# CONES, POSITIVITY AND ORDER UNITS

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# CONTENTS

# 1 INTRODUCTION

An ordered vector space $E$ is a vector space endowed with a partial order which is 'compatible' with the vector space operations (in some sense). If the order structure of $E$ is a lattice, then $E$ is called a Riesz space. This order structure leads to a notion of a (positive) cone, which is the collection of all 'positive elements' in $E$.

In many applications positivity of normed (function) spaces plays an important role. The most natural example is the 'standard machinery', a tool which is frequently used in measure and integration theory.

The aim of this thesis is to provide a construction of a normed Riesz space, given a cone. This requires a definition of a cone outside the context of ordered vector spaces and will be given by a modification of the Grothendieck group construction.

Another goal is to find an example of a Riesz space which does not have a weak order unit. There are several existence results of Riesz spaces that do not have such units, but there are not many explicit examples.

## 1.1 ASSUMPTIONS AND NOTATIONS

Before reading any further, the reader should be familiar with basic measure theory and functional analysis. However, most of the material will be developed along the way.

Throughout this thesis, the following assumptions, conventions and notations will be used without further ado:

1. The Axiom of Choice (AC) is accepted;
2. A sequence $\{x_n : n \in \mathbf{N}\}$ in a topological space $X$ will be denoted as $(x_n)$;
3. Let $S$ be a metric space, $x \in S$ and $r > 0$. Now define

$$\begin{aligned} \mathrm{B}(x;r) &= \{y \in S : \mathrm{d}(x,y) < r\}, \\ \overline{\mathrm{B}}(x;r) &= \{y \in S : \mathrm{d}(x,y) \leqslant r\}, \\ \mathbf{B}(x;r) &= \{y \in S : \mathrm{d}(x,y) = 1\}; \end{aligned}$$

4. If $A$ is a subset of a topological space $X$, then the set $\overline{A}$ denotes the closure of $A$;
5. The dual of a vector space $X$ will be denoted by $X^*$.

# 2

# RIESZ SPACES

This chapter will provide an elementary introduction to the theory of Riesz spaces. Later on, this theory will be used to describe certain vector spaces which consist of sets.

Note that this chapter is in no way a complete introduction in the field of positivity. This chapter only provides the bare necessities in order to describe the results of this thesis. The interested reader can find a detailed exposition on this subject in the literature, e.g. in [5] and [2].

## 2.1 DEFINITIONS

2.1.1 **Definition.** A partially ordered set $(X, \preccurlyeq)$ is a *lattice* if each pair of elements $x, y \in X$ has a least upper bound (a supremum) and a greatest lower bound (an infimum). The supremum and infimum of any two elements $x, y \in X$ is denoted by

$$\sup\{x, y\} = x \vee y \quad \text{and} \quad \inf\{x, y\} = x \wedge y.$$

Note that both the supremum and infimum are unique, provided that they exist.

The maps

$$\begin{array}{rcl} \vee \colon X \times X & \longrightarrow & X \\ (x, y) & \longmapsto & x \vee y \end{array} \quad \text{and} \quad \begin{array}{rcl} \wedge \colon X \times X & \longrightarrow & X \\ (x, y) & \longmapsto & x \wedge y \end{array}$$

are the *lattice operations* on $X$.

If, in addition, $X$ is a vector space over $\mathbf{R}$, then $X$ is called an *ordered vector space* if for all $x, y \in X$ the following holds:

1. $x + z \preccurlyeq y + z$ for all $z \in X$;
2. $\alpha x \preccurlyeq \alpha y$ for all $\alpha > 0$.

A *Riesz space* (or *vector lattice*) is an ordered vector space that is simultaneously a lattice. A *normed Riesz space* is a Riesz space endowed with a norm $\|\cdot\|$ such that

$$0 \preccurlyeq x \preccurlyeq y \implies \|x\| \leqslant \|y\| \quad \text{and} \quad \||x|\| = \|x\|$$

for any $x, y \in E$. A complete normed Riesz space is called a *Banach lattice*. «

From this point on, the letter $E$ will be used to denote either a Riesz space or an ordered vector space $(E, \preccurlyeq)$. Additionally, the standard ordering on $\mathbf{R}$ will be denoted by the symbol $\leqslant$.

2.1.2 **Example.** Let $1 \leqslant p \leqslant \infty$.

1. Let $n \in \mathbf{N}$ and consider $\mathbf{R}^n$ under the usual vector operations. Let $x, y \in \mathbf{R}^n$ and write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then $\mathbf{R}^n$ is a Riesz space under the partial ordering $\preccurlyeq$ defined by

   $$x \preccurlyeq y \iff x_k \leqslant y_k \quad \text{for } k \in \{1, \ldots n\}.$$

   By a similar reasoning, the sequence spaces $c_0$ and $\ell_p$ are Riesz spaces.

2. Let $X$ be a topological space, then $C(X)$, the space of continuous functions on $X$, is a Riesz space under the (pointwise) ordering $\preccurlyeq$ given by

$$f \preccurlyeq g \iff f(x) \leqslant g(x) \quad \text{for all } x \in X.$$

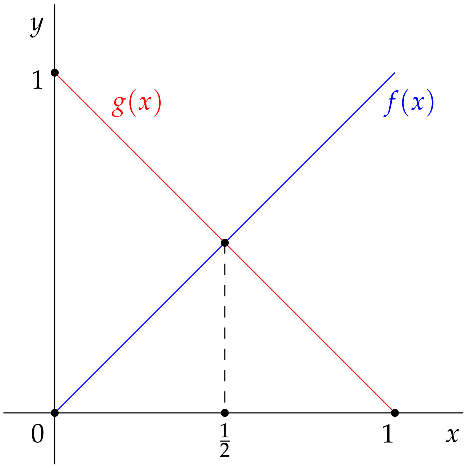
Note that by slightly modifying the previous ordering, the Lebesgue spaces are also examples of Riesz spaces. Suppose $(X, \Sigma, \mu)$ is an arbitrary measure space, then $\mathrm{L}_p(X, \Sigma, \mu)$ is a Riesz space under the ordering $\preccurlyeq$ defined as

$$f \preccurlyeq g \iff f(x) \leqslant g(x) \quad \text{for almost all } x \in X.$$ «

The previous examples indicate that Riesz spaces appear in a natural fashion when studying function spaces. However, not every function space is a Riesz space:

2.1.3 **Example.** Consider $E = \mathrm{C}^1[0,1]$, the class of continuously differentiable functions on the interval $[0,1] \subseteq \mathbf{R}$. Observe that $E$ is an ordered vector space under pointwise ordering. However, $E$ is not a Riesz space.

Indeed, consider the functions $f, g \in E$ given by $f(x) = x$ and $g(x) = 1 - x$. The figure below depicts the graphs of both $f$ and $g$.




Note that both $f \wedge g$ and $f \vee g$ are not differentiable at $x = \frac{1}{2}$. Therefore, $E$ is not a Riesz space. «

## 2.2 POSITIVE CONES AND RIESZ HOMOMORPHISMS

2.2.1 **Definition.** Let $(E, \preccurlyeq)$ be an ordered vector space. The set

$$E^+ = \{x \in E \colon x \succcurlyeq 0\}$$

is the *positive cone* of $E$. The members of $E^+$ are said to be the *positive elements* of $E$. The *strictly positive elements* of $E$ are all the non-zero members of $E^+$. «

2.2.2 **Definition.** Let $E$ and $F$ be Riesz spaces and $T$ a linear map $E \to F$. The map $T$ is called *positive* if $T[E^+] \subseteq F^+$. «

2.2.3 **Definition.** Let $T \colon E \to F$ be a linear map between Riesz spaces $E$ and $F$, then $T$ is a *Riesz homomorphism* (or *lattice homomorphism*) if $T$ preserves the lattice operations, that is,

$$T(x \vee y) = Tx \vee Ty \quad \text{and} \quad T(x \wedge y) = Tx \wedge Ty$$

for all $x, y \in E$.

A *Riesz isomorphism* is a bijective Riesz homomorphism. «

Note that any Riesz homomorphism is a positive map, since $x \in E^+$ if and only if $x = x \vee 0$.

The next proposition lists some important properties of the lattice operations on a Riesz space $E$. The proof is straightforward and will be omitted.

2.2.4 **Proposition.** *Let $E$ be a Riesz space and $A \subseteq E$ a non-empty subset.*

1. *Let $T \colon E \to E$ be a Riesz isomorphism. If $A$ has a supremum, then so does $T[A]$. Furthermore,*
$$\sup T[A] = T(\sup A) \quad \text{and} \quad \inf T[A] = T(\inf A).$$
2. *If $x, y \in E$, then*
$$-(x \vee y) = (-x) \wedge (-y).$$
3. *Define $\lambda A = \{\lambda a \colon a \in A\}$ for $\lambda \in \mathbf{R}^+$ and suppose that $\sup A$ and $\inf A$ exists. Then*
$$\sup(\lambda A) = \lambda \sup(A) \quad \textit{for all } \lambda \in \mathbf{R}^+$$
*and*
$$\inf(\lambda A) = \lambda \inf(A) \quad \textit{for all } \lambda \in \mathbf{R}^+.$$
4. *For any $x, y \in E$ and all $\lambda \in \mathbf{R}^+$,*
$$\lambda(x \vee y) = (\lambda x) \vee (\lambda y).$$
5. *Define for $x_0 \in E$, the collection $A + x_0 = \{a + x_0 \colon a \in A\}$. Then*
$$\sup(A + x_0) = (\sup A) + x_0 \quad \text{and} \quad \inf(A + x_0) = (\inf A) + x_0,$$
*provided that both $\sup A$ and $\inf A$ exist.*
6. *For all $x, y, z \in E$ the identity*
$$(x + z) \vee (y + z) = (x \vee y) + z$$
*holds.*
7. *The lattice operations are distributive, i.e. the operations $\wedge$ and $\vee$ satisfy*
$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) \quad \text{and} \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$
*for all $x, y, z \in E$.* «

2.2.5 **Definition.** Let $E$ be a Riesz space and $x \in E$. The *positive part* $x^+$ and the *negative part* $x^-$ of $x$ are defined by
$$x^+ = x \vee 0 \quad \text{and} \quad x^- = (-x) \wedge 0.$$
The *absolute value* $|x|$ of an element $x \in X$ is given by
$$|x| = x \vee (-x).$$
«

The next proposition follows directly from Proposition 2.2.4:

2.2.6 **Proposition.** *Let $E$ be a Riesz space and let $x \in E$. Then*
$$x = x^+ - x^- \quad \text{and} \quad |x| = x^+ + x^-.$$
«

2.3.1 **Definition.** Let $E$ be a Riesz space and $x \in E$. If

$$\inf_{n\in\mathbf{N}} \frac{1}{n} \cdot x = 0$$

for any $x \in E^+$, then $E$ is called *Archimedean*. «

There is another way to define the Archimedean property, one that is frequently used in the literature. The proof can be found in [2, p .14].

2.3.2 **Lemma.** *Let E be a Riesz space, then the following statements are equivalent:*

*(a)* $\inf\{\frac{1}{n}x \colon n \in \mathbf{N}\} = 0$ *for all* $x$ *in* $E^+$.

*(b) If* $x, y \in E^+$ *and* $nx \preccurlyeq y$ *for all* $n \in \mathbf{N}$, *then* $x = 0$. «

The next proposition provides infinitely many examples of Archimedean Riesz spaces:

2.3.3 **Proposition.** *Any normed Riesz space is Archimedean.* «

Proof. Let $E$ be a normed Riesz space and let $x, y \in E$ such that $n \cdot x \preccurlyeq y$ for any $n \in \mathbf{N}$. This means $n \cdot x \preccurlyeq y \preccurlyeq y^+$ for all $n \in \mathbf{N}$, which implies $0 \preccurlyeq n \cdot x^+ \preccurlyeq y^+$ for any $n \in \mathbf{N}$. Hence

$$n\|x^+\| = \|n \cdot x^+\| \leqslant \|y^+\| \quad \text{for all } n \in \mathbf{N}.$$

Since $\|y^+\|$ is a finite real number, it follows that $\|x^+\| = 0$. On the other hand, $x^+ = 0$ by definition ofi the norm. Therefore $x \preccurlyeq x^+ = 0$. This shows $E$ is Archimedean. ■

The previous proposition implies that each Riesz space from Example 2.1.2 is Archimedean.

At this point a remark is in order. The Archimedean property from Definition 2.3.1 differs from the Archimedean property in the set of real numbers, which states that for any $x, y \in \mathbf{R}$ there is a $n \in \mathbf{N}$ such that

$$|y| \leqslant n|x|.$$

Note that this property is implied by both Lemma 2.3.2 and Definition 2.3.1, but it need not be its equivalent:

2.3.4 **Example.** Consider the space $C(0,1)$ and endow this space with the pointwise ordering. Observe that this space is Archimedean in the sense of Definition 2.3.1.

Indeed, let $0 \preccurlyeq f \in \mathrm{C}(0,1)$. Then

$$\forall x \in \mathbf{R} \colon \tfrac{1}{n} f(x) \downarrow 0 \quad \text{in } \mathbf{R},$$

therefore

$$\tfrac{1}{n} f \downarrow 0 \quad \text{in } \mathrm{C}(0,1).$$

This shows that $C(0,1)$ is Archimedean. To prove that it does not satisfy the Archimedean property of the set of real numbers, consider $f, g \in C(0,1)$ given by $f(x) = \frac{1}{x}$ and $g(x) = 1$. Then there is no $n \in \mathbf{N}$ such that $f \leqslant ng$. «

2.4.1 **Definition.** Let $E$ be a Riesz space and let $x \in E$. The *principal band* generated by $x \in E$, denoted by $B_x$, is the collection given by

$$B_x = \Big\{ y \in E \colon |y| = \sup_{n \in \mathbf{N}} |y| \wedge n|x| \Big\}.$$

The *principal ideal* generated by an element $x \in E^+$, denoted by $E_x$, is the set

$$E_x = \{ y \in E \colon \text{there is a } \lambda \geqslant 0 \text{ such that } |y| \preccurlyeq \lambda x \}.$$

If there is an element $e^+ \in E$ such that $B_e = E$, then $e$ is called a *weak order unit* and if $E_e = E$, then $e$ is a *strong order unit*. «

It is clear from the definition that any strong order unit is also a weak order unit. The converse statement is not true.

A natural question to ask is whether a Riesz spaces has any order unit at all (strong or weak). There are very few explicit examples of Riesz spaces that do not have a weak order unit. One of the goals of this thesis is to add a new example to that list.

Let's state some known results. The proof of the next lemma is somewhat involved and will therefore be omitted, but can be found in Meyer-Nieberg (see [15, p. 20]).

2.4.2 **Lemma.** *Let* $p \in [1, \infty)$.

1. *The Riesz spaces* $c_0$, $\ell_p$ *do not have a strong order unit.*
2. *Let* $(X, \Sigma, \mu)$ *be a measure space, then* $\mathrm{L}_p(X, \Sigma, \mu)$ *does not have a strong order unit.*
3. *The dual of* $\ell_\infty$ *does not have a weak order unit.* «

2.4.3 **Proposition.** *Let* $(X, \Sigma, \mu)$ *be a measure space. Then the constant function* $\mathbf{1}_X$ *is a strong order unit in* $\mathrm{L}_\infty(X, \Sigma, \mu)$. «

Proof. Since any $f \in \mathrm{L}_\infty$ is essentially bounded, it follows that $|f| \leqslant \|f\|_\infty$. Therefore,

$$-\|f\|_\infty \cdot \mathbf{1}_X \leqslant f \leqslant \|f\|_\infty \cdot \mathbf{1}_X \quad \text{for any } f \in \mathrm{L}_\infty.$$

Hence $\mathbf{1}_X$ is a strong order unit in $\mathrm{L}_\infty$. ∎

2.4.4 **Proposition.** *Let* $K$ *be a compact Hausdorff space. Then any strictly positive function* $f \in C(K)$ *is a strong order unit.* «

Proof. Let $f \in C[K]$ be strictly positive. Consider

$$\alpha = \min_{x \in K} f(x).$$

Note that $\alpha$ is well-defined since continuous functions on compacts sets attain a minimum. Moreover, $\alpha > 0$ and $f \leqslant \alpha \cdot \mathbf{1}_K$. This implies $f$ is a strong order unit. ∎

2.4.5 **Theorem.** *Let* $(X, \Sigma, \mu)$ *be a measure space and* $p \in [1, \infty)$. *Suppose* $(f_n)$ *is a sequence of* $\mathrm{L}_p$*-functions which converges in norm to a* $f \in \mathrm{L}_p$. *If* $f_n \leqslant f$, *then*

$$f = \sup_{n \in \mathbf{N}} f_n.$$

«

Proof. Define $F_n = f - f_n$ and consider the sequence $(F_n)$. Then, by using the assumptions, it follows that $F_n(x) \geqslant 0$ for all $x \in X$ and

$$\lim_{n\to\infty} \int F_n^p \, d\mu = \lim_{n\to\infty} \|F_n\|_p^p = 0.$$

By applying Fatou's lemma,

$$\begin{aligned} 0 &\leqslant \int \liminf_{n\to\infty} F_n^p \, d\mu \\ &\leqslant \liminf_{n\to\infty} \int F_n^p \, d\mu \\ &= \lim_{n\to\infty} \int F_n^p \, d\mu \\ &= 0. \end{aligned}$$

This means $\liminf F_n^p = 0$ almost everywhere.

On the other hand, note that for any $n \in \mathbf{N}$ and any $x \in X$

$$0 \leqslant \inf_{n\in\mathbf{N}} F_n(x)^p \leqslant \lim_{k\to\infty} \inf_{n\geqslant k} F_k(x)^p = \liminf_{k\to\infty} F_k(x)^p.$$

Hence,

$$\inf_{n\in\mathbf{N}} F_n(x) = 0$$

almost everywhere on X. Therefore,

$$f(x) = \sup_{n\in\mathbf{N}} f_n(x)$$

for almost all $x \in X$. ■

2.4.6 **Theorem.** *Let $(X, \Sigma, \mu)$ be a measure space, then any strictly positive $\mathrm{L}_p$ function is a weak order unit.* «

Proof. Let $f, g \in \mathrm{L}_p$ such that $f$ is strictly positive and $g \geqslant 0$. Then, by Theorem 2.4.5, it suffices to show

$$g = \sup_{n\in\mathbf{N}} g \wedge (nf).$$

Recall that the collection of step functions is dense in $\mathrm{L}_p$. So for any $\varepsilon > 0$ there is a step function $s \in \mathrm{L}_p$ such that $\|g - s\|_p < \varepsilon$. Since $s$ is bounded, there exists a $M \in \mathbf{N}$ such that $Mf \geqslant s$ almost everywhere. Furthermore,

$$\|g \wedge (nf) - s \wedge (nf)\|_p^p = \int |g \wedge (nf) - s \wedge (nf)|^p \, d\mu$$

for all $n \in N$. Let $x \in X$ and note that if $g(x) \leqslant nf(x)$, which means $s(x) \geqslant nf(x)$. So in this case, $g(x) - nf(x) \leqslant nf(x) - nf(x) = 0$. Therefore, $nf(x) - g(x) \leqslant s(x) - g(x)$.

This yields

$$|g(x) \wedge (nf(x)) - s(x) \wedge (n(f(x))| \leqslant |g(x) - s(x)| \quad \text{for all } x \in S.$$

Then

$$\begin{aligned} \|g \wedge (nf) - s \wedge (nf)\|_p^p &= \int |g \wedge (nf) - s \wedge (nf)|^p \, d\mu \\ &= \int |g - s)|^p \, d\mu \\ &= \|g - s\|_p^p \\ &< \varepsilon^p. \end{aligned}$$

Hence, for all $n \geqslant N$,

$$\begin{aligned}\|g - g \wedge (nf)\| &\leqslant \|g - s\|_p + \|s - s \wedge (nf)\|_p + \|s \wedge (nf) - g \wedge (nf)\|_p \\ &\leqslant \varepsilon + 0 + \varepsilon.\end{aligned}$$

This shows that $g \wedge (nf) \to g$ (w.r.t $\|\cdot\|_p$) as $n \to \infty$. In addition, $(g \wedge nf) \leqslant g$ for any $n \in \mathbf{N}$. Therefore, by applying Theorem 2.4.5, the result follows. ∎

# 3

# ABSTRACT CONES

The Grothendieck group construction is a tool used in abstract algebra that constructs an abelian group from a commutative monoid in a universal way. This construction was develloped in the 1950s by Alexander Grothendieck (see [3] and [7]) and has played an important role in the devellopment of $K$-theory.

This chapter provides a definition of a positive cone outside the context of ordered vector space and a tool to construct a Riesz space from the cone. The construction is based on a technique similar to the construction of the Grothendieck group.

## 3.1 DEFINITIONS

3.1.1 **Definition.** A non-empty set $\mathcal{C}$ with a binary operation

$$\begin{aligned} +\colon \mathcal{C}\times\mathcal{C} &\longrightarrow \mathcal{C} \\ (a,b) &\longmapsto a+b \end{aligned}$$

such that

(M1) $(a+b)+c = a+(b+c)$ for all $a,b,c\in\mathcal{C}$;

(M2) There is a $e\in\mathcal{C}$ such that $e+a=a+e=a$ for all $a\in\mathcal{C}$;

(M3) $a+b=b+a$ for all $a,b\in\mathcal{C}$;

is said to be a *commutative monoid*. If the operation $+$ satisfies only (M1) and (M2), then $\mathcal{C}$ is a *monoid*. The operation $+$ is called *the addition on* $\mathcal{C}$ or simply *addition*. The element $e$ from (M2) is called the *unit element for addition on* $\mathcal{C}$ and will be denoted by the symbol 0 for convenience. «

Observe that the unit element of a monoid is unique. Indeed, suppose that there are two elements $e\in\mathcal{C}$ and $u\in\mathcal{C}$ such that

$$a+e=a \quad \text{and} \quad a+u=a$$

for all $a\in\mathcal{C}$. Then, for $a=e$, it follows that $u=u+e=e$. Hence $e=u$.

3.1.2 **Example.** The collection of natural numbers including 0 is a commutative monoid under the usual addition. «

3.1.3 **Definition.** A commutative monoid $\mathcal{C}$ equipped with a map

$$\begin{aligned} \cdot\colon \mathbf{R}^+\times\mathcal{C} &\longrightarrow \mathcal{C} \\ (\lambda,a) &\longmapsto \lambda\cdot a \end{aligned}$$

such that

(SM1) $1\cdot a=a$ and $0\cdot a=0$ for all $a\in\mathcal{C}$;

(SM2) $\lambda\cdot a+\mu\cdot a=(\lambda+\mu)\cdot a$ for all $\lambda,\mu\in\mathbf{R}^+$ and all $a\in\mathcal{C}$;

(SM3) $\lambda\cdot a+\lambda\cdot b=\lambda\cdot(a+b)$ for all $\lambda\in\mathbf{R}^+$ and all $a,b\in\mathcal{C}$;

is said to be a *(positive) (convex) cone*. The map $\cdot$ is called the *scalar multiplication* on $\mathcal{C}$. If no confusion can arise, the symbol $\cdot$ will be left out. «

3.1.4 **Definition.** Let $\mathcal{C}$ be a cone. If there exists a partial ordering $\preccurlyeq$ on $\mathcal{C}$ such that

$$\begin{aligned} a \preccurlyeq b &\implies \lambda a \preccurlyeq \lambda b \text{ for all } \lambda \in \mathbf{R}^+ \text{ and all } a, b \in \mathcal{C}; \\ a \preccurlyeq b &\implies a + c \preccurlyeq b + a \text{ for all } a, b, c \in \mathcal{C}, \end{aligned}$$

then $\mathcal{C}$ is called an *ordered cone*. «

Any cone $\mathcal{C}$ with the property

$$a + b = e \implies a = b = e \quad \text{for all } a, b \in \mathcal{C}$$

can be ordered by defining $a \leqslant b$ if and only if there is a $c \in \mathcal{C}$ such that

$$a + c = b.$$

A routine verification shows that $\leqslant$ defines a partial order that meets all the requirements of Definition 3.1.4. From now on, this order $\leqslant$ will be called the *standard order on a cone* $\mathcal{C}$.

## 3.2 CONSTRUCTING A VECTOR SPACE FROM A CONE

Let $\mathcal{C}$ be a cone and define the relation $\sim$ on $\mathcal{C} \times \mathcal{C}$ by

$$(a, b) \sim (x, y) \iff \text{there is a } c \in \mathcal{C} \text{ such that } a + y + c = x + b + c.$$

Then, by a straightforward verification, $\sim$ is an equivalence relation on the Cartesian product $\mathcal{C} \times \mathcal{C}$.

Consider the quotient space

$$E_{\mathcal{C}} = \mathcal{C} \times \mathcal{C} / \sim.$$

In order to avoid confusing notation, an element of $E_{\mathcal{C}}$ will be denoted by $[a, b]$. The quotient space $E_{\mathcal{C}}$ is a vector space by defining the following (vector) operations

$$\begin{aligned} \lambda[a, b] &= [\lambda a, \lambda b] && \text{for all } \lambda \in \mathbf{R}^+ \text{ and all } a \in \mathcal{C}; \\ [a, b] + [x, y] &= [a + x, b + y] && \text{for all } a, b, x, y \in \mathcal{C}; \\ [a, b] &= -[b, a] && \text{for all } a, b \in \mathcal{C}. \end{aligned}$$

The cone $\mathcal{C}$ can be mapped into $E_{\mathcal{C}}$ under the quotient map

$$\begin{aligned} q \colon \mathcal{C} &\longrightarrow E_{\mathcal{C}} \\ a &\longmapsto [a, 0] \end{aligned}$$

This map preserves the cone structure of $\mathcal{C}$, but it need not necessarily be an embedding. The cone operation $+$ must satisfy

$$a + c = b + c \implies a = b \quad \text{for all } a, b, c \in \mathcal{C}.$$

This is called the *cancellation law of the cone operation* $+$.

Indeed, suppose that $q(a) = q(b)$ for $a, b \in \mathcal{C}$, then $[a, 0] = [b, 0]$. By the definition of $\sim$, there is a $c \in \mathcal{C}$ such that $a + c = b + c$. By the cancellation law, $q$ is indeed injective hence an embedding.

From this point on, elements of the form $[a, 0]$ are called *positive elements of* $E_{\mathcal{C}}$ and elements of the form $[0, b]$ are said to be the *negative elements of* $E_{\mathcal{C}}$. This convention ensures that the members of $\mathcal{C}$ correspond to the positive elements of $E_{\mathcal{C}}$.

Observe that the comments above agree with Proposition 2.2.6. That is, if $c \in E_{\mathcal{C}}$, then there are elements $a, b \in \mathcal{C}$ such that $c = a - b$.

Given a cone $\mathcal{C}$ in a metric space $S$, it is possible to construct a norm on $E_{\mathcal{C}}$ under certain extra assumptions on the metric.

3.3.1 **Definition.** A metric space $\mathcal{C} = (\mathcal{C}, \mathrm{d})$ which is simultaneously a cone is called a *metric cone*. «

3.3.2 **Theorem.** *Let $\mathcal{C}$ be a metric cone. Assume that the metric* d *is translation invariant and homogeneous in both arguments, that is,*

$$\begin{aligned} \mathrm{d}(a,b) &= \mathrm{d}(a+c,b+c) && \text{for all } a,b,c \in \mathcal{C}; \\ \lambda \mathrm{d}(a,b) &= \mathrm{d}(\lambda a, \lambda b) && \text{for all } \lambda \in \mathbf{R}^+ \text{ and all } a,b \in \mathcal{C}. \end{aligned}$$

*Then there is a norm $\|\cdot\|$ on $E_{\mathcal{C}}$, which induces* d. «

Proof. Let $a,b,c \in \mathcal{C}$ such that $a+c = b+c$. Then, by translation invariance,

$$0 = \mathrm{d}(a+c,b+c) = \mathrm{d}(a,b).$$

This proves the cancellation law.

Next, let $a,b \in \mathcal{C}$ and put

$$\|[a,b]\| = \mathrm{d}(a,b).$$

The map $\|\cdot\|$ is well-defined. Indeed, suppose $(a,b) \sim (x,y)$, then there is a $c \in \mathcal{C}$ such that $a+y+c = x+b+c$. Then $a+y = x+b$, by using the cancellation law. Since d is translation invariant, it follows that

$$\begin{aligned} \mathrm{d}(a,b) &= \mathrm{d}(a+x+y, b+x+y) \\ &= \mathrm{d}(x,y). \end{aligned}$$

Therefore, $\|\cdot\|$ is well-defined.

The mapping $\|\cdot\|$ is indeed a norm on $E_{\mathcal{C}}$. Suppose that $\lambda \in \mathbf{R}^+$ and $a,b \in \mathcal{C}$, then, by homogeneity,

$$\lambda\|[a,b]\| = \lambda\mathrm{d}(a,b) = \mathrm{d}(\lambda a, \lambda b) = \big\|[\lambda a, \lambda b]\big\| = \big\|\lambda[a,b]\big\|.$$

If $\lambda \leqslant 0$, then

$$\big\|\lambda[a,b]\big\| = \big\||\lambda|[b,a]\big\| = |\lambda|\|[b,a]\| = |\lambda|\mathrm{d}(b,a) = |\lambda|\mathrm{d}(a,b) = |\lambda|\big\|[a,b]\big\|.$$

Therefore, the norm is homogeneous.

Let $a,b,x,y \in \mathcal{C}$, then by translation invariance,

$$\begin{aligned} \|[a,b]+[x,y]\| &= \|[a+x,b+y]\| \\ &= \mathrm{d}(a+x,b+y) \\ &\leqslant \mathrm{d}(a+x,a+y) + \mathrm{d}(a+y,b+y) \\ &= \mathrm{d}(x,y) + \mathrm{d}(a,b). \end{aligned}$$

Therefore

$$\|[a,b]+[x,y]\| \leqslant \|[a,b]\| + \|[x,y]\|,$$

which proves the triangle inequality.

Lastly, note that $\|[a,b]\| = 0$ if and only if $\mathrm{d}(a,b) = 0$. Hence, $a = b$ and consequently, $[a,b] = [0,0]$. ∎

3.3.3 **Corollary.** *Let $(\mathcal{C}, \mathrm{d})$ be a metric cone such that* d *is homogeneous and translation invariant. Then there is an isometric embedding of $\mathcal{C}$ into $E_{\mathcal{C}}$.* «

Proof. By the proof of Theorem 3.3.2, $\mathcal{C}$ has the cancellation law, so the map $\psi\colon \mathcal{C} \to E_{\mathcal{C}}$ where $\psi(a) = [a, 0]$ is a well-defined embedding. By using the definition of the norm on $E_{\mathcal{C}}$ (from the proof of Theorem 3.3.2), $\psi$ is an isometry. ■

3.3.4 **Corollary.** *Let $X$ be a vector space over $\mathbf{R}$ and $\mathcal{C} \subseteq X$ a metric cone in $X$ such that the metric on $\mathcal{C}$ is homogeneous and translation invariant. Then there is a linear embedding $\psi\colon E_{\mathcal{C}} \to X$.*

*If in addition, $X$ is a normed space and $\mathrm{d}$ is the metric on $\mathcal{C}$ induced by the norm on $X$, then $\psi$ is an isometric embedding.* «

Proof. For the first part, put $\psi([a, b]) = a - b$. The definition of the norm on $E_{\mathcal{C}}$ ensures that $E_{\mathcal{C}} \subseteq X$ is a subspace. Explicitly,

$$E_{\mathcal{C}} = \{a - b \colon a, b \in \mathcal{C}\}.$$

For the second part, observe that

$$\|a - b\| = \mathrm{d}(a, b) = \|[a, b]\| \quad \text{for all } a, b \in \mathcal{C}.$$

Therefore $\psi$ is isometric. ■

The previous theorem and its corollaries, lead to the next definition:

3.3.5 **Definition.** A metric cone $(\mathcal{C}, \mathrm{d})$, is said to be a *normed (convex) cone* whenever there is a linear isometric embedding into some normed space $(X, \|\cdot\|)$. «

## 3.4 Lattice structures and completeness

3.4.1 **Definition.** An ordered cone $\mathcal{C}$ is a *lattice cone* if both $a \vee b$ and $a \wedge b$ exist for any $a, b \in \mathcal{C}$. «

Given an ordered cone $\mathcal{C}$, it is possible to put an ordering on the associated vector space $E_{\mathcal{C}}$. This is done by defining

$$[a, b] \leqslant [x, y] \iff a + y \leqslant x + b.$$

It turns out that $E_{\mathcal{C}}$ inherits the lattice cone structure of $\mathcal{C}$:

3.4.2 **Proposition.** *If $\mathcal{C}$ is a lattice cone, then $E_{\mathcal{C}}$ is a Riesz space.* «

Proof. Let $a, b \in \mathcal{C}$. It suffices to show that $[a \vee b, b]$ is the supremum of $[a, b]$ and 0.

Because $a \leqslant a \vee b$, it follows that $[a, b] \leqslant [a \vee b, b]$. Due to $b \leqslant a \vee b$, it follows that $[a \vee b, b]$ is positive. So $[a \vee b, b$ is an upper bound of $[a, b]$ and 0.

It remains to show that $[a, b] \vee 0$ is indeed the supremum of $[a, b]$ and 0. To this end, let $x, y \in \mathcal{C}$ such that $[x, y] \geqslant 0$. Then there is a $c \in \mathcal{C}$ such that

$$(x, y) = (c, 0).$$

Suppose $[a, b] \leqslant [c, 0]$, then $a + 0 \leqslant b + c$. Since $b \leqslant b + c$, it follows that $a \vee b \leqslant b + c$. Therefore,

$$[c, 0] = [b + c, b] \geqslant [a \vee b, b],$$

which proves that $[a \vee b, b] \vee 0$ is the supremum of $[a, b]$ and 0. ■

There is an expression for the absolute value of elements in $E_{\mathcal{C}}$:

3.4.3 **Proposition.** *Let $\mathcal{C}$ be a normed cone. Then*

$$\big|[a,b]\big| = \big[|a-b|,0\big] \quad \textit{for all } [a,b] \in E_{\mathcal{C}}.$$ «

Proof. Note that $[a,b] \leqslant [|a-b|,0]$, since $a+0 \leqslant b+|a-b|$. Similarly, $[b,a] = -[a,b] \leqslant [|a-b|,0]$, because $b+0 \leqslant a+|a-b|$.

Let $[x,y] \in E_{\mathcal{C}}$ and suppose that $[x,y] \geqslant [a,b]$ and $[x,y] \geqslant -[a,b]$. Then

$$a+x \leqslant y+b \text{ and } b+y \leqslant x+a.$$

This yields

$$a-b \leqslant x-y \text{ and } b-a \leqslant x-y.$$ ■

Completeness of $\mathcal{C}$ does not imply that the associated vector space $E_{\mathcal{C}}$ is complete:

3.4.4 **Remark.** Let $\mathcal{C}$ be a complete normed cone, then $E_{\mathcal{C}}$ need not be complete.

Proof. Consider a normed space $E$ which is not complete. Define the trivial ordering

$$x \leqslant y \iff x = y.$$

Then $E$ is an ordered vector space and the only positive element is $0$, which means $E^+ = \{0\}$. This is clearly a complete normed space. ■

Recall the following result from Banach space theory (see [14, p. 20] for a proof):

3.4.5 **Lemma.** *Let $X$ be a normed space. Then $X$ is a Banach space if and only if every absolutely convergent series converges in $X$.* «

3.4.6 **Theorem.** *Let $\mathcal{C}$ be a normed lattice cone such that its norm is a non-decreasing map. Then the associated normed space $(E_{\mathcal{C}}, |||\cdot|||)$ where*

$$|||x||| = \|\,|x|\,\|$$

*is a Banach lattice whenever $\mathcal{C}$ is a Banach space.* «

Proof. First note that $E_{\mathcal{C}}$ is a lattice under its natural ordering.

Let $(x_n)$ be a $|||\cdot|||$-absolutely convergent sequence in $X$.

Note that $0 \leqslant x_n^+ \leqslant |x_n|$ for all $n \in \mathbf{N}$. Therefore, because the norm on $\mathcal{C}$ is assumed to be non-decreasing,

$$\|x_n^+\| \leqslant \|\,|x_n|\,\| = |||x_n||| \quad \text{for all } n \in \mathbf{N}.$$

Consequently, the series over all positive parts is absolutely convergent. So, by completeness of $\mathcal{C}$, the series $\sum x_n^+$ is convergent. By a similar reasoning, the sum over all negative parts is also convergent.

Then, for $N \in \mathbf{N}$ sufficiently large,

$$\begin{aligned}\Big|\Big|\Big| \sum_{n \leqslant N} x_n^- - \Big(\sum_{n\in\mathbf{N}} x_n^+ - \sum_{n\in\mathbf{N}} x_n^-\Big)\Big|\Big|\Big| &\leqslant \Big|\Big|\Big| \sum_{n\leqslant N} x_n^+ - \sum_{n\in\mathbf{N}} x_n^+ \Big|\Big|\Big| \\ &\quad + \Big|\Big|\Big| \sum_{n\leqslant N} x_n^- - \sum_{n\in\mathbf{N}} x_n^- \Big|\Big|\Big| \\ &\leqslant \Big|\Big|\Big| \sum_{n\leqslant N+1} x_n^+ \Big|\Big|\Big| + \Big|\Big|\Big| \sum_{n\leqslant N+1} x_n^- \Big|\Big|\Big| \\ &\leqslant \sum_{n\leqslant N+1} |||x_n^+||| + \sum_{n\leqslant N+1} |||x_n^-|||.\end{aligned}$$

By letting $N \to \infty$, the series $\sum_{n\leqslant N} x_n$ converges in $E_{\mathcal{C}}$ with respect to $|||\cdot|||$. Therefore, $E_{\mathcal{C}}$ is complete with respect to $|||\cdot|||$, which means that it is a Banach lattice by Lemma 3.4.5. ■

# 4 THE HAUSDORFF DISTANCE

The main goal of this chapter is to define a distance between subsets in a Banach space $X = (X, \|\cdot\|)$. Since the power set of the entire space $X$ can be quite 'large', it is hard to define a notion of distance on the whole power set. It is therefore necessary to make a restriction to a suitable subcollection of the power set instead.

## 4.1 DISTANCE BETWEEN POINTS AND SETS

In a metric space $S$, a distance is assigned to every pair of elements $x, y \in S$.

4.1.1 **Definition.** Let $S$ be a metric space, $x \in S$ and $A \subseteq S$. The *distance from $x$ to $A$* is then given by

$$\mathrm{d}(x, A) = \inf_{a \in A} \mathrm{d}(x, a).$$

«

Loosely speaking, the notion of distance between a point $x \in S$ and a subset $A \subseteq S$ is the distance from $x$ to the 'nearest' point $a \in A$. There might be some difficulties when $A$ is empty. Note that the distance between a point and a non-empty subset is a non-negative element in $\mathbf{R}$, by definition. To avoid complications with the empty set, define $\inf \varnothing = \infty$ so that $\mathrm{d}(x, \varnothing) = \infty$ for each $x \in S$.

To illustrate this definition, consider the following example:

4.1.2 **Example.**

1. Consider $\mathbf{R}$ equipped with the Euclidean metric. Take $x \in \mathbf{R}$, then $\mathrm{d}(x, \mathbf{Q}) = 0$, because $\mathbf{Q}$ is dense in $\mathbf{R}$.
2. Consider $\mathbf{R}^2$ with the Euclidean metric. Take the point $x_0 = (-1, 2)$ and the line $A = \{(x, y) \colon y = x\} = \{(x, x) \colon x \in \mathbf{R}\}$. Then

$$\mathrm{d}(x_0, A) = \inf_{y \in A} \|x_0 - y\| = \inf_{x \in \mathbf{R}} \sqrt{(-1-x)^2 + (2-x)^2}.$$

Put $f(x) = (-1-x)^2 + (2-x)^2$, then, by using the derivative of $f$, it follows that the point in $A$ nearest to $x_0$ is $(1/2, 1/2) \in A$. Therefore,

$$\mathrm{d}(x_0, A) = \|(-1, 2) - (\tfrac{1}{2}, \tfrac{1}{2})\| = \tfrac{3}{2}\sqrt{2}.$$

«

4.1.3 **Proposition.** *Let $S$ be a metric space and let $A \subseteq S$ be a fixed non-empty subset. Then the function*

$$\begin{aligned} f_A \colon S &\longrightarrow \mathbf{R}^+ \\ x &\longmapsto \mathrm{d}(x, A) \end{aligned}$$

*is Lipschitz continuous (as a function of $x$) with Lipschitz constant* 1. «

PROOF. Let $x, y \in S$, then

$$\mathrm{d}(x, A) \leqslant \mathrm{d}(x, a) \leqslant \mathrm{d}(x, y) + \mathrm{d}(y, x) \quad \text{for all } a \in A.$$

This yields

$$\mathrm{d}(x, A) - \mathrm{d}(x, y) \leqslant \inf_{a \in A} \mathrm{d}(y, A) = f_A(y),$$

which gives

$$\mathrm{d}(x, A) - \mathrm{d}(y, A) \leqslant \mathrm{d}(x, y).$$

■

The map $f_A$ has another important property:

4.1.4 **Proposition.** *Let $S$ be a metric space and $A \subseteq S$ a non-empty subset. If $f_A$ is not identically zero, then $f_A(x) = 0$ if and only if $x \in \overline{A}$.* «

Proof. Let $A \subseteq S$. Suppose that $\mathrm{d}(x, A) = 0$. Then $x \in A$ or $x \notin A$. Note that if $x \notin A$ holds, then $x$ must be in the closure of $A$.

Indeed, let $\varepsilon > 0$. The definition of $\mathrm{d}(x, A)$ implies that there is a $y \in A$ such that $\mathrm{d}(x, y) < \varepsilon$. That is, $y \in \mathrm{B}(x; \varepsilon)$. So every ball with centre $x$ and radius $\varepsilon$ contains a point of $A$. This implies $\mathrm{B}(x; \varepsilon) \cap A \neq \emptyset$, which means that $x$ is a limit point of $A$. Therefore, $x \in \overline{A}$.

For the converse, assume $x \in \overline{A}$. Then $\mathrm{d}(x, A) = 0$. Indeed, let $\varepsilon > 0$ then $\mathrm{B}(x; \varepsilon)$ contains a point $y \in A$. So $\mathrm{d}(x, A) < \varepsilon$. Since $\varepsilon$ was arbitrary, $\mathrm{d}(x, A) = 0$. ■

## 4.2 Distance between sets

Let $S$ be a metric space and $A, B \subseteq S$. In the previous section the concept of 'distance from point to subset' was introduced and discussed. This concept can be extended in order to find a suitable definition of distances between subsets of $S$.

### *The Hausdorff semi-distance*

Definition 4.1.1 can be modified to 'measure' the distance between subsets in the following way:

4.2.1 **Definition.** Let $S$ be a metric space and $A, B \subseteq S$ non-empty subsets. The *distance from $A$ to $B$* or *Hausdorff semi-distance* is defined as

$$\delta(A, B) = \sup_{a \in A} \mathrm{d}(a, B) = \sup_{a \in A} \inf_{b \in B} \mathrm{d}(a, b).$$ «

Note that $\delta(A, B)$ may be infinite for unbounded sets $A, B \subseteq S$ according to the conventions from Definition 4.1.1. Since this might lead to some technical difficulties later on, it is therefore convenient (and sometimes necessary) to consider a 'suitable' collection of subsets in $S$.

Recall that a subset $A$ of a metric space $S$ is said to be *bounded* if and only if $A$ has finite diameter, that is,

$$\operatorname{diam} A = \sup_{x, y \in A} \mathrm{d}(x, y) < \infty.$$

The supremum of the empty set is defined to be $-\infty$.

4.2.2 **Definition.** Let $S$ be a metric space, then define the collection

$$\mathbf{H}(S) = \{A \subseteq X \colon A \text{ is bounded, closed and non-empty}\}.$$

This set is known as the *hyperspace of all non-empty bounded, closed subsets of $S$*. The set of all non-empty and compact subsets in $S$ will be denoted by $\mathcal{K}(S)$.«

There is a relation between $\mathcal{K}(S)$ and $\mathbf{H}(S)$. Clearly, $\mathcal{K}(S) \subseteq \mathbf{H}(S)$ for any metric space $S$. Recall that a metric space $S$ is said to have the *Heine-Borel property*, if any closed and bounded set is compact. If this happens to be the case, then $\mathcal{K}(S) = \mathbf{H}(S)$.

It turns out that $\delta$ is a pseudometric on $\mathbf{H}(S)$:

4.2.3 **Proposition.** *Let $S$ be a metric space and $A, B, C \in \mathbf{H}(S)$, then*

1. $\delta(A, B) = 0$ *if and only of* $A \subseteq \overline{B}$.

2. $\delta(A, B) \leqslant \delta(A, C) + \delta(C, B)$. «

Proof. Let $A, B, C \in \mathbf{H}(S)$.

1. Note that if $A \subseteq B$, then $\delta(A, B) = 0$. Conversely, suppose that $\delta(A, B) = 0$, then $\mathrm{d}(a, B) = 0$ for all $a \in A$. Then $a \in \overline{B}$, by using the result of Proposition 4.1.4

2. By using the 'ordinary' metric of the underlying space $S$, for all $a \in A$, $b \in B$ and $c \in C$ the distance satisfies

$$\mathrm{d}(a, b) \leqslant \mathrm{d}(x, c) + \mathrm{d}(c, b) \leqslant \mathrm{d}(x, c) + \mathrm{d}(c, B).$$

This estimate is valid for each $b \in B$, so

$$\mathrm{d}(a, B) \leqslant \mathrm{d}(a, c) + \mathrm{d}(c, B) \leqslant \mathrm{d}(a, c) + \delta(C, B).$$

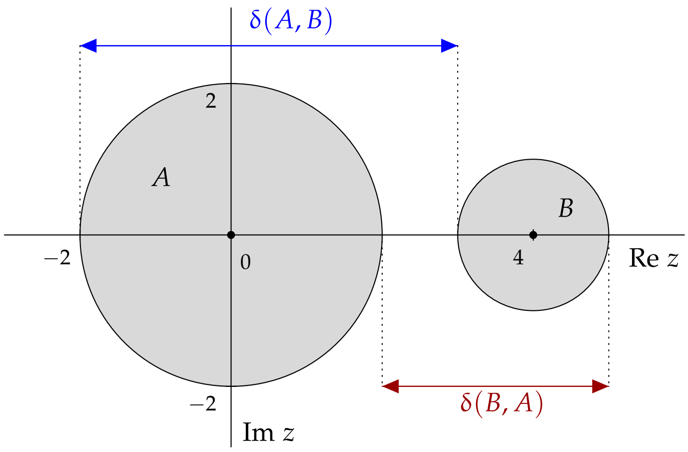
for all $a \in A$ and $c \in C$. Note that this estimate is valid for any $c \in C$. Therefore,

$$\mathrm{d}(a, B) \leqslant \mathrm{d}(a, C) + \delta(C, B)$$

for any $a \in A$. By taking the supremum over all $a \in A$, the result follows. ■

Note that $\delta$ need not be a symmetric map:

4.2.4 **Example.** Let $S = \mathbf{C}$ and consider the subsets $A = \{z \in \mathbf{C} \colon |z| \leqslant 2\}$ and $B = \{z \in \mathbf{C} \colon |z - 4| \leqslant 1\}$. Both of these sets are bounded, closed and non-empty.




Then $\delta(A, B) = 5$ and $\delta(B, A) = 3$, which yields $\delta(A, B) \neq \delta(B, A)$. «

The previous example show that $(\mathbf{H}(S), \delta)$ is not a metric space. However, this problem can be fixed, by 'symmetrizing' $\delta$. This yields a metric on $\mathbf{H}(S)$.

4.2.5 **Definition.** Let $S$ be a metric space and $A, B \in \mathbf{H}(S)$. The mapping

$$\begin{aligned} \mathrm{d_H}\colon \mathbf{H}(S) \times \mathbf{H}(S) &\longrightarrow \mathbf{H}(S) \\ (A,B) &\longmapsto \delta(A,B) \vee \delta(B,A) \end{aligned}$$

is the *Hausdorff metric* or *Hausdorff distance* on $\mathbf{H}(S)$. «

The word 'metric' from the previous definition needs some verification:

4.2.6 **Theorem.** $\mathrm{d_H}$ *is a metric on* $\mathbf{H}(S)$. «

Proof. Since both $A$ and $B$ are bounded and non-empty, both $\delta(A,B)$ and $\delta(B,A)$ are finite. The triangle inequality follows from Proposition 4.2.3. Note that $\mathrm{d_H}$ is symmetric by definition. By symmetry and Proposition 4.2.3, it follows that

$$\mathrm{d_H}(A,B) = 0 \iff A \subseteq \overline{B} \subseteq \overline{A}.$$

If both $A$ and $B$ are closed, then they are equal to their respective closures, which means $A = B$. ∎

Now that $\mathbf{H}(S) = (\mathbf{H}(S), \mathrm{d_H})$ is indeed a metric space, one could wonder whether there is an embedding of $S$ into $\mathbf{H}(S)$.

4.2.7 **Theorem.** *Let $S$ be a metric space, then $S$ can be isometrically embedded into $\mathbf{H}(S)$ by the map*

$$\begin{aligned} J\colon S &\longrightarrow \mathbf{H}(S) \\ x &\longmapsto \{x\} \end{aligned}$$

*Furthermore, the set $J[S]$ is closed in $\mathbf{H}(S)$.* «

Proof. Note that for any $x, y \in S$ the distance between $x$ and $y$ satisfies

$$\mathrm{d}(x,y) = \mathrm{d_H}(\{x\}, \{y\}).$$

This shows that $S$ can be isometrically identified with $J[S] \subseteq \mathbf{H}(S)$.

Let $A \in \overline{J[S]}$, then the distance from $A$ to $J[S]$ is 0 by Proposition 4.1.4. Therefore, for any $\varepsilon > 0$, there is an element $x \in S$ such that $\mathrm{d_H}(A, \{x\}) < \varepsilon$. This yields $\mathrm{d}(x,y) < \varepsilon$ for any $y \in A$. In addition, $\operatorname{diam} A \leqslant \varepsilon$, because $A$ is non-empty. This shows that $A$ must be a set that contains only one element, which means $A \in J[S]$. Therefore, $J[S]$ contains all of its limit points, which implies $J[S]$ is closed. ∎

The following result will be used freely without mention in this thesis:

4.2.8 **Proposition.** *Let $S$ be a metric space and $K, L \in \mathcal{K}(S)$. Then there is an element $a \in A$ and $b \in B$ such that*

$$\mathrm{d}(a,b) = \mathrm{d_H}(A,B).$$ «

Proof. Assume without loss of generality that $\mathrm{d_H}(A,B) = \delta(A,B)$. Since the map $f_B\colon A \to \mathbf{R}^+ \colon x \mapsto \mathrm{d}(x,B)$ is continuous and $A$ is compact, there is an element $a \in A$ such that $\mathrm{d}(a,B) = \mathrm{d_H}(A,B)$.

Similarly, by continuity of the map $x \mapsto \mathrm{d}(a,x)$ and compactness of $B$, it follows that there is a $b \in B$ such that

$$\mathrm{d}(a,b) = \inf_{x \in B} \mathrm{d}(a,x) = \mathrm{d}(a,B) = \mathrm{d_H}(A,B).$$ ∎

The Hausdorff distance between non-compact set might not be attained:

4.2.9 **Example.** Consider the metric space $\ell_2$ and consider the collection of unit elements $U = \{u_n \colon n \in \mathbf{N}\}$ where

$$u_n(k) = \begin{cases} 1 & \text{if } k = n; \\ 0 & \text{otherwise.} \end{cases}$$

Next, define $A = \{x\} \cup U$ where $x$ is the sequence $\{-1/n \colon n \in \mathbf{N}\}$. Note that both $A$ and $U$ are closed, bounded and non-empty subsets of $\ell_2$, but not compact. Observe that $\mathrm{d_H}(A, U) = \mathrm{d}(x, U)$, because $\delta(U, A) = 0$.

On the other hand, by using Euler's identity,

$$\mathrm{d}(x, u_n) = \left(1 + \tfrac{\pi^2}{6} + \tfrac{2}{n}\right)^{1/2} \quad \text{for all } n \in \mathbf{N}.$$

Therefore,

$$\mathrm{d_H}(A, U) = \inf_{u_n \in U} \mathrm{d}(x, u_n) = \left(1 + \tfrac{\pi^2}{6}\right)^{1/2}.$$

But then,

$$\mathrm{d}(x, u_n) > \left(1 + \tfrac{\pi^2}{6}\right)^{1/2} \quad \text{for all } n \in \mathbf{N}.$$ «

## 4.3 COMPLETENESS AND COMPACTNESS

This section will investigate sufficient conditions to ensure that $\mathbf{H}(S)$ is complete and $\mathcal{K}(S)$ is compact, for a given a metric space $S$.

4.3.1 **Theorem.** *$S$ is a complete metric space if and only if $(\mathbf{H}(S), \mathrm{d_H})$ is complete.* «

PROOF. $[\Longleftarrow]$ Suppose $\mathbf{H}(S)$ is complete, then the collection $\{\{x\} \colon x \in S\}$ is closed in $\mathbf{H}(S)$ by Theorem 4.2.7 and complete by completeness of $\mathbf{H}(S)$. By utilizing the isometric map from Theorem 4.2.7, it follows that $S$ is complete.

$[\Longrightarrow]$ Assume $S$ is complete. Let $(A_n)$ be a Cauchy sequence in $\mathbf{H}(S)$ and put

$$A = \bigcap_{m \in \mathbf{N}} \overline{\bigcup_{n \geqslant m} A_n}.$$

Note that $A$ is closed by definition. The goal is to show that $A$ is the limit of $(A_n)$ and that $A \in \mathbf{H}(S)$.

*Claim 1.* Let $\varepsilon > 0$, then by the Cauchy property, there is a $N \in \mathbf{N}$ such that

$$\mathrm{d_H}(A_n, A_m) < \varepsilon \quad \text{whenever } n, m \geqslant N.$$

Then $\mathrm{d}(a, A_N) \leqslant \varepsilon$ for any $a \in A$ and $A$ is bounded.

Indeed, note that $\delta(A_n, A_N) < \varepsilon$ for all $n \geqslant N$ (by the Cauchy property). Therefore

$$\overline{\bigcup_{n \geqslant N} A_n} \subseteq \{x \in X \colon \mathrm{d}(x, A_N) \leqslant \varepsilon\},$$

so $A \subseteq \{x \in X \colon \mathrm{d}(x, A_N) \leqslant \varepsilon\}$. Thus $\mathrm{d}(x, A_N) \leqslant \varepsilon$ for all $x \in A$. Next, in order to prove that $A$ is bounded, take $x, y \in A$. Then there are $a, b \in A_N$ such that

$$\mathrm{d}(a, x) \leqslant 2\varepsilon \quad \text{and} \quad \mathrm{d}(b, y) \leqslant 2\varepsilon.$$

Then, by the triangle inequality,

$$d(x,y) \leqslant d(x,a) + d(a,b) + d(b,y) \leqslant 4\varepsilon + \operatorname{diam} A_N.$$

Therefore $\operatorname{diam} A \leqslant 4\varepsilon + \operatorname{diam} A_N$, which implies that $A$ is bounded. Hence $A \in \mathbf{H}(S)$, which concludes this claim.

*Claim 2.* Let $\varepsilon > 0$, then $d(y, A) \leqslant \varepsilon$ for any $y \in A_N$.

In order to prove the claim, it suffices to show that given $\varepsilon > 0$, there is an element $a \in A$ such that that $d(y, a) \leqslant \varepsilon$. This will be done by building a Cauchy sequence $(a_n)$ in $X$ for each $y \in A$. Then, by completeness of $X$, there is an element $a \in X$ such that $a_n \to a$.

To this end, take $y \in A_N$ and put $N = N_1$. By repeatedly applying the Cauchy property, for each $k \in \mathbf{N}$ there is a $N_k \in \mathbf{N}$ such that $N_k < N_{k+1}$ and

$$d_H(A_n, A_m) < \varepsilon \cdot 2^{1-k} \quad \text{whenever } n, m \geqslant N_k.$$

Set $a_1 = y$, then $a_1 \in A_{N_1}$. Now recursively take $a_k \in A_{N_k}$ such that

$$d(a_{k+1}, a_k) < \varepsilon \cdot 2^{1-k}.$$

Observe that each $a_k$ can be chosen in this way, due to the 'clever' choice of the indices $N_k$. In addition, by repeatedly applying the triangle inequality,

$$\begin{aligned} d(a_{k+n}, s_k) &\leqslant \sum_{\ell=0}^{n-1} d(a_{k+\ell+1}, a_{k+\ell}) \\ &< \sum_{\ell=0}^{n-1} \varepsilon \cdot 2^{1-k-\ell} \\ &\leqslant \varepsilon \cdot 2^{1-k} \end{aligned} \tag{1}$$

for all $k, n \in \mathbf{N}$ where $n \geqslant 0$. This proves that $(a_n)$ is Cauchy in $X$ and by completeness, it converges to some $a \in X$. Let $n \to \infty$, then by the estimate in (1),

$$d(a, a_k) < \varepsilon \cdot 2^{2-k}$$

for all $k \in \mathbf{N}$ and consequently,

$$d(a, y) = d(a, a_1) \leqslant \varepsilon.$$

In order to show $a \in A$, note that $a$ is defined as the limit of the sequence $(a_k)$ and each term of this sequence is contained in the corresponding set $A_{N_k}$. Therefore $a \in \overline{\{a_k : k \geqslant m\}}$ for all $m \in \mathbf{N}$. Since $k < N_k$ for all $k \in \mathbf{N}$, it follows that

$$a \in \overline{\{a_k : k \geqslant m\}} \subseteq \overline{\bigcup_{k \geqslant m} A_{N_k}} \subseteq \overline{\bigcup_{n \geqslant m} A_n}$$

for all $m \in \mathbf{N}$. So $a \in A$, which finishes the proof of the claim.

By combining the results of the previous two claims, $d_H(A, A_N) \leqslant 2\varepsilon$. So by using the triangle inequality,

$$d_H(A, A_n) \leqslant d_H(A, A_N) + d_H(A_N, A_n) \leqslant \varepsilon \quad \text{for all } n \geqslant N. \qquad \blacksquare$$

4.3.2 **Corollary.** *If $S$ is a complete metric space, then $\mathcal{K}(S)$ is complete.* «

Proof. Let $(K_n)$ be a Cauchy sequence in $\mathcal{K}(S)$. Since $\mathcal{K}(S) \subseteq \mathbf{H}(S)$, the limit of $(K_n)$ exists in $\mathbf{H}(X)$ by Theorem 4.3.1. Therefore, by completeness of $S$, it remains to show that the limit

$$K = \bigcap_{m \in \mathbf{N}} \overline{\bigcup_{n \geqslant m} K_n}$$

is totally bounded.

To this end, let $\varepsilon > 0$. By the proof of Theorem 4.3.1, there exists a $N \in \mathbf{N}$ such that

$$K \subseteq \{x \in X \colon \mathrm{d}(x, K_N) \leqslant \varepsilon/3\}.$$

Furthermore, $K_N$ is compact by definition. This means there is a $n \in \mathbf{N}$ and $x_1, \ldots x_n \in A_N$ such that

$$K_N \subseteq \bigcup_{k=1}^{n} \mathrm{B}(x_k; \varepsilon/2).$$

Take $x \in K$, then $\mathrm{d}(x, K_N) \leqslant \varepsilon/3$. Therefore there is a point $y \in K_N$ with $\mathrm{d}(x, y) < \varepsilon/2$ and there exist $k$ such that $y \in \mathrm{B}(x_k; \varepsilon/2)$. Thus

$$\mathrm{d}(x, x_k) \leqslant \mathrm{d}(x, y) + \mathrm{d}(y, x_k) \leqslant \varepsilon.$$

Therefore

$$K \subseteq \bigcup_{k=1}^{n} \mathrm{B}(x_k; \varepsilon),$$

which means $K$ is totally bounded. ∎

4.3.3 **Corollary.** *If $S$ is a compact metric space, then $\mathcal{K}(S)$ is compact.* «

Proof. Recall that any compact metric space is complete. Therefore, $\mathcal{K}(S)$ is complete by Corollary 4.3.2. It remains to show that $\mathcal{K}(S)$ is totally bounded.

Observe that $S$ is totally bounded. So, given $\varepsilon > 0$, there is a $n \in \mathbf{N}$ and there exist $x_1, \ldots x_n \in S$ such that

$$\inf_{1 \leqslant k \leqslant n} \mathrm{d}(x, x_k) = \min_{1 \leqslant k \leqslant n} \mathrm{d}(x, x_k) < \varepsilon \quad \text{for all } x \in X.$$

Let $K \subseteq X$ be a compact non-empty subset and define

$$B = \{x_k \colon \mathrm{d}(x_k, K)\} < \varepsilon.$$

Then $\mathrm{d}_{\mathrm{H}}(B, K) < \varepsilon$. Therefore, any $K \in \mathcal{K}(S)$ is at Hausdorff distance less then $\varepsilon$ of a subset of the (finite) collection $\{x_k \colon 1 \leqslant k \leqslant n\}$. Hence $\mathcal{K}(S)$ is totally bounded and complete, which implies that $\mathcal{K}(S)$ is compact. ∎

## 4.4 Cone and order structure on closed bounded sets

If $S = \big(S, \mathrm{d}\big)$ is a metric space, then $\mathbf{H}(S)$ is a partially ordered space of sets under the inclusion relation '$\subseteq$'.

By definition, the inclusion is a partial order on $\mathbf{H}(S)$. The addition in this vector space is given by the Minkowski sums. Explicitly, let $A, B \in \mathbf{H}(S)$ then define

$$A + B = \overline{\{a + b \colon a \in A \text{ and } b \in B\}}.$$

Define the scalar multiplication on $\mathbf{H}(S)$ by

$$\lambda A = \{\lambda a \colon a \in A\}$$

for any $\lambda \in \mathbf{R}$ and $A \in \mathbf{H}(S)$. By definition, both addition and scalar multiplication are well-defined.

Under these assumptions, $\mathbf{H}(S)$ is indeed an ordered vector space and an ordered cone (according to Definition 3.1.3). Moreover, inclusion is the standard order as defined in section 3.1. From now on, the cone $\mathbf{H}(S)$ shall be investigated in more detail.

Note that all of the previously defined operations and observations all hold for $\mathcal{K}(S)$ as well.

There is way to incorporate the Minkowski sums into the definition of Hausdorff distance:

4.4.1 **Lemma.** *Let $A, B \in \mathbf{H}(X)$, then*

$$\mathrm{d_H}(A, B) = \inf\{\varepsilon > 0 \colon A \subseteq B + \varepsilon\mathbf{B} \text{ and } B \subseteq A + \varepsilon\mathbf{B}\}. \quad «$$

Proof. Let $\varepsilon > 0$ be given arbitrarily. Suppose $A \subseteq B + \varepsilon\mathbf{B}$, then $\mathrm{d}(a, B) \leqslant \varepsilon$ for all $a \in A$. Therefore

$$\delta(A, B) = \sup_{a \in A} \inf_{b \in B} \mathrm{d}(a, b) \leqslant \varepsilon$$

and by an analogous reasoning, $\delta(B, A) < \varepsilon$. Hence $\mathrm{d_H}(A, B) < \varepsilon$.

To show the other inequality, suppose that $A$ is not contained in $B + \varepsilon\mathbf{B}$. Then there exists an element $a \in A$ such that $a \notin B + \varepsilon\mathbf{B}$. Therefore

$$\mathrm{d}(a, b) > \varepsilon \quad \text{for all } b \in B,$$

so that $\mathrm{d}(a, B) \geqslant \varepsilon$. Then, by taking the supremum of all $a \in A$, it follows that $\mathrm{d_H}(A, B) \geqslant \varepsilon$. So if $A$ is not contained in $B + \varepsilon\mathbf{B}$, then $\mathrm{d_H}(A, B) \geqslant \varepsilon$. This concludes the proof. ∎

The expression of the Hausdorff distance from Lemma 4.4.1 will be used without mention from this point on. The 'power' of this expression will become apparent in the next chapter.

In order to obtain a Banach lattice from the cone $\mathbf{H}(S)$, the cancellation law (of the Minkowski addition) must be verified. It turns out that the cancellation law holds in a very general setting, namely the setting of topological vector spaces.

Recall that a subset $A$ of a topological vector space $X$ over a field $\mathbf{F}$ is *bounded* if for every neighbourhood $N$ of the zero vector there exists a scalar $\lambda \in \mathbf{F}$ so that $A \subseteq \lambda N$.

4.4.2 **Theorem.** *Let $X$ be a topological vector space, then*

$$A + B \subseteq C + B \implies A \subseteq C,$$

*for any non-empty subsets $A, B, C \subset X$ such that $B$ is bounded and $C$ closed and convex.* «

Proof. Let $N_0$ be a base of neighbourhoods of the zero-element in $X$. Let $U_0 \in N_0$ be a given and define a sequence $(V_n)$ in $N_0$ such that $V_0 + V_0 \subseteq U_0$ and

$$V_n + V_n \subseteq V_{n-1} \quad \text{for all } n \geqslant 1.$$

Now assume that $A + B \subseteq C + B$, then

$$A + B \subseteq C + B + V \quad \text{for any } V \in N_0.$$

Therefore
$$A + B \subseteq C + B + V_n \quad \text{for all } n \geqslant 1.$$

Next, let $a \in A, b_1 \in B$ and let $n \in \mathbf{N}$ be a large, fixed number. Then there exists $c_1 \in C, b_2 \in B$ and $v_1 \in V$ such that

$$a + b_1 = c_1 + b_2 + v_1.$$

By proceeding inductively, there are $c_k \in C, b_{k+1} \in B$ and $v_k \in V$ such that

$$a + b_k = c_k + b_{k+1} + v_k \quad \text{for } k \in \{1, \dots, n\}.$$

Then, by adding up all the equations, rearranging the terms and using the telescope sum on all the $b_k$'s, it follows that

$$a = \frac{1}{n} \sum_{k=1}^{n} c_k + \frac{1}{n}(b_{n+1} - b_1) + \frac{1}{n} \sum_{k=1}^{n} v_k.$$

Therefore, since $C$ is convex and $B$ is bounded, $a \in C + V_0 + \ldots + V_n$. Then, for $n$ large and $U_0 \in N_0$,

$$a \in C + V_1 + \ldots + V_n \subseteq C + U_0.$$

Hence $A \subseteq C + U_0$ for all $U_0 \in N_0$, which means $A \subseteq C$. ∎

If $X$ is a normed space and the sets $A, B$ and $C$ from the previous theorem are convex, bounded, closed and non-empty, an alternative geometric proof can be given by using a Hahn-Banach argument.

Recall the following separation theorem (see [21, p. 140] for a proof):

4.4.3 **Theorem** (Hahn-Banach separation theorem). *Let $X$ be a normed linear space and let $A, B \subseteq X$ be convex and non-empty subsets. If $A$ is compact and $B$ is closed, then there exists a functional $\phi \in X^*$ and $s, t \in \mathbf{R}$ such that*

$$\phi(a) < t < s < \phi(b) \quad \textit{for all } a \in A \textit{ and } b \in B. \quad «$$

4.4.4 **Lemma.** *Let $X$ be a normed space and $A, B, C \subseteq X$ be non-empty subsets such that $A$ and $B$ are bounded, closed and convex and $C$ is both closed and convex. Then*

$$A + C \subseteq B + C \implies A \subseteq B. \quad «$$

Proof. Let $a \in A \setminus B$, then the singleton $\{a\}$ is both compact and convex. Note that $B$ is closed and convex. Therefore, by the Hahn-Banach separation theorem, there is a continuous linear functional $\phi \colon X \to \mathbf{R}$ which strictly separates $\{a\}$ and $B$. In other words, there exists a $t \in \mathbf{R}$ such that

$$\phi(a) < t < \phi(b) \quad \text{for all } b \in B.$$

Additionally, since $C$ is bounded and $\phi$ is continuous, the set $\phi[C]$ is bounded. Therefore
$$\phi(a) + \phi[C] \not\subseteq \phi[B] + \phi[C],$$
so that $a + C \not\subseteq B + C$. Therefore $A + C \not\subseteq B + C$. ∎

The cancellation law of the Minkowski sums is a direct consequence of Theorem 4.4.2:

4.4.5 **Corollary.** *Let $X$ be a topological vector space, then*

$$A + B = C + B \implies A = C,$$

*for any non-empty subsets $A, B, C \subset X$ such that $B$ is bounded and $A$ and $C$ closed and convex.* «

Proof. Suppose $A + C = B + C$. Then $A + C \subseteq B + C$ and $A + C \supseteq B + C$. So by using Theorem 4.4.2, $A \subseteq B$ and $A \supseteq B$. Therefore $A = B$, which proves the cancellation law. ■

The remainder of this section is devoted to showing the link between the ordering on $\mathbf{H}(S)$ and the Hausdorff metric and the possibility of constructing a Riesz space from the cone $\mathbf{H}(S)$ by utilizing the results established in chapter 3.

4.4.6 **Lemma.** *Let $S$ be a metric space. Let $(A_n)$ be a nested sequence in $\mathbf{H}(S)$ where $A_n \supseteq A_{n+1}$ for all $n \in \mathbf{N}$ and assume there is a number $N \in \mathbf{N}$ such that $A_N$ is compact. Then*

$$A = \bigcap_{n \in \mathbf{N}} A_n \in \mathcal{K}(S).$$

«

Proof. Note that arbitrary intersections of closed sets are closed and a closed subset of a compact set is compact. It remains to show that $A$ is non-empty.

Pick for each $n \in \mathbf{N}$ an element $a_n \in A_n$. This provides a sequence $(a_n)$ with a convergent subsequence $(a_{n_k})_k$ and with limit $a$, since every term after $a_N$ is contained in the compact set $A_N$.

For each $\ell \geqslant N$, $(a_{n_k})_{k \geqslant \ell}$ is contained in the compact set $A_\ell$. So $a \in A_\ell$ and consequently, $a \in A$. This means $A$ is non-empty. ■

4.4.7 **Theorem** (Order continuity of the Hausdorff distance)**.** *Let $S$ be a metric space and $\{A_n\}$ be a sequence in $\mathcal{K}(S)$ such that $A_n \supseteq A_{n+1}$ for all $n \in \mathbf{N}$. Then*

$$A = \bigcap_{n \in \mathbf{N}} A_n \in \mathcal{K}(S)$$

*and $\mathrm{d_H}(A, A_n) \downarrow 0$.* «

Proof. By Lemma 4.4.6, $A \in \mathcal{K}(S)$. For the other part, note that the sequence $\{\mathrm{d_H}(A, A_n) \colon n \in \mathbf{N}\}$ is non-increasing since $A_n$ contains all $A_m$ for $m < n$. From Proposition 4.2.8, it follows that for any $n \in \mathbf{N}$ there is an $a_n \in A_N$ such that

$$\mathrm{d}(a_n, A) = \mathrm{d_H}(A_n, A).$$

There is a subsequence $(a_{n_k})_k$ of $(a_n)$ which converges to a point $a \in A$. Then

$$\mathrm{d_H}\big(A_{n_k}, A\big) = \mathrm{d}\big(a_{n_k}, A\big) \leqslant \mathrm{d}\big(a_{n_k}, a\big).$$

Let $k \to \infty$, then $\mathrm{d}\big(a_{n_k}, a\big) \downarrow 0$. ■

In order to turn $\mathbf{H}(S)$ into a normed space, the Hausdorff distance needs to be homogeneous and translation invariant, according to the construction of Theorem 3.3.2.

In order to obtain homogeneity of the Hausdorff distance, it is convenient to work with a normed space instead of a metric space. The main reason for this assumption is the lack of linear structure of metric spaces.

For the remainder of this thesis, $X = (X, \|\cdot\|)$ will denote a normed space over the field of real numbers. In particular, it follows that

$$d_H(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\| \quad \text{for } A, B \in \mathbf{H}(X).$$

This extra assumption yields the homogeneity of the Hausdorff distance on $\mathbf{H}(X)$:

4.4.8 **Proposition.** *Let $X$ be a normed space, then $d_H$ is a homogeneous map on $\mathbf{H}(X)$.* «

Proof. Let $A, B \in \mathbf{H}(X)$ and $\lambda \in \mathbf{R}$, then, by using the homogeneity of the norm,

$$d_H(\lambda A, \lambda A) = \sup_{a \in A} \inf_{b \in B} \|\lambda a - \lambda b\| = |\lambda| \sup_{a \in A} \inf_{b \in B} \|a - b\| = \lambda d_H(A, B). \quad \blacksquare$$

Unfortunately, the Hausdorff distance is not necessarily translation invariant on every subset of $\mathbf{H}(X)$ or $\mathcal{K}(X)$:

4.4.9 **Example.** Consider $X = \mathbf{R}$ and take $A = C = [-1, 1]$ and $B = \{-1, 1\}$. Then $A + C = [-2, 2]$ and $B + C = [-2, 2]$. Note that $\delta(B, A) = 0$, since $B \subseteq A$ and $\delta(A, B) = 1$. Then

$$d_H(A, B) = 1 \quad \text{and} \quad d_H(A + C, B + C) = 0,$$

by definition of the Hausdorff distance. This shows that $d_H$ is not translation invariant. «

This last example proves that neither $\mathbf{H}(X)$ nor $\mathcal{K}(X)$ can be turned into a normed Riesz space (by using the construction of chapter 3). It turns out that both $\mathbf{H}(X)$ and $\mathcal{K}(X)$ are too 'large'. The construction is possible on the space of convex, closed and bounded subsets of a normed space $X$ that contain 0. This collection will be studied in more detail in the next chapters.

# 5 CONVEXITY

This chapter will provide additional conditions to insure translation invariance of the Hausdorff distance. The key is to consider a suitable subspace of the hyperspace of closed bounded non-empty subsets of a normed space $X$. This will be the space of convex, bounded and non-empty subsets of $X$. It turns out that the ensuing Riesz space has a strong order unit.

## 5.1 THE RIESZ SPACE OF CONVEX SETS

5.1.1 **Definition.** Let $X$ be a normed space and $A \subseteq X$. The set of all convex combinations of points in $A$, denoted by $\operatorname{co}(A)$ is said to be the *convex hull* of $A$. The set $\overline{\operatorname{co}}(A)$ is called the *closed convex hull* of $A$ and is defined as the closure of $\operatorname{co}(A)$. «

5.1.2 **Definition.** Let $X$ be a Banach space, then $\operatorname{Conv}(X)$ denotes the collection of all non-empty, bounded, closed and convex subsets of $X$. «

For the remainder of this chapter, $X$ will denote a normed space (unless stated otherwise).

Note that $\operatorname{Conv}(X)$ is a metric cone when endowed with the Hausdorff metric and the Minkowski sums. From this point on, the metric space $\operatorname{Conv}(X)$ will be partially ordered by inclusion. The lattice operations $\vee$ and $\wedge$ are given by

$$A \vee B = \overline{\operatorname{co}}(A \cup B) \quad \text{and} \quad A \wedge B = A \cap B \quad \text{for } A, B \in \operatorname{Conv}(X).$$

Note that $A \wedge B$ might not always exist, as $A \cap B$ may be empty. To avoid this possible complication, consider $\operatorname{Conv}_0(X)$ instead, where

$$\operatorname{Conv}_0(X) = \{A \in \operatorname{Conv}(X) \colon 0 \in A\}.$$

The collection $\operatorname{Conv}(X)$ inherits some structure from the underlying space normed space $X$.

5.1.3 **Lemma.** *Let $X$ be a normed space and $A, B \subseteq X$ bounded, convex and non-empty subsets. Then*

$$\operatorname{co}(A \cup B) = \bigcup_{\lambda \in [0,1]} [\lambda A + (1-\lambda)B].$$

«

Proof. Let $A, B \subseteq X$ be bounded and convex. Define

$$L = \operatorname{co}(A \cup B) \text{ and } R = \bigcup_{\lambda \in [0,1]} [\lambda A + (1-\lambda)B].$$

Note that $R = \{\lambda a + (1-\lambda)b \colon \lambda \in [0,1], a \in A \text{ and } b \in B\}$. With this terminology, it remains to show that $L = R$.

[$\subseteq$] Let $x \in L$. Note that if $x \in A \cup B$, then, by definition of the convex hull, $x \in R$.

Suppose $x \notin A \cup B$, then there are elements $x_1, \ldots, x_n \in A \cup B$ and scalars $\lambda_1, \ldots, \lambda_n$ in the unit interval that sum up to 1 such that

$$x = \sum_{k=1}^{n} \lambda_k x_k.$$

Define index-sets $\Lambda_1$ and $\Lambda_2$ by $\Lambda_1 = \{k \colon x_k \in A\}$ and $\Lambda_2 = \{k \colon x_k \in B\}$ and define

$$\lambda = \sum_{k \in \Lambda_1} \lambda_k.$$

Then

$$1 - \lambda = \sum_{k \in \Lambda_2} \lambda_k.$$

Note that both $\lambda$ and $1-\lambda$ are both non-zero, because $x \notin A \cup B$. In addition, $\lambda \leqslant 1$, since all $\lambda_k$ are positive for $1 \leqslant k \leqslant n$ and sum up to 1. Therefore, the sum over all elements of $\Lambda_1$ cannot exceed 1.

By using that both $A$ and $B$ are convex, there is an element $a \in A$ and an element $b \in B$ such that $x = \lambda a + (1-\lambda) b$ where

$$a = \frac{1}{\lambda} \sum_{k \in \Lambda_1} \lambda_k x_k \quad \text{and} \quad b = \frac{1}{1-\lambda} \sum_{k \in \Lambda_2} \lambda_k x_k.$$

This shows $x \in R$.

$[\supseteq]$ Let $x \in R$, then there is an element $a \in A$ and an element $b \in B$ such that $x = \lambda a + (1-\lambda) b$ for $\lambda \in [0,1]$. Then, by definition of the convex hull, $x \in L$. ■

5.1.4 **Theorem.** *If $X$ is a Banach space, then* $\operatorname{Conv}(X)$ *is complete.* «

Proof. Let $(A_n)$ be a Cauchy sequence in $\operatorname{Conv}(X)$. Since $\mathbf{H}(X)$ is complete (by Theorem 4.3.1) and $\operatorname{Conv}(X)$ is contained in $\mathbf{H}(X)$, there is a $A \in \mathbf{H}(X)$ such that

$$A_n \xrightarrow{d_H} A \text{ as } n \to \infty.$$

Because $A$ is closed, bounded and non-empty by definition, it remains to show that $A$ is convex.

To this end, let $a, b \in A$, $\lambda \in [0,1]$ and put $x = \lambda a + (1-\lambda) b$. Then, for any $\varepsilon > 0$ there is a $N \in \mathbf{N}$ such that for any $n \geqslant N$,

$$A_n \subseteq A + \varepsilon \mathbf{B} \quad \text{and} \quad A \subseteq A_n + \varepsilon \mathbf{B}.$$

Note that $A_n + \varepsilon \mathbf{B}$ is convex for all $n \geqslant N$, since it is a sum of convex sets. Therefore,

$$x \in A_n + \varepsilon \mathbf{B} + 2\varepsilon \mathbf{B},$$

which yields $x \in A$. ■

Observe that all results of Section 4.4 are valid for $\operatorname{Conv}(X)$, since Theorem 5.1.4 implies that it is a closed subspace of $\mathbf{H}(X)$.

Note that the Hausdorff distance is still a homogeneous map on the elements of $\operatorname{Conv}(X)$. Moreover, the Hausdorff distance is translation invariant. The proof of this assertion requires some tools from the theory of convex analysis.

## 5.2 SUPPORT FUNCTIONS

5.2.1 **Definition.** Let $X$ be a normed space and let $A \subseteq X$ be a fixed non-empty subset. The map

$$\begin{aligned} h_A \colon X^* &\longrightarrow [0, \infty] \\ \phi &\longmapsto \sup_{a \in A} \phi(a) \end{aligned}$$

is called the *support function of $A$*. «

For any $A \subseteq X$, the support function preserves lattice structure. The proof of the next proposition is available in [1, p. 292].

5.2.2 **Proposition.** *Let $X$ be a normed space and $A, B \in \mathrm{Conv}(X)$ and let $h_A$ and $h_B$ denote the support function of $A$ and $B$ respectively. Then*

1. $h_{A\vee B} = h_A \vee h_B$ *and* $h_{A\wedge B} = h_A \wedge h_B$;
2. $h_{A+B} = h_A + h_B$;
3. $h_{\lambda A} = \lambda h_A$ *for all* $\lambda > 0$;
4. *If* $A \subseteq B$, *then* $h_A \leqslant h_B$. «

5.2.3 **Proposition.** *There is a* 1-1 *correspondence between support functions and the elements of* $\mathrm{Conv}(X)$. «

Proof. Let $A \in \mathrm{Conv}(X)$ and let $h_A$ be the support function of $A$. It suffices to show that

$$A = \{x \in X \colon \phi(x) \leqslant h_A(\phi) \text{ for all } \phi \in X^*\}.$$

[$\subseteq$] If $x \in A$, then

$$\phi(x) \leqslant \sup_{a\in A} \phi(a) = h_A(\phi) \quad \text{for all } \phi \in X^*.$$

[$\supseteq$] Suppose there is an element

$$x \in \{a \in X \colon \phi(a) \leqslant h_A(\phi) \text{ for all } \phi \in X^*\} \setminus A.$$

Note that $\{x\}$ is compact and $A$ is closed and convex. In addition, $\{x\}$ and $A$ are disjoint. Therefore, by the Hahn-Banach separation theorem, there is a functional $\phi \in X^*$ and $s \in \mathbf{R}$ such that

$$\phi(x) > s \quad \text{and} \quad \phi(a) \leqslant s \quad \text{for all } a \in A.$$

Then

$$\sup_{a\in A} \phi(a) \leqslant s,$$

which yields $h_A(\phi) \leqslant s$. Hence

$$\phi(x) \leqslant h_A(\phi) \leqslant s,$$

which leads to a contradiction. ∎

By using support functions (given a fixed non-empty closed, bounded and convex set), it is possible to obtain another expression for the Hausdorff distance:

5.2.4 **Theorem.** *Let $X$ be normed and $A, B \in \mathrm{Conv}(X)$, then*

$$d_H(A, B) = \sup_{\|\phi\|=1} \left|h_A(\phi) - h_B(\phi)\right|.$$ «

Proof. The case $A = B$ is clear.

Suppose $A \neq B$ and let $\varepsilon > 0$ such that $A \subseteq B + \mathrm{B}(0;\varepsilon)$ and $B \subseteq A + \mathrm{B}(0,\varepsilon)$. Then, for all $\phi \in X^*$ with $\|\phi\| = 1$,

$$h_A(\phi) \leqslant h_B(\phi) + \varepsilon \quad \text{and} \quad h_B(\phi) \leqslant h_A(\phi) + \varepsilon.$$

So $|h_A(\phi) - h_B(\phi)| < \varepsilon$. This means

$$\sup_{\|\phi\|=1} |h_A(\phi) - h_B(\phi)| \leqslant \mathrm{d_H}(A, B).$$

To show the other inequality, note that if

$$\varepsilon = \sup_{\|\phi\|=1} |h_A(\phi) - h_B(\phi)| > 0,$$

then

$$\phi(a) \leqslant \sup_{b \in B} \phi(b) + \varepsilon \quad \text{for every } \phi \in X^* \text{ with } \|\phi\| = 1.$$

So for all $\eta > 0$ there is a $b \in B$ with $\phi(a - b) \leqslant \varepsilon + \eta$. Therefore

$$\|a - b\| \leqslant \varepsilon + \eta.$$

This yields $A \subseteq B + \overline{\mathrm{B}}(0;\varepsilon)$. An analogous reasoning yields $B \subseteq A + \overline{\mathrm{B}}(0;\varepsilon)$. So $\mathrm{d_H}(A, B) \leqslant \varepsilon$. ∎

If the space $X$ is uniformly convex (see appendix), then the proof of the previous theorem can be simplified:

5.2.5 **Lemma.** *Let $X$ be a uniformly convex normed space and $A, B \in \mathrm{Conv}(X)$. Let $h_A$ and $h_B$ denote the support function of $A$ and $B$ respectively. Then*

$$\mathrm{d_H}(A, B) = \sup_{\|\phi\|=1} |h_A(\phi) - h_B(\phi)|.$$

«

Proof. Note that the inequality $\leqslant$ has been established in Theorem 5.2.4. It remains to prove the other inequality.

Let $\varepsilon > 0$. There exists an element $a_0 \in A$ be at distance $\mathrm{d_H}(A, B) - \varepsilon$ from $B$. By using the Corollary A.10, there is a $b_0 \in B$ with minimal distance to $a_0$. Now consider

$$\phi = \frac{a_0 - b_0}{\|a_0 - b_0\|}.$$

Then $\|\phi\| = 1$ and by a similar reasoning as in the proof of Theorem 5.2.4, it follows that

$$h_A(\phi) - h_B(\phi) > \mathrm{d_H}(A, B) - \varepsilon.$$

∎

The translation invariance of the Hausdorff distance with respect to the Minkowski sums follows directly from Proposition 5.2.2 and Theorem 5.2.4. Therefore, it is possible to construct a Riesz space from the metric cone $\mathrm{Conv}_0(X)$, by using the construction of Chapter 3.

## 5.3 AN RIESZ SPACE WITH A STRONG ORDER UNIT

Let $\mathfrak{C} = \mathfrak{C}(X)$ denote the Riesz space obtained from $\mathrm{Conv}_0(X)$ by the construction in Chapter 3. By Theorem 5.2.4, the Hausdorff distance is translation invariant and the Hausdorff distance is homogeneous in both arguments by definition. Therefore, by Theorem 3.3.2, $\mathfrak{C}$ is a normed space such that its norm induces the Hausdorff distance.

The Riesz space $\mathfrak{C}$ has a strong order unit. The idea behind the proof is that any non-empty bounded set $A$ is contained in a (closed) ball. So by 'stretching out the unit ball far enough', the set $A$ will eventually be contained in this 'stretched ball'. The convexity-stucture ensures that the 'stretching-process' is well-defined.

5.3.1 **Theorem.** *The Riesz space $\mathfrak{C}$ is Archimedean and he closed unit ball $\mathbf{B}$ is a strong order unit in $\mathfrak{C}$.* «

Proof. Let $[A, B] \in \mathfrak{C}$. If there there is a $n \in \mathbf{N}$ such that

$$-n[\mathbf{B}, 0] \leqslant [A, B] \leqslant n[\mathbf{B}, 0],$$

then the proof is complete.

Suppose $[A, B], [C, D] \in \mathfrak{C}$ such that $n \cdot [A, B] \leqslant [C, D]$ for all $n \in \mathbf{N}$. Then $[A, B] \leqslant [0, 0]$, since $\mathfrak{C}$ is Archimedean. Indeed, for any $a \in A$ and $d \in D$ there is a $b_n \in B$, a $c_n \in C$ and an $x_n \in \mathbf{B}$ such that

$$na + d = nb_n + c_n + x_n.$$

This show that $a \in B$, because $B$ is closed and both $C$ and $D$ are bounded. Hence the implication

$$nA + D \subseteq nB + C \quad \text{for all } n \in \mathbf{N} \implies A \subseteq B$$

holds true.

Now take $N \in \mathbf{N}$ such that both $A$ and $B$ are contained in $N\mathbf{B}$. This is possible since both $A$ and $B$ have a finite diameter. Then for $n \geqslant N$ and the fact that both $A$ and $B$ contain 0,

$$A \subseteq B + n\mathbf{B} \quad \text{and} \quad B \subseteq A + n\mathbf{B} = A + (-n)\mathbf{B}.$$

Therefore, $\mathbf{B}$ is indeed a strong order unit in $\mathfrak{C}$. ∎

# 6 SPACES OF CONVEX AND COMPACT SETS

The goal of this chapter is to show that for a given normed space $X$, that the Riesz space generated by the collection of all compact and convex subsets of $X$ containing 0, need not have a weak order unit.

## 6.1 THE HILBERT CUBE

6.1.1 **Definition.** Let $X$ be a normed space, and define the collection

$$\mathbf{K}(X) = \{A \subseteq X \colon \text{A is convex, compact and } 0 \in A\} \subseteq \operatorname{Conv}(X). \quad «$$

Observe that $\mathbf{K}(X)$ is a cone under the Minkowski addition. If the underlying normed space is complete, then so is $\mathbf{K}(X)$:

6.1.2 **Lemma.** *If $X$ is a Banach space, then $\mathbf{K}(X)$ is a complete metric space.* «

Proof. By combining the statements of both Corollary 4.3.2 and Theorem 5.1.4, the result follows. ∎

The previous lemma implies that $\mathbf{K}(X)$ is a closed subspace of $\operatorname{Conv}_0(X)$. Let $\mathfrak{c} = \mathfrak{c}(X)$ denote the Riesz space obtained from $\mathbf{K}(X)$ (by using the construction in chapter 3). The lattice operations are given by

$$A \vee B = \overline{\operatorname{co}}(A \cup B) \quad \text{and} \quad A \wedge B = A \cap B.$$

Note that $\mathfrak{c}$ is closed under these lattice operations.

Before studying order units on $\mathfrak{c}$, a remark is order. The idea behind the proof of Theorem 5.3.1 will not work in $\mathfrak{c}$, because the unit ball is not compact in a general normed space. The 'next best thing' is to consider a set which keeps getting 'thinner' in each coordinate. The natural candidate to describe this behaviour in $\ell_p$ is the *Hilbert cube*.

For the remainder of this chapter, $p$ is assumed to be a constant in the interval $[1, \infty)$.

6.1.3 **Definition.** Let $\alpha = (\alpha_n) \in \ell_p$ be a fixed sequence. The subset given by

$$H_\alpha = \{(x_n) \in \ell_p \colon |x_n| \leqslant |\alpha_n| \text{ for all } n \in \mathbf{N}\}$$

is the *Hilbert cube in $\ell_p$*. «

The Hilbert cube $H_\alpha$ is by definition non-empty, since it contains the zero-sequence. In addition, $H$ is both convex and compact. The latter assertion does not follow directly from the definition. Its proof requires some results from topology:

6.1.4 **Theorem.** *Let $\{[a_n, b_n] \colon n \in \mathbf{N}\}$ be a collection of non-empty compact intervals in $\mathbf{R}$, then the (countable) Cartesian product*

$$\prod_{n \in \mathbf{N}} [a_n, b_n]$$

*is compact in the product topology.* «

Proof. This is a direct consequence of Tikhonov's theorem, but it is possible to prove this theorem without using the Axiom of Choice (see [9, p. 28]). ∎

6.1.5 **Lemma.** *Let $X, Y$ be topological spaces such that $X$ is compact. Then for any continuous map $f\colon X \to Y$, the set $f[X]$ is compact in $Y$.* «

6.1.6 **Lemma.** *The Hilbert cube is a compact and convex subset of $\ell_p$.* «

Proof. Consider the (bijective) map

$$\begin{aligned} f\colon [-1,1]^{\mathbf{N}} &\longrightarrow \ell_p \\ (x_n) &\longmapsto (\alpha_n \cdot x_n) \end{aligned}$$

Note that $H_\alpha$ is the image of $[-1,1]^{\mathbf{N}}$ under $f$. By Theorem 6.1.4, $[-1,1]^{\mathbf{N}}$ is compact (in the product topology). Therefore, by Lemma 6.1.5, it suffices to show that $f$ is continuous. Recall that

$$\mathrm{d}(x,y) = \sum_{n\in\mathbf{N}} 2^{-n}|x_n - y_n|$$

is a metric for the topology on $[-1,1]^{\mathbf{N}}$.

Let $\varepsilon > 0$. Since $\alpha \in \ell_p$, there is a $N \in \mathbf{N}$ such that

$$\sum_{n\geqslant N+1} |\alpha_n|^p < \frac{\varepsilon}{2^{p+1}}.$$

Let $x, y \in [-1,1]^{\mathbf{N}}$ and take

$$0 < \delta = \varepsilon \cdot \frac{2^{-N}}{2\|\alpha\|_p^p}.$$

If $x, y \in [-1,1]^{\mathbf{N}}$ with $\mathrm{d}(x,y) < \delta$, then

$$\|x-y\|_p^p < \delta \quad \text{and} \quad |x_n - y_n| < \frac{\varepsilon}{2\|\alpha\|_p^p} \quad \text{for all } n \leqslant N.$$

Observe that $|x_n - y_n| \leqslant 2$ for $n \geqslant N$, which yields

$$\begin{aligned} \|f(x) - f(y)\|_p^p &= \sum_{n\in\mathbf{N}} |\alpha_n|^p |x_n - y_n|^p \\ &= \sum_{n\leqslant N} |\alpha_n|^p |x_n - y_n|^p + \sum_{n\geqslant N+1} |\alpha_n|^p |x_n - y_n|^p \\ &< \frac{\varepsilon}{2\|\alpha\|_p^p} \sum_{n\leqslant N} |\alpha_n|^p + 2^p \sum_{n\geqslant N+1} |\alpha_n|^p \\ &< \frac{\varepsilon}{2\|\alpha\|_p^p} \cdot \|\alpha\|_p^p + 2^p \cdot \frac{\varepsilon}{2^{p+1}} \\ &< \varepsilon. \end{aligned}$$

So $f$ is indeed continuous and therefore $H_\alpha$ is compact.

To show that $H_\alpha$ is convex, let $(x_n), (y_n) \in H_\alpha$ and $\lambda \in [0,1]$. Then,

$$\begin{aligned} |\lambda x_n + (1-\lambda) y_n| &\leqslant \lambda |x_n| + (1-\lambda)|y_n| \\ &\leqslant \lambda \alpha_n + (1-\lambda)\alpha_n \\ &= \alpha_n \end{aligned}$$

for all $n \in \mathbf{N}$. ∎

In order to find order units in a Riesz space, it suffices to only consider the positive cone, by Definition 2.4.1. Hence for Riesz spaces as constructed from hyperspaces, it suffices to consider the hyperspaces. By different choices, it is possible to get all variations of space with or without order units.

The aim of this chapter is to prove that the Hilbert cube is not a weak order unit in $\mathfrak{c}(\ell_p)$.

By Definition 2.4.1, the Hilbert cube $H_\alpha \in \mathfrak{c}(\ell_p)$ is a weak order unit whenever any element $A \in \mathfrak{c}(\ell_p)$ can be expressed as

$$A = \overline{\text{co}}\Big[\bigcup_{n\in\mathbf{N}} (nH_\alpha \cap A)\Big]. \qquad (*)$$

Intuitively, it is not possible to express every $A \in \mathfrak{c}(\ell_p)$ in this manner. The problem is that given a fixed $\ell_p$-sequence $(\alpha_n)$, there is an $\ell_p$-sequence $(\beta_n)$ which converges at a slower rate. In other words, for any $(\alpha_n) \in \ell_p$ there exists a $(\beta_n)$ such that $\beta_n/\alpha_n \to \infty$ as $n \to \infty$. This observation can be used to show that for a given $\ell_p$-sequence $(\alpha_n)$ and a given $A \in \mathfrak{c}(\ell_p)$, the intersection $nH_\alpha \cap A$ can be empty for all $n \in \mathbf{N}$.

Additionally, if $A \subseteq X$ is a convex set which contains 0, then

$$A \subseteq \lambda A \quad \text{for all } \lambda \geqslant 1.$$

Indeed, let $a \in A$, then write

$$\tfrac{1}{\lambda} \cdot (\lambda \cdot a) + (1 - \tfrac{1}{\lambda}) \cdot 0.$$

This shows that $a \in \lambda A$.

Combining all these observations yields sufficient material to disprove $(*)$.

*Disproving* $(*)$

6.2.1 **Theorem.** *There exists an element $A \in \mathfrak{c}(\ell_p)$ such that $(*)$ does not hold.* «

Proof. Take a sequence $(\beta_n)$ in $\ell_p$ such that $\beta_n/\alpha_n \to \infty$ as $n \to \infty$ and consider

$$A = \{\lambda\beta : \lambda \in [0,1]\}.$$

Then $A \in \mathbf{K}(\ell_p)$ and

$$nH_\alpha \cap A = \{0\} \quad \text{for all } n \in \mathbf{N}.$$

This shows that $(*)$ does not hold. ∎

The previous theorem shows that the Hilbert cube is not a weak order unit in $\mathfrak{c}(\ell_p)$.

There is another cone in which the Hilbert cube is a weak order unit. The proof requires the Baire Category theorem. There are several versions of this theorem in circulation. The statement of the version that will be used in this theses is presented below. The proof can be found in [17, p. 296].

*The Baire Category theorem*

6.2.2 **Definition.** A *nowhere dense set* in a topological space $X$ is a set whose closure has empty interior. «

In the setting of metric spaces there is another characterisation of nowhere dense sets:

6.2.3 **Proposition.** *Let $S$ be a metric space and $A \subseteq S$ a closed subset. Then $A$ is nowhere dense if and only if there is no open ball which is contained in $A$.* «

Proof. Assume there is no open ball contained in $A$. This assumption is no restriction, for if there is an open ball contained in $A$, then $A$ is not nowhere dense.

Let $U \subseteq S$ be a non-empty open subset. Then $A$ is not contained in $U$, since $U$ contains an open ball and $A$ does not. Let $x \in U$ such that $x \notin A$. Because $A$ is closed, there exists $r > 0$ such that $\mathrm{B}(x; r)$ does not intersect $A$, that is,

$$\mathrm{B}(x; r) \cap A = \varnothing.$$

Since $U$ is open, there exists $s > 0$ such that $\mathrm{B}(x; s) \subseteq U$.

Next, take $R = \min(r, s)$. Then $\mathrm{B}(x; R)$ is contained in $U$ and $\mathrm{B}(x; R)$ does not intersect $A$. Therefore, $A$ is nowhere dense. ■

The next proposition presents a common example of a nowhere dense set in a normed space of infinite dimension.

6.2.4 **Proposition.** *Let $X$ be an infinite dimensional normed space. Then any non-empty compact subset of $X$ is nowhere dense.* «

Proof. Suppose $K$ is not nowhere dense. Then there is an element $x \in K$ and a $r > 0$ such that $\mathrm{B}(x; r) \subseteq K$. Define $s = r \cdot (\mathrm{e}^{\pi} - \pi)/42$, then $s < r$.

If $X$ is infinite dimensional, then the closed ball centered at $x$ with radius $r$ is not compact. For if it were, the closed ball $\overline{\mathrm{B}}(x, s)$ would be compact as well since it is a closed subspace of $K$. This contradicts the fact that closed balls in infinite dimensional normed spaces are not compact. ■

6.2.5 **Theorem** (Baire Category theorem). *Any non-empty complete metric space is not equal to a countable union of nowhere-dense closed sets.* «

## 6.3 A Riesz space with a weak order unit

It turns out that the Hilbert cube is a weak order in a certain subspace of $\mathbf{K}(\ell_p)$.

6.3.1 **Definition.** Let $K \in \mathbf{K}(\ell_p)$, $x \in \ell_p$ and $y \in K$. If $|x_n| \leqslant |y_n|$ for all $n \in \mathbf{N}$ implies $x \in K$, then $K$ is a *set of type (A)*. The collection of all $K \in \mathbf{K}(\ell_p)$ of type (A) will be abbreviated by $\mathbf{A}(\ell_p)$. That is,

$$\mathbf{A}(\ell_p) = \{K \in \mathbf{K}(\ell_p) \colon K \text{ is of type } (A)\}.$$ «

6.3.2 **Example.** For a fixed $\ell_p$-sequence $\alpha$, the Hilbert cube $H_\alpha$ is a set of type (A), according to the results of the previous section. «

6.3.3 **Theorem.** *Let $\alpha \in \ell_p$ be a fixed sequence. The Hilbert cube $H_\alpha$ is a weak order unit in $\mathbf{A}(\ell_p)$, that is,*

$$K = \overline{\mathrm{co}} \bigcup_{n \in \mathbf{N}} [(nH_\alpha) \cap K] \quad \text{for all } K \in \mathbf{A}(\ell_p).$$ «

Proof. [$\subseteq$] Let $K \in \mathbf{A}(\ell_p)$ and $a \in K$. Define for all $N \in \mathbf{N}$ the sequence

$$a^N = (a_1, \ldots, a_N, 0, \ldots).$$

This sequence is contained in $(nH_\alpha) \cap K$ for $n$ sufficiently large. Therefore,

$$a^N \in \operatorname{co} \bigcup_{n \in \mathbf{N}} [(nH_\alpha) \cap K] \quad \text{for all } N \in \mathbf{N}.$$

And since $a^N \to a$ with respect to the $p$-norm, it follows that

$$a \in \overline{\operatorname{co}} \bigcup_{n \in \mathbf{N}} [(nH_\alpha) \cap K].$$

[$\supseteq$] Let $n \in \mathbf{N}$, then $(nH_\alpha) \cap K \subseteq K$. This implies

$$\bigcup_{n \in \mathbf{N}} [(nH_\alpha) \cap K] \subseteq K.$$

Since $K$ is convex and closed, it follows that

$$\overline{\operatorname{co}} \bigcup_{n \in \mathbf{N}} [(nH_\alpha) \cap K] \subseteq K.$$

Therefore, $H_\alpha$ is a weak order unit in $\mathbf{A}(\ell_p)$. ∎

Note that the Hilbert cube need not be a strong order unit in $\mathbf{A}(\ell_p)$. This follows from the fact that for any fixed $\ell_p$-sequence $\alpha$, there is a sequence which converges at a faster rate.

6.3.4 **Proposition.** *For $p = 2$ and $\alpha = \{1/n \colon n \in \mathbf{N}\}$, the Hilbert cube $H_\alpha$ is not a strong order unit in $\mathbf{A}(\ell_p)$.* «

Proof. Consider the set

$$K = \{x \in \ell_2 \colon |x_n| \leqslant \log(n)/n\}.$$

Note that $K$ contains the zero-sequence. Because the sequence $(\beta_n)$ where $\beta_n = \log(n)/n$ for all $n \in \mathbf{N}$ is square summable, it follows that $K$ is compact and convex. Therefore $K \in \mathbf{A}(\ell_2)$.

Now note that the sequence $(\beta_n)$ is contained in $K$, but not contained in $kH_\alpha$ for all $k \in \mathbf{N}$. Therefore, $H_\alpha$ is not a strong order unit. ∎

6.3.5 **Proposition.** *$\mathbf{A}(\ell_p)$ does not have a strong order unit.* «

Proof. Suppose $K \in \mathbf{A}(\ell_p)$ is a strong order unit. If $z \in \ell_p$, then

$$\{x \in \ell_p \colon |x_n| \leqslant |z_n| \text{ for all } n\} \in \mathbf{A}(\ell_p).$$

Therefore, there exists a $\lambda \in \mathbf{R}^+$ such that

$$\{x \in \ell_p \colon |x_n| \leqslant |z_n| \text{ for all } n\} \subseteq \lambda K.$$

This yields $z \in \lambda K$ and therefore $\frac{1}{\lambda} z \in K$. This shows that for any $z \in \ell_p$ exists a number $n \in \mathbf{N}$ such that $z \in nK$. Hence

$$\bigcup_{n \in \mathbf{N}} nK = \ell_p.$$

Observe that each $nK$ is compact and therefore closed and $\ell_p$ is a complete metric space. By applying the Baire category theorem, not every $nK$ is nowhere dense. This means there is a natural number $m$ such that $mK$ is not nowhere dense. In other words, $mK$ does not have nonempty interior. This contradicts Proposition 6.2.4.

This concludes the proof. ∎

Consider the collection

$$\mathcal{A}(\ell_p) = \{K \in \mathbf{K}(\ell_p) \colon \text{there is an } A \in \mathbf{A}(\ell_p) \text{ such that } K \subseteq A\}.$$

This collection is a cone and the Riesz space generated by it does not have a weak order unit.

6.4.1 **Theorem.** *$\mathcal{A}(\ell_p)$ does not have a weak order unit.* «

Proof. Let $H \in \mathcal{A}(\ell_p)$. Then there is an $A \in \mathcal{A}(\ell_p)$ such that $H \subseteq A$. Then there is a sequence $z \in \ell_p$ such that

$$A = \{x \in \ell_p \colon |x| \leqslant |z|\}.$$

Therefore $|x| \leqslant |z|$ for all $x \in H$.

Now take $y \in \ell_p$ such that $y_n/x_n \to \infty$ as $n \to \infty$ and define

$$K = \{\lambda y \colon \lambda \in [0,1]\}.$$

Then $K \subseteq \{x \colon |x| \leqslant |y|\}$, so $K \in \mathcal{A}(\ell_p)$. Then

$$nH \cup K = \{0\} \quad \text{for each } n \in \mathbf{N}.$$

Therefore

$$\overline{\text{co}} \bigcup_{n \in \mathbf{N}} nH \cap K = \{0\} \neq K,$$

which shows that $H$ is not a weak order unit. ∎

# A
# UNIFORM CONVEXITY

This is a self-contained chapter which explains the structure on uniformly convex normed spaces. It can be used to simplify certain proofs presented in the previous chapters.

Let $X$ be a normed space over a field $\mathbf{F}$. The closed unit ball in $X$, defined as

$$\mathbf{B}(X, \|\cdot\|) = \{x \in X \colon \|x\| \leqslant 1\},$$

will be denoted by $\mathbf{B}$ if no confusion can arise.

A.1 **Definition.** Let $X$ be a normed space over $\mathbf{F}$. Then $X$ is called *uniformly convex* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left\|\frac{x+y}{2}\right\| \leqslant 1 - \delta$$

for all $x, y \in \mathbf{B}$ such that $\|x - y\| \geqslant \varepsilon$. «

Loosely speaking, the center of a line segment inside the unit ball must lie 'deep inside' the unit ball unless the segment is 'short'.

Note that uniform convexity is a property of the norm, not a property of the underlying vector space. It can happen that a vector space is uniformly convex when endowed with some norm $\|\cdot\|$, but not uniformly convex when endowed with another norm, even when the two norms are equivalent!

A.2 **Example.** Let $n \geqslant 1$ then $(\mathbf{R}^n, \|\cdot\|_1)$ is not uniformly convex, but $(\mathbf{R}^n, \|\cdot\|_2)$ is.

Put

$$\mathbf{B}_1 = \mathbf{B}(\mathbf{R}^n, \|\cdot\|_1) \quad \text{and} \quad \mathbf{B}_2 = \mathbf{B}(\mathbf{R}^n, \|\cdot\|_2).$$

For the first part, consider $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$ and take $\varepsilon = 2$. Then $x, y \in \mathbf{B}_1$ and $\|x - y\|_1 = 2$. On the other hand,

$$\left\|\frac{x+y}{2}\right\|_1 = \tfrac{1}{2} \cdot 2 = 1 > 1 - \delta$$

for any $\delta > 0$. This shows that $(\mathbf{R}^n, \|\cdot\|_1)$ is not uniformly convex.

To prove that $(\mathbf{R}^n, \|\cdot\|_2)$ is uniformly convex, let $x, y \in \mathbf{R}^n$ and write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|_2^2 - \left\|\frac{x-y}{2}\right\|_2^2 &= \frac{1}{4}\Big[\sum_{k=1}^n (x_k + y_k)^2 + \sum_{k=1}^n (x_k - y_k)^2\Big] \\ &= \frac{1}{4}\sum_{k=1}^n (2x_k^2 + 2y_k^2) \\ &= \frac{1}{2}\Big(\|x\|_2^2 + \|y\|_2^2\Big). \qquad (*) \end{aligned}$$

Let $\varepsilon > 0$ and take $\delta > 0$ such that

$$(1 - \delta)^2 = 1 - \frac{\varepsilon^2}{4}.$$

Let $x, y \in \mathbf{B}_2$ such that $\|x-y\| \geqslant \varepsilon$, then, by $(*)$,

$$\begin{aligned}\left\|\frac{x+y}{2}\right\|_2^2 &= \frac{\|x\|_2 + \|y\|_2}{2} - \left\|\frac{x-y}{2}\right\|_2^2 \\ &\leqslant 1 - \left\|\frac{x-y}{2}\right\|_2^2 \\ &\leqslant 1 - \frac{\varepsilon^2}{4} \\ &= (1-\delta)^2.\end{aligned}$$

This shows that $(\mathbf{R}^n, \|\cdot\|_2)$ is indeed uniformly convex. «

The previous example can be generalized:

A.3 **Theorem.** *Let $(X, \Sigma, \mu)$ be a positive measure space, then $\mathrm{L}_p(X, \Sigma, \mu)$ is uniformly convex for $1 < p < \infty$.* «

The proof of this theorem is quite involved and will not be presented in this thesis. The interested reader can find a proof in the literature (see [4] or [8]).

Recall the following result from linear algebra:

A.4 **Lemma** (Parallelogram law). *Let $X$ be an inner product space over $\mathbf{F}$. Then, for $x, y \in X$, the norm on $X$ satisfies*

$$\|x+y\|^2 + \|x-y^2\| = 2 \cdot \|x\|^2 + 2 \cdot \|y\|^2. \quad «$$

A.5 **Theorem.** *Let $X$ be an inner product space over $\mathbf{F}$. Then $(X, \|\cdot\|)$ is a uniformly convex normed space.* «

Proof. Let $\varepsilon > 0$ and take $\delta > 0$ such that

$$1 - \delta = \sqrt{1 - \frac{\varepsilon^2}{4}}.$$

Let $x, y \in \mathbf{B}$ such that $\|x-y\| \geqslant \varepsilon$, then

$$\begin{aligned}\left\|\frac{x+y}{2}\right\|^2 &= 2 \cdot \left\|\frac{x}{2}\right\|^2 + 2 \cdot \left\|\frac{y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2 \\ &\leqslant 1 - \frac{\varepsilon^2}{4} \\ &= (1-\delta)^2.\end{aligned}$$

Therefore, $X$ is uniformly convex. ■

It turns out that any uniformly convex Banach space is reflexive. The proof of this statement requires a result from the theory on the weak topology. The proof is omitted, but can be found in e.g. Dunford and Schwartz (see [6, p. 424]).

A.6 **Theorem.** *Let $X$ be a normed space, then $\mathbf{B}$ is $\sigma(X^{**}, X^*)$ dense in the closed unit of ball $X^{**}$.* «

A.7 **Theorem** (Milman-Pettis). *Any uniformly convex Banach space $X$ over $\mathbf{F}$ is reflexive.* «

Proof. Suppose $X$ is a uniformly convex Banach space which is not reflexive. Let $\mathbf{B}^{**}$ denote the closed unit ball in $X^{**}$ and consider the isometric duality maps

$$J\colon X \to X^{**} \quad \text{and} \quad J^{*}\colon X^{*} \to X^{***}.$$

By Theorem A.7,

$$\mathbf{B}^{**} = \overline{J[\mathbf{B}]}^{\text{wk}*},$$

where $\overline{J[\mathbf{B}]}^{\text{wk}*}$ denotes the closure of $J[\mathbf{B}]$ in the $\sigma(X^{**}, J^{*}[X^{*}])$ topology. Since $X$ is not reflexive, $J[X] \neq X^{**}$ and so there must be a $\psi \in X^{**}$ that is not in $J[\mathbf{B}]$ with $\|\psi\| = 1$.

Since $J$ is an isometry, $J[\mathbf{B}]$ is (norm) closed in $X^{**}$. So

$$\mathrm{d}(\psi, J[\mathbf{B}]) = 2\varepsilon$$

for some given $\varepsilon > 0$. On the other hand, $\psi \in \overline{J[\mathbf{B}]}^{\text{wk}*}$. So

$$\psi \in \overline{U \cap J[\mathbf{B}]}^{\text{wk}*}$$

for any $\sigma(X^{*}, J^{*}[X^{*}])$ neighbourhood $U$ of $\psi$.

Next, let $\phi \in X^{*}$ such that $\|\phi\| = 1$ and $|\psi(\phi) - 1| < \delta$. Note that this follows from the fact that $\|\psi\| = 1$. Put

$$U = \{\psi \in X^{**}\colon |\psi(\phi) - 1| < \delta\},$$

then

$$|\psi_1(\phi) - \psi_2(\phi)| < 2\delta$$

for all $\psi_1, \psi_2 \in U \cap J[\mathbf{B}]$. Additionally,

$$\begin{aligned} \|\psi_1 + \psi_2\| &\geqslant |\psi_1(\phi) + \psi_2(\phi)| \\ &= |2 + \psi_1(\phi) - 1 + \psi_2(\phi) - 1| \\ &\geqslant 2 - 2\delta, \end{aligned}$$

for all $\psi_1, \psi_2 \in U \cap J[\mathbf{B}]$. Therefore, $\|\psi_1 - \psi_2\| < \varepsilon)$ for all $\psi_1, \psi_2 \in U \cap J[\mathbf{B}]$ since $X$ is uniformly convex. Now fix $\psi_1$, then

$$U \cap J[\mathbf{B}] \subseteq \psi_1 + \varepsilon\big(\overline{U \cap J[\mathbf{B}]}^{\text{wk}*}\big).$$

Note that $\overline{U \cap J[\mathbf{B}]}^{\text{wk}*}$ is closed so $\psi \in \overline{U \cap J[\mathbf{B}]}^{\text{wk}*}$. But this contradicts the assumption $\mathrm{d}(\psi, J[\mathbf{B}]) = 2\varepsilon$. Hence, $X$ is reflexive. ∎

The Milman-Pettis theorem was proven independently by Milman [16] and Pettis [18]. Shortly after publishing these papers, Kakutani came up with a simplified proof (see [12]). About twenty years later, Ringrose [20] published a shorter proof.

A.8 **Lemma.** *Let $X$ be a uniformly convex normed space over* $\mathbf{F}$ *and let $(x_n)$ be a sequence in $X$ such that*

*(1)* $\|x_n\| \to 1$ *as* $n \to \infty$.

*(2) For* $\varepsilon > 0$ *there is a* $N \in \mathbf{N}$ *such that*

$$\big|\|x_n + x_m\| - 2\big| \leqslant \varepsilon$$

*for all* $n, m > N$.

*Then $(x_n)$ is a Cauchy sequence.*

*In particular, if $X$ is Banach space, then $(x_n)$ is convergent.* «

Proof. Assume that $x_n \in \mathbf{B}$ for all $n$. Let $\varepsilon > 0$, then there is a $N \in \mathbf{N}$ such that $\|x_n - x_m\| < \varepsilon$ for all $n, m > N$. This is possible by using (2). So if the sequence $(x_n)$ is contained in $\mathbf{B}$, then the sequence is Cauchy.

If not all of $(x_n)$ are in the unit ball, assume (without loss of generality and (1)) that $\|x_n\| \neq 0$ for all $n \in \mathbf{N}$. Next, define the sequence $(y_n)$ by setting

$$y_n = \frac{x_n}{\|x_n\|} \qquad \text{for all } n \in \mathbf{N}.$$

Then $(y_n)$ satisfies (1) and since the terms of $(y_n)$ are in $\mathbf{B}$ for all $n$, the reasoning from the first part yields that $(y_n)$ is a Cauchy sequence. And then $(x_n)$ is Cauchy due to (1). ■

A.9 **Theorem.** *Let $X$ be a uniformly convex Banach space and $C \subseteq X$ a non-empty closed and convex set. Then there is a unique $x \in C$ such that*

$$\|x\| = \inf_{z \in C} \|z\|.$$

«

Proof. Note that if $0 \in C$, take $x = 0$ and the result follows. So assume that $0 \notin C$.

Put $d = \inf\{\|z\| \colon z \in C\}$, then $d > 0$ by definition of the norm and the fact that $0 \notin C$. Let $(x_n)$ be a sequence in $C$ such that $\|x_n\| \to d$ as $n \to \infty$. Define a sequence $(y_n)$ by putting

$$y_n = \frac{x_n}{d} \qquad \text{for all } n \in \mathbf{N}.$$

Then, by the triangle inequality,

$$\|y_n + y_m\| = \left\|\frac{x_n + x_m}{d}\right\| \leqslant \frac{\|x_n\| + \|x_m\|}{d}$$

for all $n, m \in \mathbf{N}$. Next, since $C$ is convex, $d \leqslant \frac{1}{2}\|x_n + x_m\|$ for all $n, m \in \mathbf{N}$. Therefore,

$$\begin{aligned}
\|y_n + y_m\| &= \left\|\frac{x_n + x_m}{d}\right\| \\
&= \frac{2}{d}\left\|\frac{x_n + x_m}{2}\right\| \\
&\geqslant 2 \cdot \frac{2}{\|x_n + x_m\|} \cdot \frac{\|x_n + x_m\|}{2} \\
&= 2
\end{aligned}$$

for all $n, m \in \mathbf{N}$. These estimates prove that the sequence $(y_n)$ satisfy the conditions of Lemma A.8, which means $(y_n)$ is a Cauchy sequence. Note that is implies that $(x_n)$ is also a Cauchy sequence. Since $X$ is Banach and $C$ is closed, there is a $x \in C$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Since $\|x_n\| \to d$, it follows that $\|x\| = d$. This proves existence.

To prove uniqueness, let $\varepsilon > 0$ and assume there are $x, y \in C$ such that $\|x\| = \|y\| = d$ and $\|x - y\| = \varepsilon$. Since $X$ is uniformly convex, there is a $\delta > 0$ with

$$\left\|\frac{x + y}{2}\right\| \leqslant (1 - \delta) \cdot d < d.$$

On the other hand, the fact that $C$ is convex yields

$$\frac{x + y}{2} \geqslant \inf_{z \in C} \|z\| = d.$$

This is clearly a contradiction. Hence $x = y$, which establishes uniqueness.■

A.10 **Corollary** (Nearest point projection). *Let $X$ be a uniformly convex Banach space over $\mathbf{F}$ and $C \subsetneq X$ be non-empty closed and convex. Let $y \in X \setminus C$, then there is a unique $x \in C$ such that*

$$\|y - x\| = \inf_{z \in C} \|y - z\|.$$ «

Proof. Set $D = C - y = \{z - y \colon z \in C\}$. This set is non-empty, closed and convex. By applying Theorem A.9, there is a unique $x \in D$ such that

$$\begin{aligned} \|x\| &= \inf_{z \in D} \|z\| \\ &= \inf_{v \in C} \|v - y\|. \end{aligned}$$

Take $u = x + y$, then $u \in C$ and

$$\|u - y\| = \|x\| = \inf_{v \in C} \|v - y\|.$$

This implies

$$\|y - u\| = \inf_{v \in C} \|y - v\|.$$ ∎

A.11 **Corollary.** *Let $X$ be a uniformly convex normed space over $\mathbf{F}$. If $\phi \in X^*$ is a non-zero linear functional, then there is a unique $x \in X$ with $\|x\| = 1$ such that $\phi(x) = \|x\|$.* «

The proof of this corollary requires some results from functional analysis.

A.12 **Definition.** Let $X$ be a normed linear space over $\mathbf{R}$ and $H \subsetneq X$. Then $H$ is a *hyperplane* if and only if there is a non-zero $\phi \in X^*$ and a $\lambda \in \mathbf{R}$ such that

$$H = \{x \in X \colon \phi(x) = \lambda\}.$$ «

Recall the following consequence of the Hahn-Banach theorem (see [23, p. 107-108]):

A.13 **Theorem.** *Let $X$ be a reflexive Banach space over $\mathbf{F}$ and let $\phi \in X^*$. Then there is an element $x \in X$ with $\|x\| = 1$ such that*

$$\phi(x) = \|\phi\|.$$ «

At this point there is enough material available to prove Corollary A.11.

Proof (Corollary A.11). Consider for a non-zero linear functional $\phi \in X$ the collection $H = \{x \in X \colon \phi(x) = \|\phi\|\}$. Then $0 \notin H$ and according to Definition A.12, $H$ is a hyperplane in $X$. In addition, $H$ is convex (by linearity of $\phi$) and closed since $\phi$ is continuous. Therefore, by Theorem A.9, there is a unique $x \in H$ such that

$$\|x\| = \inf_{z \in H} \|z\| > 0.$$

Since $\phi(x) = \|\phi\|$, it follows that $\|x\| \geqslant 1$.

Next, since $X$ is uniformly convex, it is reflexive due to Theorem A.7. By applying Theorem A.13, there is a $y \in X$ with $\|y\| = 1$ such that $\phi(y) = \|\phi\|$. This means $y \in H$ and $\|y\| \leqslant \|x\|$. Additionally $x = y$, since $x$ was the unique element in $H$ with minimal norm. Hence $\|x\| = 1$. ∎

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