown f_1 &= (a,b)\,, \\ \triangledown f_2 &= (c,d)\,, \end{aligned}$$

where

$$\begin{aligned} a &= -1.7e^{-(x_1-x_2)^2}(x_1-x_2) + \frac{1}{2}(1 + \frac{x_1-x_2}{\sqrt{1+(x_1-x_2)^2}} + \frac{x_1+x_2}{\sqrt{1+(x_1+x_2)^2}}), \\ b &= 1.7e^{-(x_1-x_2)^2}(x_1-x_2) + \frac{1}{2}(-1 - \frac{x_1-x_2}{\sqrt{1+(x_1-x_2)^2}} + \frac{x_1+x_2}{\sqrt{1+(x_1+x_2)^2}}), \\ c &= -1.7e^{-(x_1-x_2)^2}(x_1-x_2) + \frac{1}{2}(-1 + \frac{x_1-x_2}{\sqrt{1+(x_1-x_2)^2}} + \frac{x_1+x_2}{\sqrt{1+(x_1+x_2)^2}}), \\ d &= 1.7e^{-(x_1-x_2)^2}(x_1-x_2) + \frac{1}{2}(1 - \frac{x_1-x_2}{\sqrt{1+(x_1-x_2)^2}} + \frac{x_1+x_2}{\sqrt{1+(x_1+x_2)^2}}). \end{aligned}$$

The reader can see that computation of $\langle \frac{\triangledown f_1}{|\triangledown f_1|}, \frac{\triangledown f_2}{|\triangledown f_2|} \rangle$ will be long and tedious. In this case application of the probabilistic method is preferred to find the Efficient Set and then to see the Pareto Front.

Solving this problem by mean of probabilistic gradient method, Wolfram Mathematica program was used to give the values of x_1, x_2, such that the value of $\langle \frac{\nabla f_1}{|\nabla f_1|}, \frac{\nabla f_2}{|\nabla f_2|}\rangle \leq -0.995$, which is already pretty close to -1, but one can ask for closer values. The results that we got were

x_1	-0.1963	0.6653	0.6878	0.8186	0.612	0.1695	1.7226
x_2	0.1770	-0.6701	-0.6742	-0.8106	-0.585	-0.147	-1.7201

x_1	-0.0868	0.8349	0.5763	0.5144	1.0898	-0.7619
x_2	-0.0426	-0.818	-0.5533	-0.5367	-1.0849	0.7371

This suggests that the Efficient Set may be $x_1 = -x_2$.

Let us proceed with this solution and check analytically if it is the proper Efficient Set for the Dent Problem. Substituting $x_1 = -x_2$ into the expressions for $\nabla f_1(x_1, x_2)$ and gradient of $\nabla f_2(x_1, x_2)$ we get:

$$\begin{cases} f_1 = \frac{1}{2}(1 + \sqrt{1+4x_2^2} - 2x_2) + 0.85e^{-4x_2^2}, \\ f_2 = \frac{1}{2}(1 + \sqrt{1+4x_2^2} + 2x_2) + 0.85e^{-4x_2^2} \end{cases}$$

and consequently

$$\begin{cases} \nabla f_1 = \left(3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(1 - \frac{2x_2}{\sqrt{1+4x_2^2}}\right), -3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(-1 + \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right), \\ \nabla f_2 = \left(3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(-1 - \frac{2x_2}{\sqrt{1+4x_2^2}}\right), -3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(1 + \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right). \end{cases}$$

Moreover

$$\begin{aligned} |\nabla f_1| &= \sqrt{\left(3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(1 - \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right)^2 + \left(-3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(-1 + \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right)^2} \\ &= \sqrt{2}\left(\frac{1}{2} + 3.4e^{-4x_2^2}x_2 - \frac{x_2}{\sqrt{1+4x_2^2}}\right) \end{aligned}$$

and

$$\begin{aligned} |\nabla f_2| &= \sqrt{\left(3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(-1 - \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right)^2 + \left(-3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(1 + \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right)^2} \\ &= \sqrt{2}\left(\frac{1}{2} - 3.4e^{-4x_2^2}x_2 + \frac{x_2}{\sqrt{1+4x_2^2}}\right). \end{aligned}$$

Hence

$$\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \frac{\nabla f_2}{|\nabla f_2|} \right\rangle = \frac{\left(3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(-1 - \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right)\left(3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(1 - \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right)}{2\left(\frac{1}{2} + 3.4e^{-4x_2^2}x_2 - \frac{x_2}{\sqrt{1+4x_2^2}}\right)\left(\frac{1}{2} - 3.4e^{-4x_2^2}x_2 + \frac{x_2}{\sqrt{1+4x_2^2}}\right)} +$$
$$\frac{\left(-3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(-1 + \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right)\left(-3.4e^{-4x_2^2}x_2 + \frac{1}{2}\left(1 + \frac{2x_2}{\sqrt{1+4x_2^2}}\right)\right)}{2\left(\frac{1}{2} + 3.4e^{-4x_2^2}x_2 - \frac{x_2}{\sqrt{1+4x_2^2}}\right)\left(\frac{1}{2} - 3.4e^{-4x_2^2}x_2 + \frac{x_2}{\sqrt{1+4x_2^2}}\right)} = -1.$$

Hence we proved that indeed the points (x_1, x_2) with $x_1 = -x_2$, where $x_1, x_2 \in \mathbb{R}$ are an Efficient Set. From the parametric form of the initial functions we get the picture of Pareto Frontier for the Dent Problem:

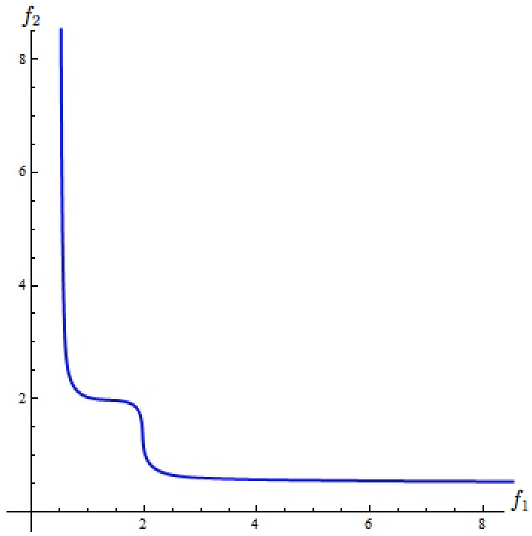

Figure 16: The Pareto Frontier for the Dent Problem for the critical value $\lambda = 0.85$.

Figure 16 shows Pareto Frontier in case when $\lambda = 0.85$, but when increasing λ there will become a dent in it, therefore Figure 17 shows a case when $\lambda = 20$:

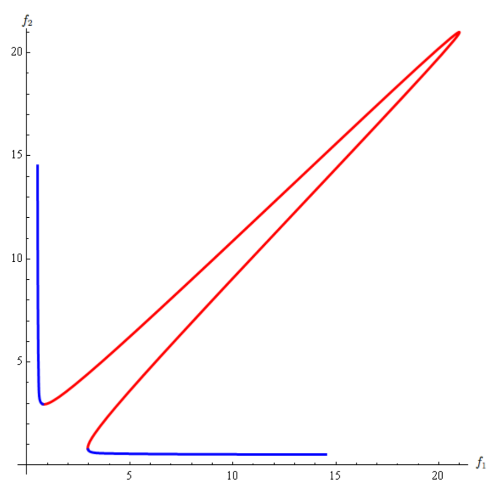

Figure 17: Candidate Set in case when $\lambda = 20$ and $-7 < x_2 < 7$.

Figure 17 represents a Candidate Set, Pareto Frontier is depicted with a blue colour. On this picture Pareto Frontier is for $-7 < x_2 < -1.1$ then $2.96645 < f_1 < 14.5178$, $0.5178 < f_2 < 0.7664$ and for $1.1 < x_2 < 7$ then $2.96645 < f_2 < 14.5178$, $0.5178 < f_1 < 0.7664$. As we can see there is a significant curve segment in the middle. Worth to notice is the fact that using the knowledge of the definition of Pareto Front, some points that belong to that curve cannot be the points from the Pareto Frontier because, on these points, both functions f_1 and f_2 are or both increasing or both decreasing.
Graphs of both functions in only one variable:

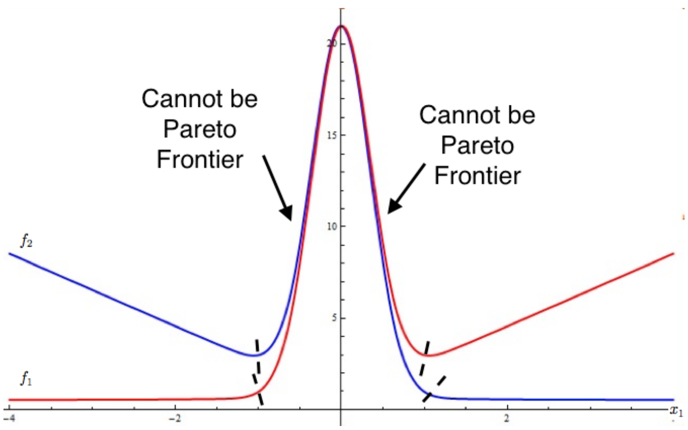

Figure 18: Both functions f_1 and f_2, when depending only from one variable.

On the Figure 18 it can be easily seen that the middle part cannot be Pareto Frontier because both functions are increasing or decreasing simultaneously.

3 Finding a Pareto Frontier for Implicit Functions

In this chapter a new method is proposed that can be used to find the Pareto Frontier of implicitly given objective functions.

3.1 The Implicit Function

An explicit function is a function $f : D \in \mathbb{R}^n \to \mathbb{R}$ for which the value $y = f(x)$ is given by an explicitly computable expression, y is called the dependent variable and x the independent variable (argument).
Let now g is a continuously differentiable real function in the plane, then the equation $g(x, y) = 0$ can be solved for y in terms of x in a neighbourhood of any point (a, b) at which $g(a, b) = 0$ and $\frac{\partial g}{\partial y} \neq 0$. Likewise, one can solve for x in terms of y near (a, b) if $\frac{\partial g}{\partial x} \neq 0$ at (a, b), in other worlds y is a function (implicit) of x. For a simple example which illustrates the need for assuming $\frac{\partial g}{\partial y} \neq 0$, consider

$$g(x, y) = x^2 + y^2 - 1.$$

The pairs of x and y which satisfy the first relationship will also satisfy the second relationship and vice versa. In theory, any implicit function could be converted in principal into an explicit function by solving for y in terms of x. In practice, this may be rather challenging, though and might not yield a closed form expression. Consider for example:

$$\ln(x + y) + xy - 12 = 0.$$

In this case it is impossible to isolate either of the variables. But anyway the implicit function exists and can be analysed.

3.2 The Implicit Function Theorem

1. Assume that we have an implicit function of the form

$$g(x, y) = 0, \quad x, y \in \mathbb{R}.$$

Theorem 1. [12, pp.8] *(The Implicit Function Theorem) Let g be a real-valued continuously differentiable function defined in a neighbourhood of $(x_0, y_0) \in \mathbb{R}^2$.*
Suppose that g satisfies the two conditions:

$$g(x_0, y_0) = z_0,$$

$$\frac{\partial g}{\partial y}(x_0, y_0) \neq 0.$$

Then there exist open intervals U and V, with $x_0 \in U$, $y_0 \in V$, and a unique function $f : U \to V$ satisfying $g[x, f(x)] = z_0$, for all $x \in U$, and this function F is continuously differentiable with

$$\frac{dy}{dx}(y_0) = f'(y_0) = -\left[\frac{\partial g}{\partial x}(x_0, y_0)\right] / \left[\frac{\partial g}{\partial y}(x_0, y_0)\right].$$

2. Version for many y variables and many x variables.
Let we have m functions of $n+m$ variables $g_i(x_1,...,x_n,y_1,...,y_m), \quad i=1,...,m$. Consider the system of equations:

$$\begin{aligned} g_1(x_1,x_2,....,x_n,y_1,y_2,...,y_m) &= 0, \\ g_2(x_1,x_2,....,x_n,y_1,y_2,...,y_m) &= 0, \\ &\vdots \\ g_m(x_1,x_2,....,x_n,y_1,y_2,...,y_m) &= 0. \end{aligned}$$

If we denote $x=(x_1,...,x_n)\in\mathbb{R}^n, \quad y=(y_1,...,y_m)\in\mathbb{R}^m$ then we can rewrite the system in the vector form:

$$g(x,y)=0,$$

where $g:\mathbb{R}^{n+m}\to\mathbb{R}^m$.
Theorem 2. [12, pp. 43][3] *(Multivariable version of the Implicit Function Theorem) Suppose that g_i are real-valued functions defined on a domain D and continuously differentiable on an open set $D^1\subset D\subset\mathbb{R}^{n+m}$, where $m>0$ and*

$$\begin{aligned} g_1(x_1^0,x_2^0,...,x_n^0,y_1^0,...,y_m^0) &= 0, \\ g_2(x_1^0,x_2^0,...,x_n^0,y_1^0,...,y_m^0) &= 0, \\ &\vdots \\ g_m(x_1^0,x_2^0,...,x_n^0,y_1^0,...,y_m^0) &= 0. \end{aligned} \tag{7}$$

We often write the equation (7) as follows

$$g_i(x^0,y^0)=0, \quad i=1,2,...,n \quad \textit{and} \quad (x^0,y^0)\in D^1.$$

Assume the Jacobian matrix $D_y g(x,y)=\left[\frac{\partial g_i(x^0,y^0)}{\partial y_j}\right]$ has rank m. Then there exists a neighbourhood $N_\delta(x^0,y^0)\subset D^1$, an open set $D^2\subset\mathbb{R}^n$ containing x^0 and real valued functions y_k, $k=1,2,...,m$, continuously differentiable on D^2, such that the following conditions are satisfied:

$$\begin{aligned} y_1^0 &= y_1(x^0) \\ y_2^0 &= y_2(x^0) \\ &\vdots \\ y_m^0 &= y_m(x^0) \end{aligned} \tag{8}$$

For every $x\in D^2$, we have

$$g_i(x_1,x_2,...,x_n,y_1(x),y_2(x),...,y_m(x))\equiv 0, \quad i=1,2,....,n$$

or

$$g_i(x,y(x))\equiv 0, \quad i=1,2,...,n.$$

[3]The formulation of the theorem is given using the notation of this thesis work.

If we write this in differential form, we obtain:

$$D_x g(x,y)dx + D_y g(x,y)dy = 0.$$

One can rewrite this equation to get one multivariate formulation of the theorem:

$$dy = -D_y g(x,y)^{-1} D_x g(x,y)dx,$$

meaning:

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} = - \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}.$$

From here we have:

$$\begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_m} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_i} \\ \vdots \\ \frac{\partial y_m}{\partial x_i} \end{pmatrix} = - \begin{pmatrix} \frac{\partial g_1}{\partial x_i} \\ \vdots \\ \frac{\partial g_m}{\partial x_i} \end{pmatrix}.$$

This is a system of linear equations with respect to $\frac{\partial y_k}{\partial x_i}$, where $k = 1, ..., m$ and $i = 1, ..., n$. Using Cramer's rule we get:

$$\frac{\partial y_k}{\partial x_l} = - \frac{\det \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial x_i} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial x_i} & \cdots & \frac{\partial g_m}{\partial y_m} \end{pmatrix}}{\det \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial y_1} & \cdots & \frac{\partial g_m}{\partial y_m} \end{pmatrix}}.$$

Remark. *Theorems given in this section have local character i.e. implicit function is considered only in some neighbourhood. Later on we propose global character Theorems (Theorem 3., Theorem. 4.)*

3.3 First example

Given explicit forms of $\frac{\partial y_i}{\partial x_j}$, $i \in \{1, ..., m\}$, $j \in \{1, ..., k\}$, we can now use methods introduced in Chapter 1-2 to compute the Pareto Frontier for y_1 and y_2.
Let

$$\begin{cases} g_1(x_1, x_2, y_1) = x_1^2 + x_2^2 - y_1 - 0.05y_1^3 = 0, \\ g_2(x_1, x_2, y_2) = (x_1 - 1)^2 + (x_2 - 1)^2 - y_2 - 0.05y_2^3 = 0, \end{cases}$$

and

$$F = (x_1^2 + x_2^2 - y_1 - 0.05y_1^3)^2 + ((x_1 - 1)^2 + (x_2 - 1)^2 - y_2 - 0.05y_2^3)^2.$$

Solving it using the program to compute all random values of (y_1, y_2), with 500000 iterations and for $x_1, x_2, y_1, y_2 \in [-2, 2]$.

We got the points:

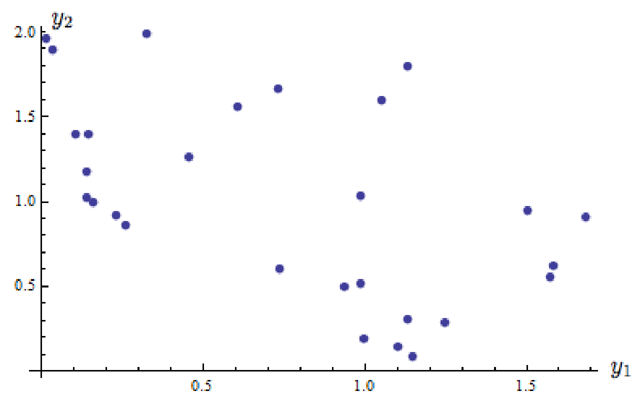

Figure 19: Points that can belong to the Pareto Frontier

We would like to solve more general example of this type deterministically, using the theorems from the introduction to this chapter to show at the end how all 'non-dominated' points from a random set in Figure 19 are actually close to the proper Pareto Frontier.
Let $a \geq 0$ and consider

$$\begin{cases} g_1(x_1, x_2, y_1) = x_1^2 + x_2^2 - y_1 - ay_1^3 = 0 \\ g_2(x_1, x_2, y_2) = (x_1 - 1)^2 + (x_2 - 1)^2 - y_2 - ay_2^3 = 0. \end{cases} \tag{9}$$

So we have:

$$\begin{cases} \partial g_1(x_1, x_2, y_1)/\partial x_1 = 2x_1, \\ \partial g_1(x_1, x_2, y_1)/\partial x_2 = 2x_2, \\ \partial g_1(x_1, x_2, y_1)/\partial y_1 = -1 - 3ay_1^2, \\ \partial g_2(x_1, x_2, y_2)/\partial x_1 = 2(x_1 - 1), \\ \partial g_2(x_1, x_2, y_2)/\partial x_2 = 2(x_2 - 1), \\ \partial g_2(x_1, x_2, y_2)/\partial y_2 = -1 - 3ay_2^2. \end{cases}$$

Hence

$$D_y g = \begin{bmatrix} -1 - 3ay_1^2 & 0 \\ 0 & -1 - 3ay_2^2 \end{bmatrix},$$

$$D_x g = \begin{bmatrix} 2x_1 & 2x_2 \\ 2(x_1 - 1) & 2(x_2 - 1) \end{bmatrix}$$

so

$$D_y g^{-1} = \begin{bmatrix} \frac{1}{-1-3ay_1^2} & 0 \\ 0 & \frac{1}{-1-3ay_2^2} \end{bmatrix}$$

and

$$\frac{\partial y}{\partial x} = -D_y g^{-1} \cdot D_x g = -\begin{bmatrix} \frac{2x_1}{-1-3ay_1^2} & \frac{2x_2}{-1-3ay_1^2} \\ \frac{2(x_1-1)}{-1-3ay_2^2} & \frac{2(x_2-1)}{-1-3ay_2^2} \end{bmatrix} = \begin{bmatrix} \frac{2x_1}{1+3ay_1^2} & \frac{2x_2}{1+3ay_1^2} \\ \frac{2(x_1-1)}{1+3ay_2^2} & \frac{2(x_2-1)}{1+3ay_2^2} \end{bmatrix}.$$

Let us denote by $y_1(x_1, x_2) = f_1$ and $y_2(x_1, x_2) = f_2$ are solutions of the equations (9). We get:

$$\nabla f_1 = \left(\frac{2x_1}{1 + 3ay_1^2}, \frac{2x_2}{1 + 3ay_1^2} \right),$$

$$\nabla f_2 = \left(\frac{2(x_1 - 1)}{1 + 3ay_2^2}, \frac{2(x_2 - 1)}{1 + 3ay_2^2} \right),$$

$$|\nabla f_1| = \sqrt{\frac{4x_1^2 + 4x_2^2}{(1 + 3ay_1^2)^2}} = \frac{2\sqrt{x_1^2 + x_2^2}}{1 + 3ay_1^2}, \quad \text{since} \quad a \geq 0,$$

$$|\nabla f_2| = \sqrt{\frac{4(x_1 - 1)^2 + 4(x_2 - 1)^2}{(1 + 3ay_2^2)^2}} = \frac{2\sqrt{(x_1 - 1)^2 + (x_2 - 1)^2}}{1 + 3ay_2^2}.$$

Therefore

$$\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \frac{\nabla f_2}{|\nabla f_2|} \right\rangle = \frac{((x_1 - 1)x_1 + (x_2 - 1)x_2)}{(\sqrt{x_1^2 + x_2^2}\sqrt{(x_1 - 1)^2 + (x_2 - 1)^2})}.$$

The last should be equal to -1 in order to have Pareto Optimality, based on the 'gradient method' from Chapter 2 Section 1. Solving the resulting equation yields:

$$x_1(x_1 - 1) + x_2(x_2 - 1) = -\sqrt{x_1^2 + x_2^2} \cdot \sqrt{(x_1 - 1)^2 + (x_2 - 1)^2},$$

hence

$$x_1^2(x_1 - 1)^2 + 2x_1x_2(x_1 - 1)(x_2 - 1) + x_2^2(x_2 - 1)^2 = (x_1^2 + x_2^2)\left[(x_1 - 1)^2 + (x_2 - 1)^2\right])$$

$$(\text{and} \quad x_1(x_1 - 1) + x_2(x_2 - 1) \leq 0)$$

$$\text{which is equivalent to} \quad 2x_1x_2(x_1 - 1)(x_2 - 1) = x_2^2(x_1 - 1)^2 + x_1^2(x_2 - 1)^2$$

$$(\text{and} \quad x_1(x_1 - 1) + x_2(x_2 - 1) \leq 0).$$

So

$$(x_2(x_1 - 1) - x_1(x_2 - 1))^2 = 0$$

$$(\text{and} \quad x_1(x_1 - 1) + x_2(x_2 - 1) \leq 0),$$

hence

$$x_2(x_1 - 1) = x_1(x_2 - 1) \quad (\text{and} \quad x_1(x_1 - 1) + x_2(x_2 - 1) \leq 0),$$

$$x_1 = x_2 \quad (\text{and} \quad x_1(x_1 - 1) + x_2(x_2 - 1) \leq 0),$$

$$\text{which yields} \quad x_1 = x_2 \quad 2x_1(x_1 - 1) \leq 0.$$

$$\text{That is,} \quad x_1 = x_2 \quad \text{and} \quad 0 \leq x_1 \leq 1.$$

Hence $g_1(x_1, x_2, y_1) = 2x_2^2 - y_1 - ay_1^3 = 0$ and $g_2(x_1, x_2, y_2) = 2(x_2 - 1)^2 - y_2 - ay_2^3 = 0$. From $g_1(x_1, x_2, y_1) = 2x_2^2 - y_1 - ay_1^3 = 0$ we get that $x_2 = \sqrt{\frac{y_1 + ay_1^3}{2}}$, substituting these into $g_2(x_1, x_2, y_2) = 0$ we get $g_2(x_1, x_2, y_2) = 2\left(\sqrt{\frac{y_1 + ay_1^3}{2}} - 1\right)^2 - y_2 - ay_2^3 = 0$. So the Pareto

Frontier is $2\left(\sqrt{\frac{y_1+ay_1^3}{2}}-1\right)^2-y_2-ay_2^3=0$. Let $a=0.05$ the Pareto Frontier for y_1, y_2 is $2\left(\sqrt{\frac{y_1+0.05y_1^3}{2}}-1\right)^2-y_2-0.05y_2^3=0$, we get the picture:

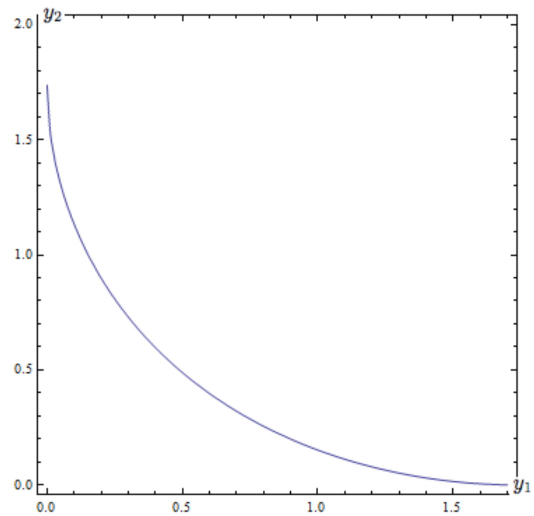

Figure 20: Analytically computed Pareto Frontier

Combining the Figure 19 and Figure 20 we obtain the following figure:

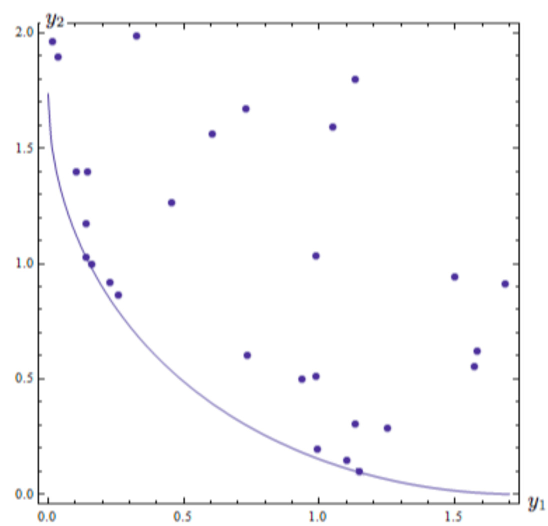

Figure 21: Deterministically calculated Pareto Frontier and probabilistically found points that can be in Pareto Frontier

3.4 Second example

Let us consider

$$\begin{cases} g_1(x_1,x_2,y_1,y_2)=x_1^2+x_2^2-y_1-ay_1^3-y_2-by_2^3=0, \\ g_2(x_1,x_2,y_1,y_2)=(x_1-1)^2+(x_2-1)^2-y_1-by_1^3-y_2-ay_2^3=0, \end{cases} \tag{10}$$

then

$$\begin{cases} \partial g_1(x_1,x_2,y_1,y_2)/\partial x_1 = 2x_1, \\ \partial g_1(x_1,x_2,y_1,y_2)/\partial x_2 = 2x_2, \\ \partial g_1(x_1,x_2,y_1,y_2)/\partial y_1 = -1-3ay_1^2, \\ \partial g_1(x_1,x_2,y_1,y_2)/\partial y_2 = -1-3by_2^2, \\ \partial g_2(x_1,x_2,y_1,y_2)/\partial x_1 = 2(-1+x_1), \\ \partial g_2(x_1,x_2,y_1,y_2)/\partial x_2 = 2(-1+x_2), \\ \partial g_2(x_1,x_2,y_1,y_2)/\partial y_1 = -1-3by_1^2, \\ \partial g_2(x_1,x_2,y_1,y_2)/\partial y_2 = -1-3ay_2^2. \end{cases}$$

In order to make calculations more clear we will denote $g_1(x_1,x_2,y_1,y_2)$ as g_1 and $g_2(x_1,x_2,y_1,y_2)$ as g_2.
Using the multivariable version of the Implicit Function Theorem, we know:

$$D_y g = \begin{bmatrix} -1-3ay_1^2 & -1-3by_2^2 \\ -1-3by_1^2 & -1-3ay_2^2 \end{bmatrix},$$

$$D_x g = \begin{bmatrix} 2x_1 & 2x_2 \\ 2(x_1-1) & 2(x_2-1) \end{bmatrix}.$$

Let we assume $\det D_y g \neq 0$,

$$\det D_y g = (1+3ay_1^2)(1+3ay_2^2) - (1+3by_1^2)(1+3by_2^2) \neq 0$$

so

$$(1+3ay_1^2)(1+3ay_2^2) \neq (1+3by_1^2)(1+3by_2^2),$$
$$\frac{1+3ay_1^2}{1+3by_1^2} \neq \frac{1+3by_2^2}{1+3ay_2^2}.$$

We have

$$D_y g^{-1} = \frac{1}{\det D_y g} \begin{bmatrix} -1-3ay_2^2 & 1+3by_2^2 \\ 1+3by_1^2 & -1-3ay_1^2 \end{bmatrix},$$

$$\frac{\partial y}{\partial x} = -D_y g^{-1} \cdot D_x g = \frac{1}{|J|} \begin{bmatrix} 1+3ay_2^2 & -1-3by_2^2 \\ -1-3by_1^2 & -1-3ay_1^2 \end{bmatrix} \cdot \begin{bmatrix} 2x_1 & 2x_2 \\ 2(x_1-1) & 2(x_2-1) \end{bmatrix} =$$
$$= \frac{1}{|J|} \begin{bmatrix} 2x_1(1+3ay_2^2) - 2(1+3by_2^2)(x_1-1) & 2x_2(1+3ay_2^2) - 2(1+3by_2^2)(x_2-1) \\ 2x_1(1+3ay_1^2) - 2(1+3by_1^2)(x_1-1) & 2x_2(1+3ay_1^2) - 2(1+3by_1^2)(x_2-1) \end{bmatrix}$$

so

$$\frac{\partial y}{\partial x} = -D_y g^{-1} \cdot D_x g =$$
$$\frac{-1}{|J|} \begin{bmatrix} -2(1+3ay_2^2) + 2x_1(1+3by_2^2) - 2x_1(1+3ay_2)^2 & -2(1+3ay_2^2) + 2x_2(1+3by_2^2) - 2x_2(1+3ay_2)^2 \\ -2(1+3ay_1^2) + 2x_1(1+3by_1^2) - 2x_1(1+3ay_1)^2 & -2(1+3ay_1^2) + 2x_2(1+3by_1^2) - 2x_2(1+3ay_1)^2 \end{bmatrix}$$
$$= \frac{1}{|J|} \begin{bmatrix} -2(1+3by_2^2) + 6x_1(by_2^2 - ay_2^2) & -2(1+3by_2^2) + 6x_2(by_2^2 - ay_2^2) \\ -2(1+3by_1^2) + 6x_1(by_1^2 - ay_1^2) & -2(1+3by_1^2) + 6x_2(by_1^2 - ay_1^2) \end{bmatrix}.$$

Therefore

$$\nabla f_1(x_1, x_2) = (-\frac{a_1}{|J|}, -\frac{a_2}{|J|}),$$

$$\nabla f_2(x_1, x_2) = (-\frac{b_1}{|J|}, -\frac{b_2}{|J|}),$$

$$(\nabla f_1(x_1, x_2), \nabla f_2(x_1, x_2)) = \frac{a_1 b_1 + a_2 b_2}{|J|^2}$$

and

$$|\nabla f_1| = \sqrt{\frac{a_1^2 + a_2^2}{|J|^2}},$$

$$|\nabla f_2| = \sqrt{\frac{b_1^2 + b_2^2}{|J|^2}}.$$

Thus,

$$\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \frac{\nabla f_2}{|\nabla f_2|} \right\rangle = \frac{a_1 b_1 + a_2 b_2}{\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}}.$$

Computing a_1, a_2, b_1, b_2 we get:

$$\begin{cases} a_1 = -2 - 6by_2^2 - 6ax_1y_2^2 + 6bx_1y_2^2 = -2(1 + 3by_2^2) + 6x_1(by_2^2 - ay_2^2), \\ a_2 = -2 - 6by_2^2 - 6ax_2y_2^2 + 6bx_2y_2^2 = -2(1 + 3by_2^2) + 6x_2(by_2^2 - ay_2^2), \\ b_1 = 2 + 6ay_1^2 - 6ax_1y_1^2 + 6bx_1y_1^2 = 2(1 + 3ay_1^2) + 6x_1(by_1^2 - ay_1^2), \\ b_2 = 2 + 6ay_1^2 - 6ax_2y_1^2 + 6bx_2y_1^2 = 2(1 + 3ay_1^2) + 6x_2(by_1^2 - ay_1^2). \end{cases}$$

Since using our "gradient method"

$$\left\langle \frac{\nabla f_1}{|\nabla f_1|}, \frac{\nabla f_2}{|\nabla f_2|} \right\rangle = \frac{a_1 b_1 + a_2 b_2}{\sqrt{(a_1^2 + a_2^2)(b_1^2 + b_2^2)}},$$

should be equal to -1 we obtain $(a_1b_1 + a_2b_2)^2 - (a_1^2 + a_2^2)(b_1^2 + b_2^2) = 0$, and $(a_1b_1 + a_2b_2) \leq 0$, from which follows the equation

$$2a_1a_2b_1b_2 - a_1^2b_2^2 - a_2^2b_1^2 = 0,$$

hence

$$(a_1b_2 - a_2b_1)^2 = 0$$

so

$$a_1b_2 - a_2b_1 = 0.$$

Substituting the values of a_1, a_2, b_1, b_2 we get:

$$\left(-2(1 + 3by_2^2) + 6x_1(by_2^2 - ay_2^2)\right)\left(2(1 + 3ay_1^2) + 6x_2(by_1^2 - ay_1^2)\right) -$$
$$\left(-2(1 + 3by_2^2) + 6x_2(by_2^2 - ay_2^2)\right)\left(2(1 + 3ay_1^2) + 6x_1(by_1^2 - ay_1^2)\right) = 0.$$

If we denote:

$$\begin{cases} k = -2(1+3by_2^2), \\ l = by_2^2 - ay_2^2, \\ m = by_1^2 - ay_1^2, \\ n = 2(1+3ay_1^2), \end{cases}$$

we get:

$$(k + 6x_1 l)(n + 6x_2 m) - (k + 6x_2 l)(n + 6x_1 m) = 0,$$

hence

$kn + 6x_1 ln + 6x_2 km + 36x_1 x_2 lm - kn - 6x_1 km - 6x_2 ln - 36x_1 x_2 lm =$
$6x_1(ln - km) + 6x_2(km - ln) = 6(x_1 - x_2)(ln - km) = 0.$

So either $x_1 = x_2$ or $ln = km$. If $ln = km$ we get:

$$2(1 + 3ay_1^2)(by_2^2 - ay_2^2) = -2(1 + 3by_2^2)(by_1^2 - ay_1^2),$$

therefore

$$3a^2 y_1^2 y_2^2 + ay_1^2 + ay_2^2 = 3by_1^2 y_2^2 + by_1^2 + by_2^2,$$

is equivalent to

$$3(3a^2 y_1^2 y_2^2 + ay_1^2 + ay_2^2) + 1 = 3(3by_1^2 y_2^2 + by_1^2 + by_2^2) + 1$$

which is equivalent to

$$\frac{1 + 3ay_1^2}{1 + 3by_1^2} = \frac{1 + 3by_2^2}{1 + 3ay_2^2}$$

which interdicts the assumption that

$$\frac{1 + 3ay_1^2}{1 + 3by_1^2} \neq \frac{1 + 3by_2^2}{1 + 3ay_2^2}.$$

Therefore $x_1 = x_2$, then because given $g_1(x_1, x_2, y_1, y_2)$ and $g_2(x_1, x_2, y_1, y_2)$ are equal to zero:
$g_1 = 2x_2^2 - y_1 - ay_1^2 - y_2 - by_2^3 = 0,$
$g_2 = 2(-1 + x_2)^2 - y_1 - by_1^3 - y_2 - ay_2^3 = 0.$
From the first equation we have $x_2 = \pm\frac{\sqrt{y_1 + ay_1^3 + y_2 + by_2^3}}{\sqrt{2}}$, from the second equation $g_2 = 0$ when $x_2 = \frac{\sqrt{y_1 + ay_1^3 + y_2 + by_2^3}}{\sqrt{2}}$ we get candidate for a Pareto Frontier: $2 + a(y_1^3 - y_2^3) + b(-y_1^3 + y_2^3) - 2\sqrt{2}\sqrt{y_1 + ay_1^3 + y_2 + by_2^3} = 0$,
when $x_2 = -\frac{\sqrt{y_1 + ay_1^3 + y_2 + by_2^3}}{\sqrt{2}}$ we have candidate for a Pareto Frontier: $2 + a(y_1^3 - y_2^3) + b(-y_1^3 + y_2^3) + 2\sqrt{2}\sqrt{y_1 + ay_1^3 + y_2 + by_2^3} = 0$, if we take for example $a = 0.5$, $b = 0.7$. Pareto Frontier is determined by the equation: $2 + 0.5(y_1^3 - y_2^3) + 0.7(-y_1^3 + y_2^3) + 2\sqrt{2}\sqrt{y_1 + 0.5y_1^3 + y_2 + 0.7y_2^3} = 0$. In the Figure 22 is depicted the Candidate Set which we get from this equation.

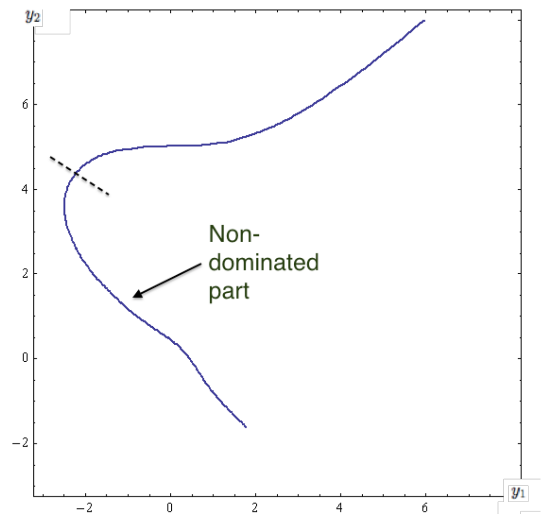

Figure 22: Candidate Set

and the proper Pareto Frontier is the bottom part of this graphic, where either one function is decreasing and the other increasing, as by the definition in the introduction. The picture of it is shown on the Figure 23:

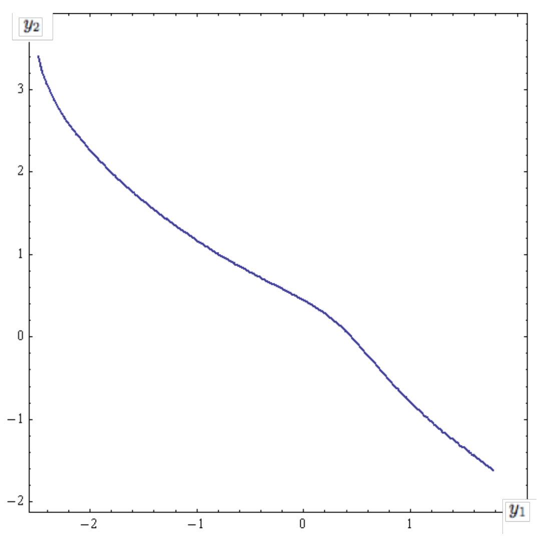

Figure 23: Pareto Frontier

3.5 Third example

This example is constructed in such way that it will be really hard to solve it analytically, therefore the Wolfram Mathematica program was instructed to give us many points "almost" satisfying our gradient requirement, meaning that the error of the final points will be pretty small (in this case ≤ 0.01). Additionally the computer was used to make a million iterations in order to get more points.
Let

$$\begin{cases} g_1(x_1, x_2, y_1, y_2) = x_1^2 + x_2^2 + a\sin(x_1 + y_1 + y_2) - y_1^3 - y_1, \\ g_2(x_1, x_2, y_1, y_2) = (x_1 - 1)^2 + (x_2 - 1)^2 + b\cos(x_2 + y_1 + y_2) - y_2^3 - y_2, \end{cases}$$

$$\frac{\partial y_1}{\partial x_1} = -\frac{\begin{vmatrix} \partial g_1/\partial x_1 & \partial g_1/\partial y_2 \\ \partial g_2/\partial x_1 & \partial g_2/\partial y_2 \end{vmatrix}}{\begin{vmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial y_2 \\ \partial g_2/\partial y_1 & \partial g_2/\partial y_1 \end{vmatrix}} = -\frac{a_1}{|J|},$$

where

$$\begin{pmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial y_2 \\ \partial g_2/\partial y_1 & \partial g_2/\partial y_1, \end{pmatrix}$$

is a Jacobian matrix, which will be denoted as J.
Analogically:

$$\frac{\partial y_1}{\partial x_2} = -\frac{\begin{vmatrix} \partial g_1/\partial x_2 & \partial g_1/\partial y_2 \\ \partial g_2/\partial x_2 & \partial g_2/\partial y_2 \end{vmatrix}}{|J|} = -\frac{a_2}{|J|},$$

$$\frac{\partial y_2}{\partial x_1} = -\frac{\begin{vmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial x_1 \\ \partial g_2/\partial y_1 & \partial g_2/\partial x_1 \end{vmatrix}}{|J|} = -\frac{b_1}{|J|},$$

$$\frac{\partial y_2}{\partial x_2} = -\frac{\begin{vmatrix} \partial g_1/\partial y_1 & \partial g_1/\partial x_2 \\ \partial g_2/\partial y_1 & \partial g_2/\partial x_2 \end{vmatrix}}{|J|} = -\frac{b_2}{|J|}.$$

Then
$a_1 = -2x_1 - 6x_1y_2^2 + a\cos(x_1+y_1+y_2) - 2ax_1\cos(x_1+y_1+y_2) - 3ay_2^2\cos(x_1+y_1+y_2) - 2bx_1\sin(x_2+y_1+y_2) - ab\cos(x_1+y_1+y_2)\sin(x_2+y_1+y_2)$,

$a_2 = -2x_2 - 6x_2y_2^2 + 2a\cos(x_1+y_1+y_2) - 2ax_2\cos(x_1+y_1+y_2) - 2bx_2\sin(x_1+y_1+y_2) + ab\cos(x_1+y_1+y_2)\sin(x_2+y_1+y_2)$,

$b_1 = 2 - 2x_1 + 6y_1^2 - 2a\cos(x_1+y_1+y_2) + 2ax_1\cos(x_1+y_1+y_2) + 2bx_1\sin(x_2+y_1+y_2) + ab\cos(x_1+y_1+y_2+)\sin(x_2+y_1+y_2)$,

$b_2 = 2 - 2x_2 + 6y_1^2 - 6x_2y_1^2 - 2a\cos(x_1+y_1+y_2) + 2ax_2\cos(x_1+y_1+y_2) + b\sin(x_2+y_1+y_2) + 2bx_2\sin(x_2+y_1+y_2) + 3by_1^2\sin(x_2+y_1+y_2) - ab\cos(x_1+y_1+y_2)\sin(x_2+y_1+y_2)$.

As we can see by now, the equations are pretty long and $\langle \frac{\triangledown f_1}{|\triangledown f_1|}, \frac{\triangledown f_2}{|\triangledown f_2|} \rangle$ is much worse and takes one whole page to write it down. Therefore we compute it using the Wolfram Mathematica program. The Pareto Frontier is shown on a Figure 24.

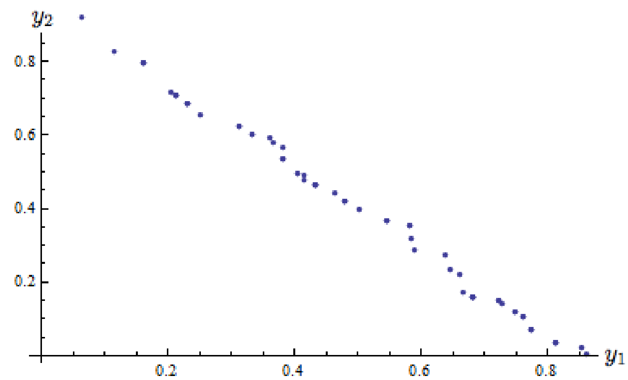

Figure 24: Pareto Frontier, by analytic method numerical approximation

Efficient Set:

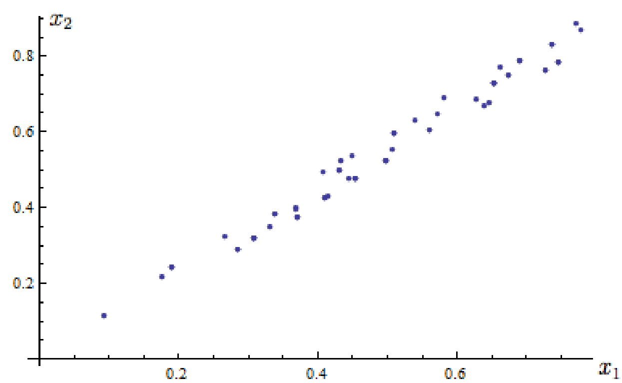

Figure 25: Efficient Set, by analytic method numerical approximation.

3.6 Global Theorems on Implicit Functions

Here we would like to take a look at the implicit functions from a bit new perspective.
Let us consider the following system of implicit functions:

$$\begin{cases} g_1(x_1, ..., x_n, y_1, y_2) = 0, \\ g_2(x_1, ..., x_n, y_1, y_2) = 0. \end{cases}$$

We will assume that functions $g_1(x_1, ..., x_n, y_1, y_2) = 0$, $g_2(x_1, ..., x_n, y_1, y_2) = 0$ are of the following form:

$$\begin{cases} g_1(x_1, ..., x_n, y_1, y_2) = f_1(x_1, ..., x_n) - \phi_1(y_1, y_2) = 0, \\ g_2(x_1, ..., x_n, y_1, y_2) = f_2(x_1, ..., x_n) - \phi_2(y_1, y_2) = 0. \end{cases}$$

and are differentiable everywhere. This class of functions we call 'separable', since the dependance on x and y separate into two distinct functions.
Our next steps are:
1. We find an Efficient Set and a Pareto Frontier for the following functions:

$$\begin{cases} z_1 = f_1(x_1, ..., x_n), \\ z_2 = f_2(x_1, ..., x_n), \end{cases} \tag{11}$$

we do this using methods proposed in Section 2.
2. After the previous step we find Pareto Frontier for the implicitly defined functions (y_1, y_2). We do it from the equation:

$$\begin{cases} z_1 = \phi_1(y_1, y_2), \\ z_2 = \phi_2(y_1, y_2). \end{cases} \tag{12}$$

By this way, in order to find Pareto Frontier we need to solve the system of non-linear equations. In order to have unique solvability of this system we require some properties for functions ϕ_1, ϕ_2, f_1, f_2. We also want the equation (11) to have the solution for all values $z_1, z_2 \in \mathbb{R}$. But the Implicit Function Theorem given above has the local character, hence we cannot use it. Therefore we use the following Theorem:

Theorem 3. [13, pp.91] *Given $H : E^n \to E^n$, where E^n is n-dimensional Euclidean Space, H is differentiable with Jacobian matrix $J(x)$. If there exists a constant $\epsilon > 0$ such that the absolute values of the leading principal minors $\triangle_1, \triangle_2, ..., \triangle_n$ of $J(x)$ satisfy*

$$\mid \triangle_1 \mid \geq \epsilon, \quad \frac{\mid \triangle_2 \mid}{\mid \triangle_1 \mid} \geq \epsilon, ..., \frac{\mid \triangle_n \mid}{\mid \triangle_{n-1} \mid} \geq \epsilon$$

for all $x \in E^n$, then H is one-to-one from E^n onto E^n.

We formulate the sufficient conditions with which the equation (12) has unique, continuous solution $y_1(z_1, z_2)$ and $y_2(z_1, z_2)$.

Theorem 4. *Let $\mid \triangle_1 \mid \geq \epsilon > 0$, $\quad \frac{|\triangle_2|}{|\triangle_1|} \geq \epsilon > 0$, where $\triangle_1 = \frac{\partial \phi_1}{\partial y_1}$ and $\triangle_2 = \frac{\partial \phi_1}{\partial y_1}\frac{\partial \phi_2}{\partial y_2} - \frac{\partial \phi_2}{\partial y_1}\frac{\partial \phi_1}{\partial y_2}$ then the equation (12) has the unique, continuous solution $y_1(z_1, z_2)$ and $y_2(z_1, z_2)$.*

Proof. Let us consider mapping $H : E^2 \to E^2$, where E^2 is a two dimensional Euclidean Space, such that $H(y) = \begin{bmatrix} \phi_1(y_1, y_2) \\ \phi_2(y_1, y_2) \end{bmatrix}$. Then fulfilment of conditions of our theorem leads to fulfilment of the Theorem 3, hence it provides a unique solution $y_1(z_1, z_2)$ and $y_2(z_1, z_2)$. Therefore for any point (z_1, z_2) there exists a unique point (y_1, y_2) such that the equation (12) holds.
The continuity of the solution follows from the Implicit Function Theorem, considered in points (z_1^0, z_2^0), (y_1^0, y_2^0), where $z_1^0 = \phi_1(y_1^0, y_2^0)$ and $z_2^0 = \phi_2(y_1^0, y_2^0)$. In fact, take $g_i(z_1, z_2, y_1, y_2) = z_i - \phi_i(y_1, y_2)$. Then $D_y g(z, y) = -D_y \phi(y)$. The condition $\frac{|\triangle_2|}{|\triangle_1|} \geq \epsilon > 0$ implies that the condition for the Implicit Function Theorem hold.

3.7 Final approach in finding the Pareto Frontier for the Implicit Functions

Now we give the main result of this work.

For a vector function $f : \mathbb{R}^n \to \mathbb{R}^2$, the Candidate (Efficient) Set E_f^{can} is the collection of all $x \in \mathbb{R}^n$ at which the gradient condition for optimality holds, i.e.: there exists $\lambda > 0$ s.t. $\triangledown f_1(x) = -\lambda \triangledown f_2(x)$.

Theorem 5. *Let $f : \mathbb{R}^n \to \mathbb{R}^2$ be continuously differentiable and $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ differentiable such that it satisfies the conditions of Theorem 4. Then:*

i) *There exist unique functions $y_1(x_1, ..., x_n)$ and $y_2(x_1, ..., x_n)$ such that:*

$$\begin{cases} \phi_1(y_1(x_1, ..., x_n), y_1(x_1, ..., x_n)) = f_1(x_1, ..., x_n), \\ \phi_2(y_1(x_1, ..., x_n), y_2(x_1, ..., x_n)) = f_2(x_1, ..., x_n). \end{cases} \tag{13}$$

for all $x \in \mathbb{R}^n$

ii) *The Candidate Efficient Set E_f^{can} of f is equal to the Candidate Efficient Set E_y^{can} of the implicitly defined functions y_1, y_2. In particular, E_y^{can} is independent of ϕ.*

iii) *If the functions f_1 and f_2 are convex and the functions y_1 and y_2 are convex, then instead of Candidate Sets one can consider Efficient Sets.*

Proof.
(i)
Accordingly to Theorem 4. (and 3), the function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is bijective. So

$$y(x) = \phi^{-1}(f(x)) \quad \text{for} \quad x \in \mathbb{R}^n.$$

(ii)
Let us consider the system of equations (13). In the points of the Candidate Set for the functions:

$$\begin{cases} z_1 = f_1(x_1, ..., x_n), \\ z_2 = f_2(x_1, ..., x_n), \end{cases}$$

should overlap the equation: $\nabla f_1 = -\lambda \nabla f_2$, where $\lambda > 0$, i.e. $\frac{\partial f_1}{\partial x_1} / \frac{\partial f_2}{\partial x_1} = \frac{\partial f_1}{\partial x_2} / \frac{\partial f_2}{\partial x_2} = ... = \frac{\partial f_1}{\partial x_n} / \frac{\partial f_2}{\partial x_n} = -\lambda$, where $\lambda > 0$. We will show that on the other hand $\nabla y_1 = -\lambda \nabla y_2$, where y_1, y_2 are the solutions of the initial system (13). In order to find the gradients we will differentiate the initial system of the equations (13) by x_i and x_j, where $i \neq j$ and $i, j = 1, ..., n$. Differentiating it by x_i we get:

$$\begin{cases} \frac{\partial \phi_1}{\partial y_1} \frac{\partial y_1}{\partial x_i} + \frac{\partial \phi_1}{\partial y_2} \frac{\partial y_2}{\partial x_i} = \frac{\partial f_1}{\partial x_i}, \\ \frac{\partial \phi_2}{\partial y_1} \frac{\partial y_1}{\partial x_i} + \frac{\partial \phi_2}{\partial y_2} \frac{\partial y_2}{\partial x_i} = \frac{\partial f_2}{\partial x_i}. \end{cases}$$

Therefore by the formula of Cramer we get:

$$\frac{\partial y_1}{\partial x_i} = \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x_i} & \frac{\partial \phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_i} & \frac{\partial \phi_2}{\partial y_2} \end{vmatrix}}{\triangle},$$

$$\frac{\partial y_2}{\partial x_i} = \frac{\begin{vmatrix} \frac{\partial \phi_1}{\partial y_1} & \frac{\partial f_1}{\partial x_i} \\ \frac{\partial \phi_2}{\partial y_1} & \frac{\partial f_2}{\partial x_i} \end{vmatrix}}{\triangle},$$

where $\triangle= \begin{vmatrix} \frac{\partial\phi_1}{\partial y_1} & \frac{\partial\phi_1}{\partial y_2} \\ \frac{\partial\phi_2}{\partial y_1} & \frac{\partial\phi_2}{\partial y_2} \end{vmatrix}$.
When differentiating the initial system of equations (13) by x_j, we get:

$$\begin{cases} \frac{\partial\phi_1}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \frac{\partial\phi_1}{\partial y_2}\frac{\partial y_2}{\partial x_j} = \frac{\partial f_1}{\partial x_j}, \\ \frac{\partial\phi_2}{\partial y_1}\frac{\partial y_1}{\partial x_j} + \frac{\partial\phi_2}{\partial y_2}\frac{\partial y_2}{\partial x_j} = \frac{\partial f_2}{\partial x_j}. \end{cases}$$

Analogously using the Cramer formula:

$$\frac{\partial y_1}{\partial x_j} = \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x_j} & \frac{\partial\phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_j} & \frac{\partial\phi_2}{\partial y_2} \end{vmatrix}}{\triangle},$$

$$\frac{\partial y_2}{\partial x_j} = \frac{\begin{vmatrix} \frac{\partial\phi_1}{\partial y_1} & \frac{\partial f_1}{\partial x_j} \\ \frac{\partial\phi_2}{\partial y_1} & \frac{\partial f_2}{\partial x_j} \end{vmatrix}}{\triangle}.$$

We will show that: $\frac{\partial y_1}{\partial x_i}/\frac{\partial y_2}{\partial x_i} = \frac{\partial y_1}{\partial x_j}/\frac{\partial y_2}{\partial x_j}$. Therefore we will show:

$$\frac{\partial y_1}{\partial x_i}/\frac{\partial y_2}{\partial x_i} = \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x_i} & \frac{\partial\phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_i} & \frac{\partial\phi_2}{\partial y_2} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x_i} & \frac{\partial\phi_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_i} & \frac{\partial\phi_2}{\partial y_1} \end{vmatrix}} = \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x_j} & \frac{\partial\phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_j} & \frac{\partial\phi_2}{\partial y_2} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x_j} & \frac{\partial\phi_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_j} & \frac{\partial\phi_2}{\partial y_1} \end{vmatrix}} = \frac{\partial y_1}{\partial x_j}/\frac{\partial y_2}{\partial x_j}.$$

In the left side of the equation we can divide first column by $\frac{\partial f_1}{\partial x_i}$. In the right side of the equation we can divide the first column by $\frac{\partial f_1}{\partial x_j}$. Then we get:

$$\frac{\begin{vmatrix} 1 & \frac{\partial\phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i} & \frac{\partial\phi_2}{\partial y_2} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{\partial\phi_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i} & \frac{\partial\phi_2}{\partial y_1} \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \frac{\partial\phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j} & \frac{\partial\phi_2}{\partial y_2} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{\partial\phi_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j} & \frac{\partial\phi_2}{\partial y_1} \end{vmatrix}}.$$

Hence it is true because:

$$\frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i} = \frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j},$$

which is identical to:

$$\frac{\partial f_1}{\partial x_i}/\frac{\partial f_2}{\partial x_i} = \frac{\partial f_1}{\partial x_j}/\frac{\partial f_2}{\partial x_j}.$$

Which indeed means

$$\nabla y_1 = -\lambda \nabla y_2.$$

On the other hand.

Let point $x \in E_y^{can}$ i.e. holds

$$\frac{\partial y_1}{\partial x_i}/\frac{\partial y_2}{\partial x_i} = \frac{\partial y_1}{\partial x_j}/\frac{\partial y_2}{\partial x_j} \Rightarrow \frac{\begin{vmatrix} 1 & \frac{\partial \phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i} & \frac{\partial \phi_2}{\partial y_2} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{\partial \phi_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i} & \frac{\partial \phi_2}{\partial y_1} \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \frac{\partial \phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j} & \frac{\partial \phi_2}{\partial y_2} \end{vmatrix}}{\begin{vmatrix} 1 & \frac{\partial \phi_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j} & \frac{\partial \phi_2}{\partial y_1} \end{vmatrix}},$$

hence

$$\begin{vmatrix} 1 & \frac{\partial \phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i} & \frac{\partial \phi_2}{\partial y_2} \end{vmatrix} \begin{vmatrix} 1 & \frac{\partial \phi_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j} & \frac{\partial \phi_2}{\partial y_1} \end{vmatrix} = \begin{vmatrix} 1 & \frac{\partial \phi_1}{\partial y_2} \\ \frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j} & \frac{\partial \phi_2}{\partial y_2} \end{vmatrix} \begin{vmatrix} 1 & \frac{\partial \phi_1}{\partial y_1} \\ \frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i} & \frac{\partial \phi_2}{\partial y_1} \end{vmatrix},$$

i.e.

$$\left(\frac{\partial \phi_2}{\partial y_2} - \frac{\partial \phi_1}{\partial y_2}\left(\frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i}\right)\right)\left(\frac{\partial \phi_2}{\partial y_1} - \frac{\partial \phi_1}{\partial y_1}\left(\frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j}\right)\right) =$$
$$= \left(\frac{\partial \phi_2}{\partial y_2} - \frac{\partial \phi_1}{\partial y_2}\left(\frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j}\right)\right)\left(\frac{\partial \phi_2}{\partial y_1} - \frac{\partial \phi_1}{\partial y_1}\left(\frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i}\right)\right),$$

from where follows:

$$\frac{\partial \phi_2}{\partial y_2}\frac{\partial \phi_1}{\partial y_1}\left(\frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j}\right) + \frac{\partial \phi_1}{\partial y_2}\frac{\partial \phi_2}{\partial y_1}\left(\frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i}\right) = \frac{\partial \phi_2}{\partial y_2}\frac{\partial \phi_1}{\partial y_1}\left(\frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i}\right) + \frac{\partial \phi_1}{\partial y_2}\frac{\partial \phi_2}{\partial y_1}\left(\frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j}\right),$$

we get

$$\frac{\partial \phi_2}{\partial y_2}\frac{\partial \phi_1}{\partial y_1}\left(\frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j} - \frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i}\right) + \frac{\partial \phi_1}{\partial y_2}\frac{\partial \phi_2}{\partial y_1}\left(\frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i} - \frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j}\right) = 0,$$

and

$$\left(\frac{\partial \phi_2}{\partial y_2}\frac{\partial \phi_1}{\partial y_1} - \frac{\partial \phi_1}{\partial y_2}\frac{\partial \phi_2}{\partial y_1}\right)\left(\frac{\partial f_2}{\partial x_j}/\frac{\partial f_1}{\partial x_j} - \frac{\partial f_2}{\partial x_i}/\frac{\partial f_1}{\partial x_i}\right) = 0,$$

but from the Theorem 4. we know

$$\frac{\partial \phi_2}{\partial y_2}\frac{\partial \phi_1}{\partial y_1} - \frac{\partial \phi_1}{\partial y_2}\frac{\partial \phi_2}{\partial y_1} \neq 0.$$

Therefore

$$\frac{\partial f_1}{\partial x_i}/\frac{\partial f_2}{\partial x_i} = \frac{\partial f_1}{\partial x_j}/\frac{\partial f_2}{\partial x_j}.$$

This means

$$\nabla f_1 = -\lambda \nabla f_2,$$

which indeed means that $x \in E_f^{can}$.

(iii)

The case when functions are convex f_1, f_2, y_1, y_2 from the Theorem on page 5 directly implies the third statement iii), that one can consider Efficient Sets instead of Pre-Efficient ones.

3.8 Example

Let us consider the following example

$$\begin{cases} g_1(x_1, x_2, y_1, y_2) = x_1^2 + x_2^2 - y_1 - y_1^3 - y_2, \\ g_2(x_1, x_2, y_1, y_2) = (x_1 - 1)^2 + (x_2 - 1)^2 - y_1 - y_2 - y_2^3. \end{cases}$$

It is not hard to show that the Efficient Set as in the first example of this section will be $x_1 = x_2$ for $z_1 = x_1^2 + x_2^2$ and $z_2 = (x_1 - 1)^2 + (x_2 - 1)^2$. Basing on that knowledge, the program was constructed in order to get the Pareto Frontier for those implicit functions.
The Wolfram Mathematica program, and the Pareto Frontier for the $y_1(x_1, x_2)$, $y_2(x_1, x_2)$:

```
f1 = x1^2 + x2^2 - y1 - y1^3 - y2;
f2 = (x1 - 1)^2 + (x2 - 1)^2 - y1 - y2 - y2^3;
y2 → x1^2 + x2^2 - y1 - y1^3; h = 0.001;
p1 = x1^2 + x2^2 - y1 - y1^3 /. x2 → x1 /. x1 → h i;
p2 = Expand[f2 /. y2 → x1^2 + x2^2 - y1 - y1^3 /. x2 → x1 /. x1 → h i];
Do[
 {Subscript[k, i] = {Subscript[m, i] = Root[p2, 1],
    Subscript[n, i] = p1 /. y1 → Root[p2, 1]}}, {i, 0, 1000}]
yy1 = ListPlot[Table[Subscript[k, i], {i, 0, 1000}],
  PlotStyle → PointSize[0.01]]
```

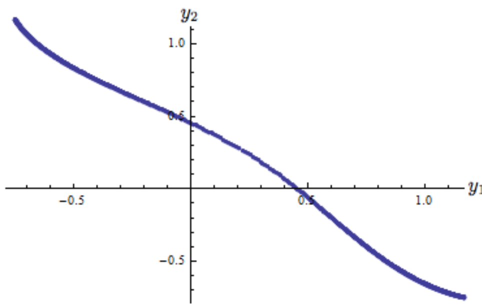

Figure 26: Pareto Frontier

Efficient Set in this example is $x_1 = x_2$ then in this case we can show how level curves of the functions z_1 and z_2 behave. Let for example $x_1 = x_2 = 0.1$ then level curves of z_1 and z_2 will look as depicted below:

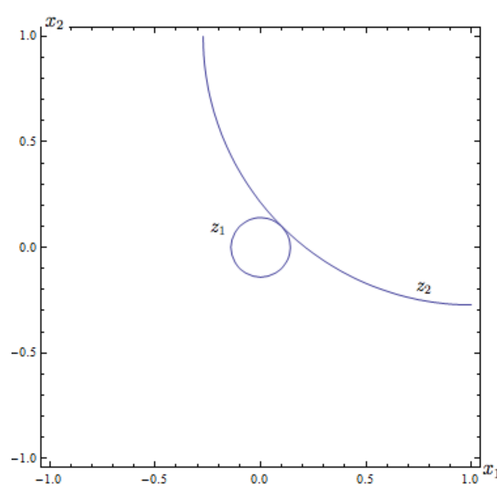

Figure 27: Level curves.

In this case the the point where level curves touch is $(0.1, 0.1)$ gives us the values of y_1 and y_2, which are $y_1 = -0.70558$ and $y_2 = 1.07685$.

4 Conclusions and outlook

In this work we presented different approaches to finding the Efficient Set and the Pareto Frontier. Among those: the gradient method, Karush-Kuhn-Tucker Condition. We proposed a method for finding the Pareto Frontier for implicitly defined objective functions. This method can be used in various types of problems. This interesting approach can be used in the case of so-called 'The Black Box problem'. In science, computing, and engineering, a Black Box is a device, system or object which can be viewed in terms of its inputs and outputs (or transfer characteristics), without any knowledge of its internal workings.
Let us interpret our problem in terms of two Black Boxes:

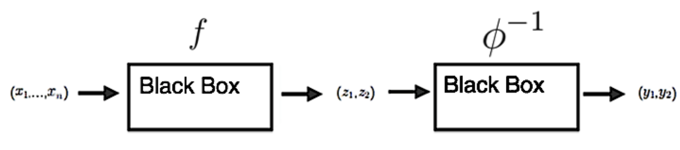

Figure 28: Black Boxes.

First one allows to find a Pareto Frontier for a couple of explicit functions $z_1 = f_1(x_1, x_2, ..., x_n)$ and $z_2 = f_2(x_1, x_2, ..., x_n)$, which form one cannot see, since they are wrapped in a first Black Box. We can find the Pareto Frontier and the Efficient Set for them using the Expected Hypervolume Improvement Algorithm method of M.T.M. Emmerich and give to the second Black Box the information of the Pareto Frontier for f_1, f_2.

Whereas the second Black Box simply solves the non-linear system $z_1 = \phi_1(y_1, y_2)$ and $z_2 = \phi_2(y_1, y_2)$, Where one cannot see the functions ϕ_1, ϕ_2. Due to Theorem 5. the Candidate Efficient Set E_f^{can} stays the same for implicit functions y_1, y_2 so $E_f^{can} = E_y^{can}$. Therefore on the output of the second Black Box we obtain the set of pairs y_1, y_2 which is the Candidate Set for the Pareto Frontier for implicitly given objective functions.

The implicit functions that our problem could be solved for, are defined from the following system:

$$\begin{cases} g_1(x_1, ..., x_n, y_1, y_2) = f_1(x_1, ..., x_n) - \phi_1(y_1, y_2) = 0, \\ g_2(x_1, ..., x_n, y_1, y_2) = f_2(x_1, ..., x_n) - \phi_2(y_1, y_2) = 0, \end{cases}$$

and ϕ is bijective on $\mathbb{R}^2$.
By this approach enabled by our approach to objective functions that are implicitly defined, for a new class of objective functions partially analytically, partially numerically the problem of finding the Pareto Frontier can be solved.

4.1 Acknowledgment

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$\square$