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On the Geometric Realization of Dendroidal Sets

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Et tu ouvriras parfois ta fenêtre, comme ça, pour le plaisir. . .
Et tes amis seront bien étonnés de te voir rire en regardant le ciel.
Alors tu leur diras: "Oui, les étoiles, ça me fait toujours rire!"
Et ils te croiront fou. Je t'aurai joué un bien vilain tour. . .

A Irene, Lorenzo e Paolo

a chi ha fatto della propria vita poesia
a chi della poesia ha fatto la propria vita

Contents

Introduction

Tolto dunque un uovo, tutti qu' maestri si provarono
per farlo star ritto, ma nessuno trovò il modo.
Onde, essendo detto a Filippo ch'e' lo fermasse,
egli con grazia lo prese, e datoli un colpo
del culo in sul piano del marmo, lo fece star ritto.
G. Vasari

At the core of this work are *Multicategories* and the theory of *Dendroidal Sets*.
Operads and multicategories (of which operads are the one object version) were introduced about at the same time by May ([Ma1]) for the study of *infinite loop spaces*, and by Lambek ([La]) in order to describe *deductive systems*. Multicategories are, roughly, an extension of categories obtained by allowing arrows with any finite number of inputs, so to get an adjunction

$$j_! : Cat \rightleftarrows Multicat : j^* \tag{1}$$

between the category of (small) categories and that of (symmetric) multicategories.
Following this line, during the last few years Ieke Moerdijk and Ittay Weiss developed the theory of *Dendroidal Sets* ([MW1],[MW2]), which extends the well known theory of Simplicial Sets. More in detail, dendroidal sets are presheaves over a certain category Ω, in which the simplicial category Δ naturally embeds, leading to an adjunction

$$i_! : SSet \rightleftarrows dSet : i^* \tag{2}$$

The notion of a *dendroidal nerve* of a multicategory, analogous to that of the *nerve* of a category, finally provides the following square which links the adjunctions 1 and 2

$$\begin{tikzcd}
Cat \arrow[r, shift left, "j_!"] \arrow[d, shift left, "N"] & Multicat \arrow[l, shift left, "j^*"] \arrow[d, shift left, "N_d"] \\
SSet \arrow[r, shift left, "i_!"] \arrow[u, shift left, "\tau"] & dSet \arrow[l, shift left, "i^*"] \arrow[u, shift left, "\tau_d"]
\end{tikzcd}$$

Motivations and main contributions

It is well known that simplicial sets can be transformed into topological spaces by means of the *geometric realization* functor

$$|\cdot| : SSet \to Top \tag{3}$$

When I was first reading of dendroidal sets, the absence of a "dendroidal" analog of the above functor immediately caught my eyes, becoming the subject of this work and one of my obsessions during the last few months.

Not much later I became aware of the works of May ([Ma1]) and of Thomason ([Th2]); their results convinced me of the fact that a geometric realization is not just a question of consistency with respect to simplical sets, but really a need, a must, and a natural port for the theory developed by Moerdijk and Weiss.
In fact, multicategories (or, better, operads) are intimately related to infinite loop spaces; similarly, there is a tight link between infinite loop spaces and symmetric monoidal categories. The triangle is closed once we notice that any (symmetric) monoidal category can be naturally viewed as a (symmetric) multicategory. In practice, one should have in mind a sort of Trinity of the form

$$\begin{tikzcd} \text{Monoidal Categories} \arrow[rr, dash] \arrow[dr, dash] & & \text{Multicategories} \arrow[dl, dash] \\ & \text{Infinite loop spaces} & \end{tikzcd}$$

In order to make things clear and disappoint the reader, I have to admit that no geometric realization has been constructed yet (as far as I know, and for sure not by me). Unfortunately, the problem turned out to be much more difficult than what I originally thought.
On the other hand, the difficulties that I encountered, together with the above remarks, have been a source of ideas in order to pave the way towards a "dendroidal geometric realization", and suggested a new perspective from which consider dendroidal sets.

In section 6.1 I conjecture the existence of a category *StrTop* of *topological spaces with structure*, which should extend the category of topological spaces; I will describe some properties that such category should satisfy, in the form of diagrams relating *StrTop* to the category *Top* of topological spaces and to the category E_∞-*Spaces* of E_∞-*spaces*. Subsequently I will sketch in 6.1.1 the shape of a "dendroidal geometric realization" functor $|\cdot|_d : dSet \to StrTop$ with target this new category. Under these assumptions, it is easily seen (6.1.3) that the dendroidal classifying space of a monoidal category (i.e. the dendroidal classifying space of the multicategory underlying it) is an E_∞-space.

The main result of this work appears in section 6.3. Having in mind that simplicial sets are a good replacement for topological spaces, in this section it should become clear what I mean by a category of topological spaces with structures.
In 6.3.1 I will present dendroidal sets from a new perspective. Any dendroidal set X has an underlying simplicial set i^*X, the *simplicial part* of X; the question is to understand the role of those "gadgets" that the functor i^* forgets. It turns out that the *dendroidal part* of X (i.e. the complement of i^*X in X), provides sort of operations on the simplicial set i^*X. In particular, as explained in 6.3.2, when X is the dendroidal nerve of a monoidal category $(\mathcal{M}, \otimes)$, such operations are intimately related to the product defined by the tensor $\otimes$ on the usual (simplicial) nerve $N\mathcal{M}$.
The philosophy behind the point of view I propose, is that dendroidal sets should be thought of as *simplicial sets endowed with an algebraic structure*, and such structure is already encoded in the dendroidal set.

In other words, a dendroidal set X forms a *unicum*, it suffices to itself; it is able to sustain itself, just like Brunelleschi's egg.

Chapter 1

Category Theory

I recall here few concepts and results from category theory; this is mostly an opportunity to fix notations, nothing should be new. Throughout this thesis I will often make use of theorem 1.4.4 and of the notion of a monoidal category, defined in section 1.5; monads and their algebras, defined in 1.3, will be also of importance.

1.1 Categories, functors, natural transformations

Definition 1.1.1. A *category* $\mathcal{C}$ consists of:

(i) a collection $\mathcal{C}_0$ (or $ob(\mathcal{C})$) of *objects* denoted by $A, B, C, \ldots, X, Y, \ldots$ or $a, b, c, \ldots$

(ii) a class $\mathcal{C}_1$ of *arrows* (or *morphisms*) denoted by $f, g, h \ldots$ To each arrow f are associated a unique object $s(f)$ and a unique object $t(f)$, the *source* and *target* of f. I denote by $\mathcal{C}(A, B)$ the set of arrows having source A and target B

(iii) for each object A there is given an arrow 1_A in $\mathcal{C}(A, A)$, the *identity* of A

(iv) for all objects A, B, C a composition law $\mu : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \to \mathcal{C}(A, C)$ sending a pair (g, f) of arrows to their composite, written $g \circ f$ or simply gf.

The above data are subject to axioms:

1. Associativity: given arrows $h \in \mathcal{C}(C, D), g \in \mathcal{C}(B, C), f \in \mathcal{C}(A, B)$ one has $(h \circ g) \circ f = h \circ (g \circ f)$

2. Identity: for all arrows $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$ hold $1_B \circ f = f$ and $g \circ 1_B = g$

In the following, for a category $\mathcal{C}$, I will simply write $A \in \mathcal{C}$ to say that A is an object of the category. In the above definition is assumed that a category is *locally small*, i.e each $\mathcal{C}(X, Y)$ is a set; one says that a category $\mathcal{C}$ is *small* if both the classes $\mathcal{C}_0$ of objects and $\mathcal{C}_1$ of arrows are sets.

Example 1.1.2. The following categories will occur throughout this work:

(i) *Set*: the category of *sets*, whose objects are sets and arrows the maps of sets

(ii) *Top*: the category of *topological spaces* and continuous functions

(iii) *Top*$_*$: the category of *pointed topological spaces* and continuous base-point preserving functions

(iv) given a category $\mathcal{C}$, denote by $\mathcal{C}^{op}$ the *opposite category* of $\mathcal{C}$. Then $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$ has, but $\mathcal{C}^{op}(X,Y) = \mathcal{C}(Y,X)$ for any pair of objects X, Y

For a category $\mathcal{C}$, one says that an arrow $f : X \to Y$ is a *monomorphism* (*mono*) if whenever $fg = fh$ for arrows $g, h : Z \to X$, one has $g = h$. The dual notion is that of *epimorphism* (*epi*), while an *isomorphism* (*iso*) is an arrow $X \xrightarrow{f} Y$ having both right and left inverse.

Definition 1.1.3. Given categories $\mathcal{C}$ and $\mathcal{D}$ a (*covariant*) *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of

- a map $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ taking an object X to $F_0(X)$ or just F_0X.
- for each pair of objects X and Y of $\mathcal{C}$, a map $F_1 : \mathcal{C}(X,Y) \to \mathcal{D}(F_0(X), F_0(Y))$, taking an arrow f to $F_0(f)$ or simply F_0f.

These data respect the category structure, in the sense that

1. for every pair of composable arrows $g \in \mathcal{C}(Y,Z), f \in \mathcal{C}(X,Y)$, holds the equality
$$F_1(g \circ f) = F_1(g) \circ F_1(f)$$
2. for every object X of $\mathcal{C}$, $F_1(1_X) = 1_{F_0X}$

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$.

I will suppress indexes and write simply F both for F_0 and F_1.

Definition 1.1.4. Given functors $F, G : \mathcal{C} \to \mathcal{D}$ a *natural transformation* $\alpha : F \Rightarrow G$ is a collection of arrows $(\alpha_X)_{X \in \mathcal{C}_0}$, $\alpha_X : FX \to GX$ of $\mathcal{D}$, such that that, for every arrow $f : X \to Y$ in $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[d, "\alpha_Y"] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}$$

For natural transformations $\alpha : F \Rightarrow G, \beta : G \Rightarrow H$, there is an obvious way of composing them, the *vertical composition*, given at the component X by $(\alpha \circ \beta)_X = \alpha_X \circ \beta_X$.

Example 1.1.5. With the last two definitions in mind we obtain

(i) *Cat*: the category of *small categories* and functors.

(ii) For a small category $\mathcal{J}$ and a category $\mathcal{D}$, denote by $\mathcal{D}^{\mathcal{J}}$, the *functor category* of covariant functors $F : \mathcal{J} \to \mathcal{D}$ and natural transformations.

Recall that

Definition 1.1.6. A functor $F : \mathcal{C} \to \mathcal{D}$ is said

(i) *full* if $\forall X, Y \in \mathcal{C}$ $F : \mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$ is surjective

(ii) *faithful* if $\forall X, Y \in \mathcal{C}$ $F : \mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$ is injective

(iii) *essentially surjective* if every object D in $\mathcal{D}$ is isomorphic to an object FC for C in $\mathcal{C}$

(iv) an *isomorphism* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that $FG = 1_{\mathcal{D}}$ and $GF = 1_{\mathcal{C}}$; then the categories are *isomorphic*, in symbols $\mathcal{C} \cong \mathcal{D}$

(v) an *equivalence* of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\varepsilon : FG \to 1_{\mathcal{D}}$ and $\eta : 1_{\mathcal{C}} \to GF$; then the categories are said to be *equivalent*, $\mathcal{C} \simeq \mathcal{D}$

It is well known that (v) is equivalent to F being *full, faithful* and *essentially surjective.*

A notable kind of functor categories appears when $\mathcal{J}$ is the opposite category $\mathcal{C}^{op}$ of a small category $\mathcal{C}$ and $\mathcal{D} = Set$. In this case the functor category $Set^{\mathcal{C}^{op}}$ of contravariant functors $F : \mathcal{C} \to Set$ is often denoted by $\hat{\mathcal{C}}$ and called the *presheaf category* of $\mathcal{C}$. In particular, any small category $\mathcal{C}$ can be embedded into its presheaf category by means of the *Yoneda embedding*

$$\mathcal{C}[\cdot] : \mathcal{C} \to Set^{\mathcal{C}^{op}}$$

which takes an object X in $\mathcal{C}$ to the *representable* functor $\mathcal{C}[X] = \mathcal{C}(\cdot, X)$.
Functor categories, presheaf categories and representable functors have remarkable properties as we shall see in the next section, and will be central in this work.
The fact that the Yoneda embedding is actually an *embedding* (i.e. full, faithful and injective on objects), follows from

Yoneda Lemma 1.1.7. *For every object F in $Set^{\mathcal{C}^{op}}$ and every object X in $\mathcal{C}$, there is a bijection*

$$y : Set^{\mathcal{C}^{op}}(\mathcal{C}(\cdot, X), F) \cong F(X)$$

sending a natural tranformation $\alpha \in Set^{\mathcal{C}^{op}}(\mathcal{C}(\cdot, X), F)$ to the element $\alpha_X(1_X)$ of FX.

Proof. Let $\alpha : \mathcal{C}(\cdot, X) \Rightarrow F$ a natural transformation. Then α is uniquely determined by the element $a = \alpha_X(1_X)$; in fact, for any element $f \in \mathcal{C}(Y, X)$, $f = 1_X \circ f = \mathcal{C}(f, X)(1_X)$ and by naturality of α we have that $\alpha_Y(f) = \alpha_Y \circ \mathcal{C}(f, X)(1_X) = Ff \circ \alpha_X(1_X) = Ff(a)$ as the following diagram shows

$$\begin{tikzcd}
\mathcal{C}(X,X) \arrow[r, "\mathcal{C}(f,X)"] \arrow[d, "\alpha_X"'] & \mathcal{C}(Y,X) \arrow[d, "\alpha_Y"] \\
FX \arrow[r, "Ff"'] & FY
\end{tikzcd}$$

Then, the isomorphism y is specified by sending α to the element $\alpha_X(1_X) \in FX$, while y^{-1} takes an element $a \in FX$ to the unique natural transformation determined by the datum $\alpha_X(1_X) = a$. $\square$

Corollary 1.1.8. *The functor $\mathcal{C}[\cdot] : \mathcal{C} \to Set^{\mathcal{C}^{op}}$, $\mathcal{C}[X] = \mathcal{C}(\cdot, X)$ is full, faithful and injective on objects.*

Proof. By the Yoneda's Lemma we have

$$Set^{\mathcal{C}^{op}}(\mathcal{C}(\cdot, X), \mathcal{C}(\cdot, Y)) \cong \mathcal{C}(X, Y)$$

Injectivity on objects follows from 1.1.1,(ii). $\square$

1.2 Adjoint functors, limits, colimits

In the previous section I defined a functor $F : \mathcal{C} \to \mathcal{D}$ to be an equivalence of categories if it has a *pseudo inverse* $G : \mathcal{D} \to \mathcal{C}$. This is part of a more general situation, where the pair of functors (F, G) are said to be *adjoint*, in the following sense.

Definition 1.2.1 (Adjoint functors)**.** Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. Then F is said *left adjoint* to G, $F \dashv G$, and G *right adjoint* to F if one of the following equivalent conditions holds

(i) for every $X \in \mathcal{C}$, $Y \in \mathcal{D}$, there exist an isomorphism $\phi : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$, natural in both variables X, Y in the sense that the square

$$\begin{tikzcd}
\mathcal{D}(FX,Y) \arrow[r, "\phi"] \arrow[d, "\mathcal{D}(Ff,g)"'] & \mathcal{C}(X,GY) \arrow[d, "\mathcal{C}(f,Gg)"] \\
\mathcal{D}(FX',Y') \arrow[r, "\phi"'] & \mathcal{C}(X',GY')
\end{tikzcd}$$

is commutative for every $X, X' \in \mathcal{C}$, $Y, Y' \in \mathcal{D}$ and arrows $X' \xrightarrow{f} X \in \mathcal{C}$, $Y \xrightarrow{g} Y' \in \mathcal{D}$

(ii) there are natural tranformations $\varepsilon : FG \Rightarrow 1$ and $\eta : 1 \Rightarrow GF$ such that the following diagrams commute

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "\eta_G"] \arrow[dr, Rightarrow, "1_G"'] & GFG \arrow[d, Rightarrow, "G\varepsilon"] \\
& G
\end{tikzcd}
\qquad
\begin{tikzcd}
F \arrow[r, Rightarrow, "F\eta"] \arrow[dr, Rightarrow, "1_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon_F"] \\
& F
\end{tikzcd}$$

In the above cases one says that F and G are part of an adjunction, denoted by (F, G, ϕ), $(F, G, \varepsilon, \eta)$ or $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$. The natural transformations ε and η are called the *counit* and *unit* of the adjunction.
Maps $f : FX \to Y$ and $g : X \to GY$ corresponding under ϕ are said *transposes*.
The isomorphisms $\phi : \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$ are completely determined by the maps $\eta_X = \phi(1_{FX}) : X \to GFX$, which in fact define the unit of the adjunction. Using an argument similar to that of the Yoneda's Lemma, we find that for an arrow $f : FX \to Y$, $\phi(f)$ is given by the composite

$$X \xrightarrow{\eta_X} GFX \xrightarrow{Gf} GY$$

Conversely, the counit ε is given by the maps $\phi^{-1}(1_{GY}) : FGY \to Y$, and for $g : X \to GY$, $\phi^{-1}(g)$ is

$$FX \xrightarrow{Fg} FGY \xrightarrow{\varepsilon_Y} Y$$

I will briefly recall the definition of *limits* and *colimits*; then, after listing some of them, I will recall some important properties of adjoint pairs of functors.
Let $\mathcal{C}$ and $\mathcal{D}$ categories. For an object X of $\mathcal{D}$, denote by ΔX the constant functor at X, sending every object of $\mathcal{C}$ to X and every arrow to de identity arrow on X. Then, the assignment $X \mapsto \Delta X$ defines a functor $\Delta : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$.

Definition 1.2.2. Let $F : \mathcal{C} \to \mathcal{D}$ a functor.
If it exists, the limit of F is an object $\lim F$ of $\mathcal{D}$ such that

$$\mathcal{D}^{\mathcal{C}}(\Delta X, F) \cong \mathcal{D}(X, \lim F)$$

and the above isomorphism is natural in the variable X.

Dually, we have the notion of colimit.

Definition 1.2.3. Let $F : \mathcal{C} \to \mathcal{D}$ a functor.
If it exists, the colimit of F is an object $\operatorname{colim} F$ of $\mathcal{D}$ such that

$$\mathcal{D}(\operatorname{colim} F, X) \cong \mathcal{D}^{\mathcal{C}}(F, \Delta X)$$

and the above isomorphism is natural in the variable X.

One says that

Definition 1.2.4. A category $\mathcal{D}$ has all (co)limits of type $\mathcal{C}$ if every functor $F : \mathcal{C} \to \mathcal{D}$ admits a (co)limit.

In the above situation, the assignments $F \mapsto \lim F$ and $F \mapsto \operatorname{colim} F$ give rise to functors $\mathcal{D}^{\mathcal{C}} \to \mathcal{C}$, so that the previous definition can be rephrased as

Definition 1.2.5. A category $\mathcal{D}$ has all (co)limits of type $\mathcal{C}$ if the functor $\Delta : \mathcal{C} \to \mathcal{D}^{\mathcal{C}}$ has a (left) right adjoint.

Finally, we say that

Definition 1.2.6. A category $\mathcal{D}$ is *complete* if the functor $\Delta : \mathcal{C} \to \mathcal{D}^{\mathcal{C}}$ has a right adjoint, for every small category $\mathcal{C}$. A category $\mathcal{D}$ is *cocomplete* if the functor $\Delta : \mathcal{C} \to \mathcal{D}^{\mathcal{C}}$ has a left adjoint, for every small category $\mathcal{C}$. A category $\mathcal{D}$ is *bicomplete* if it is both complete and cocomplete.

Example 1.2.7. Many classical limits will occur throughout this work, such as *products* and *pullbacks*.
For objects X, Y in a category $\mathcal{C}$, the *product* $X \times Y$ is the limit of the functor F from the category $\mathbf{2}$ consisting of only two objects $0, 1$ and identity arrows

$$\begin{tikzcd} 0 \arrow[loop left, "id"] & 1 \arrow[loop right, "id"] \end{tikzcd}$$

assigning X to 0 and Y to 1.
Then, the natural isomorphism in 1.2.2 gives us the projections π_X and π_Y and the universal property

$$\begin{tikzcd} & & Y \\ Z \arrow[r, dashed] \arrow[urr, bend left] \arrow[drr, bend right] & X \times Z \arrow[ur, "\pi_Y"] \arrow[dr, "\pi_X"'] & \\ & & X \end{tikzcd}$$

An *equalizer* for maps $f, g : X \to Y$ in a category $\mathcal{C}$ is a limit K for the functor F taking the category

$$\begin{tikzcd} 0 \arrow[loop left, "id"] \arrow[r, shift left] \arrow[r, shift right] & 1 \arrow[loop right, "id"] \end{tikzcd}$$

to the diagram $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, so to get the usual universal property depicted as

$$\begin{tikzcd} K \arrow[r] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \\ Z \arrow[u, dashed] \arrow[ur] & & \end{tikzcd}$$

Similarly, a *pullback* (or *fiber product*) is a limit for a functor F on the category

$$\begin{tikzcd} & 1 \arrow[loop right, "id"] \arrow[d, "p"] \\ 2 \arrow[loop left, "id"] \arrow[r, "q"'] & 0 \arrow[loop right, "id"] \end{tikzcd}$$

and one obtains the well known diagram

$$\begin{tikzcd} W \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ & Y \arrow[r, "q"'] & Z \end{tikzcd}$$

By reversing arrows in the above diagrams one has then the dual notions of *coproduct*, *coequalizer* and *pushout*.
Recall the known criterion for a category to be (co)complete:

Proposition 1.2.8. *A category $\mathcal{C}$ is (co)complete if, and only if, it has (co)products and (co)equalizers.*

A remarkable property of adjoint functors is that they preserve limits, as is made precise by the following

Proposition 1.2.9. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ a pair of adjoint functors. Then F preserves colimits existing in $\mathcal{C}$, while G preserves limits existing in $\mathcal{D}$.*

Proof. Let $H : \mathcal{E} \to \mathcal{C}$ a functor and assume that $\operatorname{colim} H$ exists in $\mathcal{C}$; we want to prove that $\operatorname{colim} FH$ exits and can be chosen to be $F(\operatorname{colim} H)$.
We have

$$\begin{aligned} \mathcal{D}^{\mathcal{E}}(FH, \Delta X) \cong & \mathcal{C}^{\mathcal{E}}(H, G\Delta X) = \mathcal{C}^{\mathcal{E}}(H, \Delta GX) \cong \\ \cong & \mathcal{C}(\operatorname{colim} H, GX) \cong \mathcal{D}(F \operatorname{colim} H, X) \end{aligned}$$

where the isomorphism $\mathcal{D}^{\mathcal{E}}(FH, \Delta X) \cong \mathcal{C}^{\mathcal{E}}(H, G\Delta X)$ is induced by the isomorphism ϕ : $\mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$ of 1.2.1 and makes sense by the naturality assumption. This proves the first part of the proposition; the fact that G preserves limits is dual. $\square$

Very important examples of adjunction are provided by the product in *Set* and *Cat*. It is well known that there is an adjunction between the product functor $\cdot \times B$ and the *exponential* $(\cdot)^B$ for any set $B \in Set$

$$Set(A \times B, C) \cong Set(A, C^B)$$

obtained by sending a function $f : A \times B \to C$ to the map $\tilde{f} : A \to C^B$ defined by

$$a \mapsto (b \mapsto f(a, b))$$

Recall that the product $\mathcal{C} \times \mathcal{D}$ of two categories is the category with set of objects $\mathcal{C}_0 \times \mathcal{D}_0$ and arrows the pairs (f, g) for $f \in \mathcal{C}$ and $g \in \mathcal{D}$ and the obvious composition $(f, f')(g, g') = (fg, f'g')$. Noting that a set can be viewed as a small category with only identity arrows, the above adjunction becomes just a particular case of the following

Example 1.2.10. For any small category $\mathcal{D}$ the functor $\cdot \times \mathcal{D} : Cat \to Cat$ has as right adjoint the functor $(\cdot)^{\mathcal{D}}$ assigning to a small category $\mathcal{E}$ the functor category $\mathcal{E}^{\mathcal{D}}$.
One easily constructs the isomorphism

$$Cat(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong Cat(\mathcal{C}, \mathcal{E}^{\mathcal{D}})$$

Let $H : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ a functor. Define the functor $\hat{H} : \mathcal{C} \to \mathcal{E}^{\mathcal{D}}$ sending an object C to the functor H_C

$$D \xmapsto{H_C} H(C, D)\ , \qquad (f : D \to D') \xmapsto{H_C} H(1_C, f)$$

while an arrow $f : C \to C'$ gives the natural transformation $\phi = \hat{H}(f)$

$$\phi_D = H(f, 1_D) : H(C, D) \to H(C', D)$$

On the other hand, given a functor $G : \mathcal{C} \to \mathcal{E}^{\mathcal{D}}$, one constructs the functor $\tilde{G} : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ by sending an object (C, D) to $\tilde{G}(C, D) = G(C)(D)$, while for an arrow $(f, g) : (C, D) \to (C', D')$, $\tilde{G}(f, g)$ is just the diagonal in the naturality square

$$\begin{tikzcd}
G(C)(D) \arrow[r, "G(C)(g)"] \arrow[d, "G(f)_D"'] \arrow[dr, dashed, "\tilde{G}(f,g)"] & G(C)(D') \arrow[d, "G(f)_{D'}"] \\
G(C')(D) \arrow[r, "G(C')(g)"'] & G(C')(D')
\end{tikzcd}$$

The above constructions are then obviously inverses one to the other, so that we have the adjunction.

With the language of section 1.5 this makes *Cat* and *Set* into *closed symmetric monoidal categories.*

1.3 Monads

In this section I will recall a couple of definitions and constructions concerning monads and their algebras. We will need the concepts explained below only in chapter 5 but, due to the purely categorical flavour, I find this the most appropriate place to introduce them.

Definition 1.3.1. Let $\mathcal{C}$ a category. A *monad* on $\mathcal{C}$ consists of a functor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow T$, $\mu : T^2 \Rightarrow T$ such that the following diagrams commute

$$\begin{tikzcd}
T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\
T^2 \arrow[r, Rightarrow, "\mu"'] & T
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, Rightarrow, "T\eta"] \arrow[dr, Rightarrow, "1_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "\eta_T"'] \arrow[dl, Rightarrow, "1_T"] \\
& T &
\end{tikzcd}$$

As a first example, I recall that any pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ gives rise to a monad GF on $\mathcal{C}$ where η is the unit of the adjunction and $\mu = G\varepsilon_F$. The requirements on μ and η are in fact fulfilled since

$$\begin{tikzcd}
GFGFGF \arrow[r, Rightarrow, "GFG\varepsilon_F"] \arrow[d, Rightarrow, "G\varepsilon_{FGF}"'] & GFGF \arrow[d, Rightarrow, "G\varepsilon_F"] \\
GFGF \arrow[r, Rightarrow, "G\varepsilon_F"'] & GF
\end{tikzcd}$$

simply express naturality of $G\varepsilon : GFG \Rightarrow G$ with respect to $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$. Similarly the diagrams

$$\begin{tikzcd}
GF \arrow[r, Rightarrow, "GF\eta"] \arrow[dr, Rightarrow, "1_{GF}"'] & GFGF \arrow[d, Rightarrow, "G\varepsilon_F"] & GF \arrow[l, Rightarrow, "\eta_{GF}"'] \arrow[dl, Rightarrow, "1_{GF}"] \\
& GF &
\end{tikzcd}$$

commute by definition of unit and counit of the adjunction in 1.2.1.

One can obviously make monads into a category (or, even better, into a strict 2-category).

Definition 1.3.2. Let $(T, \eta, \mu), (T', \eta', \mu')$ be monads on $\mathcal{C}$ and $\mathcal{C}'$. A *lax map of monads* between them is a pair (Q, ϕ) where $Q : \mathcal{C} \to \mathcal{C}'$ is a functor and $\phi : T'Q \Rightarrow QT$ a natural transformation, such that the following diagrams commute

$$\begin{tikzcd}
T'^2Q \arrow[r, Rightarrow, "T'\phi"] \arrow[d, Rightarrow, "\mu_Q"'] & T'QT \arrow[r, Rightarrow, "\phi_T"] & QT^2 \arrow[d, Rightarrow, "Q\mu"] \\
T'Q \arrow[rr, Rightarrow, "\phi"'] & & QT
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, equal] \arrow[d, Rightarrow, "\eta'_Q"'] & Q \arrow[d, Rightarrow, "Q\eta"] \\
T'Q \arrow[r, Rightarrow, "\phi"'] & QT
\end{tikzcd}$$

One can also define *colax* and *weak* maps of monads, by inverting in the above definition the direction of ϕ or choosing it to be an isomorphism.

The idea behind a monad is that of encoding a certain algebraic structure in the data of a functor, a unit element and a multiplication. Given such a "skeleton" for an algebraic theory in the shape of a monad, models are then given by the algebras for the monad, as made precise by the following

Definition 1.3.3. Let (T, η, μ) a monad on a category $\mathcal{C}$.
An *algebra for T* is a pair (X, ξ) with $X \in \mathcal{C}$ and $\xi : TX \to X$ an arrow in $\mathcal{C}$ such that the

following diagrams commute

$$
\begin{tikzcd}
T^2X \arrow[r, "T\xi"] \arrow[d, "\mu_X"'] & TX \arrow[d, "\xi"] \\
TX \arrow[r, "\xi"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, equal] & TX \arrow[d, "\xi"] \\
& X
\end{tikzcd}
$$

Given algebras (X, ξ) and (X', ξ') for the monad T, an arrow between them is an arrow $X \xrightarrow{f} X'$ in $\mathcal{C}$ making the following square commute

$$
\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "\xi"'] & TX' \arrow[d, "\xi'"] \\
X \arrow[r, "f"'] & X'
\end{tikzcd}
$$

Denote by T-*Alg* the category of algebras for the monad T and their morphisms as just defined.

It is easily seen that there is an adjunction

$$F : \mathcal{C} \rightleftarrows T\text{-}Alg : U$$

where U sends an algebra (X, ξ) to X, while F sends an object X of $\mathcal{C}$ to the *free algebra* (TX, μ_X).

I end the section with a standard example, which will make clear the above concepts and will be useful later on.

Example 1.3.4. (Free monoid monad)
Consider on *Set* the following functor

$$X \mapsto TX = \coprod_{n=0}^{\infty} X^n$$

TX is the collection of *words* on the set X. One can make T into a monad with unit η given by

$$\eta_X : x \mapsto \langle x \rangle$$

sending an element of X to the word $\langle x \rangle$ of lenght 1 consisting of the letter x. The multiplication

$$T^2X = \coprod_{k=0}^{\infty} (\coprod_{m=0}^{\infty} X^m)^k \xrightarrow{\mu_X} TX = \coprod_{n=0}^{\infty} X^n$$

amounts to "eliminating" parenthesis, so that an element $\langle\langle x_1^1, \ldots, x_{n_1}^1\rangle, \ldots, \langle x_1^k, \ldots, x_{n_k}^k\rangle\rangle$ of T^2X is sent to $\langle x_1^1, \ldots, x_{n_1}^1, \ldots, x_1^k, \ldots, x_{n_k}^k\rangle$. An algebra (X, ξ) for the monad T means the choice of a set X and a *product* ξ on X. The multiplication μ of the monad ensures associativity of the product ξ, while the image $e = \xi(\langle\rangle)$ of the unique element $\langle\rangle$(the empty word) in X^0 is the unit for the product, in fact:

$$\xi(\langle x, e\rangle) = \xi(\langle \xi(\langle x\rangle), \xi(\langle\rangle)\rangle) = \xi T\xi(\langle\langle x\rangle, \langle\rangle\rangle) = \xi\mu(\langle\langle x\rangle, \langle\rangle\rangle) = \xi(\langle x\rangle) = x$$

where we used the axioms for the algebras

$$\xi \circ T\xi = \xi \circ \mu_X \text{ and } \xi \circ \eta_X = 1_X$$

It can be shown then that monoids are exactly the algebras for the monad T.

1.4 More on categories of functors

I already mentioned categories of functors at the end of the first section. I will now concentrate more on them and give some fundamental results, some of which will be essential tools in the following chapters. Recall that, for categories $\mathcal{C}$ and $\mathcal{D}$, with $\mathcal{C}$ small, $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are the covariant functors $F : \mathcal{C} \to \mathcal{D}$ and morphisms are the natural transformations with the vertical composition.
As usual, the first thing one cares of is the existence of limits. A sufficient criterion is given by the following.

Proposition 1.4.1. *Let $\mathcal{C}$ and $\mathcal{D}$ categories, and assume that $\mathcal{D}$ has (co)limits of type $\mathcal{J}$ for a category $\mathcal{J}$. Then so does $\mathcal{D}^{\mathcal{C}}$.*

Proof. The result is easily proved by means of universal diagrams.
Let $H : \mathcal{J} \to \mathcal{D}^{\mathcal{C}}$. Then H amounts to functors $H_j : \mathcal{C} \to \mathcal{D}$ for $j \in \mathcal{J}$ and natural transformations $\gamma : H_j \Rightarrow H_{j'}$ for $j \xrightarrow{g} j'$ in $\mathcal{J}$.
Define for C in $\mathcal{C}$ $H_C : \mathcal{J} \to \mathcal{D}$ taking j to $H_j(C)$.
By the assumption on $\mathcal{D}$ it is possible to define a functor $\overline{H}$ in $\mathcal{D}^{\mathcal{C}}$ as $\overline{H}(C) = \lim H_C$. I claim that $\overline{H}$ is the limit of H.
Let F in $\mathcal{D}^{\mathcal{C}}$ be a functor together with natural transformations $\phi_j : F \Rightarrow H_j$ for every $j \in \mathcal{J}$ and compatible with the transformations $\gamma = H(g)$, in the sense that

$$\begin{tikzcd}
 & H_j \arrow[dd, Rightarrow, "\gamma=H(g)"] \\
F \arrow[ur, Rightarrow, "\phi_j"] \arrow[dr, Rightarrow, "\phi_{j'}"'] & \\
 & H_{j'}
\end{tikzcd}$$

commutes for every j, j' and $j \xrightarrow{g} j'$ in $\mathcal{J}$.
Then at each component C the arrows $\phi_{j,C}$ factor as

$$\begin{tikzcd}
 & & H_j(C) \arrow[dd, "\gamma_C"] \\
F(C) \arrow[r, "p_C"] \arrow[urr, bend left, "\phi_{j,C}"] \arrow[drr, bend right, "\phi_{j',C}"'] & \overline{H}(C) \arrow[ur, "\pi_{j,C}"] \arrow[dr, "\pi_{j',C}"'] & \\
 & & H_{j'}(C)
\end{tikzcd}$$

so to get the desired natural transformation p depicted below

$$\begin{tikzcd} & & H_j \arrow[dd, Rightarrow, "\gamma"] \\ F \arrow[r, Rightarrow, "p"] \arrow[urr, Rightarrow, bend left, "\phi_j"] \arrow[drr, Rightarrow, bend right, "\phi_{j'}"'] & \overline{H} \arrow[ur, Rightarrow, "\pi_j"] \arrow[dr, Rightarrow, "\pi_{j'}"] & \\ & & H_{j'} \end{tikzcd}$$

The proof for colimits is dual. □

Corollary 1.4.2. *Any presheaf category $Set^{\mathcal{C}^{op}}$ is both complete and cocomplete.*

I shall now specialize to presheaf categories, and prove two important results, to which I will refer quite often in the following pages. They provide the tools for *realization* and a way to describe presheaves as colimits over a suitable category. The proofs follow those in [MM]. Recall that for a category $\mathcal{C}$, I denote by $\mathcal{C}[C]$ the *representable presheaf* $\mathcal{C}(\cdot, C)$. Let me also define the category mentioned above.

Definition 1.4.3. Let $P : \mathcal{C}^{op} \to Set$ be a functor. The *category of elements of P* is the category $\int_{\mathcal{C}} P$, whose objects are the pairs (C, p) for $c \in \mathcal{C}$ and $p \in P(C)$ and an arrow $(C, p) \xrightarrow{f} (C', p')$ is an arrow $C \xrightarrow{f} C'$ of $\mathcal{C}$ such that $Pf(p') = p$.

There is then a "projection" functor

$$\int_{\mathcal{C}} P \xrightarrow{\pi_P} \mathcal{C} \qquad (C, p) \mapsto C$$

Theorem 1.4.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a small category $\mathcal{C}$ to a cocomplete category $\mathcal{D}$.*
Then the functor $R : \mathcal{D} \to Set^{\mathcal{C}^{op}}$ defined by

$$R(D)(C) = \mathcal{D}(F(C), D)$$

has a left adjoint $L : Set^{\mathcal{C}^{op}} \to \mathcal{D}$ sending a presheaf P to the colimit

$$L(P) = \mathrm{colim}(\int_{\mathcal{C}} P \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{F} \mathcal{D})$$

Proof. We have to prove the isomorphism

$$\mathcal{D}(L(P), D) \cong Set^{\mathcal{C}^{op}}(P, R(D))$$

Let $\phi : P \Rightarrow R(D)$ a natural tranformation. This means having arrows $(\phi_C)_{C \in \mathcal{C}}$ satisfying naturality squares as below for every $C \xrightarrow{f} C'$

$$\begin{tikzcd} P(C') \arrow[r, "\phi_{C'}"] \arrow[d, "Pf"'] & \mathcal{D}(F(C'), D) \arrow[d, "(\cdot)\circ Ff"] \\ P(C) \arrow[r, "\phi_C"'] & \mathcal{D}(F(C), D) \end{tikzcd}$$

Now note that each ϕ_C gives arrows $\phi_C(p) : F(C) \to D$ varying $p \in P(C)$, so we can consider ϕ as a family of arrows $\phi_C(p)$ indexed over the elements (C, p) of $\int_{\mathcal{C}} P$. Finally the above square gives commutative triangles

$$
\begin{tikzcd}
F\pi_P(C, Pf(p')) = & FC \arrow[d, "Ff"'] \arrow[dr, "\phi_C(Pf(p'))"] & \\
& & D \\
F\pi_P(C', p') = & FC' \arrow[ur, "\phi_{C'}(p')"'] &
\end{tikzcd}
$$

and by universality of the colimit there is a unique arrow $\hat{\phi} : L(P) \to D$, giving the adjunction isomorphism. $\square$

Remark 1.4.5. Notice that in the previous theorem we have for $P = \mathcal{C}[X]$ that $L(P) \cong FX$. Suppose in fact that we are given a category $\mathcal{T}$ with a terminal object 1 and a functor $F : \mathcal{T} \to \mathcal{C}$; then $\operatorname{colim} F$ exists and can be chosen to be $F1$.
In fact for all $C \in \mathcal{T}$ we have a map $FC \xrightarrow{F!} F1$, where ! is the unique map in $\mathcal{T}$ from C to the terminal object 1; in particular $F1 \to F1$ is the identity. Now suppose Z is a colimit of F and let ϵ_1 the map $F1 \to Z$; then the usual diagram expressing Z as a colimit of F has the following form

$$
\begin{tikzcd}
FC \arrow[dd, "Ff"'] \arrow[dr, "F!"] \arrow[drrr, bend left, "\epsilon_C"] & & & \\
& F1 \arrow[r, "1"] & F1 \arrow[r, "\epsilon_1"] & Z \\
FC' \arrow[ur, "F!"'] \arrow[urrr, bend right, "\epsilon_{C'}"'] & & &
\end{tikzcd}
$$

This shows that $F1$ is a limit for F, and ϵ_1 is iso.
In the particular case when $P = \mathcal{C}[X]$ the category of elements of P, $\int_{\mathcal{C}} P$, has objects the pairs (A, p) with $A \in \mathcal{C}$ and $p \in P(A) = \mathcal{C}(A, X)$. An arrow $(A, p) \to (B, q)$ in $\int_{\mathcal{C}} P$ is then an arrow $A \xrightarrow{r} B$ of $\mathcal{C}$ making the following triangle commute

$$
\begin{tikzcd}
A \arrow[rr, "r"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\
& X &
\end{tikzcd}
$$

so that $\int_{\mathcal{C}} P$ is nothing but the *slice category* $\mathcal{C}/X$ of objects over X, with terminal object $(X, 1_X)$. The argument above proves then the claim.

Corollary 1.4.6. *Every presheaf is a colimit of representable presheaves.*

Proof. In the previous theorem let $\mathcal{D}$ be the presheaf category $Set^{\mathcal{C}^{op}}$ and $F = \mathcal{C}[\cdot]$ the Yoneda embedding. Then, letting $D = P$ a presheaf we have by the Yoneda Lemma 1.1.7

$$R(P)(C) = Set^{\mathcal{C}^{op}}(\mathcal{C}[C], P) \cong P(C)$$

so that R is isomorphic to the identity. By the uniqueness(up to iso) of adjoints we have that also L is isomorphic to the identity and

$$P \cong L(P) = \operatorname{colim}(\int_{\mathcal{C}} P \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{F} \mathcal{D}) = \operatorname*{colim}_{(C,p)\in\int_{\mathcal{C}} P} \mathcal{C}[C]$$

□

1.5 Monoidal Categories

We close the chapter with a glance at *monoidal categories.* A monoidal category is, roughly, a category with a notion of product.
Most of the categories we are used to are monoidal (e.g. *Set* and *Top* with the cartesian product, *Ab* with the tensor product), and most category of interest are *closed* monoidal or in some sense "equivalent" to a *closed monoidal* one.
The cases I just mentioned (and many others) motivate the rest of this section and the following definitions.

Definition 1.5.1. A *monoidal category* is a category $\mathcal{M} = (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ equipped with

(i) a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}, (X, Y) \mapsto X \otimes Y$, the *tensor*

(ii) a distinguished object I of $\mathcal{M}$, the *unit* of the tensor

(iii) natural isomorphisms

$$\alpha = (\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z))_{X,Y,Z\in\mathcal{M}}$$
$$\lambda = (\lambda_X : I \otimes X \to X)_{X\in\mathcal{M}}$$
$$\rho = (\rho_X : X \otimes I \to X)_{X\in\mathcal{M}}$$

The above isomorphisms are such that the following diagrams commute

$$\begin{tikzcd}
W \otimes (X \otimes (Y \otimes Z)) \arrow[r, "\alpha"] \arrow[d, "1\otimes\alpha"'] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "\alpha"] & ((W \otimes X) \otimes Y) \otimes Z \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\alpha"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[u, "\alpha\otimes 1"']
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (I \otimes Y) \arrow[rr, "\alpha"] \arrow[dr, "1\otimes\lambda"'] & & (X \otimes I) \otimes Y \arrow[dl, "\rho\otimes 1"] \\
& X \otimes Y &
\end{tikzcd}$$

for every W, X, Y, Z in $\mathcal{M}$. Also, one requires that

$$\lambda_I = \rho_I : I \otimes I \to I$$

Sometimes monoidal categories, as just defined, are referred to as *weak monoidal categories*, as opposed to the notion of a *strict monoidal category*, that is a weak monoidal category in which the natural isomorphisms α, λ and ρ are identities.
Such distinction is not so necessary, since every weak monoidal category is equivalent to a

strict one as proved in [JS].
There are also notions of *lax, colax* monoidal categories, but this doesn't need to bother us here; more about them can be found in [Le].
Before passing to *symmetric* and *closed* monoidal categories, which are for us of more interest, let me recall the notions of *monoidal functors* and *transformations* between them.

Definition 1.5.2. Let $\mathcal{M} = (\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ and $\mathcal{M}' = (\mathcal{M}', \otimes, I, \alpha, \lambda, \rho)$ monoidal categories.
A *lax monoidal functor* $F = (F, \phi) : \mathcal{M} \to \mathcal{M}'$ consists of a functor $F : \mathcal{M} \to \mathcal{M}'$ and arrows

$$\phi_{X,Y} : FX \otimes FY \to F(X \otimes Y) \qquad \phi : I \to FI$$

the first of which is natural in X, Y, and such that for every $X, Y, Z \in \mathcal{M}$ the following diagrams commute

$$\begin{tikzcd}
FX \otimes (FY \otimes FZ) \arrow[r, "1\otimes\phi_{X,Y}"] \arrow[d, "\alpha_{FX,FY,FZ}"'] & FX \otimes F(Y \otimes Z) \arrow[r, "\phi_{X,Y\otimes Z}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F\alpha_{X,Y,Z}"] \\
(FX \otimes FY) \otimes FZ \arrow[r, "\phi_{X,Y}\otimes 1"'] & F(X \otimes Y) \otimes FZ \arrow[r, "\phi_{X\otimes Y,Z}"'] & F((X \otimes Y) \otimes Z)
\end{tikzcd}$$

$$\begin{tikzcd}
FX \otimes I \arrow[r, "1\otimes\phi"] \arrow[dr, "\rho_{FX}"'] & FX \otimes FI \arrow[r, "\phi_{X,I}"] & F(X \otimes I) \arrow[dl, "F\rho_X"] \\
& FX &
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes FX \arrow[r, "\phi\otimes 1"] \arrow[dr, "\lambda_{FX}"'] & FI \otimes FX \arrow[r, "\phi_{I,X}"] & F(I \otimes X) \arrow[dl, "F\lambda_X"] \\
& FX &
\end{tikzcd}$$

The obvious notions of *colax, weak, strict* functors are given by inverting the direction of the arrows or requiring them to be isos or identities.

With the notion of *(strict) monoidal categories* and *monoidal functors* we can define the categories

$MonCat_{str}$: monoidal categories and strict functors

$MonCat_{wk}$: monoidal categories and weak functors

$MonCat_{lax}$: monoidal categories and lax functors

$MonCat_{colax}$: monoidal categories and colax functors

and similarly for the strict monoidal categories, so to get categories $StrMonCat_{str}$, $StrMonCat_{wk}$, $StrMonCat_{lax}$, $StrMonCat_{colax}$.
Of course, one can define natural transformations of monoidal functors.

Definition 1.5.3. Let $(F, \phi), (G, \psi) : \mathcal{M} \to \mathcal{M}'$ be lax monoidal functors. A *monoidal transformation* $(F, \phi) \Rightarrow (G, \psi)$ is a natural transformation $\theta : F \Rightarrow G$ such that the following diagrams commute for all $X, Y \in \mathcal{M}$

$$\begin{tikzcd} FX \otimes FY \arrow[r, "\theta_X \otimes \theta_Y"] \arrow[d, "\phi_{X,Y}"'] & GX \otimes GY \arrow[d, "\psi_{X,Y}"] \\ F(X \otimes Y) \arrow[r, "\theta_{X\otimes Y}"] & G(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} & I \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ FI \arrow[rr, "\theta_I"'] & & GI \end{tikzcd}$$

Thinking of strict monoidal categories as categories with a sort of monoid structure on objects, naturally leads to the notion of *free strict monoidal category.* More precisely , we can force a monoidal structure on a category $\mathcal{C}$, getting an adjunction

$$F : Cat \rightleftarrows StrMonCat_{str} : U$$

The right adjoint U is of course the forgetful functor. The functor F simply constructs a strict monoidal category $F\mathcal{C}$ with objects the free monoid on $\mathcal{C}_0$ and forces it to agree with arrows so that we have a map $A_1 \dots A_n \xrightarrow{f} B_1 \dots B_m$ if and only if $n = m$ and is given by arrows $A_i \xrightarrow{f_i} B_i$; of course $A_1 \dots A_n$ is the word on the objects $A_1, \dots, A_n$ and the product is given by concatenation of words.
As one could imagine, the above *free monoidal category* functor is actually induced by the usual *free monoid* functor on *Set*. Such link is probably better understood when considering *generalized multicategories* ([Le]) or *monoids* ([ML]), but it is not the purpose of this work to deal with such generality.

In the same fashion one can consider free *commutative* monoids, the categorical version of which is given by the following.

Definition 1.5.4. A *symmetric monoidal category* is a monoidal category $\mathcal{M}$ equipped with natural isomorphisms

$$\tau_{X,Y} : X \otimes Y \to Y \otimes X$$

making the following diagrams commute

$$\begin{tikzcd} X \otimes Y \arrow[r, "\tau_{X,Y}"] \arrow[dr, equal] & Y \otimes X \arrow[d, "\tau_{Y,X}"] \\ & X \otimes Y \end{tikzcd} \qquad \begin{tikzcd} I \otimes X \arrow[d, "\lambda_X"'] & X \otimes I \arrow[l, "\tau_{X,I}"'] \arrow[dl, "\rho_X"] \\ X & \end{tikzcd}$$

$$\begin{tikzcd} X \otimes (Y \otimes Z) \arrow[r, "\alpha"] \arrow[d, "1\otimes\tau"'] & (X \otimes Y) \otimes Z \arrow[r, "\tau"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha"] \\ X \otimes (Z \otimes Y) \arrow[r, "\alpha"'] & (X \otimes Z) \otimes Y \arrow[r, "\tau\otimes 1"'] & (Z \otimes X) \otimes Y \end{tikzcd}$$

Again, after defining suitable functors, we obtain a list of categories as above, just adding the property of being *symmetric* and varying on the theme *SymmMonCat* and *SymmStrMonCat*

Definition 1.5.5. A lax symmetric monoidal functor $F : \mathcal{M} \to \mathcal{M}'$ between symmetric monoidal categories $\mathcal{M}$ and $\mathcal{M}'$ is a lax monoidal functor $F = (F, \phi)$ such that the following diagram commutes

$$\begin{tikzcd}
FX \otimes FY \arrow[r, "\tau'"] \arrow[d, "\phi_{X,Y}"'] & FY \otimes FX \arrow[d, "\phi_{Y,X}"] \\
F(X \otimes Y) \arrow[r, "F\tau"'] & F(Y \otimes X)
\end{tikzcd}$$

where τ and τ' are the twist maps in $\mathcal{M}$ and $\mathcal{M}'$

Clearly one has then the notion of *colax*,*weak* and *strict* symmetric monoidal functor.

At the beginning of the section I mentioned as examples of monoidal categories *Set* and *Ab*. The monoidal structure on them is given respectively by the cartesian product (so that *Set* is actually *cartesian monoidal*) and by the tensor product; it is well known that both functors $\cdot \times X$ on *Set* and $\cdot \otimes X$ on *Ab* have a right adjoint, given by *hom-sets*. Such phenomena happen in many other categories and are of great importance (think just of the *suspension-loop* adjunction in topology); they go under the name of *closed monoidal categories*.

Definition 1.5.6. A *right closed monoidal category* $\mathcal{M}$ is a monoidal category $\mathcal{M}$ such that for every object $Y \in \mathcal{M}$ the functor $\cdot \otimes Y$ has a right adjoint $\underline{\mathcal{M}}_r(Y, \cdot)$, so to obtain an isomorphism

$$\pi : \mathcal{M}(X \otimes Y, Z) \cong \mathcal{M}(X, \underline{\mathcal{M}}_r(Y, Z)) \tag{1.1}$$

A monoidal category is *left closed* if for every object $X \in \mathcal{M}$ the functor $X \otimes \cdot$ has a right adjoint $\underline{\mathcal{M}}_l(X, \cdot)$ A monoidal category is *closed* if both the functors $\cdot \otimes Y$ and $X \otimes \cdot$ have a right adjoint $\underline{\mathcal{M}}_r(Y, \cdot)$ and $\underline{\mathcal{M}}_l(X, \cdot)$, for every X, Y.

Note that a right (left) closed symmetric monoidal category is closed and the two above functors are isomorphic, $\underline{\mathcal{M}}_r(X, \cdot) \cong \underline{\mathcal{M}}_l(X, \cdot) = \underline{\mathcal{M}}(X, \cdot)$.
In the literature *closed monoidal category* often stands for *closed symmetric monoidal category* and I will concentrate on the last one.
As I mentioned, commonly known cases of closed monoidal categories are *Set* and *Ab*; there, the right adjoint to a product $X \otimes Y$ is exactly the *hom* functor. With this example in mind, the more general right adjoint $\underline{\mathcal{M}}(X, \cdot)$, the *internal hom*, should be a good replacement for the *hom* functor. In more categorical terms, this says that a *closed symmetric monoidal category* can be *enriched* over itself ([Ke]).
I will try to now to give a brief explanation of the argument above.
Recall that the usual hom sets of a small category $\mathcal{C}$ and rules amount to have for objects X, Y in $\mathcal{C}$ a set $\mathcal{C}(X, Y)$. For every triple of objects X, Y, Z a map $m : \mathcal{C}(Y, Z) \times \mathcal{C}(X, Y) \to \mathcal{C}(X, Z)$ and for every object X a distinguished arrow, the *identity*, given by $\iota_X : \{*\} \to \mathcal{C}(X, X)$. Note that $\{*\}$ is the identity for the product in *Set*. The axioms for *associativity* and *identity* can be stated in terms of the following commutative diagrams, where a denotes the associativity map

for the cartesian product of sets, l and r are the isomorphisms $\{*\}\times X \to X$ and $X\times\{*\} \to X$

$$\begin{tikzcd}
(\mathcal{C}(Z,W)\times\mathcal{C}(Y,Z))\times\mathcal{C}(X,Y) \arrow[rr, "a"] \arrow[d, "m\times 1"'] & & \mathcal{C}(Z,W)\times(\mathcal{C}(Y,Z)\times\mathcal{C}(X,Y)) \arrow[d, "1\times m"] \\
\mathcal{C}(Y,W)\times\mathcal{C}(X,Y) \arrow[dr, "m"] & & \mathcal{C}(Z,W)\times\mathcal{C}(X,Z) \arrow[dl, "m"'] \\
& \mathcal{C}(X,W) &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(Y,Y)\times\mathcal{C}(X,Y) \arrow[r, "m"] & \mathcal{C}(X,Y) & \mathcal{C}(X,Y)\times\mathcal{C}(X,X) \arrow[l, "m"'] \\
\{*\}\times\mathcal{C}(X,Y) \arrow[u, "\iota_Y\times 1"] \arrow[ur, "l"'] & & \mathcal{C}(X,Y)\times\{*\} \arrow[u, "1\times\iota_X"'] \arrow[ul, "r"]
\end{tikzcd}$$

The idea now is that the above hom sets should be replaced by the internal homs $\underline{\mathcal{M}}(\cdot,\cdot)$, and the composition and identity maps recovered from the adjunction 1.1. Of course the cartesian product is replaced by the tensor product $\otimes$, $\{*\}$ by the unit element I of $\mathcal{M}$ and the associativity is the associativity α of the tensor.
The "internal" composition law is the map corresponding under our adjunction to

$$(\underline{\mathcal{M}}(Y,Z)\otimes\underline{\mathcal{M}}(X,Y))\otimes X \xrightarrow{\alpha} \underline{\mathcal{M}}(Y,Z)\otimes(\underline{\mathcal{M}}(X,Y)\otimes X) \xrightarrow{1\otimes\varepsilon} (\underline{\mathcal{M}}(Y,Z)\otimes X) \xrightarrow{\varepsilon} Z$$

where ε is the counit of the adjunction 1.1. The identity is then the map

$$I \xrightarrow{\tilde{\lambda}} \underline{\mathcal{M}}(X,X)$$

obtained by adjunction from the map

$$I\otimes X \xrightarrow{\lambda} X$$

Chapter 2

Simplicial Sets

In this chapter I will review the basic theory of Simplicial Sets, the category of contravariant *Set*-valued functors on the category of finite totally ordered sets, Δ. As we shall see in Chapter 4, Simplicial Sets are the basis of the theory of Dendroidal Sets, and some of their properties motivated my point of view on the latter.
Simplicial sets provide a very pervasive tool in various branches of mathematics, from topology to homological algebra and category theory itself. Probably one of the most beautiful aspects is how they capture the classical homotopy theory of topological spaces, leading then to the general (categorical) homotopy theory.
Unfortunately, this is not the place to treat and discuss a too large topic such as *model categories*; roughly speaking, a *model category* is a category $\mathcal{C}$ admitting a particular structure which allows a "homotopy theory" in a fashion similar to that of topological spaces. In particular, simplicial sets do carry such a structure and it is possible to prove that, at the homotopy level, the category of simplicial sets is equivalent to that of topological spaces. For more about this fascinating subject I refer to [GJ],[Ho] and [Qu1].

2.1 The Simplicial Category Δ

As I mentioned, the category of simplicial sets is the category of presheaves on the category Δ. There are three possible equivalent definitions of Δ, which I list below.

Definition 2.1.1. Algebraic definition
Δ is the category with objects the finite totally ordered sets $[n] = \{0, \ldots, n\}$ and arrows the monotone maps.

It is well known that any *partially ordered set* P can be naturally viewed as a category $\mathcal{P}$ having as set of objects the set underlying P and for objects p, q there exists a unique arrow $p \to q$ if, and only if, $p \leq q$ as elements of P.
It is then natural to think of Δ in the following way

Definition 2.1.2. Categorical definition
Δ is the full subcategory of *Cat* consisting of the categories $[n]$ with $n + 1$ objects and only one arrow between them

$$0 \to 1 \to \cdots \to n$$

The algebraic definition translates then quite naturally in the following

Definition 2.1.3. Topological definition
Δ is the category with objects the standard topological *n-simplexes*

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | \sum x_i = 1, x_i \geq 0\}$$

that is, the convex hull of the points $\mathbf{v}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, for $i = 0, \ldots, n$ ($\mathbf{v}_i$ has $n+1$ entries all 0 but for the i-th one); arrows are the linear maps induced by the data $\mathbf{v}_i \mapsto \mathbf{v}_{f(i)}$ for f a monotone map $f : [m] \to [n]$.

I shall now describe two special classes of maps in Δ, which are essential to the stucture of Δ and of simplicial sets.

Definition 2.1.4. Among the maps $f : [n] \to [m]$ in Δ there are distinguished ones

the *cofaces* $d^i : [n-1] \to [n]$, $0 \leq i \leq n$, defined by $d^i(j) = j$ if $j < i$ and $d^i(j) = j+1$ if $j \geq i$

the *codegeneracies* $s^i : [n+1] \to [n]$, $0 \leq i \leq n$, defined by $s^i(j) = j$ if $j \leq i$ and $s^i(j) = j-1$ for $j > i$

Thinking by means of the categorical definition, the coface d^i skips i and for $0 < i < n$ composes the arrows $i-1 \to i \to i+1$ giving

$$0 \to 1 \to \ldots \to i-1 \to i+1 \to \ldots \to n$$

while s^i sends $i+1$ to i and the arrow $i \to i+1$ to the identity

$$0 \to 1 \to \ldots \to i \xrightarrow{1} i+1 \to \ldots \to n$$

For example, consider $d^2 : [2] \to [3]$ and $s^2 : [3] \to [2]$

$$\begin{tikzcd}
 & 3 & \\
0 \arrow[ur] \arrow[rr] \arrow[dr] & & 2 \arrow[ul] \\
 & 1 \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

then d^2 embeds $0 \to 1 \to 2$ as the face $0 \to 1 \to 3$ while s^2 collapses $0 \to 1 \to 2 \to 3$ onto $0 \to 1 \to 2$. In particular, applying s^2 after d^2 gives the identity on [2].
This is just a particular instance of the following relations, verified by the codegeneracies and cofaces, the *cosimplicial identities*

$$\begin{array}{rcll}
d^j d^i & = & d^i d^{j-1} & i < j \\
s^j s^i & = & s^i s^{j+1} & i \leq j \\
s^j d^i & = & d^i s^{j-1} & i < j \\
s^j d^i & = & 1 & i = j, j+1 \\
s^j d^i & = & d^{i-1} s^j & i > j+1
\end{array} \tag{2.1}$$

We end the section by showing how maps in Δ can be described by means of faces and degeneracies.

Proposition 2.1.5. *Every monotone map $f : [m] \to [n]$ can be written in a unique way as*

$$f = d^{i_1} \dots d^{i_t} s^{j_1} \dots s^{j_u}$$

where $n \geq i_1 \dots \geq i_t \geq 0$, $m \geq j_u \dots \geq j_1 \geq 0$ and $m + t = n + u$.

Proof. A monotone function f is determined by the sets $\{j | f(j) = f(j+1)\}$ (that is, those intervals $[j_k, \dots, j_l]$ that f collapses to a point) and the image $f([m]) \subseteq [n]$ (or equivalently its complement $\{i_t \leq \dots \leq i_1\}$, i.e. the set of elements in $[n]$ skipped by f). Then f can be rewritten as above by first collapsing each interval $[j_k \leq \dots \leq j_l]$ and then embedding via the maps d^i. The choice of the order in the indexes assures that the above representation for f is unique. □

Thanks to the previous proposition and equations 2.1 we see that arrows in Δ are generated by the coface and codegeneracy map, fact that will be very useful when studying simplicial sets.

2.2 The category *SSet* of Simplicial Sets

Definition 2.2.1. The category *SSet* of *simplicial sets* is the presheaf category $Set^{\Delta^{op}}$ of contravariant set-valued functors on the simplicial category Δ.

For a simplicial set $X \in SSet$, one usually denotes its value at $[n]$ by X_n instead of $X([n])$. Then, thanks to 2.1.5 and the equations 2.1, a simplicial set X amounts to data

- sets X_n, $n \in \mathbb{N}$, whose elements are called the *n-simplices*
- maps $d_i : X_n \to X_{n-1}$, the *face maps*
- maps $s_i : X_{n-1} \to X_n$, the *degeneracy maps*

where the maps d_i, s_i satisfy the *simplicial identities*, induced by the cosimplicial ones.

$$\begin{array}{rcll} d_i d_j & = & d_{j-1} d_i & i < j \\ s_i s_j & = & s_{j+1} s_i & i \leq j \\ d_i s_j & = & s_{j-1} d_i & i < j \\ d_i s_j & = & 1 & i = j, j+1 \\ d_i s_j & = & s_j d_{i-1} & i > j+1 \end{array} \tag{2.2}$$

For a simplex x of a simplicial set X we call *face* of x the image of x under iterations of face maps. On the other hand we say that a simplex x is *degenerate* if there is a simplex y such that x is the image of y under an iteration of degeneracy maps, so that x is a *degeneracy* of y; it is easy to prove that any simplex is the degeneracy of a unique *non-degenerate* simplex.

We give a first important result on simplicial sets

Proposition 2.2.2. *The category SSet admits the structure of a closed symmetric monoidal category.*

Proof. As a presheaf category in fact, $SSet$ admits finite products, which commute with colimits. This provides us with a symmetric product
Now consider, for a simplicial set K, the functor

$$K \times \Delta[\cdot] : \Delta \to SSet$$

Thanks to 1.4.4 we obtain the pair of adjoint functors $K \times \cdot \dashv \underline{SSet}(K, \cdot)$. For a simplicial set X, $K \times X$ is simply the product , while 1.4.4 tells that $\underline{SSet}(K, X)$ is given by $\underline{SSet}(K, X)_n = SSet(K \times \Delta[n], X)$. □

Before going further I recall some easy concepts and important constructions in the category $SSet$, which will be crucial in the next session.

Definition 2.2.3. Let X, Y simplicial sets. We say that X is a *subsimplicial set* of Y if there are monomorphisms $f_n : X_n \to Y_n$ which are compatible with the face and degeneracy maps, i.e. both X and Y are presheaves on Δ and there is a natural transformation $f : X \Rightarrow Y$ which is componentwise injective.
If X' is a set of simplices of a simplicial set X , the simplicial set *generated* by X' is the subsimplicial set given by taking all the degeneracies and faces of the simplices in X'.

Recall, that for a presheaf category $Set^{\mathcal{C}^{op}}$, there are distinguished functors, the representable presheaves $\mathcal{C}[X] = \mathcal{C}(\cdot, X)$.
In the case of simplicial sets, the representable presheaf $\Delta[n]$ is sometimes called the *standard simplicial n-simplex*. There are other notable kinds of simplicial sets:

Definition 2.2.4. The *boundary* of $\Delta[n]$ is the subsimplicial set $\partial\Delta[n]$ of $\Delta[n]$ generated by the faces $d_i(1_{[n]})$, $0 \le i \le n$, that is by the $(n-1)$-simplices $\Delta[n](d^i)(1_{[n]}) = 1_{[n]} \circ d^i$.
The k^{th} *horn* $\Lambda^k[n]$ is the subsimplicial set of $\Delta[n]$ generated by the faces $d_i(1_{[n]})$ except the k^{th} face $d_k(1_{[n]})$; that is, $\Lambda^k[n]$ is generated by those faces which do not miss the vertex k.

Anticipating results from *realization*, one should think of $\partial\Delta[n]$ as the boundary of the standard topological n-simplex.
To make this clearer, note that one can express the boundary $\partial\Delta[n]$ as a coequalizer

$$\coprod_{0 \le i < j \le n} \Delta[n-2] \rightrightarrows \coprod_{0 \le i \le n} \Delta[n-1] \to \partial\Delta[n] \tag{2.3}$$

where the two maps are induced by $\Delta[d^{j-1}] : \Delta[n-2]^{i,j} \to \Delta[n-1]^i$ and $\Delta[d^i] : \Delta[n-2]^{i,j} \to \Delta[n-1]^j$, while the coequalizer map is induced by the maps $\Delta[d^i] : \Delta[n-1]_i \to \partial\Delta[n]$. The result follows by the cosimplicial identity $d^j d^i = d^i d^{j-1}, i < j$. This is just the same as gluing the faces of a topological n-simplex Δ^n along their edges, so to obtain its boundary $\partial\Delta^n$.

We know from 1.4.6 that any simplicial set X can be expressed as a colimit of representable functors $\Delta[n]$, indexed over the natural transformations $\Delta[n] \Rightarrow X$. There is another way of describing X which will be useful later.

Definition 2.2.5. Let X a simplicial set. The n^{th} *skeleton* of X is the subsimplicial set $Sk_n X$ of X generated by the simplices of degree $\le n$.
For every n, $Sk_{n-1}X$ embeds into $Sk_n X$ and X turns out to be the union of its skeleta.

Denote by NX_n the set of non degenerate n-simplices of X. One could give a more illuminating and useful way of describing the skeleta of X, as the following proposition shows.

Proposition 2.2.6. *Let X a simplicial set. For every n there is a pushout diagram*

$$\begin{tikzcd} \coprod_{NX_n} \partial\Delta[n] \arrow[r] \arrow[d] & Sk_{n-1}X \arrow[d] \\ \coprod_{NX_n} \Delta[n] \arrow[r] & Sk_n X \end{tikzcd}$$

Proof. From 1.4.1 we prove it levelwise, that is by proving that for each m

$$\begin{tikzcd} \partial\Delta[n]_m \arrow[r] \arrow[d] & Sk_{n-1}X_m \arrow[d] \\ \Delta[n]_m \arrow[r] & Sk_n X_m \end{tikzcd}$$

is a pushout. For $m < n$ it is, since the vertical arrows are isos by definition of skeleta. For $m = n$ the complement of $\partial\Delta[n]_n$ in $\Delta[n]_n$ is the singleton $\{1_{[n]}\}$ so that the complement of $\coprod_{NX_n} \partial\Delta[n]_n$ in $\coprod_{NX_n} \Delta[n]_n$ is isomorphic to NX_n. On the other hand, $Sk_nX_n = Sk_nX_{n-1} \cup NX_n$. Thus the above diagram is a pushout also at level n. □

The above proposition tells that one can construct the skeleta of a simplicial set X by induction, in a way very similar to that of CW complexes; that is, by attaching "cells" to the n^{th} skeleton in a suitable way in order to construct the $(n+1)^{th}$ skeleton.

To conclude this section, note that one can define for any category $\mathcal{C}$ the functor category $\mathcal{C}^{\Delta^{op}}$; an object of $\mathcal{C}^{\Delta^{op}}$ is said a *simplicial object* in $\mathcal{C}$. I stress the following

Example 2.2.7. Let $\mathbb{Z} : SSet \to Ab$ be the functor taking a simplicial set X to the *simplicial abelian group* $\mathbb{Z}X$, where $\mathbb{Z}X_n$ is the *free abelian group* on X_n. Then to $\mathbb{Z}X$ is associated a chain complex

$$\cdots \xrightarrow{d} \mathbb{Z}X_2 \xrightarrow{d} \mathbb{Z}X_1 \xrightarrow{d} \mathbb{Z}X_0$$

where

$$d = \sum_{i=0}^{i=n} (-1)^i d_i$$

When $X = SY$ for a topological space Y (as explained below), by applying homology we obtain the *homology groups* of Y with coefficients in $\mathbb{Z}$, $H_*(Y;\mathbb{Z})$.

2.3 Geometric Realization

Here we finally show how simplicial sets are related to topological spaces, in particular to CW complexes.

Recall, for example from [Ma3], that

Definition 2.3.1. A *CW complex* is a space X constructed as the colimit of a sequence of spaces $X^0 \subset X^1 \subset X^2 \subset \cdots$ such that

X^0 is a discrete set of points

X^{n+1} is obtained from X^n by attaching to it disks D^{n+1} along *attaching maps* $j : S^n \to X^n$.
That is, X^{n+1} is the pushout

$$\begin{array}{ccc} \coprod_{j\in J_{n+1}} S^n & \xrightarrow{\coprod j} & X^n \\ \downarrow & & \downarrow \\ \coprod_{j\in J_{n+1}} D^{n+1} & \longrightarrow & X^{n+1} \end{array}$$

where J_{n+1} is the discrete set of the attaching maps j.

The topology on the colimits appearing above is the *compactly generated topology*. A subset of the CW complex X is closed if and only if it is closed in every X^n.
Note that in general J_{n+1} is not a subset of $Top(S^n, X^n)$: we can have more copies of the same map j, i.e. more disks attached along the same boundary.

The topological definition 2.1.3 of Δ gives us a functor $|\cdot| : \Delta \to Top$, sending $[n]$ to the *affine n-simplex*

$$\Delta^n = \{\sum_{i=0}^{n} t_i e_i | t_i \geq 0, t_0 + \cdots + t_n = 1\} \subset \mathbb{R}^{n+1}$$

and a map $f : [m] \to [n]$ to the linear map induced by $e_i \mapsto e_{f(i)}$, where the e_i's clearly are the vectors of the standard basis of $\mathbb{R}^{n+1}$.

By 1.4.4, we obtain an adjunction

$$|\cdot| : SSet \rightleftarrows Top : S \tag{2.4}$$

The right adjoint S is the *singular complex functor*, which takes a topological space X to its singular complex $SX_n = Top(\Delta^n, X)$.

The functor $|\cdot|$ is the *geometric realization* functor, on which this section concentrates.
I will first prove that it takes simplicial sets to CW complexes and then prove the invaluable fact that it preserves finite products.
Let me first show that the geometric realization $|\partial\Delta[n]|$ is the boundary of the affine n-simplex Δ^n. By 1.4.5 we know that $|\Delta[n]| = \Delta^n$. Having a right adjoint $|\cdot|$ preserves colimits, in particular the realization of 2.3 is the coequalizer in Top

$$\coprod_{0\leq i<j\leq n} \Delta^{n-2} \underset{|v|}{\overset{|u|}{\rightrightarrows}} \coprod_{0\leq i\leq n} \Delta^{n-1} \xrightarrow{|p|} |\partial\Delta[n]|$$

Now observe that $|u|(|v|)$ embeds $\Delta^{n-2}_{i,j}$ in $\Delta^{n-1}_i(\Delta^{n-1}_j)$ as the face opposite to the $(j-1)^{th}(i^{th})$ vertex. The coequalizer of $|u|$ and $|v|$ is then the quotient of $\coprod_{0\leq i\leq n} \Delta^{n-1}$ obtained by gluing along such faces, that is the boundary $\partial\Delta^n$ of the *n-simplex*.

It follows now that the geometric realization of a simplicial set is a CW complex. Consider in fact a simplicial set as the colimit of its skeleta. Using 2.2.5 the realization $|X|$ is the colimit of spaces $X^n = |Sk_n X|$ given as pushouts

$$\begin{tikzcd} \coprod_{NX_n} \partial\Delta^{n+1} \arrow[r] \arrow[d] & X^n \arrow[d] \\ \coprod_{NX_n} \Delta^{n+1} \arrow[r] & X^{n+1} \end{tikzcd}$$

Since the pairs $(D^{n+1}, S^n), (\Delta^{n+1}, \partial\Delta^{n+1})$ are homeomorphic and $X^0 = |Sk_0 X|$ is discrete, we have that $|X|$ is a CW complex.
It is then reasonable to restrict the codomain of the realization functor to the category $\mathcal{U}$ of *compactly generated spaces*, to wich CW-complexes belong. Recall that

Definition 2.3.2. A space X is said *compactly generated* if it is a *weak Hausdorff k-space*. That is

i) $g(K)$ is closed in X for every $K \xrightarrow{g} X$ with K compact

ii) every subspace $A \subset X$ such that $f^{-1}(A)$ is closed in K for any $K \xrightarrow{f} X$ with K compact, is closed in X.

The advantage of the category $\mathcal{U}$ of compactly generated spaces over *Top* is that $\mathcal{U}$ is cartesian closed (i.e. the product functor has a right adjoint), but still it is equivalent to *Top* at the homotopy level.
Once we restrict our attention to $\mathcal{U}$, we have that geometric realization preserves finite products: this is the statement of theorem 2.3.5. The proof I give here follows both the one in [GZ] and that given in [Ho], each one having its advantages over the other. We use the fact that $|\cdot|$ preserves colimits, and that the product of representable simplicial sets can be expressed as a coequalizer. Finally, one needs commutativity of the product with respect to colimits; this doesn't usually happen in *Top*, but it holds in $\mathcal{U}$, hence the restriction to compactly generated spaces.

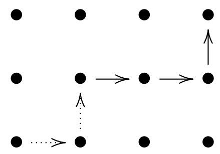
The first step towards the proof of theorem 2.3.5 is to understand the product $\Delta[m] \times \Delta[n]$ of representables.

Lemma 2.3.3. *The product $\Delta[m] \times \Delta[n]$ of the representable simplicial sets $\Delta[m], \Delta[n]$ can be expressed as a coequalizer*

$$\coprod \Delta[n_{c(i) \cap c(j)}] \rightrightarrows \coprod \Delta[n_{c(i)}] \to \Delta[m] \times \Delta[n]$$

for certain indexing sets $c(i)$.

Proof. Consider the presheaves $\Delta[m], \Delta[n]$. A *p-simplex* x of $\Delta[m] \times \Delta[n]$ is an element of $\Delta([p],[m]) \times \Delta([p],[m]) \cong Cat([p],[m]) \times Cat([p],[n]) \cong Cat([p],[m] \times [n])$. Then x can be viewed as a chain of lenght p in the partially ordered set $[m] \times [n]$; we know it is sufficient to consider non-degenerate simplex.
The idea is that, as for CW complexes, the p-simplices of $\Delta[m] \times \Delta[n]$ should come from pairs of *q-simplices* and *r-simplices* such that $q + r = p$.
Also notice that any chain of lenght p in $[m] \times [n]$ can be expanded to a chain of lenght $m + n$; that is any non degenerate p-simplex is a face of an (m+n)-simplex. As an example let $m = 3$,$n = 2$



The picture above shows a 3-simplex, which should be thought as arising from the horizontal 2-simplex of [3] and the vertical 1-simplex of [2] . The dotted lines show a possible way of extending it to a 5-simplex. To understand better what goes on below, consider again the rectangle $[3] \times [2]$. Now, labelling the lattice $[m] \times [n]$ we have that a maximal chain is determined by the labels at the end of its horizontal segments . Then, maximal chains are in 1-1 correspondence with the subsets of $\{1, \ldots, m+n\}$ of cardinality m, for a total of $\binom{m+n}{m}$. Finally, denote by $c(i)$, $0 \leq i \leq \binom{m+n}{m}$ the maximal chains in $[m] \times [n]$; for a chain c let n_c be the number of edges of c. Consider the diagram

$$\coprod_{0\leq i<j\leq\binom{m+n}{m}} \Delta[n_{c(i)\cap c(j)}] \overset{u}{\underset{v}{\rightrightarrows}} \coprod_{0\leq i\leq\binom{m+n}{m}} \Delta[n_{c(i)}] \xrightarrow{p} \Delta[m] \times \Delta[n] \tag{2.5}$$

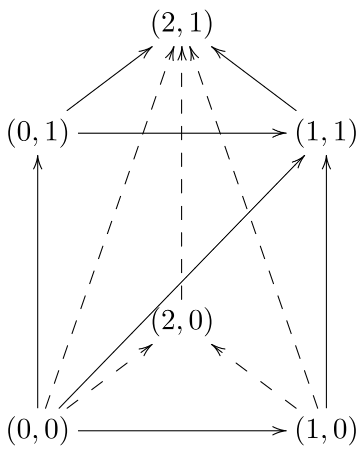
here u and v are induced by the inclusions of $c(i) \cap c(j)$ in $c(i)$ or $c(j)$. Let X be a coequalizer of u and v, then the *(m+n)-simplices* of X are given by the identities $1_{n_{c(i)}}$ glued along the faces $u(1_{n_{c(i)\cap c(j)}})$ and $v(1_{n_{c(i)\cap c(j)}})$; all the other identifications of cells are induced by these and the other simplices of X are just obtained from them as faces and degeneracies. Comparing with the discussion above, follows that $\Delta[m] \times \Delta[n]$ is the coequalizer of u and v. $\square$

I guess it would be worth to have an example in mind.

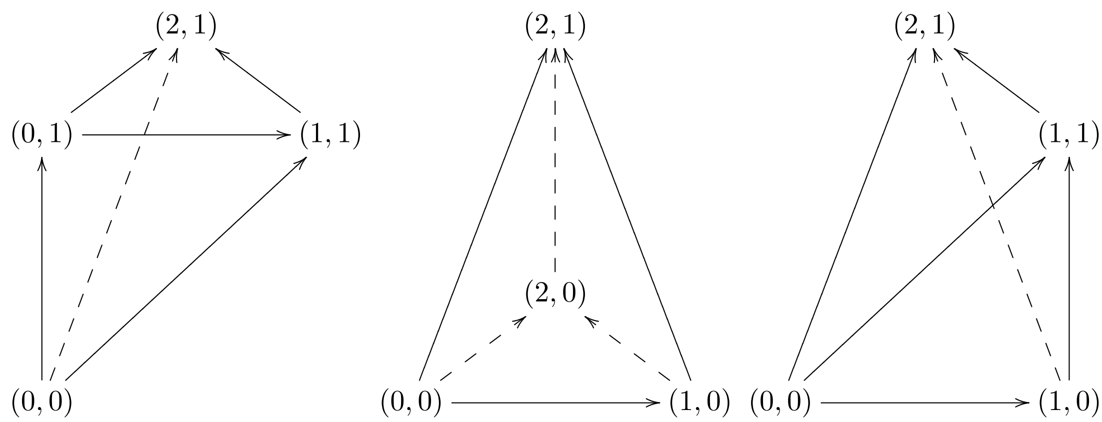
Example 2.3.4. Take the case $m = 2$ and $n = 1$. Pictorially, $\Delta[2]$ and $\Delta[1]$ are the *triangle* and the *unit interval*

$$\begin{tikzcd} & 2 & & 1 \\ 0 \arrow[ur] \arrow[rr] & & 1 \arrow[ul] & 0 \arrow[u] \end{tikzcd}$$

Intuitively, the product $\Delta[2] \times \Delta[1]$ should look like



Going through the argument above, we have in the retangle $[2] \times [1]$ three maximal chains, given by the paths $\{(0,0)(0,1)(1,1)(2,1)\}$, $\{(0,0)(1,0)(2,0)(2,1)\}$ and $\{(0,0)(1,0)(1,1)(2,1)\}$ to which correspond three copies of $\Delta[3]$



The intersection of the maximal chains give one 1-cell $\{(0,0)(2,1)\}$ and 2-cells $\{(0,0)(1,0)(2,1)\}$ and $\{(0,0)(1,1)(2,1)\}$. Gluing along these cells we obtain us the prism representing $\Delta[2]\times\Delta[1]$.

In order to see that geometric realization preserves products, let me define the n-simplex Δ_n as the convex hull of the points $(1,\ldots,1),(0,1,\ldots,1),\ldots,(0\ldots0)$, that is the set of points $(u_1,\ldots,u_n)\in\mathbb{R}^n$ such that $0\le u_1\le\ldots\le u_n\le 1$. It is clear that Δ_n is homeomorphic to Δ^n defined before.
Recall that a maximal chain $c(i)$ is a sequence $i_1\le\cdots\le i_m$ in $\{1,\ldots,m+n\}$; denote its complement by $j_1\le\cdots\le j_n$. Define now the maps $f_i:\Delta_{c(i)}\to\Delta_m\times\Delta_n$ by $(u_1,\ldots,u_{m+n})\mapsto(u_{i_1},\ldots,u_{i_m};u_{j_1},\ldots,u_{jn})$. We obtain a coequalizing diagram

$$\coprod_{0\le i<j\le\binom{m+n}{m}}\Delta_{n_{c(i)\cap c(j)}}\underset{|v|}{\overset{|u|}{\rightrightarrows}}\coprod_{0\le i\le\binom{m+n}{m}}\Delta_{n_{c(i)}}\xrightarrow{f}\Delta_m\times\Delta_n$$

where f is induced by the maps f_i. Then f is onto and the points identified by f correspond to the images under $|u|$ and $|v|$ of the same point in $\Delta_{n_{c(i)\cap c(j)}}$.
Finally, having restricted the codomain of $|\cdot|$ to $\mathcal{U}$, since the product in $\mathcal{U}$ commutes with colimits the following equalities hold

$$\begin{aligned}|X\times Y| = |(\operatorname{colim}\Delta[m])\times(\operatorname{colim}\Delta[n])| = |\operatorname{colim}(\Delta[m]\times\Delta[n])| = \operatorname{colim}(|\Delta[m]\times\Delta[n]|) =\\ = \operatorname{colim}(|\Delta[m]|\times|\Delta[n]|) = (\operatorname{colim}|\Delta[m]|)\times(\operatorname{colim}|\Delta[n]|) = |X|\times|Y|\end{aligned}$$

and we conclude that

Theorem 2.3.5. *The geometric realization functor $|\cdot|:SSet\to\mathcal{U}$ preserves finite products.*

In particular $|\cdot|:SSet\to\mathcal{U}$ is a monoidal functor, where on $\mathcal{U}$ the symmetric monoidal structure is given by the product, while the internal hom is given by the space $\mathcal{U}(X,Y)$ with the *compact-open topology* (cf. [ML] or [Ma3]).

2.4 Classifying Spaces

We are now left to investigate the categorical definition 2.1.2 of the simplicial category Δ. As usual, thanks to 1.4.4 and the inclusion $\Delta \hookrightarrow Cat$ we obtain a pair of adjoint functors

$$\tau : SSet \rightleftarrows Cat : N \tag{2.6}$$

The functor N above is the *nerve* functor, taking a category $\mathcal{C}$ to the simplicial set $N\mathcal{C}$, whose n-simplices are chains of length n of composable arrows in $\mathcal{C}$

$$N\mathcal{C}_n = \{C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n\}$$

The face and degeneracies operators are given, as one would expect, by composition of arrows and by inserting identities.
The *nerve functor* is probably better known when associated to the *geometric realization functor* of the previous section. By this I mean the composition $|\cdot| \circ N : Cat \to Top$, the *classifying space functor*, usually denoted by B, as studied by Segal in [Se1].
Then, for a category $\mathcal{C}$, $B\mathcal{C}$ is a CW complex with vertices the objects of $\mathcal{C}$, 1-cells the morphisms, 2-cells the commutative triangles and so on...
As I shall recall later, the classifying space is an important tool in the theory of *infinite loop spaces* and *spectra* ([Ad2],[Ma1],[Se2],[Th1],[Th2]).
Moreover, the *classifying space* is also fundamental in *algebraic K-theory* as spelled out in [Qu2].

Of course, in this section I am far from investigating classifying spaces or any of the above results. I only recall a simple result from [Qu2] that caught my eyes, on which I will comment later.
Let $\mathcal{C}$ a small category and E a covering of its classifying space $B\mathcal{C}$. Denoting by $E(X)$ the fiber of E over a vertex X, an arrow $X \xrightarrow{f} X'$ defines a path in $B\mathcal{C}$ and hence an isomorphism $E(X) \to E(X')$. The assignment $X \mapsto E(X)$ clearly defines a functor $E : \mathcal{C} \to Set$ which is *morphism-inverting*. More in detail, one has

Proposition 2.4.1. *The category of covering spaces of $B\mathcal{C}$ is canonically equivalent to the category of morphism-inverting functors $F : \mathcal{C} \to Set$.*

In a very naive way, one could look at the CW complex $B\mathcal{C}$ as a localization of $\mathcal{C}$; here the arrows of $\mathcal{C}$ are brought to paths which can be walked both ways, giving so inverses.
To underline more the fascinating interplay between topology, homotopy, simplicial sets and (higher) category theory, one could look at a CW complex as encoding the structure of a *weak n-category*, $n \leq \infty$, given the n-morphisms by the n-cells of the CW complex. The "invertibility" translates then into the concept of ∞*-groupoid*. The relation between ∞-groupoids and homotopy types (hence CW complexes) was suggested by Grothendieck and studied by Kapranov and Voevodsky in [KV], while a description of weak n-categories by means of simplicial sets was given by Tamsamani in [Ta].

Chapter 3

Multicategory Theory

With this chapter we finally start dealing with the main subject of this work.
Multicategories are nothing mysterious; in the same way as we have been taught about maps of one or more variables, it is natural to think of "categories" where arrows have more than one input. The idea is to redefine a category so to get maps from a (finite) string of objects $(X_1, \ldots, X_n)$ to an output object X; of course one needs a few more axioms than just "n-arity", as we shall see in the following.
Notice that the output consists of only one object. One could obviously think of arrows with more outputs, i.e. of *polycategories* or PROPs, but this will not bother us here.

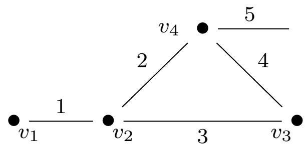
3.1 Trees

In this preliminary section I shall give an overview of the basic concepts about trees. Trees provide in fact a very nice language for describing multicategories, and will be the main tool in the theory of dendroidal sets.
Trees will be defined as certain kind of graphs; I immediately point out the fact that the definitions of graphs and trees below slightly differs from the usual one (see for example [Di]).
In the present setting a vertex of a tree should represent a multiarrow in a multicategory; to emphasize this, a graph will be defined starting from the set E of edges (which represent objects), and the set V of vertices will appear as a subset of the powerset of E. In particular in our trees an edge often happens to be incident to only one vertex. The latter fact is in contrast with the usual definition of trees and graphs, where the set E of edges is a subset of the set V^2 of pairs of vertices, so that an edge $e = (v_1, v_2)$ necessarily belongs to two vertices.

Definition 3.1.1. A *graph* G is a pair (E, V) consisting of a set E of *edges* and a set $V \subseteq P(E)$ of vertices, such that an edge $e \in E$ belongs to at most two vertices.
Edges belonging to two vertices are called *inner*, while those belonging to one vertex are called *outer*. Two edges e_1, e_2 belonging to the same vertex v are said *linked* by v.

Definition 3.1.2. Given edges e_1 and e_n in a graph G, a *path* from e_1 to e_n is a sequence $(e_1, v_1, e_2, \ldots, v_{n-1}, e_n)$ of edges e_i and vertices v_j, such that $e_i, e_{i+1} \in v_i$ and $v_i \neq v_j$ for $i \neq j$.
A path with at least three edges such that $e_1 = e_n$ is called a *cycle*. A graph G in which for any two edges e, e' there is a path from e to e' is called *connected*.

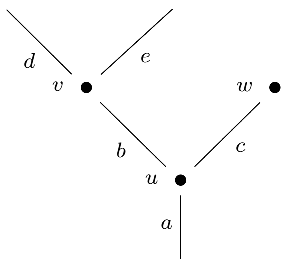
Example 3.1.3. Let $E = \{1, 2, 3, 4, 5\}$ and $V = \{v_1, v_2, v_3, v_4\}$ where $v_1 = \{1\}, v_2 = \{1, 2, 3\}, v_3 = \{3, 4\}, v_4 = \{2, 4, 5\}$. The corresponding graph is



with a cycle $(2, v_2, 3, v_3, 4, v_4, 2)$.

Definition 3.1.4. A *tree* is a connected graph with no cycles and a chosen outer edge called the *root*, such that each vertex v is a non-empty finite set. The outer edges other than the root are called *leaves.*

The choice of a root defines a direction on the tree and hence for each vertex v a set *in(v)* of incoming edges (*inputs*) and an outgoing edge *out(v)*, the *output.* Trees will be drawn with the root at the bottom, directed towards the root. For example in the tree



the vertex v has inputs the edges d, e and output b, while w has output c and no inputs, and a is the root.
A tree T is then determined by the triple $(E(T), V(T), out(T))$, where $out(T)$ denotes the root of T, while the leaves are denoted by $in(T)$.
An important operation on trees is the *grafting*

Definition 3.1.5. Let T and S be trees such that $E(S) \cap E(T) = \{r\}$ where r is the root of S and a leaf of T. The *grafting* $T \circ S$ of S on T along r is the tree $(E(S) \cup E(T), V(S) \cup V(T), out(T))$.

In particular, suppose the vertex v containing the root r of a tree T has inputs $e_1, \ldots, e_n$; denoting by T_{e_i} the subtree of T having e_i as root and by T_r the tree consisting only of the vertex v one can decompose T as the grafting

$$T = T_r \circ (T_{e_1}, \ldots, T_{e_n})$$

that is the iterated grafting of T_{e_i} along the leaf e_i of the tree $T_r \circ (T_{e_1} \circ \cdots \circ T_{e_{i-1}})$. This is known as the *fundamental decomposition of trees* and follows by definition of grafting and the fact that $E(T_{e_i}) \cap E(T_{e_j}) = \emptyset$ if $i \neq j$ while $E(T_{e_i}) \cap E(T_r) = e_i$.

A kind of tree that will occur quite often is the following

Definition 3.1.6. A *planar tree* T, is a tree T together with a linear ordering of $in(v)$ for each vertex v.

In the case of planar trees, in particular, it is possible to define an order on the leaves and hence to graft planar trees along the i-th leaf.
Given planar trees T and S, the grafting $T \circ_i S$ of S along the i-th leaf of T, is defined by renaming the edges of S so that its root coincides with the i-th leaf of T and then apply the usual grafting.

3.2 Planar Multicategories

In this section I will give the definition of *planar*, or *non-symmetric*, *multicategories* and functors between them. Some examples will also be given, though the main ones will appear later when talking of *symmetric multicategories*.

Definition 3.2.1. A planar multicategory $\mathcal{P}$ consists of

(i) a class of objects $\mathcal{P}_0$.

(ii) for each $n \in \mathbb{N}$, $n \geq 0$ and objects $p_1, \dots, p_n, p$ a set $\mathcal{P}(p_1, \dots, p_n; p)$, the *operations* (or *arrows*). It is assumed that if $\mathcal{P}(p_1, \dots, p_n; p_0) \cap \mathcal{P}(q_1, \dots, q_m; q_0) \neq \emptyset$, then $n = m$ and $p_i = q_i \forall i = 0, \dots n$. Notice that also are allowed operations of *arity* 0, whose set is denoted by $\mathcal{P}(;p)$.

(iii) for each object p there is given an operation $1_p \in \mathcal{P}(p;p)$, the identity on p.

(iv) given $p_1, \dots, p_n, p$ and for each $1 \leq i \leq n$ a sequence $p^i_1, \dots, p^i_{m_i}$, there is given a *composition map*

$$\begin{array}{c} \mathcal{P}(p_1 \dots p_n; p) \times \mathcal{P}(p^1_1, \dots, p^1_{m_1}; p_1) \times \dots \times \mathcal{P}(p^n_1, \dots, p^n_{m_n}; p_n) \\ \Big\downarrow \mu \\ \mathcal{P}(p^1_1, \dots, p^1_{m_1}, \dots, p^n_1, \dots, p^n_{m_n}; p) \end{array}$$

taking $(\psi, \psi_1, \dots \psi_n)$ to $\psi \circ (\psi_1, \dots \psi_n)$ or $\psi(\psi_1, \dots \psi_n)$

Composition and *identities* are required to satisfy the obvious axioms

1. *Identity*: each time the composition makes sense, one has that

$$1_p(\psi) = \psi \qquad \phi(1_{p_1}, \dots, 1_{p_n}) = \phi$$

2. *Associativity*: given $\psi \in \mathcal{P}(p_1 \dots p_n; p)$, arrows $\psi_i \in \mathcal{P}(p^i_1, \dots, p^i_{m_i}; p_i)$ for $1 \leq i \leq n$ and arrows $\psi^i_{j_i}$ for $1 \leq i \leq n$ and $1 \leq j_i \leq m_i$ with output $p^i_{j_i}$, the following equality holds

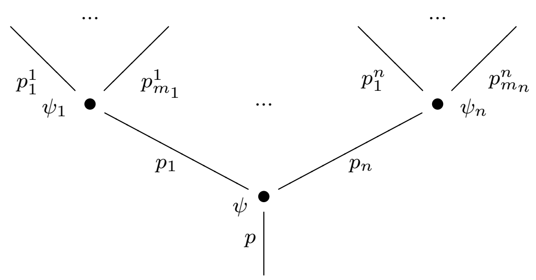
$$\psi(\psi_1(\psi^1_1, \dots, \psi^1_{m_1}), \dots, \psi_n(\psi^n_1, \dots, \psi^n_{m_n})) = (\psi(\psi_1, \dots, \psi_n))(\psi^1_1, \dots, \psi^n_{m_n})$$

There is another way of defining composition in multicategories, by means of the $\circ_i$-*composition*. The idea is that of composing operations step by step at each instance i.
More in detail, let $\phi \in \mathcal{P}(p_1, \dots, p_i, \dots, p_n; p)$ and $\psi \in \mathcal{P}(q_1, \dots, q_m; p_i)$; define the $\circ_i$-*composition* of ϕ and ψ as

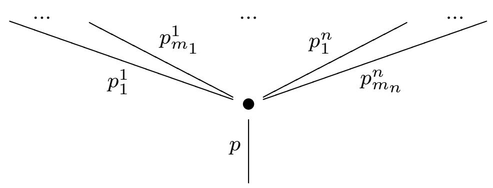
$$\phi \circ_i \psi = \phi \circ (id, \dots, id, \psi, id, \dots, id) \in \mathcal{P}(p_1, \dots, p_{i-1}, q_1, \dots, q_m, p_{i+1}, \dots, p_n; p)$$

It is clear that by iterated use of the $\circ_i$ one recovers the previous notion of composition, and the order in which the operations are composed does not affect the final result.

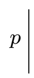
The above definitions are better understood if using *labelled planar trees*. Given a planar multicategory $\mathcal{P}$ one can create a labelled planar tree T by labeling its edges with the objects of $\mathcal{P}$ and its vertices with the operations of $\mathcal{P}$. That is, given an operation $\psi \in \mathcal{P}(p_1 \dots p_n; p)$ one constructs a node with vertex ψ, input edges $in(\psi) = (p_1 \dots p_n)$ and output edge $out(\psi) = p$. For example, the composition map μ can be pictured as taking the tree



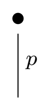
to the *corolla*



while the identity arrow at an object p has the form



In particular, an operation of *arity* 0 in $\mathcal{P}(;p)$ is depicted as



0-ary operations should be thought of as *constants* as is probably made better clear by the following example

Example 3.2.2. Let $\mathcal{M}$ be the planar multicategory with only one object X and only one operation μ_n for each *arity* n.
$\mathcal{M}$ encodes the structure of a *monoid* X with neutral element given by the 0-*ary* operation μ_0 and product by the maps $\mu_{n\geq 2}$. The fact that μ_0 gives us the identity element simply comes by the composition $\mu_2(1, \mu_0) = \mu_2(\mu_0, 1) = 1_X \in \mathcal{M}(X, X) = \{\mu_1 = 1_X\}$. Associativity of the product is imposed by the fact that each $\mathcal{M}(X, \dots, X; X)$ consists of only one element.

Example 3.2.2 introduced a particular kind of (planar) multicategories, the *planar operads*, important enough to have their own name. In our settings, operads are nothing but multi-categories with one object. The best known definition sounds a bit different, though clearly equivalent

Definition 3.2.3. A *planar operad* $\mathcal{P}$ is a collection of sets $\mathcal{P}(n), n \geq 0$ together with a distinguished element $id \in \mathcal{P}(1)$ and maps

$$\mathcal{P}(n) \times \mathcal{P}(k_1) \times \cdots \times \mathcal{P}(k_n) \xrightarrow{\mu} \mathcal{P}(k_1 + \ldots + k_n)$$

such that for maps $\psi \in \mathcal{P}(n), \psi_i \in \mathcal{P}(k_i)$ $1 \leq i \leq n, \psi^i_{j_i} \in \mathcal{P}(k_{j_i})$ $1 \leq j_i \leq k_i$

$$\begin{gathered}\mu(\psi, \mu(\psi_1, \psi^1_1, \ldots, \psi^1_{k_1}), \ldots, \mu(\psi_n, \psi^n_1, \ldots, \psi^n_{k_n})) \\ \shortparallel \\ \mu(\mu(\psi, \psi_1, \ldots, \psi_n), \psi^1_1, \ldots, \psi^1_{k_1}, \ldots, \psi^n_1, \ldots \psi^n_{k_n})\end{gathered}$$

and

$$\mu(\psi, id, \ldots, id) = \psi = \mu(id, \psi)$$

It is desirable that multicategories (and operads) form a category. To this aim, let's first define the obvious notion of functor between multicategories.

Definition 3.2.4. Let $\mathcal{P}$ and $\mathcal{Q}$ be planar multicategories. A functor $F : \mathcal{P} \to \mathcal{Q}$ between them consists of

(i) a function $F : \mathcal{P}_0 \to \mathcal{Q}_0$ taking an object p of $\mathcal{P}$ to Fp

(ii) a map $F : \mathcal{P}(p_1, \ldots, p_n; p) \to \mathcal{Q}(Fp_1, \ldots, Fp_n; Fp)$ for each sequence $(p_1, \ldots, p_n, p)$ of objects of $\mathcal{P}$, such that $F(\psi(\psi_1, \ldots \psi_n)) = F\psi(F\psi_1, \ldots F\psi_n)$, and for every object p of $\mathcal{P}$ $F(id_p) = id_{Fp}$

Denote now by $\mathit{Multicat}_\pi$ the category of planar multicategories and by Operad_π its full subcategory with objects the planar operads. It is of course possible to define *natural transformations* between functors of multicategories.

Definition 3.2.5. Let $F, G : \mathcal{P} \to \mathcal{Q}$ be functors between planar multicategories. A natural tranformation $\alpha : F \Rightarrow G$ is a collection $(\alpha_p)_{p \in \mathcal{P}}$ of unary operations $\alpha_p \in \mathcal{Q}(Fp; Gp)$ such that for any operation $\psi \in \mathcal{P}(p_1, \ldots, p_n; p)$ in $\mathcal{P}$ the following equality holds

$$G\psi(\alpha_{p_1}, \ldots, \alpha_{p_n}) = \alpha_p(F\psi)$$

When planar multicategories are replaced by symmetric ones, we will see that it is possible to define natural transformations with more inputs, so that the functors from $\mathcal{P}$ to $\mathcal{Q}$ form actually a symmetric multicategory and not merely a category.

There is an obvious adjunction

$$j_! : \mathit{Cat} \leftrightarrows \mathit{Multicat}_\pi : j^*$$

sending a multicategory $\mathcal{P}$ to its underlying category consisting of the unary operations, and a category $\mathcal{C}$ to the planar multicategory with only unary operations, so that Cat is embedded into $\mathit{Multicat}_\pi$.

A much more interesting adjunction is that between monoidal categories and planar multicategories.
Given a strict monoidal category $\mathcal{M}$, this can be regarded as a planar multicategory M with set of arrows $M(p_1,\ldots,p_n;p) = \mathcal{M}(p_1 \otimes \cdots \otimes p_n, p)$. On the other hand, given a planar multicategory M, one can construct a monoidal category $\mathcal{M}$ with set of objects the free monoid on M_0 and arrows $\mathcal{M}(p_1 \otimes \cdots \otimes p_n, p) = M(p_1,\ldots,p_n;p)$, where the tensor is clearly given by concatenation of words. We obtain an adjunction

$$F : \mathit{Multicat}_\pi \rightleftarrows \mathit{MonCat} : U$$

The above adjunction shows one of the possible ways of linking monoidal categories and multicategories. Another one was implicit in 3.2.2; the idea is that monoidal categories give models for the structures described by multicategories. More precisely

Definition 3.2.6. Let $\mathcal{P}$ a multicategory and $\mathcal{E}$ a monoidal category.
An *algebra* for $\mathcal{P}$ in $\mathcal{E}$ is a functor $\mathcal{P} \to \mathcal{E}$, where $\mathcal{E}$ is viewed as a multicategory.

For example, taking $\mathcal{E} = \mathit{Set}$ and $\mathcal{P}$ to be the monoid multicategory $\mathcal{M}$ of 3.2.2, an algebra for it is exactly a set with a monoid structure.

3.3 Symmetric multicategories

In this and the next section I review definitions and some properties of *symmetric multicategories* and *symmetric operads*, which are the target of *Dendroidal Sets*, the main subject of this work. As for the previous section, in contrast with [MW1], I prefer to keep the distinction between operads and multicategories.

Definition 3.3.1. A *(symmetric) multicategory* $\mathcal{P}$ is a planar multicategory together with a right action of the symmetric groups Σ_n on each set $\mathcal{P}(p_1,\ldots,p_n;p)$.
This means that

for a permutation $\sigma \in \Sigma_n$ and an operation $\psi \in \mathcal{P}(p_1,\ldots,p_n;p)$ there is a function $\sigma^* : \mathcal{P}(p_1,\ldots,p_n;p) \to \mathcal{P}(p_{\sigma 1},\ldots,p_{\sigma n};p)$

$(\sigma\tau)^* = \tau^*\sigma^*$ $(\sigma,\tau \in \Sigma_n)$ and $id^* = id$

The action of the symmetric groups is compatible with the composition:
given operations $\psi_0,\psi_1,\ldots,\psi_n$ (with ψ_0 of arity n, ψ_i of arity k_i) such that the composition $\psi_0(\psi_1,\ldots,\psi_n)$ is defined and permutations $\sigma_0,\ldots,\sigma_n$ $(\sigma_0 \in \Sigma_n, \sigma_i \in \Sigma_{k_i})$

$$\sigma_0^*(\psi_0)(\sigma_1^*(\psi_1),\ldots,\sigma_n^*(\psi_n) = (\sigma_0(\sigma_1,\ldots,\sigma_n))^*(\psi_0(\psi_1,\ldots,\psi_n))$$

where $\sigma = \sigma_0(\sigma_1,\ldots,\sigma_n) \in \Sigma_{k_1+\ldots k_n}$ is the *permutation product* obtained by considering $\{1,\ldots,k_1+\ldots+k_n\}$ as divided into n intervals of lenght k_i, permuting the elements of the i^{th} interval according to σ_i and the intervals according to σ_0. In formulas, noting that an element of $\{1,\ldots,k_1+\ldots k_n\}$ has the form $k_1+\ldots k_{i-1}+j$ $(1 \le i \le n, 1 \le j \le k_i)$ and letting $k_0 = 0$

$$\sigma(k_1+\ldots k_{i-1}+j) = k_{\sigma_0^{-1}(1)} + \ldots k_{\sigma_0^{-1}(\sigma_0(i))-1)} + \sigma_i(j)$$

As one would expect, the notion of functor and natural transformation extend to symmetric multicategories

Definition 3.3.2. Let $\mathcal{P}$ and $\mathcal{Q}$ be symmetric multicategories. A *functor* of symmetric multicategories $F : \mathcal{P} \to \mathcal{Q}$ consists of

(i) a function $F : \mathcal{P}_0 \to \mathcal{Q}_0$ taking an object p of $\mathcal{P}$ to Fp

(ii) a function $F : \mathcal{P}(p_1, \ldots, p_n; p) \to \mathcal{Q}(Fp_1, \ldots, Fp_n; Fp)$ for each sequence $(p_1, \ldots, p_n, p)$ of objects of $\mathcal{P}$, such that

$$F(\psi(\psi_1, \ldots \psi_n)) = F\psi(F\psi_1, \ldots F\psi_n)$$

whenever the composition $\psi(\psi_1, \ldots \psi_n))$ makes sense. Moreover for every object p of $\mathcal{P}$ $F(id_p) = id_{Fp}$ and

$$F(\sigma^*(\psi)) = \sigma^*(F(\psi))$$

for any n-ary operation ψ and $\sigma \in \Sigma_n$

Definition 3.3.3. Let $F_i : \mathcal{P} \to \mathcal{Q}$, $1 \leq i \leq n$ and $F : \mathcal{P} \to \mathcal{Q}$ be functors between symmetric multicategories. A *natural tranformation* α from $(F_1, \ldots, F_n)$ to F is a collection $(\alpha_p)_{p \in \mathcal{P}}$ with $\alpha_p \in \mathcal{Q}(F_1 p, \ldots, F_n p; Fp)$ such that for any operation $\psi \in \mathcal{P}(p_1, \ldots, p_m; p)$ we have that

$$\sigma^*_{m,n} F\psi(\alpha_{p_1}, \ldots, \alpha_{p_m}) = \alpha_p(F_1\psi, \ldots, F_n\psi)$$

where $\sigma_{m,n}$ is the permutation equating the inputs of the composite operation $F\psi(\alpha_{p_1}, \ldots, \alpha_{p_m})$ to those of $\alpha_p(F_1\psi, \ldots, F_n\psi)$.

As anticipated in the previous section, the set of functors $Func(\mathcal{P}, \mathcal{Q})$ should form a symmetric multicategory. Let $\alpha : (F_1, \ldots, F_n) \Rightarrow F$ and $\beta^i : (F^i_1, \ldots, F^i_{k_i}) \Rightarrow F_i$, $1 \leq i \leq n$ be natural transformations. Define the composite $\alpha(\beta^1, \ldots, \beta^n)$ to be the natural transformation with components

$$(\alpha(\beta^1, \ldots, \beta^n))_p = \alpha_p(\beta^1_p, \ldots, \beta^n_p)$$

To see that naturality holds, notice that for an operation $\psi \in \mathcal{P}(p_1, \ldots, p_m; p)$

$$\begin{gathered}
(\alpha_p(\beta^1_p, \ldots, \beta^n_p))(F^1_1\psi, \ldots, F^1_{k_1}\psi, \ldots, F^n_1\psi, \ldots, F^n_{k_n}\psi) \\
\shortparallel \\
\alpha_p(\beta^1_p(F^1_1\psi, \ldots, F^1_{k_1}\psi), \ldots, \beta^n_p(F^n_1\psi, \ldots, F^n_{k_n}\psi)) \\
\shortparallel \\
\alpha_p(\sigma^*_1(F_1\psi(\beta^1_{p_1}, \ldots, \beta^1_{p_m})), \ldots, \sigma^*_n(F_n\psi(\beta^n_{p_1}, \ldots, \beta^n_{p_m}))) \\
\shortparallel \\
\sigma^*(\alpha_p(F_1\psi(\beta^1_{p_1}, \ldots, \beta^1_{p_m}), \ldots, F_n\psi(\beta^n_{p_1}, \ldots, \beta^n_{p_m}))) \\
\shortparallel \\
\sigma^*(\alpha_p(F_1\psi, \ldots, F_n\psi)(\beta^1_{p_1}, \ldots, \beta^1_{p_m}, \ldots, \beta^n_{p_1}, \ldots, \beta^n_{p_m})) \\
\shortparallel \\
\sigma^*(\tau^*_\alpha(F\psi(\alpha_{p_1}, \ldots, \alpha_{p_m}))(\beta^1_{p_1}, \ldots, \beta^1_{p_m}, \ldots, \beta^n_{p_1}, \ldots, \beta^n_{p_m})) \\
\shortparallel \\
\sigma^*\tau^*(F\psi(\alpha_{p_1}, \ldots, \alpha_{p_m})(\beta^1_{p_1}, \ldots, \beta^n_{p_1}, \ldots, \beta^1_{p_m}, \ldots, \beta^n_{p_m}))
\end{gathered}$$

Here the permutations involved in the equations are those coming from the definitions of natural tranformation and composition in symmetric multicategories: σ and τ are the permutation products $id(\sigma_1, \ldots, \sigma_n)$ and $\tau_\alpha(id, \ldots, id)$.
The fact that composition of natural transformation is associative comes from the associativity in multicategories. Clearly the unit is given by the natural transformation which is the

identity at every component p.
It follows, as claimed, that the set $Func(\mathcal{P}, \mathcal{Q})$ is a symmetric multicategory.

One denotes by *Multicat* the category whose objects are the symmetric multicategories and arrows the functors between them. Thanks to the last arguments one can extend the 1-*category Multicat* to a strict 2-*category* with the peculiarity that 2-morphisms again form a multicategory; in some sense one has a "2-multicategory".
As for the planar case one denotes by *Operad* the full subcategory of *Multicat* consisting of *(symmetric) operads*, i.e. multicategories with only one object. Again, one could define operads separately without mentioning multicategories; the definition is just the same as for the planar case, with the difference that one requires an action by the symmetric groups. More in detail one has:

Definition 3.3.4. A *(symmetric) operad* $\mathcal{P}$ is a planar operad $\mathcal{P}$ together with an action of the symmetric group Σ_k on $\mathcal{P}(k)$ compatible with composition, in the sense that for $\psi \in \mathcal{P}(k), \sigma \in \Sigma_k$ and $\psi_i \in \mathcal{P}(n_i), \sigma_i \in \Sigma_{n_i}$ holds

$$\mu(\sigma^*\psi, \sigma_1^*\psi_1 \ldots, \sigma_k^*\psi_k) = (\sigma(\sigma_1, \ldots, \sigma_k))^*\mu(\psi, \psi_1, \ldots, \psi_k)$$

where $\sigma(\sigma_1, \ldots, \sigma_k) \in \Sigma_{n_1+\ldots+n_k}$ is the product defined in 3.3.1.

We have seen in the previous section that there is an adjunction between *Cat* and $Multicat_\pi$. Denoting by *Multicat* the category of symmetric multicategories and functors, we obtain a similar adjunction. To complete the triangle, I shall now describe the functor $Symm : Multicat_\pi \to Multicat$, the *symmetrization* functor.
Let $\mathcal{P}$ a planar multicategory. $Symm(\mathcal{P})$ is the symmetric multicategory with set of objects that of $\mathcal{P}$ and for objects $p_1 \ldots, p_n, p$

$$Symm(\mathcal{P})(p_1 \ldots, p_n; p) = \coprod_{\sigma \in \Sigma_n} \mathcal{P}_\sigma(p_1 \ldots, p_n; p)$$

where

$$\mathcal{P}_\sigma(p_1 \ldots, p_n; p) = \{\sigma\} \times \mathcal{P}(p_{\sigma^{-1}(1)} \ldots, p_{\sigma^{-1}(n)}; p)$$

The right action by Σ_n is given as follows:
for $(\sigma, \psi) \in \{\sigma\} \times \mathcal{P}(p_{\sigma^{-1}(1)} \ldots, p_{\sigma^{-1}(n)}; p)$ and $\tau \in \Sigma_n$

$$\tau^*(\sigma, \psi) = (\sigma\tau, \psi) \in \mathcal{P}_{\sigma\tau}(p_{\tau(1)} \ldots, p_{\tau(n)}; p) = \{\sigma\tau\} \times \mathcal{P}(p_{\sigma^{-1}(1)} \ldots, p_{\sigma^{-1}(n)}; p)$$

Composition in induced by that in $\mathcal{P}$:
suppose given operations $\psi_0 \in Symm(\mathcal{P})(p_1 \ldots, p_n; p)$ and $\psi_i \in Symm(\mathcal{P})(p_1^i \ldots, p_{k_i}^i; p_i)$, $i = 1 \ldots n$, represented by pairs (σ_0, ψ_0), $\psi_0 \in \mathcal{P}(p_{\sigma^{-1}(1)} \ldots, p_{\sigma^{-1}(n)}; p)$ and (σ_i, ψ_i), $\psi_i \in \mathcal{P}(p_{\sigma_i^{-1}(1)} \ldots, p_{\sigma_i^{-1}(k_i)}; p_i)$.
Looking at the definition of the product $\sigma = \sigma_0(\sigma_1, \ldots, \sigma_n)$, we have that the composition $\psi = \psi_0(\psi_{\sigma_0^{-1}(1)}, \ldots, \psi_{\sigma_0^{-1}(n)})$ in $\mathcal{P}$ gives the desired pair $(\sigma, \psi) \in Symm(\mathcal{P})(p_1^1, \ldots, p_{k_n}^n; p)$.
It follows that the functor *Symm* is left adjoint to the forgetful functor from symmetric to planar multicategories.

We obtain finally the diagram mentioned above

$$
\begin{tikzcd}
 & Cat \arrow[dl, shift left, "j_!"] \arrow[dr, shift right, "j_!"'] & \\
Multicat_\pi \arrow[ur, shift left, "j^*"] \arrow[rr, shift left, "Symm"] & & Multicat \arrow[ul, shift right, "j^*"'] \arrow[ll, shift left, "U"]
\end{tikzcd}
$$

in which both the inner and outer triangle commute.

Probably the reader would like now to touchons between planar and symmetric multicategories.

Example 3.3.5. Let A_∞ denote the symmetric operad given by $A_\infty(n) = \Sigma_n$ and let the symmetric groups Σ_n act freely (by multiplication). The resulting operad is nothing but $Symm(\mathcal{M})$, where $\mathcal{M}$ is the operad defined in 3.2.2, and describes *monoids*; the free action of the symmetric groups just records all the possible ways one could multiply n elements.
Let now E_∞ denote the symmetric multicategory with only one object X and one operation for each $E_\infty(n)$, but where the action of the symmetric groups Σ_n is trivial. This means that multiplication of n elements does not depend on the order, in other words one has commutative monoids.
Notice in particular that it would not have been possible to define "commutative structures" in a planar setting.

Note that any symmetric monoidal category $\mathcal{M}$ has an underlying symmetric multicategory $\tilde{\mathcal{M}}$. $\tilde{\mathcal{M}}$ is just the planar multicategory underlying $\mathcal{M}$, with action of the symmetric groups given by applying the *twist* isomorphism τ of definition 1.5.4.
We then have the symmetric analog of 3.2.6

Definition 3.3.6. Let $\mathcal{P}$ a symmetric multicategory and $\mathcal{M}$ a symmetric monoidal category. An *algebra* for $\mathcal{P}$ in $\mathcal{M}$ is a functor $\mathcal{P} \to \mathcal{M}$, where $\mathcal{M}$ is viewed as a symmetric multicategory.

For example, an algebra in *Set* for the above operad E_∞ is clearly a commutative monoid.

3.4 (co)completeness of *Multicat*

As the title suggests, in this section it will be shown that the category *Multicat* of symmetric multicategories is both complete and cocomplete. In order to do that, we first outline how to construct *free multicategories* in a way that resembles the construction of categories out of directed graphs. Such construction will be crucial also in the next section, when constructing a tensor product on *Multicat*.
Let me first give a definition, then it will be quite obvious how to proceed.

Definition 3.4.1. Given a set A, a collection C on A is a family of sets $C(a_1, \ldots, a_n; a_0)$, for $a_i \in A$ and $n \geq 0$ a natural number. Given collections $(A, C), (A', C')$ an arrow f between them consists of a family of functions

$$C(a_1, \ldots, a_n; a_0) \xrightarrow{f} C(f(a_1), \ldots, f(a_n); f(a_0))$$

Denote by *Col* the category of collections and arrows between them.

Clearly any (planar) multicategory has an underlying collection; this defines a forgetful functor $U : \mathit{Multicat}_\pi \to \mathit{Col}$. Its left adjoint F_π will be then the *free planar multicategory functor* which I will now describe.
Let (A, C) a collection. The easiest way of defining the *free multicategory* $F_\pi C$ is by using trees. The set of objects of $F_\pi C$ is clearly A.
Let T be a planar tree; label its edges by elements of A, and label any vertex v of T with inputs $in(v) = (a_1, \ldots, a_n)$ and output $out(v) = (a)$ with an element c_v of $C(a_1, \ldots, a_n; a)$. Define LT to be the set of all such labelled planar trees; for $T \in LT$ denote by $in(T)$ the (labelled) leaves of T and by $out(T)$ its (labelled) root. It is clear that these trees should be the composition schemes in our free multicategory $\mathcal{P} = F_\pi C$. The operations of $F_\pi C$ are in fact given by

$$F_\pi C(a_1, \ldots, a_n; a) = \{T \in LT | in(T) = (a_1, \ldots, a_n), out(T) = (a)\}$$

Composition in $F_\pi C$ is clearly given by grafting trees, which corresponds to the *circle-i* composition.
As one would expect, the *free symmetric multicategory* is obtained from a collection (A, C) by applying first F_π and then *Symm*.
We are able now to construct more "sophisticated" (planar) multicategories. Regard the operations obtained from a collection as *generators*. For a collection C a *relation* in the planar multicategory $F_\pi C$ is a family of sets $R = \{R_{a_1,\ldots,a_n,a_0}\}$ of relations on the sets $F_\pi C(a_1, \ldots, a_n; a_0)$. Such relation is called *normal* if each $R_{a_1,\ldots,a_n,a_0}$ is an equivalence relation and it is compatible with composition, in the sense that

$$\phi_0 \circ (\phi_1, \ldots, \phi_n) \sim \phi'_0 \circ (\phi'_1, \ldots, \phi'_n)$$

whenever $\phi_i \sim \phi'_i$.
Given any relation R, one can of course construct a normal relation R', the normal relation *generated by* R, as the intersection of all the normal relations containing R. Hence the following

Definition 3.4.2. Let C a collection on a set A and R a relation in the planar multicategory $F_\pi C$; denote by R' the normal relation generated by R. The planar multicategory $(F_\pi C/R')$ with objects A and operations $(F_\pi C/R')(a_1, \ldots, a_n; a) = F_\pi C(a_1, \ldots, a_n; a)/\sim_{R'}$, is called the planar multicategory *generated* by the *generators* C and the *relations* R.

In order to obtain a similar construction in the symmetric case, we must require an additional property. One says that a relation R in the free symmetric multicategory $Symm(F_\pi C)$ is a normal relation if it is normal in the planar sense and it respects the action of the Σ_n's; i.e. for $\phi \sim \phi'$, one has

$$\sigma^* \phi = \sigma^* \phi'$$

Defining again the sets of operations as quotients, we have

Definition 3.4.3. Let C a collection on a set A and R a relation in the symmetric multicategory $Symm(F_\pi C)$; denote by R' the normal relation generated by R. The multicategory $Symm(F_\pi C)/R'$ is called the symmetric multicategory *generated* by the *generators* C and the *relations* R.

We are now able to prove that *Multicat* is both complete and cocomplete. The idea is to construct limits in *Multicat* by limits in *Set* for the objects and sets of operations of the multicategories involved. We know also that (co)completeness is equivalent to the existence of (co)equalizers and (co)products, and this will be the way we will prove the following two theorems.

Theorem 3.4.4. *The category Multicat is complete.*

Proof. Suppose we have a diagram

$$\mathcal{P} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{Q}$$

The equalizer of f and g is the multicategory $\mathcal{R}$ constructed as follows:

The set of objects of $\mathcal{R}$ is the equalizer in *Set* of $ob(\mathcal{P}) \underset{g}{\overset{f}{\rightrightarrows}} ob(\mathcal{Q})$.

For objects $p_0, \ldots, p_n$, $\mathcal{R}(p_1, \ldots, p_n; p_0)$ is the equalizer

$$\mathcal{R}(p_1, \ldots, p_n; p_0) \xrightarrow{e} \mathcal{P}(p_1, \ldots, p_n; p_0) \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{Q}(p'_1, \ldots, p'_n; p'_0)$$

where $p'_i = f(p_i) = g(p_i)$.

The sets of objects and operations of $\mathcal{R}$ are subsets of those of $\mathcal{P}$, and the multicategory structure is induced by that of $\mathcal{P}$.
The product of a family $\{\mathcal{P}_i\}$ is computed in the obvious way, taking products of objects and vectors of arrows.
Having products and equalizers, the theorem follows. □

Theorem 3.4.5. *The category Multicat is cocomplete.*

Proof. Suppose we are given arrows in *Multicat*

$$\mathcal{P} \underset{g}{\overset{f}{\rightrightarrows}} \mathcal{Q}$$

Consider the collection C underlying $\mathcal{Q}$ and create the free symmetric multicategory constructed out of the collection C with relations R those describing $\mathcal{Q}$ and the new relations

$$f(\phi) = g(\phi)$$

for an operation ϕ of $\mathcal{P}$. Notice that this forces the identifications $f(p) = g(p)$ at the level of objects. The resulting multicategory $Symm(F_\pi C)/R$ is then the coequalizer of f and g.
For coproducts, just follow the same line. Given a collection $\{\mathcal{P}_i\}$ of multicategories one takes the free symmetric multicategory on the collection given by the union of the collections underlying the $\mathcal{P}_i$'s and relations those generated by the union of the relations describing them.
Again, the above construction ensures the cocompleteness of *Multicat*. □

3.5 Closed monoidal structure in *Multicat*

With this last section I show how the category *Multicat* admits a closed monoidal structure, which extends that of *Cat*. Let me first define the *tensor* on *Multicat*; concluding that it has a right adjoint will be then quite easy.

Definition 3.5.1 (Boardman-Vogt tensor product)**.** Let $\mathcal{P}$ and $\mathcal{Q}$ symmetric multicategories. Define the *Boardman-Vogt tensor product* $\mathcal{P} \otimes_{bv} \mathcal{Q}$ to be the symmetric multicategory given by the collection C and relations R described below. $ob(\mathcal{P} \otimes_{bv} \mathcal{Q}) = ob(\mathcal{P}) \times ob(\mathcal{Q})$; for an operation $\phi \in \mathcal{P}(p_1, \dots, p_n; p)$ and an object $q \in \mathcal{Q}$ there is a generator $\phi \otimes_{bv} q \in C((p_1, q), \dots (p_n, q); (p, q))$ and for $\psi \in \mathcal{Q}(q_1, \dots, q_n; q)$, $p \in \mathcal{P}$ there is a generator $p \otimes_{bv} \psi \in C((p, q_1), \dots (p, q_n); (p, q))$. Relations are generated by the following

(i) $(\phi \otimes_{bv} q) \circ ((\phi_1 \otimes_{bv} q), \dots, (\phi_n \otimes_{bv} q)) = (\phi \circ (\phi_1, \dots, \phi_n)) \otimes_{bv} q$

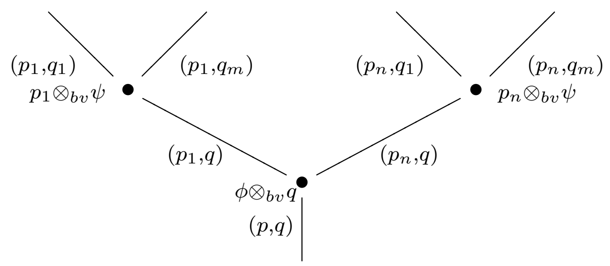
(ii) $\sigma^*(\phi) \otimes_{bv} q = \sigma^*(\phi \otimes_{bv} q)$

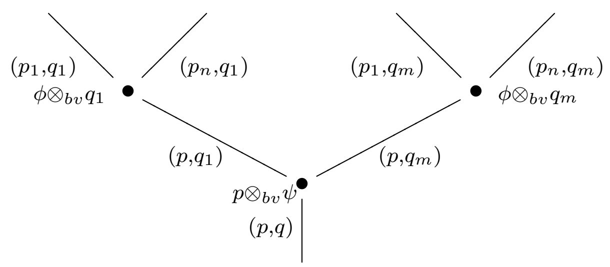
(iii) $(p \otimes_{bv} \psi) \circ ((p \otimes_{bv} \psi_1), \dots, (p \otimes_{bv} \psi_n)) = p \otimes_{bv} (\phi \circ (\psi_1, \dots, \psi_n))$

(iv) $p \otimes_{bv} \sigma^*(\psi) = \sigma^*(p \otimes_{bv} \psi)$

(v) $(\phi \otimes_{bv} q) \circ ((p_1 \otimes_{bv} \psi), \dots, (p_n \otimes_{bv} \psi)) = \sigma^*_{m,n}((p \otimes_{bv} \psi) \circ ((\phi \otimes_{bv} q_1), \dots, (\phi \otimes_{bv} q_m)))$

The relations generated by (i),(ii) and those generated by (iii),(iv) naturally give functors $\mathcal{P} \xrightarrow{F_q} \mathcal{P} \otimes_{bv} \mathcal{Q}, p \mapsto (p, q)$ and $\mathcal{Q} \xrightarrow{F_p} \mathcal{P} \otimes_{bv} \mathcal{Q}$. The permutation in (v) is the same we saw in 3.3.3; relations of type (v) state that after reordering the inputs the following composition schemes are equal





Theorem 3.5.2. *Multicat with the Boardman-Vogt tensor product is a closed symmetric monoidal category.*

Proof. The above tensor makes *Multicat* into a symmetric monoidal category. The claim now is that the internal hom is given by

$$\underline{Multicat}(\mathcal{P}, \mathcal{Q}) = Func(\mathcal{P}, \mathcal{Q})$$

In other words we need a natural isomorphism

$$Multicat(\mathcal{P} \otimes_{bv} \mathcal{Q}, \mathcal{R}) \cong Multicat(\mathcal{P}, Func(\mathcal{Q}, \mathcal{R}))$$

This goes just like the case of categories.
A functor $H : \mathcal{P} \otimes_{bv} \mathcal{Q} \to \mathcal{R}$ is taken to the functor

$$p \longmapsto H_p = (\mathcal{Q} \xrightarrow{F_p} \mathcal{P} \otimes_{bv} \mathcal{Q} \xrightarrow{H} \mathcal{R})$$

while an operation $\phi : (p_1, \ldots, p_n) \to p$ of $\mathcal{P}$ is brought to the natural transformation

$$\alpha : (H_{p_1}, \ldots, H_{p_n}) \to H_p \qquad \alpha_q = H(\phi \otimes_{bv} q)$$

The relation (v) in the previous definition and σ-invariance of H ensure that α is a natural transformation.
Now suppose we have a functor $G : \mathcal{P} \to Func(\mathcal{Q}, \mathcal{R})$. Define $H : \mathcal{P} \otimes_{bv} \mathcal{Q} \to \mathcal{R}$ by

$$H(p, q) = G(p)(q)$$

for an object $(p, q) \in \mathcal{P} \otimes_{bv} \mathcal{Q}$. For a generator $p \otimes_{bv} \psi$

$$H(p \otimes_{bv} \psi) = G(p)(\psi)$$

while for a generator $\phi \otimes_{bv} q$ take the component at q of the natural transformation $G(p)(\phi)$

$$H(\phi \otimes_{bv} q) = (G(p)(\psi))_q$$

Relations (i)-(iv) in 3.5.1 follow by functoriality of G and G_p. Relation (v) is respected by definition of natural transformation.
The two assignments just defined are then clearly inverses to each other, and the theorem follows. $\square$

Chapter 4

Dendroidal Sets

In this chapter I finally present the main subject of this thesis.
Dendroidal Sets have been recently introduced by Ieke Moerdijk and Ittay Weiss in [MW1] and [MW2]. Not long later, in [CM], Denis-Charles Cisinski and Ieke Moerdijk proved that the category of dendroidal sets admits a Quillen Model Structure, problem which was left open.
We have seen how multicategories extend categories, passing from a "linear" to a "multilinear" world. Following the same stream, dendroidal sets are an extension of simplicial sets. In fact, simplicial sets do embed in the category of dendroidal sets, in a way that agrees with the embedding of *Cat* into *Multicat* and the adjunction $SSet \rightleftarrows Cat$. Moreover, the model structure on dendroidal sets, as constructed in [CM], agrees with Joyal's model structure on *SSet*.
I can not cover here the whole theory of dendroidal sets; anyway the definitions and results in the following sections should suffice to understand the main concepts about dendroidal sets and to the purposes of the present work. I will present the basics of the theory, following as close as possible the chapter on simplicial sets. As for simplicial sets, dendroidal sets are presheaves on some category, the *dendroidal category*, which should expand the known simplicial category, and will be the subject of the first section.

4.1 The dendroidal category Ω

Recall that the definition of the simplicial category Δ made use of finite totally ordered sets, or equivalently of categories with a finite set of objects of the form $0 \to \cdots \to n$.
Such ordered sets, in the linear world of categories, should catch, and depict, a possible composition chain of morphism.
Now that we are dealing with multicategories, what we need is something that describes arrows with any number of inputs and the ways one could compose them. In the previous chapter trees made the job, and in fact one possible definition of the dendroidal category is based on trees; in order to extend the algebraic definition of Δ one needs some kind of "ramified" orderings, which should resemble to the composition schemes given by trees. They are described here below.

4.1.1 Algebraic definition of Ω

For a set A, denote by A^* the free monoid on A, i.e. the set of words on A with product given by concatenation of words and neutral element given by the empty word

$$A^* = \bigcup_{n=0}^{\infty} A^n$$

denote a word in A^* by $\overrightarrow{a}$ and write $a \in \overrightarrow{a}$ to say that a is an entry of the word $\overrightarrow{a}$. Moreover the symmetric groups act from the right on A^*: if $\overrightarrow{a} = (a_1, \ldots, a_n)$ is a word of length n and $\sigma \in \Sigma_n$, define $\sigma^*(\overrightarrow{a}) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. A *broad relation* is a pair (A, R) with $R \subset A \times A^*$; if the pair $(a, \overrightarrow{a})$ is in R one writes $aR\overrightarrow{a}$.

Definition 4.1.1. A *broad poset* is a set A together with a broad relation $(A, \leq)$ satisfying the following properties:

i) *reflexivity*: $a \leq (a)$ for all $a \in A$

ii) *transitivity*: if $a \leq (a_1, \ldots, a_n)$ and $a_i \leq \overrightarrow{b_i}$ for $1 \leq i \leq n$, then $a \leq \overrightarrow{b_1} \cdot \ldots \cdot \overrightarrow{b_n}$

iii) *anti-symmetry*: if $a \leq \overrightarrow{b}$ and $b \leq \overrightarrow{a}$ with $a \in \overrightarrow{a}, b \in \overrightarrow{b}$, then $a = b$

iv) *permutability*: if $a \leq \overrightarrow{a}$ with $\overrightarrow{a}$ of lenght n and $\sigma \in \Sigma_n$, then $a \leq \sigma^* \overrightarrow{a}$

Condition iv) is clearly the analog of symmetries for multicategories; when dropped one calls the above *planar broad poset.*

There is an obvious notion of *map of broad posets*, i.e. a map $A \xrightarrow{f} B$ such that $f(a) \leq (f(a_1), \ldots, f(a_n))$ whenever $a \leq (a_1, \ldots, a_n)$. This makes broad posets into a category, *BrdPoset*, of which Ω will be a full subcategory.

Definition 4.1.2. Let A a broad poset and a, b elements of A.
One says that a is dominated by b, $a \leq_d b$ if there is a word $\overrightarrow{b}$ such that $b \in \overrightarrow{b}$ and $a \leq \overrightarrow{b}$.
An element $r \in A$ which is minimal in A under $\leq_d$ is called a *root* of A.

Note that, by anti-symmetry, if it exists, the root is unique. Moreover the above relation makes A into a poset.
The broad relation $\leq$ also makes A^* into a poset: given $\overrightarrow{a} = (a_1, \ldots, a_n)$ and $\overrightarrow{b}$, define $\overrightarrow{a} \leq \overrightarrow{b}$ if there are $\overrightarrow{b_1}, \ldots, \overrightarrow{b_n}$ such that $a_i \leq \overrightarrow{b_i}$ and $\overrightarrow{b} = \overrightarrow{b_1} \cdot \ldots \cdot \overrightarrow{b_n}$.
For $a \in A$ define $\hat{a} = \{\overrightarrow{a} | a < \overrightarrow{a}\}$.

Definition 4.1.3. Let A a broad poset and $a \in A$. Assume that $\hat{a}$ as a sub-poset of A^* has a minimal element unique up to symmetry. One denotes it by $s(a)$ and calls it a *representative* for the *successor* of a.
In the case when $\hat{a}$ is empty, one calls a a *leaf.*

Notice that in the above definition unicity is required, for in general there can be minimal elements which are distinct even after permutations.

Definition 4.1.4. A broad poset $(A, \leq)$ is called *finite* if the set $\leq$ of relations is finite.
A is called *minimal* if whenever

$$a \leq (a_1, \ldots, a_n)$$

$a_i \neq a_j$ for $i \neq j$.

Definition 4.1.5. Let A a broad poset. A is called *dendroidally ordered* if

i) A is finite

ii) A is minimal

iii) A has a root

iv) for every $a \in A$ either a has a successor or a is a leaf.

Definition 4.1.6. The *dendroidal category* Ω is the full subcategory of *BrdPoset* consisting of the dendroidally ordered sets and maps between them.

Regarding finite totally orered sets as dendroidally ordered ones the category Δ then embeds in Ω, making Ω as just defined the multicategorical extension of the simplicial category.

4.1.2 Operadic definition of Ω

As mentioned at the beginning of the chapter, trees provide a useful tool to describe multicategories. It shouldn't surprise then that the analog of the categorical definition of Δ is based on some special kind of multicategories, those arising from trees.

Given a planar tree T, one can define a collection $C_T = (A, C)$ as follows: the set A is the set $E(T)$ of edges of T, while $C(e_1, \ldots, e_n; e_0)$ consists of only one point if there is a vertex v of T with $in(v) = (e_1, \ldots, e_n)$ and output e_0, and it is empty otherwise.

Definition 4.1.7. The *planar multicategory generated by* T, $\Omega_\pi(T)$ is the free multicategory $F_\pi C_T$ on the collection C_T.

Definition 4.1.8. Let T a non-planar tree. The multicategory $\Omega(T)$ *generated by* T is

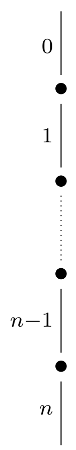
$$\Omega(T) = Symm(\Omega_\pi(\overline{T}))$$

where $\overline{T}$ is a planar representative for T.

A different choice for $\overline{T}$ above simply amounts to a different choice for the generating operation, which doesn't affect the result after symmetrization.

Definition 4.1.9. The *dendroidal category* Ω is the full subcategory of *Multicat* spanned by the multicategories of the form $\Omega(T)$, fot T a non-planar tree.

Note that Δ is embedded in Ω, by taking $[n]$ to the linear tree L_n



4.1.3 Equivalence of the definitions

Now that we have two definitions of Ω, of course one expects them to be equivalent. In order to prove it we need to develop a little bit more the algebraic definition of the dendroidal category. It should be already enough evident that the above described dendroidal orders are the algebraic analog of trees; what is needed in order to prove the equivalence is an “algebraic” notion of grafting. This will permit a decomposition of dendroidally ordered sets quite similar to that of trees, and hence a proof by induction.

Definition 4.1.10. *Grafting of dendroidally ordered sets*
Let A and B dendroidally ordered sets such that the intersection $A \cap B = \{y\}$ is the root of B and a leaf of A. The *grafting* $A \circ B$ of B on A along y, is the dendroidally ordered set $A \cup B$ with broad relation given by $x \leq (x_1, \ldots, x_n)$ if

i) $x \leq (x_1, \ldots, x_n)$ holds in A

ii) $x \leq (x_1, \ldots, x_n)$ holds in B

iii) $x \in A$ and $(x_1, \ldots, x_n) = \overrightarrow{a_1} \cdot \overrightarrow{b} \cdot \overrightarrow{a_2}$ for $\overrightarrow{a_1}, \overrightarrow{a_2} \in A^*$, $\overrightarrow{b} \in B^*$ with

$$x \leq \overrightarrow{a_1} \cdot \overrightarrow{y} \cdot \overrightarrow{a_2}$$

in A, and

$$y \leq \overrightarrow{b}$$

in B

As we did for trees, one can define an operation on dendroidally ordered sets by repeatedly grafting, so that for dendroidally ordered sets B_i with root $\{y_i\}$,$i = 1 \ldots n$ and a tree A with leaves $\{y_1, \ldots, y_n\}$ one defines a tree

$$A \circ (B_1, \ldots, B_n)$$

Following the arguments and constructions for trees, one obtains a decomposition for dendroidally ordered sets.
Let A a dendroidally ordered set; define for $a \in A$

$$A_a = \{a' \in A | a \leq_d a'\}$$

and, if the root r of A has successor $s(r) = (x_1, \ldots, x_n)$, let

$$A_r = \{r, x_1, \ldots, x_n\}$$

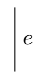
Then A clearly induces a dendroidal order on the above sets. It is now possible to state the following

Proposition 4.1.11 (Fundamental decomposition of dendroidally ordered sets)**.** *Let A a dendroidally ordered set with root r and $s(r) = (a_1, \ldots, a_n)$. Then A decomposes as*

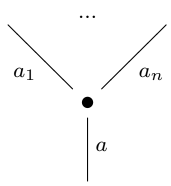
$$A = A_r \circ (A_{a_1}, \ldots, A_{a_n})$$

Proof. One must first show that the grafting in the statement is well defined, i.e. $A_i \cap A_r = \{a_i\}$. For suppose there is another $a \neq a_i$ in such intersection; then a appears in some word $\overrightarrow{a}$ with $a_i \leq \overrightarrow{a}$ and $a = a_j \in s(r)$ for $j \neq i$. Using transitivity of A we obtain $r \leq (a_1, \dots a_{i-1}, \overrightarrow{a}, a_{i+1}, \dots, a_n)$ with a_j appearing twice, a contradiction.
Secondly, one must have that, as sets, $A = A_r \cup A_1 \cup \dots \cup A_n$. One inclusion is obvious. Pick $a \in A$; then $r \leq_d a$, so that $a \in \overrightarrow{a}$ for a word $r \leq \overrightarrow{a}$. If $a \neq r$ or $a \notin s(r)$, then we must have $\overrightarrow{a} \in \hat{r}$ and $s(r) < \overrightarrow{a}$, so that $\overrightarrow{a}$ has the form $\overrightarrow{a} = \overrightarrow{a_1} \cdot \dots \overrightarrow{a_n}$ with $a_i \leq \overrightarrow{a_i}$; thus a belongs to some A_i.
Finally, by definition of grafting follows that the dendroidal orders on A and $A_r \circ (A_{a_1}, \dots, A_{a_n})$ are the same. □

We now proceed to show that the two definitions of Ω are equivalent, in the sense that the two categories defined in 4.1.6 and 4.1.9 are isomorphic. The idea is to create first a correspondence between dendroidally ordered sets and trees; this in turn will provide the required isomorphism.
Recall that by η_e one denotes the tree with only one edge e and no vertices



while an *n-corolla* C_n is the tree consisting of only one vertex with n inputs



Definition 4.1.12. Let T a finite planar tree. The dendroidally ordered set $[T]$ has as underlying set the set of edges $E(T)$ of T and is defined by induction on the number k of vertices as follows.
For $k = 0$, $T = \eta$ and $[T]$ is just the dendroidally ordered set $\{e \leq e\}$. For $k = 1$ T is an n-corolla with root r and leaves $\{a_1, \dots, a_n\}$; the corresponding dendroidally ordered set is given by the relations $a \leq \overrightarrow{a}$ where $\overrightarrow{a}$ is any possible permutation of $(a_1, \dots, a_n)$. Let now T be a tree with more than 1 vertex; decomposing it as $T_r \circ (T_{a_1}, \dots, T_{a_n})$ define $[T] = [T_r] \circ ([T_{a_1}], \dots, [T_{a_n}])$ using the grafting of dendroidally ordered sets.

The inverse construction, as one would expect, is made by induction on the broad relations. More precisely, for a dendroidally ordered set A call a pair $(a, \overrightarrow{a})$ a *link* if $a < \overrightarrow{a}$ and there is no $\overrightarrow{b}$ such that $a < \overrightarrow{b} < \overrightarrow{a}$; two links $(a, \overrightarrow{a})$ and $(a, \overrightarrow{a}')$ are said equivalent if $\sigma^* \overrightarrow{a}' = \overrightarrow{a}$ for a permutation σ. Define the *degree* of a dendroidally ordered set A, $|A|$, to be the number of equivalence classes of links in A.

Definition 4.1.13. Let A a dendroidally ordered set. Define a tree $Tr(A)$ by induction on the degree of A. If $|A| = 0$, then $A = \{a \leq a\}$ and $Tr(A) = \eta_a$; if $|A| = 1$, then $Tr(A)$ is an n-corolla. For $|A| \geq 1$ define $Tr(A)$ to be the grafting $Tr(A_r) \circ (Tr(A_{a_1}), \dots, Tr(A_{a_n}))$.

Theorem 4.1.14. *The above constructions, associating to a dendroidally ordered set A a tree $Tr(A)$ and to a tree T a dendroidally ordered set $[T]$, satisfy the following properties:*

i) $[Tr(A)] = A$

ii) $Tr([T]) = T$

iii) *Whenever one side of the equation* $[T \circ S] = [T] \circ [S]$ *is defined, so is the other; and in that case the equality holds*

iv) *Whenever one side of the equation* $Tr(A \circ B) = Tr(A) \circ Tr(B)$ *is defined, so is the other; and in that case the equality holds*

v) *The two constructions* $[\cdot]$ *and* $Tr(\cdot)$ *are unique with respect to i)-iv)*

Proof. The first point follows by induction on the degree of A and definition of the two constructions. For $|A| = 0$, $A = \{a \leq a\}$ and $[Tr(A)] = [\eta_a] = \{a \leq a\} = A$. If $|A| = 1$, $A = \{a \leq \sigma^*(a_1, \ldots, a_n) | \sigma \in \Sigma_n\}$, $Tr(A)$ is the an n-corolla with root a and leaves $\{a_1, \ldots, a_n\}$, so that $[Tr(A)]$ consists of the relations $a \leq \sigma^*(a_1, \ldots, a_n)$ and the identity holds. By induction if $|A| > 1$, $[Tr(A)] = [Tr(A_r) \circ (Tr(A_{a_1}), \ldots, Tr(A_{a_n}))] = [Tr(A_r)] \circ ([Tr(A_{a_1})], \ldots, [Tr(A_{a_n})]) = A_r \circ (A_{a_1}, \ldots, A_{a_n}) = A$. Point *ii*) is just the same.
Point *iii*) (and similarly point *iv*) again follows by induction and the equalities $[T \circ S] = [(T_r \circ (T_1, \ldots, T_n)) \circ (S_r \circ (S_1, \ldots, S_m))] = [T_r \circ (T_1, \ldots, (T_j \circ S_r \circ (S_1, \ldots, S_m)), \ldots, T_n)] = [T_r] \circ ([T_1], \ldots, ([T_j] \circ [S_r] \circ ([S_1], \ldots, [S_m])), \ldots, [T_n]) = [(T_r] \circ ([T_1], \ldots, [T_n])) \circ ([S_r] \circ ([S_1], \ldots, [S_m])) = [T] \circ [S]$.
Suppose now that $\langle \cdot \rangle$ and $Tr'(\cdot)$ satisfy $i) - iv)$. One first shows that they agree on the unit η and the n-corollas, then by *iii*) and *iv*) the constructions must concide. Suppose $|\langle \eta \rangle| > 0$; then $\langle \eta \rangle$ can be written as $T \circ S$ and $\eta = Tr'(\langle \eta \rangle) = Tr'(T) \circ Tr'(S)$ contradicting the fact that η has no vertices. The same argument shows that $\langle C_n \rangle$ consists only of n leaves and the root. It follows that $\langle \cdot \rangle = [\cdot]$; similarly we obtain that $Tr'(\cdot) = Tr(\cdot)$. □

At this point the equivalence we want to prove should be obvious.

Theorem 4.1.15. *The algebraic and operadic definition of* Ω *are equivalent.*

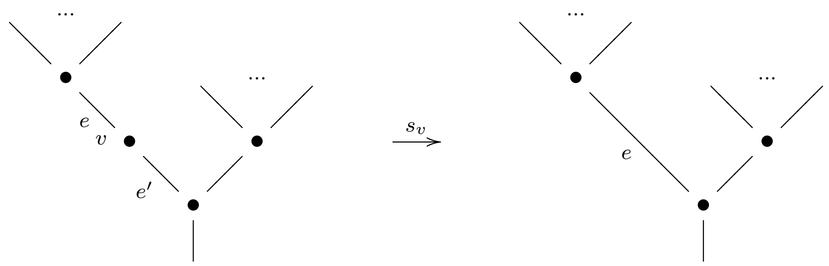
Proof. Denote by Ω_O the category Ω as defined in 4.1.9 and by Ω_A the one defined in 4.1.6. The multicategories $\Omega(T)$ are by definition in bijection with the planar trees T; such bijection composes with the ones defined above, giving an isomorphism of categories $\Omega_A \cong \Omega_O$. □

Thanks to the above results, one can consider Ω as a category of trees, just like in Δ one often states results by means of easily visualized graphs.
For a tree T, the multicategory $\Omega(T)$ will be often referred to just as T.

4.1.4 Faces and degeneracies

To end this section on the dendroidal category, I recall now how the arrows in Ω can be characterized in a way similar to that of the simplicial category. *Face* and *degeneracy* maps will be presented first following the operadic definition of Ω; the less intuitive algebraic approach is then needed to prove the analog of proposition 2.1.5.
Recall that in Δ a degeneracy map $[n+1] \to [n]$ creates a chain of morphism of lenght $n+1$ in $[n]$ by inserting an identity, while a face map $[n-1] \to [n]$ creates a chain of lenght $n-1$ in $[n]$ by composing arrows; exactly the same occurs here, in terms of $\circ_i$ composition.

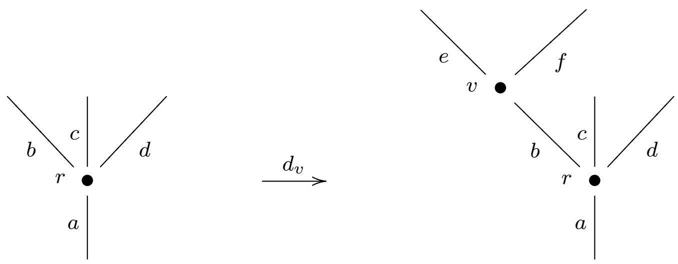
Definition 4.1.16. Let T a tree and $v \in T$ a vertex of valence 1 with input $in(v) = e$ and output $out(v) = e'$. Denote by T/v the tree obtained from T by deleting the vertex v and the edge e'. The arrow $s_v : \Omega(T) \to \Omega(T/v)$ between the corresponding multicategories in Ω, sending the object e' to e and the operation generated by v to 1_e is called a *degeneracy map*. In pictures this means



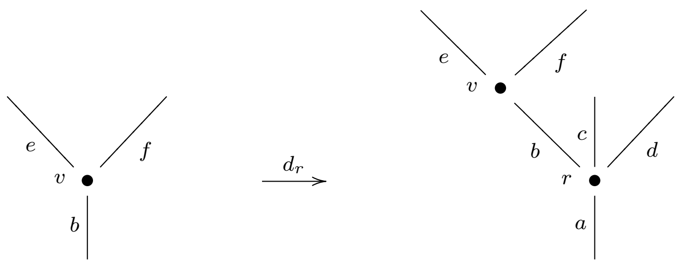
In the case of a linear tree L_n one precisely recovers the degeneracy maps of Δ.

For the *face* maps, it is useful to preliminary distinguish between *inner* and *outer* ones. Recall that an edge is said to be *inner* if it is not a leaf nor the root, otherwise it is called *outer*.

Definition 4.1.17. Let T a tree and $v \in T$ a vertex with exactly one inner edge attached to it. Create a new tree T/v by deleting from T the vertex v and all the outer edges attached to it. The inclusion $d_v : \Omega(T/v) \to \Omega(T)$ is called an *outer face* . For example

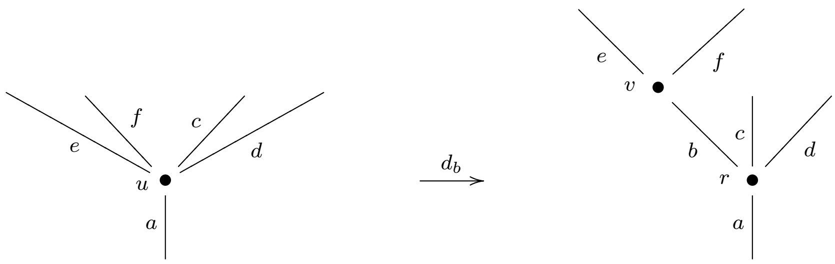


and



are outer faces.
For a tree T and an inner edge e, let T/e the tree obtained by contracting the edge e. If e is the output of a vertex v and the i-th input of a vertex u, $\Omega(T/e)$ is obtained from T by

composing $u \circ_i v$ and the inclusion $d_e : \Omega(T/e) \to \Omega(T)$ is called a *inner face*. For example



is an inner face map with $u = r \circ_1 v$.

Using the correspondence between trees and dendroidally ordered sets, it is quite easy now to understand the algebraic definitions of face and degeneracy maps.
As for the trees, given a dendroidally ordered set A, an element is called *outer* if it is either a leaf or the root, otherwise is called *inner*. Also, denote by A/B the dendroidally ordered set with underlying set $A \setminus B$ and order induced by A. If A has degree n, then for a *cluster* C(that is a set of n outer elements in a link $(a, (a_1, \ldots, a_n))$) or an inner element a, the trees A/C and $A/a := A/\{a\}$ have degree $n-1$: in fact in both cases one eliminates only one equivalence class of links, without affecting the others. On the other hand, a dendroidally ordered subset of A with degree $n-1$ must be of the form A/C or A/a, for a cluster C or an inner element a. It is also obvious that the construction $Tr(\cdot)$ brings inner elements to inner edges and clusters to clusters.
It makes sense then giving the following definitions.

Definition 4.1.18. Let A a dendroidally ordered set of degree n. Any inclusion $B \to A$ of a dendroidally ordered subset of degree $n-1$ is called a *face map* .

Definition 4.1.19. Let A a dendroidally ordered set and $v = (a_1, a_2)$ a unary link in A. The map $s_v : A \to A/a_1$

$$s_v(x) = \begin{cases} x & x \neq a_1 \\ a_2 & x = a_1 \end{cases}$$

is called a *degeneracy map* .

Note that one can "graft" degeneracies and faces, to obtain again degeneracies and faces. More generallly, suppose we are given maps of dendroidally ordered sets $f : A \to A'$ and $g : B \to B'$ such that B can be grafted on A along a and B' can be grafted on A' along $f(a) = g(a)$. Then the map $f \circ g$

$$f \circ g(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is a map of a dendroidally ordered sets.
In particular, when f is the identity and g a face (degeneracy, isomorphism), then $f \circ g$ is also a face (degeneracy, isomorphism), denoted by $A \circ g$.

It is possible now to see how maps in Ω factor as a composition of degeneracies and faces.

Theorem 4.1.20. *Any arrow $f : A \to B$ in Ω decomposes uniquely as*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\delta"'] & B \\ A' \arrow[r, "\pi"'] & B' \arrow[u, "\phi"'] \end{tikzcd}$$

where δ is a composition of degenracies, π an isomorphism and ϕ a composition of face maps.

Proof. The proof goes by induction on the degree $n = |A| + |B|$. For $n = 0$, $|A| = |B| = 0$ and A and B are isomorphic. For $n = 1$, either $A = \eta_a$ and f is the inclusion of the edge a, or A consists of the unary link $a < b$ and f is the degeneracy which identifies a and b.
Assume the theorem is proved for any $1 \leq n < m$ and $|A| + |B| = m$. There are four distinct cases.
i) $f(r_A) = b \neq r_B$ (where clearly r_A and r_B are the roots of A and B). Then f restricts to a map $\tilde{f} : A \to B_b$. By induction, $\tilde{f}$ factors as

$$\begin{tikzcd} A \arrow[r, "\tilde{f}"] \arrow[d, "\delta"'] & B \\ A' \arrow[r, "\pi"'] & B' \arrow[u, "\phi"'] \end{tikzcd}$$

On the other hand, the inclusion $B_b \hookrightarrow B$ can be factored as a sequence of faces, by repeatedly eliminating links from the root up to b.
ii) $f(r_A) = r_B$ and $f(s(r_A)) = s(r_B)$. Let $s(r_A) = (a_1, \ldots, a_k)$ and $s(r_B) = (b_1, \ldots, b_k)$. We then obtain maps $f_i : A_{a_i} \to B_{b_i}$ and $f_r : A_r \to B_r$ (where $A_r = \{r_A, a_1, \ldots, a_k\}$, $B_r = \{r_B, b_1, \ldots, b_k\}$ with order induced by A and B) such that f is the grafting $f = f_r \circ (f_1, \ldots, f_k)$. Each f_i decomposes as

$$\begin{tikzcd} A_i \arrow[r, "f_i"] \arrow[d, "\delta_i"'] & B_i \\ A'_i \arrow[r, "\pi_i"'] & B'_i \arrow[u, "\phi_i"'] \end{tikzcd}$$

By the argument above the graftings $A_r \circ (\delta_1, \ldots, \delta_k)$, $A_r \circ (\pi_1, \ldots, \pi_k)$, $B_r \circ (\phi_1, \ldots, \phi_k)$ give the desired decomposition for f.
iii) $f(s(r_A)) \neq s(r_B)$ and $f(x) \neq r_B)$ for any $x \in A$. Let $s(r_A) = (a_1, \ldots, a_k)$ and $b_i = f(a_i)$; then any element $y \in B$ with $r_B <_d y <_d b_i$ is inner and does not belong to the image of f. By repeatedly removing such elements y one obtains a dendroidally ordered subset $\hat{B}$ of B and a composition of inner face maps $\hat{\phi} : \hat{B} \to B$. Then clearly f factors as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\hat{f}"'] & B \\ & \hat{B} \arrow[u, "\hat{\phi}"'] \end{tikzcd}$$

Since $|\hat{B}| < |B|$ and $f(s(r_A)) \neq s(r_B)$, by inductive hypothesis and *ii)* $\hat{f}$ decomposes as

$$\begin{tikzcd} A \arrow[r, "\hat{f}"] \arrow[d, "\delta"'] & B \\ A' \arrow[r, "\pi"'] & B' \arrow[u, "\phi"'] \end{tikzcd}$$

Composing with $\hat{\phi}$ gives the decomposition for f.
iv) $f(r_A) = r_B$, $f(s(r_A)) \neq s(r_B)$ and there is $x \in A, x \neq r_A$ such that $f(x) = r_B$. In this case $s(r_A)$ must be a single element a; for,if not, $r_A \leq (x, a_1, \ldots, a_n)$ and $r_B \leq (r_B, f(a_1), \ldots, f(a_n))$, a contradiction. Let $s(r_A) = a$ with link (r_A, a); also, we must have $f(a) = r_B$. Let $\sigma : A \to A'$ the degeneracy collapsing a to r_A, so that f factors as

$$\begin{tikzcd} A \arrow[r, "\sigma"] \arrow[dr, "f"'] & A' \arrow[d, "f'"] \\ & B \end{tikzcd}$$

By induction, we obtain the desired decomposition of f. $\square$

4.2 The category *dSet* of Dendroidal Sets

In this session will be finally presented the category of *dendroidal sets*. Most results and constructions extend those of simplicial sets, from which get the names.

Definition 4.2.1. The category *dSet* of *dendroidal sets* is the category $Set^{\Omega^{op}}$ of presheaves on the dendroidal category Ω.

For a dendroidal set X one denotes by X_T the value of X at $\Omega(T)$; an element of X_T is then called a *dendrex of shape* T or a T*-dendrex.*

As for any presheaf category, a special role is played by the representable functors

Definition 4.2.2. For a tree T, the *representable presheaf* $\Omega(\cdot, T) : \Omega^{op} \to Set$ is denoted by $\Omega[T]$ and called the *standard* T*-dendrex.*

We know that any dendroidal set X, being a presheaf, can be written as a colimit of representables, while the T*-dendrices* of X correspond to natural transformations $\Omega[T] \Rightarrow X$. For a dendroidal set X, a *dendroidal subset* Y of X is a collection of subsets $Y_T \subset X_T$ such that the dendroidal structure of X makes Y into a dendroidal set. On the other hand, given a dendroidal set X and a collection Y of subsets $Y_T \subset X_T$, the *dendroidal set generated by* Y is the smallest dendroidal subset $\overline{Y}$ of X containing Y.
Recall from the previous section that there is an embedding i of the simplicial category Δ into Ω. This clearly defines, by restriction, a functor

$$i^* : dSet \to SSet$$

sending a dendroidal set X to its underlying simplicial set $i^*(X)_n = X_{L_n}$. In turn the left adjoint to i^* is the functor $i_!$ sending a simplicial set X to the dendroidal set $i_!(X)$ defined by

$$i_!(X)_T = \begin{cases} X_n & T = L_n \\ \emptyset & \textit{otherwise} \end{cases}$$

Just like for simplicial sets, the representable presheaves $\Omega[T]$ have two distinguished dendroidal subsets, the *boundary* and *horns*.

Definition 4.2.3. Let T a tree and $\alpha : S \to T$ a face map. The *α-face*, $\partial_\alpha \Omega[T]$, is the dendroidal subset of $\Omega[T]$ given by the image of $\Omega[S]$ through the map $\Omega[\alpha]$.

In other words $\partial_\alpha \Omega[T]$ is a copy of $\Omega[S]$, viewed as a dendroidal subset of $\Omega[T]$ thanks to $\Omega[\alpha]$, so that its dendrices of shape R are

$$\partial_\alpha \Omega[T]_R = \{R \xrightarrow{x} S \xrightarrow{\alpha} T | x \in \Omega[S]_R\}$$

When α is of the form d_e, i.e. obtained by contracting an edge e of T, the *α-face* of T is denoted by $\partial_e \Omega[T]$.

Definition 4.2.4. Let T a tree. The *boundary* of $\Omega[T]$, $\partial\Omega[T]$, is the dendroidal subset of $\Omega[T]$ generated by the union of all its faces.

$$\partial\Omega[T] = \bigcup_{\alpha \in \Phi(T)} \partial_\alpha \Omega[T]$$

where $\Phi(T)$ denotes the set of faces of T.

Definition 4.2.5. Let T a tree and α a face of T. The *α-horn* of T is the dendroidal subset $\Lambda^\alpha[T]$ of $\Omega[T]$ given by the union of all the faces of T except α

$$\Lambda^\alpha[T] = \bigcup_{\beta \neq \alpha \in \Phi(T)} \partial_\alpha \Omega[T]$$

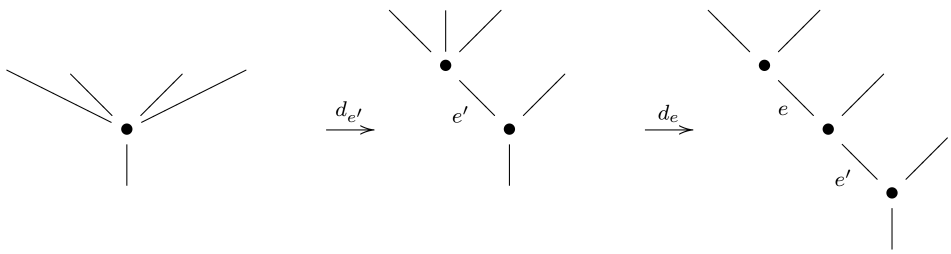
As for the boundaries, an α-horn is called *inner horn* when α is an inner face and *outer* otherwise; also, one denotes an α-horn by $\Lambda^e[T]$ when α corresponds to the contraction of an edge e.

Recall that the arrows in the simplicial category(and hence in *SSet*) are characterized by special identities, in particular we had for $i < j$ that $d^j d^i = d^i d^{j-1}$; an equality analogous to the latter holds in the case of dendroidal sets.
If $T_0 \to \cdots \to T_n$ is a sequence of n face maps, call their composition a *subface* of T_n of codimension n; also, denote by $\Phi_n(T)$ the set of subfaces of T of codimension n.
As for simplicial sets, given a map $\alpha : S \to T$ and a dendrex $x \in X_T$ for a dendroidal set X, one calls the S-dendrex $\alpha^* x = X(\alpha)(x)$

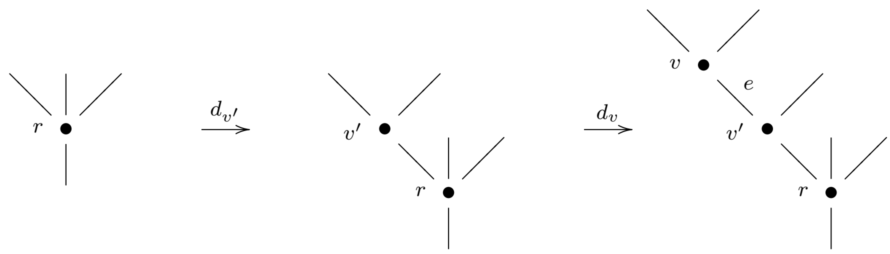
- a *face* of x if α is a *face* of T.
- a *subface* of x if α is a *subface* of T.
- a *degeneracy* of x if α is a composition of *degeneracies*. A generic dendrex y of X is called *degenerate* if it is a degeneracy of some dendrex x.

Proposition 4.2.6. *Let $S \xrightarrow{i} T$ a subface of T of codimension 2. Then i decomposes in exactly two ways as a composition of face maps.*

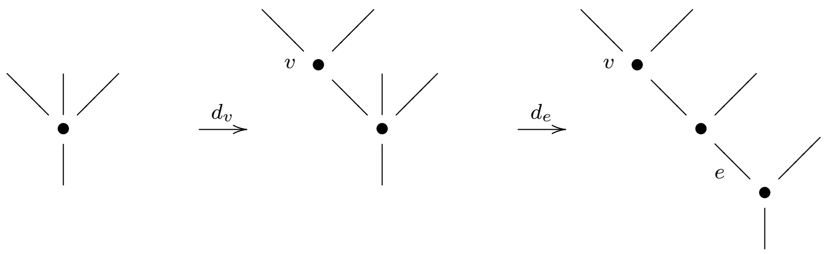
Proof. One can distinguish three ways in which S is viewed as a subtree of T. In one case, S is obtained by contracting two inner edges e, e' of T, in the second case one removes a cluster C from T and again a cluster C' from the so obtained tree T', while the last case involves the contraction of an edge and removing a cluster. In pictures, the first case is for example the following



while an example of the second case is given by



The third case can be pictured as



The two possible decompositions for i, corresponding to the above cases are then given by the following

$$
\begin{tikzcd}[column sep=small]
 & T/e \arrow[dr, "d_e"] & \\
S \arrow[ur, "d_{e'}"] \arrow[rr, "i"] \arrow[dr, "d_e"'] & & T \\
 & T/e' \arrow[ur, "d_{e'}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & T/C_v \arrow[dr, "d_v"] & \\
S \arrow[ur, "d_{v'}"] \arrow[rr, "i"] \arrow[dr, "d_u"'] & & T \\
 & T/e \arrow[ur, "d_e"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & T/e \arrow[dr, "d_e"] & \\
S \arrow[ur, "d_v"] \arrow[rr, "i"] \arrow[dr, "d_e"'] & & T \\
 & T/C_v \arrow[ur, "d_v"'] &
\end{tikzcd}
$$

$\square$

The above result simply asserts that, viewed as multicategories, a subtree S of degree $n-2$ of a tree T with $|T| = n$ is the intersection of two, and only two, subtrees T_1 and T_2 of T of degree $n-1$.

The construction of the boundary $\partial\Omega[T]$ as the union of this faces implicitly keeps track of such intersections, that is of the fact that two faces $\Omega[T_1], \Omega[T_2]$ must agree on the common subface $\Omega[S]$. This is made precise by the following, which generalizes the formula 2.3

Lemma 4.2.7. *Let T a tree. The boundary $\partial\Omega[T]$ is the coequalizer*

$$\coprod_{S \to T \in \Phi_2(T)} \Omega(S) \rightrightarrows \coprod_{R \to T \in \Phi_1(T)} \Omega(R) \to \partial\Omega[T]$$

where the two arrows are induced by the two factorizations $S \xrightarrow{\beta_i} T_i \xrightarrow{\alpha_i} T$ of an arrow $i : S \to T$.

To conclude this section, I show how dendroidal sets can be presented by means of skeleta in the same way as simplicial sets do (definition 2.2.5).
Recall that for a tree T, the degree of T is the number of vertices of T and is denoted by $|T|$; similarly, for a dendrex x of shape T define the *degree* of x to be the degree of T.

Definition 4.2.8. Let X a dendroidal set. The *n-skeleton* of X is the dendroidal set $Sk_n(X) \subseteq X$ generated by the dendrices of X of degree $\leq n$.

One has then a filtration for X, the *skeletal filtration*

$$Sk_0(X) \subseteq Sk_1(X) \subseteq \quad \cdots \quad \subseteq Sk_n(X) \subseteq \quad \cdots \quad \subseteq X = \bigcup_{n=0}^{\infty} Sk_n(X)$$

There are also commutative diagrams

$$\begin{tikzcd} \coprod_{(t,T)} \partial\Omega[T] \arrow[r] \arrow[d] & Sk_{n-1}X \arrow[d] \\ \coprod_{(t,T)} \Omega[T] \arrow[r] & Sk_n X \end{tikzcd}$$

where the pairs (t, T) are representatives for the isomorphism classes of non degenerate T-dendrices of degree n.

Unlike simplicial sets, the above diagram is not a pushout in general.

Definition 4.2.9. Let X a dendroidal set. The skeletal filtration of X is said *normal* if for every $n \geq 0$ the above square is a pushout. In that case, X is said *normal.*

4.3 Nerve of a Multicategory

As for the inclusion $\Delta \hookrightarrow Cat$, the operadic definition 4.1.9 of the dendroidal category implies that Ω embeds in *Multicat*; the usual argument, using theorem 1.4.4, provides us with the adjunction

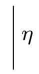
$$\tau_d : dSet \rightleftarrows Multicat : N_d$$

Definition 4.3.1. The functor $N_d : Multicat \to dSet$ is the *dendroidal nerve* functor. For a symmetric multicategory $\mathcal{P}$ one has

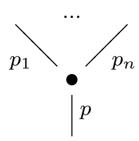
$$N_d(\mathcal{P})_T = Multicat(\Omega(T), \mathcal{P})$$

On the other hand, for a dendroidal set X, the functor τ_d constructs the multicategory $\tau_d(X)$ as follows

- the objects of $\tau_d(X)$ are the dendrices of shape η, where η is the unit tree with no vertices



- an arrow of $\tau_d(X)(p_1, \dots, p_n; p)$ is given by a dendrex of shape an n-corolla



- composition is induced by gluing along faces

It is then clear that there is an isomorphism $\tau_d N_d \cong id$, analog to the isomorphism $\tau N \cong id$.

As one would expect, the dendroidal nerve construction agrees with the nerve construction defined in 2.6, in the sense that in the square

$$\begin{tikzcd}
Cat \arrow[r, shift left, "j_!"] \arrow[d, shift left, "N"] & Multicat \arrow[l, shift left, "j^*"] \arrow[d, shift left, "N_d"] \\
SSet \arrow[r, shift left, "i_!"] \arrow[u, shift left, "\tau"] & dSet \arrow[l, shift left, "i^*"] \arrow[u, shift left, "\tau_d"]
\end{tikzcd}$$

hold the isomorphisms

$$\begin{aligned}
j_! \tau &\cong \tau_d i_! \\
N j^* &\cong i^* N_d \\
i_! N &\cong N_d j_!
\end{aligned}$$

4.4 Closed Monoidal structure on *dSet*

In this section it will be shown that the category of dendroidal sets carries a closed monoidal structure, which naturally extends that of simplicial sets and reflects the Boardman-Vogt tensor product of multicategories (3.5.1).

Proposition 4.4.1. *There is a unique (up to isomorphism) symmetric closed monoidal structure on dSet with the property that* $\Omega[S] \otimes \Omega[T] \cong N_d(\Omega(S) \otimes_{bv} \Omega(T))$ *for any* $\Omega(S), \Omega(T)$ *in* Ω.

Proof. The tensor will be constructed using theorem 1.4.4, as usual.
Let X a dendroidal set; then $X = \operatorname{colim} \Omega(T)$ where the colimit is indexed over the category of elements of X. Define the functor $\cdot \otimes X : \Omega \to dSet$ by

$$\Omega[S] \otimes X = \operatorname{colim} N_d(\Omega(S) \otimes_{bv} \Omega(T))$$

Notice first that for $X = \Omega[T]$, by 1.4.5, $\int_\Omega X$ has a terminal object and

$$\Omega[S] \otimes \Omega[T] = \operatorname{colim} N_d(\Omega(S) \otimes_{bv} \Omega(T)) \cong N_d(\Omega(S) \otimes_{bv} \Omega(T))$$

Hence the above definition of tensor is a good candidate.
Using theorem 1.4.4 we can extend the above tensor to all of *dSet* obtaining

$$Y \otimes X = \operatorname{colim} \operatorname{colim} N_d(\Omega(S) \otimes_{bv} \Omega(T))$$

where the double colimit is taken over the categories of elements of X and Y. The internal hom, right adjoint to $\cdot \otimes X$ is then given by

$$\underline{dSet}(X, Y)_S = dSet(\Omega[S] \otimes X, Y)$$

as one can check using the formula of theorem 1.4.4.
The fact that the above tensor is symmetric simply follows by symmetry of the Boardman-Vogt tensor $\otimes_{bv}$.

□

The following two results, which conclude this section, state that the functors $\tau_d : dSet \to Multicat$ and $i_! : SSet \to dSet$ are weak monoidal functors, that is they respect (up to isomorphism) the monoidal structures.

Proposition 4.4.2. *The functor $\tau_d : dSet \to Multicat$ is weak monoidal. That is, for dendroidal sets X and Y there is a natural isomorphism*

$$\tau_d(X \otimes Y) \cong \tau_d(X) \otimes_{bv} \tau_d(Y)$$

Proof. Recall the isomorphism $\tau_d N_d \cong id$ and the fact that the functors τ_d and $\otimes_{bv}$, being left adjoints, preserve colimits. Therefore we have

$$\begin{aligned}\tau_d(X \otimes Y) =& \tau_d(\operatorname{colim} \operatorname{colim} N_d(\Omega(S) \otimes_{bv} \Omega(T))) \cong \operatorname{colim} \operatorname{colim} \tau_d N_d(\Omega(S) \otimes_{bv} \Omega(T)) \cong \\ &\cong \operatorname{colim} \operatorname{colim}(\Omega(S) \otimes_{bv} \Omega(T)) \cong \operatorname{colim} \Omega(S) \otimes_{bv} \operatorname{colim} \Omega(T) \cong \tau_d(X) \otimes_{bv} \tau_d(Y)\end{aligned}$$

where the last isomorphism simply follows by the definition of $\tau_d(X)$ as the colimit $\operatorname{colim} \Omega(S)$ over the category of elements of X.

□

A similar argument proves that the functor $i_!$ is weak monoidal:
first, notice that the inclusion $j_! : Cat \to Multicat$ is also a monoidal functor: for categories $\mathcal{C}, \mathcal{D}$, the Boardman-Vogt tensor product $\mathcal{C} \otimes_{bv} \mathcal{D}$ has set of objects the product $\mathcal{C}_0 \times \mathcal{D}_0$ and arrows generated by the pairs $\phi \otimes_{bv} d, c \otimes_{bv} \psi$ for $\phi \in \mathcal{C}(c_1, c_2), \psi \in \mathcal{D}(d_1, d_2)$ and c, d objects of $\mathcal{C}$ and $\mathcal{D}$. Relations are generated by the equalities $(\phi \otimes_{bv} d) \circ (\phi' \otimes_{bv} d) = (\phi\phi' \otimes_{bv} d)$ and $(c \otimes_{bv} \psi) \circ (c \otimes_{bv} \psi') = c \otimes_{bv} \psi\psi'$, while the symmetric groups play no role being all the operations unary. Under the obvious identifications $(\phi \otimes_{bv} d) = (\phi, id_d)$ and $(c \otimes_{bv} \psi) = (id_c, \psi)$ one has

$$j_!(\mathcal{C} \times \mathcal{D}) \cong j_!(\mathcal{C}) \otimes_{bv} j_!(\mathcal{D})$$

Proposition 4.4.3. *For any two simplicial sets X, Y and any dendroidal set D, there are natural isomorphisms* *i)* $i_!(X \times Y) \cong i_!(X) \otimes_{bv} i_!(Y)$

ii) $i^*\underline{dSet}(i_!(X), D) \cong \underline{SSet}(X, i^*(D))$

iii) $i^*\underline{dSet}(i_!(X), i_!(Y)) \cong \underline{SSet}(X, Y)$

Proof. Point i) is proved by the following isomorphisms, and the fact that left adjoints preserve colimits, while right adjoints preserve products

$$\begin{aligned} i_!(X \times Y) \cong & i_!(\operatorname{colim} \Delta[n] \times \operatorname{colim} \Delta[m]) \cong \operatorname{colim} \operatorname{colim} i_!(\Delta[n] \times \Delta[m]) \cong \\ & \cong \operatorname{colim} \operatorname{colim} i_!(N[n] \times N[m]) \cong \operatorname{colim} \operatorname{colim} i_!(N([n] \times [m])) \cong \\ & \cong \operatorname{colim} \operatorname{colim} N_d j_!([n] \times [m]) \cong \operatorname{colim} \operatorname{colim} N_d j_!([n] \times [m]) \cong \\ & \cong \operatorname{colim} \operatorname{colim} N_d(j_![n] \otimes_{bv} j_![m]) \cong \operatorname{colim} \operatorname{colim} N_d(L_n \otimes_{bv} L_m) \cong \\ & \cong i_!(X) \otimes_{bv} i_!(Y) \end{aligned}$$

To prove ii), pick any simplicial set Z and consider the following isomorphism provided by the adjunctions $i_! \dashv i^*, \times \dashv \underline{SSet}(\cdot,\cdot), \otimes \dashv \underline{dSet}(\cdot,\cdot)$ and the fact that $i_!$ is monoidal

$$\begin{aligned} SSet(Z, i^*\underline{dSet}(i_!X, D)) \cong & dSet(i_!Z, \underline{dSet}(i_!X, D)) \cong dSet(i_!Z \otimes i_!X, D) \cong \\ & \cong dSet(i_!(Z \times Y), D) \cong SSet(Z \times X, i^*D) \cong \\ & \cong SSet(Z, \underline{SSet}(X, i^*D)) \end{aligned}$$

By the Yoneda's Lemma follows that $i^*\underline{dSet}(i_!(X), D) \cong \underline{SSet}(X, i^*(D))$.
The third point then is just a consequence of ii), once D is replaced by $i_!Y$ and noting that $i^*i_! \cong id$. $\square$

With the same argument one also proves that the dendroidal nerve N_d commutes with internal homs.

Proposition 4.4.4. *For any two multicategories $\mathcal{P}$ and $\mathcal{Q}$ there is a natural isomorphism*

$$\underline{dSet}(N_d(\mathcal{P}), N_d(\mathcal{Q})) \cong N_d(\underline{Multicat}(\mathcal{P}, \mathcal{Q}))$$

Proof. Again, consider a dendroidal set X and the sequence of isomorphisms

$$\begin{aligned} dSet(X, N_d(\underline{Multicat}(\mathcal{P}, \mathcal{Q}))) \cong & Multicat(\tau_d X, \underline{Multicat}(\mathcal{P}, \mathcal{Q})) \cong Multicat(\tau_d X \otimes_{bv} \mathcal{P}, \mathcal{Q}) \cong \\ & \cong Multicat(\tau_d \otimes_{bv} \tau_d N_d(\mathcal{P}), \mathcal{Q}) \cong Multicat(\tau_d(X \otimes N_d(\mathcal{P})), \mathcal{Q}) \cong \\ & \cong dSet(X \otimes N_d(\mathcal{P}), N_d(\mathcal{Q})) \cong dSet(X, \underline{dSet}(N_d(\mathcal{P}), N_d(\mathcal{Q}))) \end{aligned}$$

$\square$

Chapter 5

Infinite Loop Spaces, Spectra and May's Operads

I will recall here few facts about the theory of *infinite loop spaces*. The study of infinite loop spaces is one (probably the most known) of the branches of mathematics in which operads and multicategories arose, as we shall see in section 5.2.
The following lines are by no means a detailed exposition of the theory of loop spaces and spectra. I will only review a couple of definitions and results, which convinced me of the fact that dendroidal sets should be thought of inside a certain framework, as I will make clear in the last part of this thesis. I hope and believe that the reader, once read these few pages, will already guess my point of view and agree with me.
A nice account of the topics I will consider here is given in [Ad2], while more detailed treatments can be found in [Ad1],[Ma1],[Ma2],[Th1],[Th2].

5.1 Spectra and infinite loop spaces

In this section we will work with based spaces. That is, pairs $(X, *)$ where X is a topological space and $* \in X$ is a distinguished point. Given pointed spaces (X, x) and (Y, y), a map $f : (X, x) \to (Y, y)$ is a map $f : X \to Y$ in *Top* such that $f(x) = y$. Denote by Top_* the category of pointed spaces.

Recall the following constructions from algebraic topology.

Definition 5.1.1. Let $(X, x), (Y, y) \in \mathit{Top}_*$. The *smash product* of (X, x) and (Y, y) is the quotient space

$$X \wedge Y = \frac{X \times Y}{(X \times \{y\}) \cup (\{x\} \times Y)}$$

with base point $(X \times \{y\}) \cup (\{x\} \times Y)$.

In the case when (Y, y) is the 1-sphere $S^1 = I/\partial I$ with base point $\partial I = \partial[0, 1]$ one obtains

Definition 5.1.2. For a pointed space (X, x) the *reduced suspension* SX of X is the space

$$SX = X \wedge S^1 = \frac{X \times I}{X \times \{0, 1\} \cup \{x\} \times I}$$

On the other hand we have

Definition 5.1.3. For a based space (X, x), the *loop space* of X at x is the space

$$\Omega X = Top_*(S^1, X)$$

with base point the constant map sending S^1 to x.

The topology on ΩX is the *compact-open* topology:

Definition 5.1.4. For spaces X and Y, the *compact-open topology* on $Y^X = Top(X, Y)$ is the topology generated by the subsets

$$\{f | f(K) \subset U\}$$

for $K \subset X$ compact and $U \subset Y$ open.

Noting that $SS^n = S^{n+1}$ (where S^n is the n-sphere), the above constructions can be iterated and give rise to adjoint functors S^k and Ω^k, the *k-th suspension* and *k-fold loop space* functor .
Going back to non based spaces, recall that a map $f : X \to Y$ between topological spaces is said to be a *weak equivalence*, if it induces an isomorphism between the homotopy groups $\pi_k(X)$ and $\pi_k(Y)$ for all base points $x \in X, y \in Y$ and every $k \geq 1$ and an isomorphism between the sets $\pi_0(X)$ and $\pi_0(Y)$ of connected components of X and Y.

We now have all the terminology needed in order to define spectra and infinite loop spaces.

Definition 5.1.5. A *spectrum* is a sequence of based spaces X_i, $i \in \mathbb{Z}$, together with maps

$$\epsilon : SX_i \to X_{i+1}$$

or equivalently, by adjunction, maps

$$\tilde{\epsilon} : X_i \to \Omega X_{i+1}$$

What we are actually interested in is a particular kind of spectra:

Definition 5.1.6. A *bounded Ω-spectrum* is a sequence $\{(X_i, f_i)|i \geq 0\}$ of based spaces, together with *weak equivalences*

$$f_i : X_i \to \Omega X_{i+1}$$

Following [Ma1] and [Th2], by an abuse of terminology, I will refer to bounded Ω-spectra just as spectra, and restrict the attention to a special class among them.

Definition 5.1.7. A spectrum $\{(X_i, f_i)|i \geq 0\}$ is said -1-*connective* if each space X_i is $i-1$ connected, i.e. $\pi_j(X_i) = 1$ for $j < i$.

As always, one wishes to define maps between spectra and hence get a category. This can be easily done, so that a map $g : X \to Y$ amounts to maps $g_i : X_i \to Y_i$ which are compatible with the structure maps, in the sense that the diagram

$$\begin{tikzcd}
X_i \arrow[r, "f_i"] \arrow[d, "g_i"'] & \Omega X_{i+1} \arrow[d, "\Omega g_{i+1}"] \\
Y_i \arrow[r, "f'_i"] & \Omega Y_{i+1}
\end{tikzcd}$$

is homotopy commutative for every i, i.e. the maps $f_i' \circ g_i$ and $\Omega g_{i+1} \circ f_i$ are homotopic.

Denote then by *Spectra* the category of -1-connective spectra and maps as just defined.

The above definitions now naturally lead to the following

Definition 5.1.8. A based space X is said an *infinite loop space* if there is a spectrum $\{X_i, f_i\}$ such that $X = X_0$.

It is known that spectra play a crucial role in *generalized (co)homology theories* as well as in *stable homotopy theory*; anyway it is not the purpose of this chapter to address these topics. On the other hand, the importance of spectra and their close connection with infinite loop spaces imposes the need of determining conditions on a space X to be of the homotopy type of an infinite loop space. To this aim come in fact operads, as we shall see below.

5.2 May's machinery

J.P. May introduced operads in [Ma1], in a form that now would go under the name of *topological operads*, in order to recognize infinite loop spaces and, in case, to build a spectrum $\{X_i, f_i\}$ such that X_0 is of the homotopy type of a given space X. Such procedure is now known as *May's machinery* and basically takes advantage of the following

i) Any operad $\mathcal{P}$ in a category $\mathcal{C}$ can be transformed into a monad (P, η, μ) on $\mathcal{C}$.

ii) Given a monad (T, η, μ) and an algebra $TX \to X$, one can construct a simplicial object in $\mathcal{C}$, the simplicial resolution of X.

What May considered are certain operads $\mathcal{C}_n$ in the category $\mathcal{U}$ of compactly generated spaces, and hence the corresponding monads C_n. Roughly speaking, given an algebra X for the monad C_n, the geometric realization of the simplicial space arising as in ii) turns out to be a n-th de-looping of the given space X; iterating, one can determine whether X is of the homotopy type of an infinite loop space Y_0 and find a spectrum with zeroth space Y_0.

Definition 5.2.1 (J.P. May)**.** An *operad* $\mathcal{P}$ consists of spaces $\mathcal{P}(k) \in \mathcal{U}$ for $k \geq 0$ such that $\mathcal{P}(0) = \{*\}$ is a singleton, together with

i) *continuous* functions $\mu : \mathcal{P}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k) \to \mathcal{P}(n_1 + \ldots + n_k)$ such that the associativity formula

$$\begin{gathered} \mu(\mu(c, d_1, \ldots, d_k), d_1^1, \ldots, d_{n_1}^1, \ldots, d_1^k, \ldots, d_{n_k}^k) \\ \shortparallel \\ \mu(c, \mu(d_1, d_1^1, \ldots, d_{n_1}^1), \ldots, \mu(d_1, d_1^k, \ldots, d_{n_k}^k) \end{gathered}$$

is satisfied.

ii) An *identity element* $1 \in \mathcal{P}(1)$ such that $\mu(1, d) = d$ and $\mu(d, 1, \ldots, 1) = d$.

iii) An action of the symmetric group Σ_k on $\mathcal{P}(k)$ compatible with composition, in the sense that for $c \in \mathcal{P}(k), \sigma \in \Sigma_k$ and $d_i \in \mathcal{P}(n_i), \sigma_i \in \Sigma_{n_i}$ holds

$$\mu(\sigma^* c, \sigma_1^* d_1 \ldots, \sigma_k^* d_k) = (\sigma(\sigma_1, \ldots, \sigma_k))^* \mu(c, d_1, \ldots, d_k)$$

where $\sigma(\sigma_1, \ldots, \sigma_k) \in \Sigma_{n_1+\ldots+n_k}$ is the product defined in 3.3.1.

Let me now show how to associate to an operad $\mathcal{P}$ as defined in 3.3.4 a monad P.

Construction 5.2.2. (Monad associated to an operad)
Let X a set. Define

$$PX = \coprod_{j=0}^{\infty} (\mathcal{P}(j) \times X^j)$$

The *unit* η and *multiplication* μ are defined in the obvious way. We have

$$\begin{array}{ccc} X & \xrightarrow{\eta} & PX \\ x & \longmapsto & (1, x) \end{array}$$

where 1 is the unit of the operad and $(1, x) \in \mathcal{P}(1) \times X$.
Noting that $P^2X = \coprod \mathcal{P}(n) \times (\coprod \mathcal{P}(j) \times X^j)^n \cong \coprod (\mathcal{P}(n) \times (\mathcal{P}(j_1) \times X^{j_1}) \times \cdots \times (\mathcal{P}(j_k) \times X^{j_n}) \cong \coprod \mathcal{P}(n) \times \mathcal{P}(j_1) \times \cdots \times \mathcal{P}(j_n) \times X^k$ for $k = j_1 + \ldots + j_n$, one defines

$$\begin{array}{ccc} P^2X & \xrightarrow{\mu} & PX \\ (p, p_1, \ldots, p_n, \overrightarrow{x}) & \longmapsto & (\mu(p, p_1, \ldots, p_n), \overrightarrow{x}) \end{array}$$

The axioms for the operad finally ensure that P is a monad. It is also easily seen that maps of operads translate into maps of monads; given a map of operads $\phi = (\phi_j) : \mathcal{P} \to \mathcal{Q}$, the corrisponding map of monads is $\tilde{\phi}_X = \coprod \phi_j \times id_X^j$.
Finally, an algebra $PX \xrightarrow{f} X$ for the monad P amounts to maps $f_j : \mathcal{P}(j) \times X^j \to X$, or equivalently to maps $\tilde{f}_j : \mathcal{P}(j) \to X^{X^j}$. In other words, the algebras for the monad P are in $1:1$ correspondence with the algebras for the operad $\mathcal{P}$.

□

Definition 5.2.3. Let (T, μ, η) a monad on a category $\mathcal{C}$. A *right T-functor* is a colax map of monads $(F, \phi) : T \to 1_{\mathcal{C}'}$ from T to the identity monad on a category $\mathcal{C}'$.

Construction 5.2.4. Now let given a right T-functor $F : \mathcal{C} \to \mathcal{C}'$, a monad (T, μ, η) and an algebra $TX \xrightarrow{h} X$. We can associate to the triple (F, T, X), functorially in each variable, a simplicial object in $\mathcal{C}'$, $Bar(F, T, X)$, defined as

$$Bar(F, T, X)_q = FT^qX$$

with faces and degeneracies given by

$$\begin{array}{ll} d_0 = \lambda & \lambda : FT^qX \to FT^{q-1}X \\ d_i = FT^{i-1}\mu & \mu : T^{q-i+1}X \to T^{q-i}X, \quad 0 < i < q \\ d_q = FT^{q-1}h & h : TX \to X \\ s_i = FT^i\eta & \eta : T^{q-i}X \to T^{q-i+1}X \end{array}$$

□

Note in particular that (T,μ) itself is a right T-functor. Given a T-algebra $TX \xrightarrow{h} X$, we have then the simplicial object and augmentation h

$$\cdots \rightrightarrows T^2X \rightrightarrows TX \xrightarrow{h} X$$

More generallly, for a simplicial object Y_* and an object X in $\mathcal{C}$, there is a $1:1$ correspondence between maps $h : Y_0 \to X$ such that $hd_0 = hd_1$ and maps of simplicial objects $Y_* \to X_*$, where X_* is the constant simplicial object.

The last ingredients we need are the operads $\mathcal{C}_n$ cited above, also known as *little n-cubes operads*.

Definition 5.2.5 (Little n-cubes operad)**.** Denote by I^n the unit n-cube, and let J^n its interior. A *little n-cube* is a linear embedding of J^n into itself. Define $\mathcal{C}_n(k)$ to be the set of k-tuples $(c_1,\ldots,c_k)$ of little n-cubes with pairwise disjoint images. Then $(c_1,\ldots,c_k)$ can be regarded as a map from the disjoint union $\coprod J^n$ of k copies of J^n to J^n.
The unit if the operad is just the identity on J^n. The action of the symmetric groups is given by $\sigma^*(c_1,\ldots,c_k) = (c_{\sigma 1},\ldots,c_{\sigma k})$, while composition of operations is induced by the universal poroperty of the coproduct.
The topology on the spaces $\mathcal{C}_n(k)$ is that of a subspace of the space of continuous functions $\coprod J^n \to J^n$.

There are two facts, concerning the above operads, that are crucial in the work of May.
The operads $\mathcal{C}_n$ are connected by morphisms $\sigma_n : \mathcal{C}_n \to \mathcal{C}_{n+1}$, embedding $\mathcal{C}_n$ in $\mathcal{C}_{n+1}$. These are given by $\sigma_{n,k}(c_1,\ldots,c_k) = (c_1 \times 1,\ldots,c_k \times 1)$. Thanks to them it is possible to define an operad $\mathcal{C}_\infty$ as a colimit $\operatorname{colim}\mathcal{C}_n$.
Using the corresponding monads, one can define a morphism of algebras $C_nX \xrightarrow{\alpha_n} \Omega^nS^nX$ as the composite

$$C_nX \xrightarrow{C_n\eta} C_n\Omega^nS^nX \xrightarrow{\theta_n} \Omega^nS^nX$$

where Ω^n is the n-fold loop functor, S^n the n-fold suspension and η the unit of the corresponding adjunction; θ_n is the obvious action of the operad $\mathcal{C}_n$ on an n-fold loop space $\Omega^nX = Top_*(S^n, X)$.

Recall that given an operad $\mathcal{P}$, by a *$\mathcal{P}$-space* one means an algebra for $\mathcal{P}$ in Top_*.
Also, remind that in the present section we are dealing with topological operads and more generally with constructions over Top_*. We have seen in 5.2.2 how to associate a monad to an operad in *Set*; the same can be done with any topological operad $\mathcal{P}$, provided that we take care of the topologies already present on the $\mathcal{P}(j)$'s. Similarly, one can define a geometric realization for *simplicial spaces*, as the one constructed in 5.2.4.
We can now state the main theorem of this chapter

Recognition Theorem 5.2.6 (J.P. May)**.** *Every n-fold loop space is a $\mathcal{C}_n$-space and every connected $\mathcal{C}_n$-space has the weak homotopy type of an n-fold loop space, $n \le \infty$.*

We don't need here to see the proof in detail. Briefly, given a $\mathcal{C}_n$-space X, one uses the above constructions in order to get a diagram

$$X \xleftarrow{h} Bar(C_n, C_n, X) \xrightarrow{Bar(\alpha_n,1,1)} Bar(\Omega^nS^n, C_n, X) \to \Omega^nBar(S^n, C_n, X) \tag{5.1}$$

Applying geometric realization, the above arrows are brought to weak homotopy equivalences and $|Bar(S^n, C_n, X)|$ appears as the n-th delooping of X.
The ∞ case follows, roughly, by passage to limits.

The above result only takes care of connected spaces. Before stating the more general version it is useful to recall a few definitions from [Ma1] and [Ma2].

Definition 5.2.7. An *H-space* is a space X with a base point e and a product map $X \times X \xrightarrow{\mu} X$ such that both left and right multiplication by e are homotopic to the identity. An H-space X is said to be *grouplike* if $\pi_0(X)$ is a group under the product induced by μ.

The spaces we are iterested in (namely, algebras for the operads $\mathcal{C}_n$), are usually not grouplike, despite the fact that obviously any loop space is (in fact if $Y = \Omega X$, then $\pi_0(Y) \cong \pi_1(X)$, and ΩX is an H-space under concatenation of loops). Anyway, one can still construct a grouplike space Y out of a given H-space X, in a way that is unique up to (weak) homotopy equivalences. Such space Y is then called the *group completion* of X.

Definition 5.2.8. A morphism of operads $\psi : \mathcal{P} \to \mathcal{Q}$ is said a *local* (Σ-)*equivalence* if each ψ_j is a (Σ_j-equivariant) homotopy equivalence.

Definition 5.2.9. An operad *over* a discrete operad $\mathcal{D}$ is an operad $\mathcal{P}$ together with a morphism (an *augmentation*) $\epsilon : \mathcal{P} \to \mathcal{D}$, such that $\pi_0\epsilon : \pi_0\mathcal{P} \to \mathcal{D}$ is an isomorphism of operads.

Definition 5.2.10. Regard the operads A_∞ and E_∞ defined in example 3.3.5 as topological operads:

- An A_∞ *operad* is a Σ-free operad $\mathcal{P}$ over the operad A_∞ such that the augmentation $\epsilon : \mathcal{P} \to A_\infty$ is a local Σ equivalence. An A_∞*-space* is a $\mathcal{P}$-space for an A_∞ operad $\mathcal{P}$.

- An E_∞ *operad* is a Σ-free operad $\mathcal{P}$ over the operad E_∞ such that the augmentation $\epsilon : \mathcal{P} \to A_\infty$ is a local equivalence. An E_∞*-space* is a $\mathcal{P}$-space for an E_∞ operad $\mathcal{P}$.

It turns out in particular that the operads $\mathcal{C}_1$ and $\mathcal{C}_\infty$ are respectively A_∞ and E_∞ operads.

Finally, replacing the connected space of theorem 5.2.6 by any E_∞-space X, one has (cf. [Ma2]) that the map $Bar(\alpha_n, 1, 1)$ in the diagram 5.1 displays a group completion (and hence a weak homotopy equivalence in the case when X is grouplike), so that $|Bar(S^n, C_n, X)|$ appears as the n-th delooping of the group completion of X.
In practice, E_∞-spaces describe infinite loop spaces. In fact any infinite loop space is an E_∞-space; on the other hand an E_∞-space determines an infinite loop space uniquely up to homotopy.

5.2.1 Connection with Monoidal Categories

Recall from section 1.5 that a strict monoidal category is a commutative monoid in *Cat*; equivalently, a monoidal category $\mathcal{M}$ is an algebra in *Cat* for the operad E_∞.
This perspective allows us to place the theory of monoidal categories in the context of operads and in particular of infinite loop spaces.
We can in fact apply the classifying space functor to $\mathcal{M}$, in order to obtain a space $B\mathcal{M}$,

naturally endowed with an action by the (topological) operad E_∞. In other words we have the following

Proposition 5.2.11. *The classifying space $B\mathcal{M}$ of a strict monoidal category $\mathcal{M}$ is an E_∞-space. That is, the restriction of B to StrSymmMonCat factors as*

$$\begin{tikzcd} StrSymmMonCat \arrow[r, "\iota"] \arrow[dr] & Cat \arrow[r, "B"] & Top \\ & E_\infty - spaces \arrow[ur, "U"'] & \end{tikzcd}$$

Thanks to the above factorization and the machinery developed by May, one can construct a functor *Spt* : *StrSymmMonCat* $\to$ *Spectra*. The most interesting result, due to Thomason ([Th1], [Th2]), is that the functor *Spt* defines an equivalence between suitable localizations of the two categories *StrSymmMonCat* and *Spectra*.

Definition 5.2.12. Let (X_i, f_i) a spectrum. The weak equivalences $f_i : X_i \to \Omega X_{i+1}$ induce isomorphisms

$$\pi_{k+n}(X_n) \cong \pi_{k+n+1}(X_{n+1})$$

Define the *stable homotopy groups* of the spectrum X as

$$\pi_k^s(X) = \operatorname*{colim}_n \pi_{k+n}(X_n) \cong \pi_k(X_0)$$

Definition 5.2.13. A map $f : X \to Y$ of spectra is a *stable weak equivalence* if the induced map $\pi_k^s(f) : \pi_k^s(X) \to \pi_k^s(Y)$ is an isomorphism for all k.

We can now state Thomason's result more precisely

Theorem 5.2.14 (R.W. Thomason)**.** *The functor Spt : StrSymmMonCat $\to$ Spectra defines an equivalence of categories between the localization of Spectra with respect to the stable weak equivalences and of StrSymmMonCat with respect to the maps that Spt takes to stable weak equivalences.*

Chapter 6

Conclusions

The main subject of this final chapter is the problem of a *geometric realization* for dendroidal sets. I stress immediately the fact that the name "geometric realization" is, if not wrong, at least misleading; I simply borrow it from the theory of simplicial sets. I believe anyway that, once the meaning is made precise, none will complain. I will refer to a *dendroidal geometric realization*, dropping the adjective *dendroidal* when the context is clear; such construction should appear as an extension of the well known geometric realization of simplicial sets.

In the first section I list the *desiderata* for a geometric realization functor; I will then give a definition and deduce an easy result, behaving as if a geometric realization really exists. In section 6.2 I will present a naive approach to the problem; the comments following it will justify my choices of section 6.1 and introduce section 6.3, where I will finally present Dendroidal Sets under a new light.

6.1 The problem of Realization

Recall the main properties of the *geometric realization functor* $|\cdot| : \mathit{SSet} \to \mathit{Top}$ of section 2.3:

i) $|\cdot|$ is a monoidal functor, where the monoidal structure on *SSet* and *Top* is given by the cartesian product.

ii) $|\cdot|$ preserves finite limits.

iii) $|\cdot|$ is *conservative* (i.e. $|f|$ isomorphism $\Rightarrow f$ isomorphism).

The dendroidal geometric realization functor should obviously satisfy properties analogous to the above. What makes the difference, is that my geometric realization does **not** take values in *Top*; I will motivate this later, after illustrating my first attempt in constructing a geometric realization. By now, I simply conjecture the existence of a monoidal biclosed category of *structured topological spaces*, which I denote by *StrTop*, together with an adjunction

$$\iota : \mathit{Top} \rightleftarrows \mathit{StrTop} : \upsilon$$

so that *StrTop* comes as an extension of topological spaces, just like dendroidal sets extend simplicial ones, and multicategories extend categories; roughly, the objects of *StrTop* should be topological spaces carrying some sort of algebraic structure. It seems also appropriate to require ι to be (weak) monoidal.
I also require that topological spaces of interest, such as A_∞-*Spaces* and E_∞-*Spaces* are objects of this category, so to get the following embeddings and factorizations

$$\begin{tikzcd} A_\infty\text{-}Spaces \arrow[r, "\alpha"] \arrow[dr, "U"'] & StrTop \arrow[d, "\upsilon"] \\ & Top \end{tikzcd} \qquad \begin{tikzcd} E_\infty\text{-}Spaces \arrow[r, "\epsilon"] \arrow[dr, "U"'] & StrTop \arrow[d, "\upsilon"] \\ & Top \end{tikzcd}$$

where clearly α and ϵ are embeddings, and U is the forgetful functor.
We can pass now to our wishes, hoping that Christmas will come soon.

Desiderata 6.1.1. There is a functor, the *dendroidal geometric realization*, denoted by $|\cdot|_d$

$$|\cdot|_d : dSet \to StrTop$$

such that:

i) $|\cdot|_d$ is monoidal, where the monoidal structure on *dSet* is given by the tensor product defined in 4.4.1.

ii) $|\cdot|_d$ preserves finite limits.

iii) $|\cdot|_d$ is conservative.

Furthermore, $|\cdot|_d$ agrees with the geometric realization $|\cdot|$ of simplicial sets, in the sense that the following diagrams commute up to isomorphism

$$\begin{tikzcd} SSet \arrow[d, "|\cdot|"'] & dSet \arrow[l, "i^*"'] \arrow[d, "|\cdot|_d"] \\ Top & StrTop \arrow[l, "\upsilon"] \end{tikzcd} \qquad \begin{tikzcd} SSet \arrow[r, "i_!"] \arrow[d, "|\cdot|"'] & dSet \arrow[d, "|\cdot|_d"] \\ Top \arrow[r, "\iota"'] & StrTop \end{tikzcd}$$

Let me now assume that somewhere someone constructed a functor as the one above.

Definition 6.1.2. The *dendroidal classifying space* of a symmetric multicategory $\mathcal{P}$ is the structured space $dB\mathcal{P}$ obtained by the composite

$$Multicat \xrightarrow{N_d} dSet \xrightarrow{|\cdot|_d} StrTop$$

As I wrote, the dendroidal geometric realization should take in account the results by Thomason and May. We obtain in fact

Proposition 6.1.3. *The dendroidal classifying space of a symmetric monoidal category is an* E_∞*-space.*

Proof. The statement is analogous to say that

$$\begin{tikzcd}
StrSymmMonCat \arrow[rr, "U"] \arrow[dr] \arrow[dd, "B"'] & & Multicat \arrow[dd, "dB"] \\
& E_\infty\text{-}Spaces \arrow[dl, "U"'] \arrow[dr, "\epsilon"] & \\
Top & & StrTop \arrow[ll, "v"]
\end{tikzcd}$$

is a commutative diagram. But in fact the outer square is given by the following:

$$\begin{tikzcd}
StrSymmMonCat \arrow[r, "U"] \arrow[d, "N"'] & Multicat \arrow[d, "N_d"] \\
SSet \arrow[d, "|\cdot|"'] & dSet \arrow[l, "i^*"] \arrow[d, "|\cdot|_d"] \\
Top & StrTop \arrow[l, "v"]
\end{tikzcd}$$

and the upper rectangle commutes thanks to the following subdivision

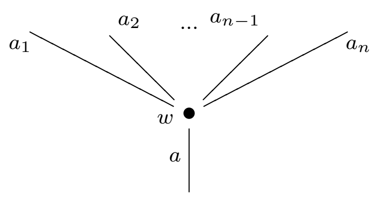
$$\begin{tikzcd}
StrSymmMonCat \arrow[rr, "U"] \arrow[dr] \arrow[dd, "N"'] & & Multicat \arrow[dl, "j^*"'] \arrow[dd, "N_d"] \\
& Cat \arrow[dl, "N"'] & \\
SSet & & dSet \arrow[ll, "i^*"]
\end{tikzcd}$$

□

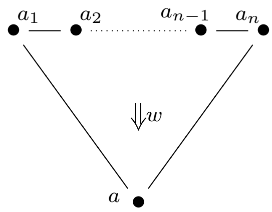
6.2 A first approach

I will now sketch a possible technique to solve our problem. It is a very naive construction, and in fact seems to fail our aim: the main question is that it closely imitates the geometric realization for simplicial sets rather than extending it; such construction was suggested by the theory of *opetopes*(cf. [Le]) and by the *Stasheff Associahedron*.
Recall that, to define a geometric realization, it suffices to give a functor on Ω.

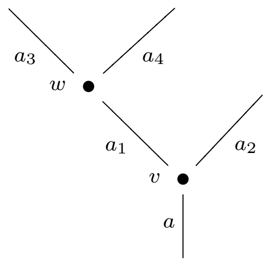
Construction 6.2.1. Define a dendroidal geometric ralization $|\cdot|_d : dSet \to Top$ to take values directly in *Top*. To understand better the construction, notice that an operation w of the form



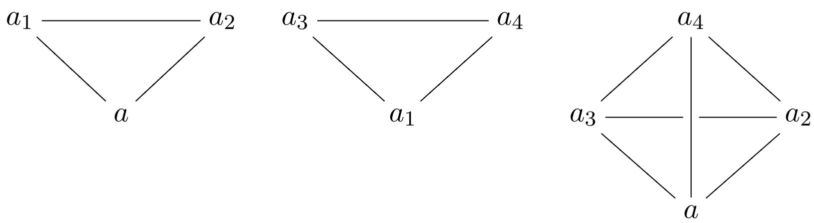
can also be represented as



It should be then quite clear how to proceed. An operation of arity n is sent to a topological n-simplex. This takes care of the n-corollas. For the other trees , we must also consider their height, which should be thought of as the length of a composition chain. For example a tree

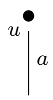


expresses the compositon $v \circ_1 w$. This should be sent to a 4-dimensional simplex with faces



□

Though complicated, the above construction still sounds reasonable. There are three points that actually can convince the reader to not even try to develop it in detail.
The first question concerns 0-ary operations: the only acceptable representative in *Top* for a tree



seems to be a point. But points should represent dendrices of shape η, that is 0-simplices. This is just a particular case of the second problem, that is the above realization does not satisfy point (iii) of 6.1.1. In fact, according to the above, the image under $|\cdot|_d$ of a dendrex of shape

is the same of one of shape an n-corolla. In practice, we can not distinguish between n-ary operations, composition chains of 1-ary arrows and composition schemes.
The last point finally should explain the reason of changing the target category. Recall Quillen's result (2.4.1) asserting that the covering spaces of the classifying space of a category $\mathcal{C}$ are in correspondence with the localizations of $\mathcal{C}$. If we try to construct a geometric realization in a way resembling to the one above, we would then get something like a "localized multicategory". There is only one simple obstruction to that: arrows in multicategories are n to 1, hence a similar result wouldn't make any sense.

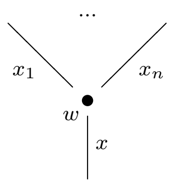
6.3 Dendroidal Sets as Simplicial Sets with structure

I will now give a picture of the second approach to our problem; it follows the slogan "multicategories describe algebraic structures".
We have seen in the previous section good motivations for considering $|\cdot|_d$ as a functor with values in a category larger than *Top*, still provided that such category contains *Top* in a reasonable way. Having in mind that topological spaces can be replaced by simplicial sets, the idea is now to view dendroidal sets as simplicial sets with some extra property.
Recall that every dendroidal set X has an underlying simplicial set i^*X; I refer to it as the "simplicial part" of X, and the dendrices of shape a linear tree will be simply called simplices. The goal is to use dendrices of shape non-linear trees in order to define operations on i^*X. Moreover, I want such operations to be as closest as possible to maps of simplicial sets; this means that I will define them level-wise, in a way that respects faces and degeneracies. Doing so, in the special case of the multicategory $\mathcal{M}$ underlying a monoidal category, we recover the monoid structure induced by the tensor on the nerve $N\mathcal{M}$.
I will work first in complete generality, then consider the special case of monoidal categories and comment on the construction.

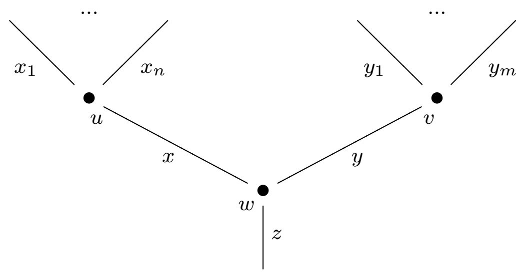
Construction 6.3.1. Let X a dendroidal set; I refer to the operations on the n-simplices as *n-operations.*

Step 1. 0-operations

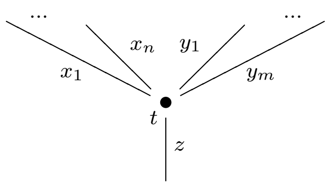
Let $x_1, \ldots, x_n$ 0-simplices; there can be a dendrex w of shape an n-corolla



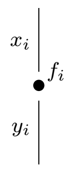
Define an operation $\otimes_w$ on $x_1, \ldots, x_n$ by $x_1 \otimes_w \cdots \otimes_w x_n = x$.
Trees with more than one vertex serve then to gather operations and control them; in some sense they provide associativity laws. For example the dendrex



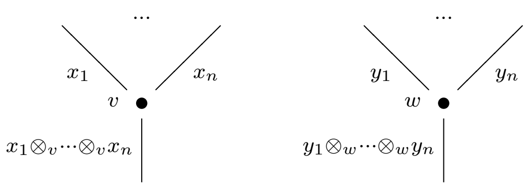
has faces (i.e. products) $x_1 \otimes_u \cdots \otimes_u x_n = x$, $y_1 \otimes_v \cdots \otimes_v y_m = y$, $x \otimes_w y = z$ and the "associativity law" $(x_1 \otimes_u \cdots \otimes_u x_n) \otimes_w (y_1 \otimes_v \cdots \otimes_v y_m) = x_1 \otimes_t \cdots \otimes_t x_n \otimes_t y_1 \otimes_t \cdots \otimes_t y_m = z$, where t is the face



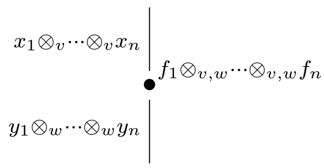
Step 2. 1-operations
Suppose we are given 1-simplices $f_1, \cdots, f_n$



and n-corollas

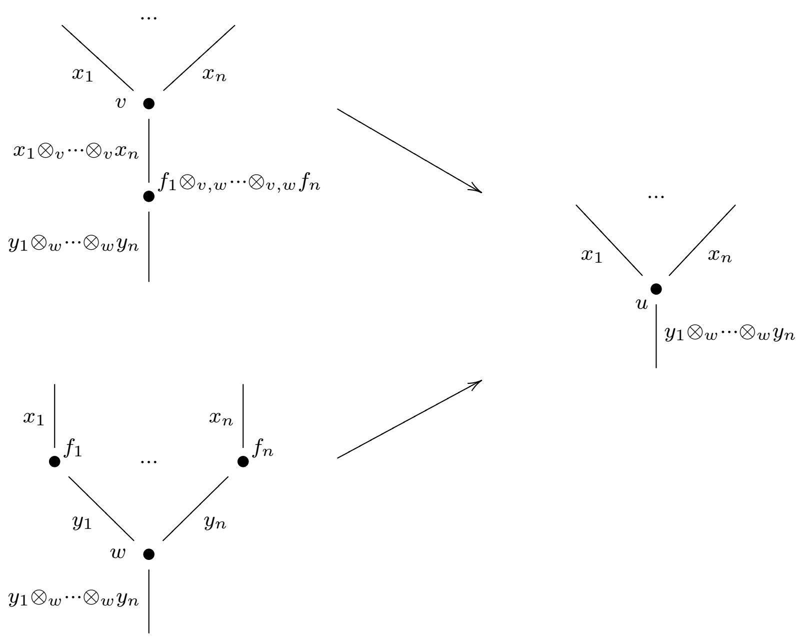


What we would like to get is a 1-simplex



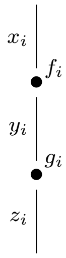
We say therefore that the (v, w)-*product* of the f_i's exists if there are dendrices and faces of the form

(6.1)

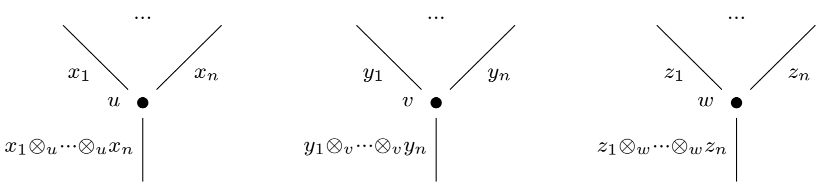


so that $u = w \circ (f_1, \ldots, f_n) = (f_1 \otimes_{v,w} \cdots \otimes_{v,w} f_n) \circ v$.

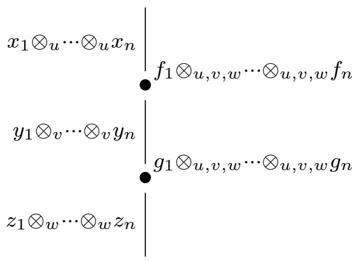
Step 3. 2-operations
Here the situation already gets a little complicated.
Suppose we are given a collection of n 2-simplices



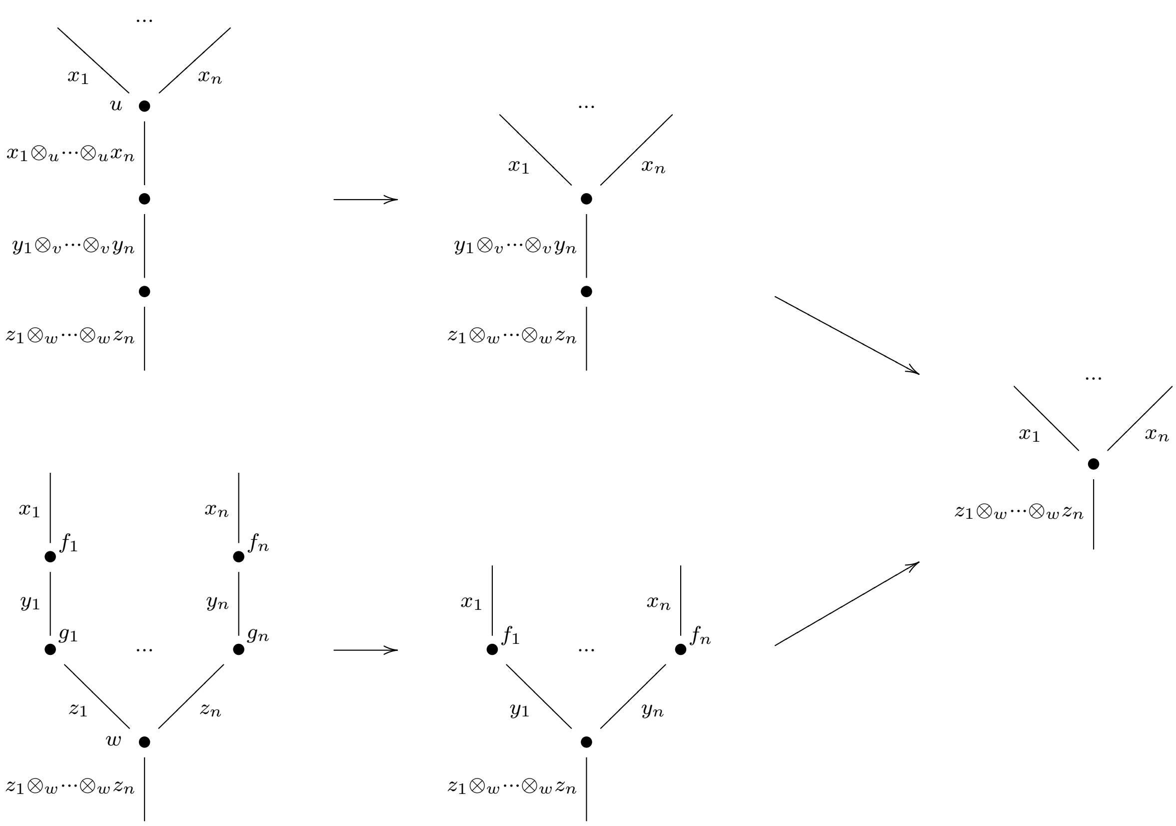
and n-corollas



Again, we want to get back a 2-simplex of the form

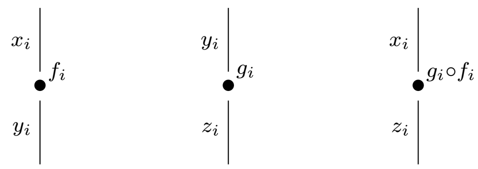


In a way similar to the 1-case, one says that the (u, v, w)-*product* of the above 2-simplices exists if we have dendrices and faces of the form

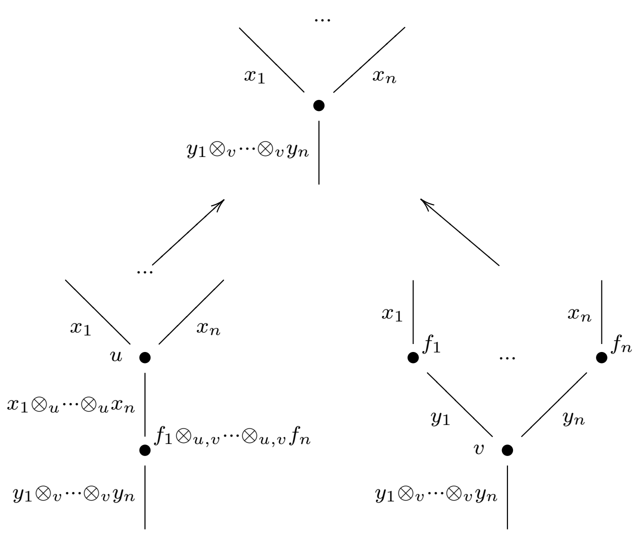


(6.2)

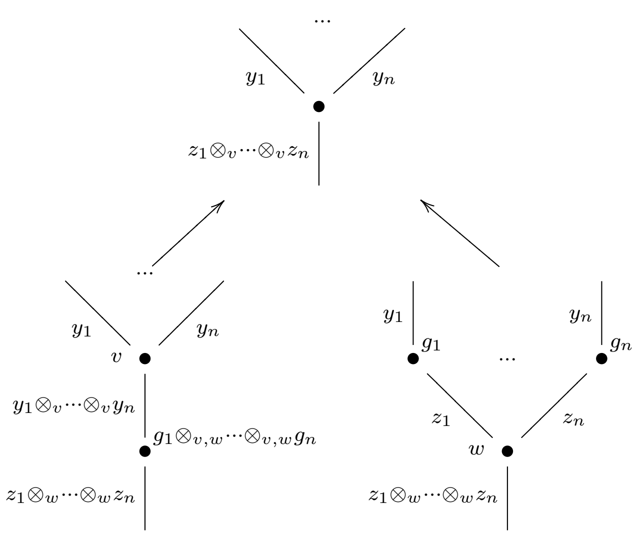
We also have to make sure that the above (u, v, w)-product agrees with the products of the faces



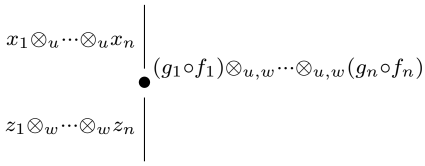
Therefore we need diagrams



for the top row of 6.2, ensuring compatibility with the (u, v)-product, while at the bottom row we grant compatibility with the (v, w)-product by requiring that



Finally, compatibility with the (u, w)-product



was already granted by diagram 6.2.

The case of higher simplices is just the same, with the only difference that one requires more compatibilities with the lower products.

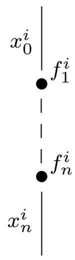
□

In favor of the above construction, let me show what happens when we consider the dendroidal nerve of a (strict) symmetric monoidal category.

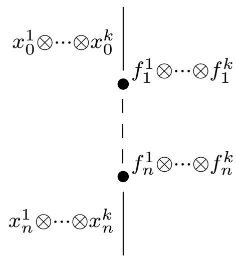
Example 6.3.2. Recall that a strict symmetric monoidal category is a category $\mathcal{M}$ together with a bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, the *tensor*, and an object $I \in \mathcal{M}$ such that the following equalities hold:

- $X \otimes Y = Y \otimes X \ \forall X, Y \in \mathcal{M}$
- $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \ \forall X, Y, Z \in \mathcal{M}$
- $I \otimes X = X \otimes I = X \ \forall X \in \mathcal{M}$

The tensor then defines a monoid structure on the nerve $N\mathcal{M}$, so that a k-tuple of n-simplices



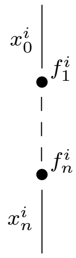
is brought to the n-simplex



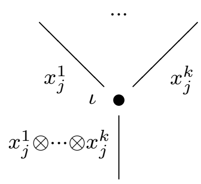
Any symmetric monoidal category $\mathcal{M}$ naturally becomes a multicategory $\tilde{\mathcal{M}}$ if we define

$$\tilde{\mathcal{M}}(x_1, \ldots, x_n; x) = \mathcal{M}(x_1 \otimes \cdots \otimes x_n, x)$$

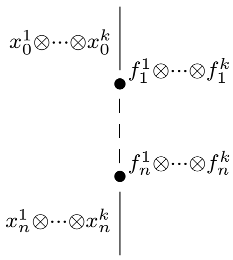
In particular, under this correspondence the identity on $x_1 \otimes \cdots \otimes x_n$ can be viewed as a multiarrow $\iota : (x_1, \ldots, x_n) \to x_1 \otimes \cdots \otimes x_n$.
The following step is obvious.
When applying the construction 6.3.1 to the dendroidal nerve $N_d\mathcal{M} = N_d\tilde{\mathcal{M}}$ of $\mathcal{M}$, these distinguished operations ι define on $i^*N_d\mathcal{M} = N\mathcal{M}$ the same monoid structure defined by the tensor product of $\mathcal{M}$. In fact, given any k-tuple of n-simplices



the k-corollas



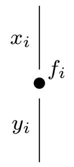
given by the operations of type ι provide the dendrices needed to get the desired output



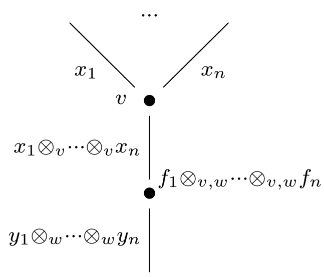
□

The above example shows a very nice situation.
To avoid unmotivated enthusiasm, notice that the system of operations that we get on i^*X is not a well behaved structure in general; given 0-simplices $(x_1, \ldots, x_n)$ there is in fact no reason for a "product" $x_1 \otimes \cdots \otimes x_n$ to exist.
The structure becomes then even poorer when we raise the dimension of the simplices, because of the many dendrices and compatibility conditions involved. Let me point out that such conditions are needed in order to determine products of simplices in a satisfactory way.

Suppose in fact that I defined the product of 1-simplices



just as the output simplex of a dendrex



instead of the combination of faces described in 6.1. Then in the case of the dendroidal nerve of a monoidal category, with v and w equal to the above operations ι, any arrow $g : x_1 \otimes \cdots \otimes x_n \to y_1 \otimes \cdots \otimes y_n$ would appear as the (tensor) product of the arrows f_i, which is not what we would like to get.

It is true that construction 6.3.1 doesn't really define an algebraic structure on i^*X; I would better say that it describes local behaviors of simplices, just like multicategories offer "algebraic-like" structures in general.

On the other hand I have shown how the product induced by the tensor on $N\mathcal{M}$ is encoded in $N_d\mathcal{M}$ and completely described by its dendrices. This is probably the main advantage of the point of view I propose, and what makes me believe that the construction I just gave is a step in the right direction.

I imagine anyway that the perspective I suggest should be placed in a larger context, not necessarily connected to algebraic topology which has no reasons to play a privileged role. After all, multicategories arose from disciplines in principle very distant: logic on one side, topology on the other.

Index

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Acknowledgements

To my advisor, Professor Ieke Moerdijk, for reminding me how beautiful algebraic topology is; for giving me the opportunity of learning about many new and various sides of mathematics, whose existence or depth I would have never suspected.
To Professor Bas Edixhoven, for encouraging and supporting my interests in mathematics, even if often quite distant from his fields; for the help during my staying at Leiden University, for reading and commenting my thesis so carefully.
A special thank to Professor Gabriele Vezzosi, who introduced me to Category Theory and the weak world, for the encouragements and help during the last year. For the much he taught me and the time he has spent on it, for the always nice and interesting talks. Thank you.

My gratitude to the professors and the administration of the ALGANT Consortium, especially to Professor Marco Garuti and Dr. Elisa Aghito, whose help and efforts have been crucial.
To Professor Carlo Casolo, the first influencing my taste in mathematics, for guiding me through more advanced things two years ago and not forgetting of me later, encouraging and helping me in finding my own way.

With love to my parents, for the efforts they have put in my education, for always feeding and stimulating my curiosity in all possible directions. To my mother, for growing in me the love for music and theoretical sciences. To my father, for reminding me that there is something real and solid out there, and beautiful; for the little things, which sometimes are so great. For tolerating me, which is probably the most difficult task.
To my older brother, Daniele, and to the younger one, Aron; to my unique brother.

To Jean-François and Toku, with whom I shared so many beers and wines, together with interests, joys, pains; thank you, for being always there, for never being afraid to criticize. To Enric and the childhood ad infinitum. To Edoardo and Anna. To Bassel, for the blueses we sang during our staying in the Netherlands, for those that hopefully are waiting for us in Paris. To Valérie Gauthier, one of the best mathematical discoveries I did in Padova. Fratribus meis Patavinis, Franz & Macu. To Herman Stel, the best fake Dutch I've ever met. И Жене; спасибо, моя улитка.
To Michela Muschietti, for showing me how does a nut in a nutcracker feel, the real meaning of the word "sacrifice", and much more. With love and gratitude to Ghigo, who has been to me like a grandfather, teaching me so much and influencing my attitude and my approach to life and people; I am sure that, though he was so far from being a mathematician, he would have understood this work much better than many scientists I know. To Dr. Poccianti, without whom I wouldn't have walked that far.
To Professor Giovanni Finzi Contini, probably the first proving me that a poet, a scientist, a wide culture and a great sense of humour can (and should) cohabit in the same person.
To Professor Paolo Chiari, for having taught me the rigor of philosophical thought and its deep breath, for the beautiful lectures he gave; I'll be always proud of having been his student and grateful to him, to the Professor and *-noch mehr!- dem Mensch.*