D. Verburg

Keeping track of time

A study of the mathematics behind historical methods

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Supervisor:
Dr. S.C. Hille

Mathematical Institute, University Leiden

Preface

Starting from my educational base of this master, the subject of this thesis originated from a mathematical walk through city of Leiden. This walk is developed in order to show everyone that mathematics is all around us. During a 4 kilometre walk through Leiden mathematical questions will be asked on paper. Those questions are linked to the environment of the city. My goal is to show that mathematics is everywhere and available for everyone.

Trying to combine mathematics with daily life became a passion for me. Talking about this walk helped to develop the idea to focus this thesis on the topic of mathematics that underlines various aspects of historical methods of time-measurement. Time is something everyone has to deal with and the ancients already developed ingenious ways to keep track of time. Moreover, the sundial turned out to be very popular. It provided a lot of opportunities for this thesis. A lot is written about sundials, especially in the field of astronomy, but looking at it from a mathematical perception gave some difficulties. For this subject I have taken several sources and used all this information to build my own view on the sundials.

Since my specialisation is the Educational Master, I have prepared this thesis with foresight. The subjects addressed in this thesis, especially the sundials, could be used for educational purposes. As such it can be considered a preliminary study. It may very well be elaborated to yield a 'Zebra-boekje', aiming at highschool students. Part of it could also be made in a useful topic in a mathematics class of 'Wiskunde D'.

Daniëlle Verburg

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1 Introduction

This thesis is written for the master *Mathematics and Education*. During the educational part of this master I have created a mathematical walk through Leiden, which links mathematics to particular locations in the city centre of Leiden. This thesis is related to a limited number of these locations, all centered around the theme of 'how to keep track of time?'. Water clocks, sundials and a pendulum on a ship will be discussed. These link to the 'Museum van Oudheden', the Astronomical Observatory and various sundials found throughout Leiden and the former 'Zeevaartschool' (the nautical college). Time-measurement on sea is essential for navigation.
All chapters have the same structure, starting with an introduction, followed by some basic mathematics and finishing with some more challenging mathematics. The chapter concerning sundials shows the educational part of my master. Everything in there uses Leiden as base and even constructions for making your own sundial are included. The thesis ends with a brief limited discussion on modern time-measurement.
Material on the topics discussed in this thesis, water clocks and sundials in particular, turned out to be discussed mainly in popular books or lecture notes available on the Internet.

Figure 1: Map of the city centre of Leiden, stars indicate the different locations.

2 Water clocks

2.1 Introduction

This subject is related to 'Museum van Oudheden', see Figure 1.
Nowadays, if someone asks 'what is the time?' we look at a clock, a watch or our mobile phone. Wanting to know the time goes back for ages: we all know the sundials for measuring time. It is hard to read the time when there is no Sun at all. Another way of measuring time in the early days was a water clock, also known as a clepsydra, the Greek word for water thief. It is not clear who invented the water clock. One of the oldest was found in the tomb of pharaoh Amniotes I. He was buried around 1500 BC. Several different kinds of water clocks have been found over the years [10]:

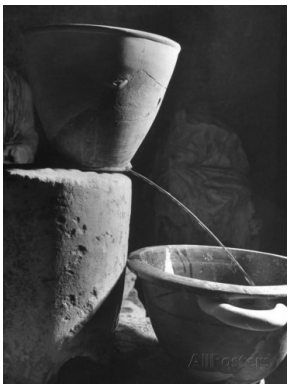


Figure 2: Water clock

- The *outflow clepsydra*; a vessel that is filled with water, loosing water by outflow from the bottom. Time is measured by the height of the water in the vessel.
- The sinking bowl clepsydra; on a surface of water a bowl with a hole is placed. The time it takes the bowl to fill and sink is taken as a unit of time.
- The *inflow clepsydra*; in this version the height of the water is not measured from the vessel from which water leaves, but from the vessel the outflowing water is caught in. This vessel measures the time.

  There are two variations on the inflow clepsydra:

  - The *inflow clepsydra with overflow tank*; This version has an extra vessel between the starting vessel and the measuring vessel. This middle vessel also has a hole in the top where water leaves the vessel. One disadvantage of this version, water is wasted.
  - The *polyvascular clepsydra*; Multiple vessels are placed in series between the beginning and the ending.

2.2 Two models for outflow clepsydra

We shall first consider the physics of water outflow from a vessel through a (small) hole or pipe near the bottom. The pressure plays a role in the velocity of the water. In order to try to keep a constant pressure there is the inflow clepsydra with overflow tank. Other features that should be taken into account are the viscosity and density of water.

ΔP	the pressure loss at the bottom of the vessel
L	the length of the nozzle
r, R	the (internal) radius of the nozzle
μ	the dynamic viscosity, $\mathrm{Pa} \cdot \mathrm{s}$[1]
Q, Φ	the volumetric flow rate
$V(t)$	the volume of liquid transferred as function of time t
v	the mean fluid velocity along the length of the tube
x	the distance in direction of the flow
ρ	the fluid density, $\mathrm{kg/m^3}$
g	the gravitational acceleration, which is $\sim 9.81\mathrm{m/s^2}$
z	the fluid's height above a reference point
P	the pressure
h	height of water column in reservoir

Let us have a look at the easiest model, the simple outflow clepsydra.

2.2.1 A model with viscosity

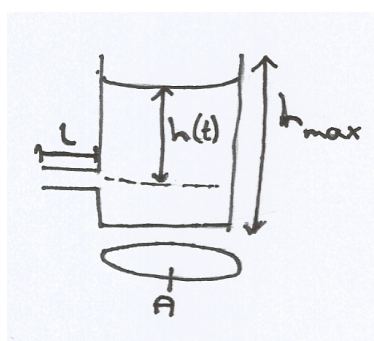


Figure 3: General situation

We need some standard fluid dynamics in order to deal with the viscosity. Poiseuille's Law gives

$$\Delta P = \frac{8\mu L Q}{\pi r^4} \Rightarrow Q = \frac{\pi r^4}{8\mu L}\Delta P. \tag{1}$$

[1]The unit $\mathrm{Pa} \cdot \mathrm{s}$ is a Pascalsecond. The Pascalsecond is derived from the unit Pascal, which is a unit for pressure. A Pascalsecond is a coefficient to express dynamic viscosity. $\mathrm{Pa} \cdot \mathrm{s}$ is equivalent to kg/m/s.

Now, $\Delta P = P_{\text{in}} - P_{\text{out}}$ where P_{in} depends on the height of the water in the cylinder. So, $\Delta P = p_{atm} + \rho h g - p_{atm} = \rho h g$. p_{atm} stands for the atmospheric pressure. We can write (1) as

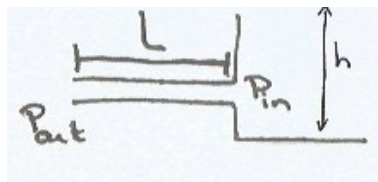


Figure 4: Pressure inside and outside

$$Q = \frac{\pi r^4 \rho g h}{8 \mu L}. \tag{2}$$

Poiseuille's Law could also be written in the physical notation, which can be derived from the Navier-Stokes equations [11].

$$\Phi = \frac{dV}{dt} = v \pi R^2 = \frac{\pi R^4}{8\mu} \left(\frac{-\Delta P}{\Delta x} \right) = \frac{\pi R^4}{8\mu} \frac{|\Delta P|}{L} \tag{3}$$

We have seen in (1) that the outflow is given by Q. This means that the outflow for the vessel will be given by $Q = \frac{\pi r^4 \rho g}{8 \mu L} h(t)$. We can say that

$$A \frac{dh(t)}{dt} = -\frac{\pi r^4 \rho g}{8 \mu L} h(t), \tag{4}$$

where A is a cross-sectional area of the vessel. We have $g = 9.81\text{m/s}^2, h, r, L, \pi$ as constants. If we look at different outflow rates for water at different temperatures we see the following [12]:

Temperature °C	μ Pa · s	ρ kg/m^3	Q
10	0.001038	1001.2817	$\frac{9.81 h \pi r^4}{L} \cdot 120578.2394$
15	0.00114	1000.6569	$\frac{9.81 h \pi r^4}{L} \cdot 109721.1513$
20	0.001005	999.7414	$\frac{9.81 h \pi r^4}{L} \cdot 124345.9453$
25	0.0008937	998.5664	$\frac{9.81 h \pi r^4}{L} \cdot 139667.4499$
30	0.0008007	997.1564	$\frac{9.81 h \pi r^4}{L} \cdot 155669.4767$
35	0.0007225	995.5317	$\frac{9.81 h \pi r^4}{L} \cdot 172237.3183$

The temperature of the water in the vessel will convert to the air temperature, which can be assumed to be between 15°C and 20°C, of course depending on the season and location.

Between these different temperatures there is a ratio of approximately 1.13.
We have (4). Let us define $y := \frac{h}{h_{\max}}$. We get

$$\frac{dy}{dt} = -\frac{\pi r^4 \rho g}{8\mu AL} y(t) = -\frac{1}{\tau} y(t), \qquad \tau = \frac{8\mu AL}{\pi r^4 \rho g}. \tag{5}$$

τ is the timescale of the emptying of the vessel (which will be exponential) and

$$y(t) = y(0)e^{-\frac{t}{\tau}}, \qquad y(0) = 1.$$

A point of discussion is, of course, the size of the vessel and the size of the nozzle. In Figure 5 we have two different versions of a vessel. A small one, which actually is a piggy bank of which the height is 110mm and the diameter equals 65mm. This means that the area of a cross-section of this vessel is given by $A = 32.5^2 \cdot \pi = 1056.25\pi\text{mm}^2$ The larger vessel is an oil drum of 200 litres and has a height of 851mm and a diameter of 572mm. These are the internal sizes of the vessel. For this vessel we can say $A = 286^2 \cdot \pi = 81796\pi\text{mm}^2$. Let us

Figure 5: Different vessels

take four different kinds of nozzles and combine these with the vessels. We will take the following nozzles for the larger vessel:

1. A long and narrow nozzle, $L = 100\text{mm}, r = 10\text{mm}$
2. A short and wide nozzle, $L = 5\text{mm}, r = 50\text{mm}$
3. A long and wide nozzle, $L = 100\text{mm}, r = 50\text{mm}$
4. A short and narrow nozzle, $L = 5\text{mm}, r = 10\text{mm}$

For the smaller vessel we take:

1. A long and narrow nozzle, $L = 13\text{mm}, r = 1\text{mm}$
2. A short and wide nozzle, $L = 0.5\text{mm}, r = 6\text{mm}$
3. A long and wide nozzle, $L = 13\text{mm}, r = 6\text{mm}$
4. A short and narrow nozzle, $L = 0.5\text{mm}, r = 1\text{mm}$

In the table below we have combined all the information in order to calculate

$$\tau = \frac{8\mu AL}{\pi r^4 \rho g}.$$

Temp °C	nozzle 1, τ s	nozzle 2, τ s	nozzle 3, τ s	nozzle 4, τ s
10	0.69	$5.5 \cdot 10^{-5}$	0.001	0.035
20	0.67	$5.4 \cdot 10^{-5}$	0.001	0.034
30	0.54	$4.3 \cdot 10^{-5}$	$8.6 \cdot 10^{-4}$	0.027

Table 1: Larger vessel in combination with its nozzles

Temp °C	nozzle 1, τ s	nozzle 2, τ s	nozzle 3, τ s	nozzle 4, τ s
10	11.6	$3.4 \cdot 10^{-4}$	0.009	0.45
20	11.26	$3.3 \cdot 10^{-4}$	0.0087	0.43
30	8.99	$2.7 \cdot 10^{-4}$	0.007	0.35

Table 2: Smaller vessel in combination with its nozzles

2.2.2 Inviscid model

Looking at (3) we encounter some problems when viscosity is low. This means either the nozzle is too wide or too short. When viscosity is low or the nozzle too wide this results in a turbulent flow and this equation is not good enough. When the nozzle is too short we can have high flow rates that are not natural. This is why the flow is bounded by Bernoulli's Principle.

$$\Phi_{\max} = \pi R^2 \sqrt{\frac{2\Delta P}{\rho}}$$

So, in case the viscosity is negligible we have to deal with another situation. The acceleration of the fluid can become extremely high due to the differences in pressure. We can use the conservation of energy along the flow lines given by Bernoulli's Law [1]:

$$\frac{1}{2}\rho v_2^2 + p_2 + g\rho z_2 = \frac{1}{2}\rho v_1^2 + p_1 + g\rho z_1. \tag{6}$$

v_1 is the velocity with which the water level in the vessel changes, v_2 is the outflow velocity, A_1 is the area of the vessel and A_2 the area of the nozzle. We have $p_1 = p_2 = p_{atm}$, the atmospheric pressure if the vessel is not extremely high. So (6) can be written as

$$\frac{1}{2}\rho v_2^2 = \frac{1}{2}\rho v_1^2 + g\rho z_1 - g\rho z_2 = \frac{1}{2}\rho v_1^2 + g\rho h. \tag{7}$$

There is conservation of mass, obviously: all water that disappears from the vessel per unit time, Av_1, must get out through the outflow opening. So $Av_1 = A_{\text{out}} v_2$. We get

$$v_2 = \sqrt{2gh \cdot \frac{A^2}{A^2 - A_{\text{out}}^2}} = \sqrt{2gh \cdot \frac{1}{1 - \left(\frac{A_{\text{out}}}{A}\right)^2}}, \tag{8}$$

(provided $A > A_{\text{out}}$). From the last expression we see that if $A_{\text{out}} \ll A$,

$$v \approx \sqrt{2gh}. \tag{9}$$

This was first observed by Torricelli [1],p12-13.

We have seen in (9) that the outflow is given approximately by $\sqrt{2gh}$. As in the viscosity model, we define $y := \frac{h}{h_{\max}}$.

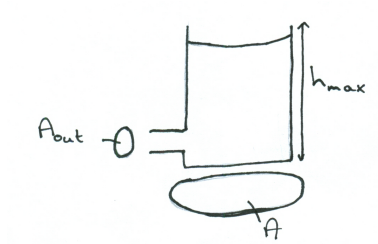


Figure 6: Inviscid model, situation visualised

$$A\frac{dy}{dt} = -\frac{A_{\text{out}}}{h_{\max}}\sqrt{2gh(t)} = -\sqrt{\frac{2g}{h_{\max}}y(t)}$$

So

$$\frac{dy}{dt} = -\frac{1}{\tau}\sqrt{y(t)}, \tag{10}$$

with

$$\tau = \frac{A}{A_{\text{out}}\sqrt{\frac{2g}{h_{\max}}}} = \frac{1}{2}\sqrt{2}\cdot\frac{A}{A_{\text{out}}}\cdot\sqrt{\frac{h_{\max}}{g}}$$

in this case.
If we make the non-linear ODE model (10) dimension free by rescaling time with τ, $\hat{t} = \frac{t}{\tau}$, then (10) becomes

$$\frac{dy}{d\hat{t}}(\hat{t}) = -\sqrt{y(\hat{t})}, \qquad y(0) = 1. \tag{11}$$

Let us write this with t instead of $\hat{t}$. Equation (11) can be solved explicitly by direct integration by means of separation of variables:

$$\begin{aligned}
\frac{dy(t)}{dt} &= -\sqrt{y(t)} \\
y^{-\frac{1}{2}}(t)\frac{dy(t)}{dt} &= -1 \\
\int_0^t y^{-\frac{1}{2}}(t)\frac{dy(t)}{dt}dt &= \int_0^t -1dt \\
\int_0^t y^{-\frac{1}{2}}(t)dy(t) &= \int_0^t -1dt \\
\left[2\sqrt{y(t)}\right]_0^t &= [-t]_0^t \\
2\left(\sqrt{y(t)} - \sqrt{y(0)}\right) &= -t \\
\sqrt{y(t)} - \sqrt{y(0)} &= -\frac{t}{2} \\
\sqrt{y(t)} &= 1 - \frac{t}{2}
\end{aligned}$$

So, finally the solution becomes

$$y(\hat{t}) = \left(1 - \frac{\hat{t}}{2}\right)^2, \qquad \text{for } 0 \leq \hat{t} \leq 2.$$

Returning to our dimension-free expression we get

$$y(t) = \left(1 - \frac{t}{2\tau}\right)^2, \qquad \text{for } 0 \leq t \leq 2\tau.$$

This gives the information that the vessels empties in 2τ time.

$$2\tau = 2 \cdot \frac{A}{A_{out}\sqrt{\frac{2g}{h_{\max}}}} = \sqrt{2} \cdot \frac{A}{A_{out}} \cdot \sqrt{\frac{h_{max}}{g}}.$$

As in the viscosity version, let us have a look at what the different sizes of vessels and nozzles mean to τ. We have taken the same vessels and the same nozzles as in the viscosity version. In this case, though, τ is given by (10).
Note that temperature plays no role. Besides temperature there is also no influence of the length of the nozzle. This has to do with the fact that there is no friction between the water and the wall of the nozzle. This is why the results above only differ for different radii.

nozzle 1, τ s	nozzle 2, τ s	nozzle 3, τ s	nozzle 4, τ s
170.35	6.81	6.81	170.35

Table 3: Larger vessel in combination with its nozzles

nozzle 1, τ s	nozzle 2, τ s	nozzle 3, τ s	nozzle 4, τ s
79	2.197	2.197	79

Table 4: Smaller vessel in combination with its nozzles

2.2.3 Viscosity and inviscid versions compared

A first observation that we can make is that the vessel will never empty completely in the viscid model. Another interesting point is

$$\begin{aligned}\left(1-\frac{t}{2\tau}\right)^2 &= 1-\frac{t}{\tau}+\frac{t^2}{4\tau^2}\\ e^{-\frac{t}{\tau}} &= 1-\frac{t}{\tau}+\frac{t^2}{2\tau^2}-\dots\end{aligned}$$

These are the solutions for $y(t)$, for the inviscid and viscosity version. For the inviscid version we know that

$$\tau = \frac{A}{A_{out}}\cdot\frac{1}{\sqrt{\frac{2g}{h_{\max}}}},$$

and for the viscosity version we have

$$\tau = \frac{8\mu AL}{\pi r^4\rho g} = \frac{8\mu L}{r^2\rho g}\cdot\frac{A}{A_{out}}.$$

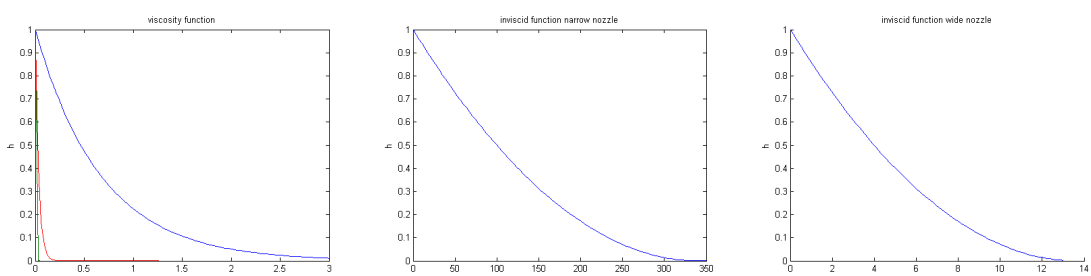


Figure 7: Viscosity and inviscid equations

2.3 Polyvascular and overflow models

In a polyvascular clepsydra, multiple vessels are put in sequence: vessels empty into the next vessel. The objective is to keep the water level constant. In [2] the authors assumed that all vessels are identical and therefore that the waterflow is equal. We are going to assume that every vessel has its own (cylindrical) form and its own timescale τ for emptying. The nozzles differ as well. We will only consider the viscosity version.

2.3.1 Polyvascular model

We have already looked at the simple outflow clepsydra. Before we are going to look at the overflow model, let us have a look at the polyvascular model.

We have seen in (5) that for the first vessel we have

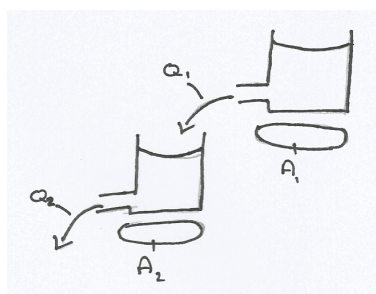


Figure 8: Polyvascular

$$\frac{dy_1(t)}{dt} = -\frac{1}{\tau_1} y_1(t), \qquad y_1(0) = 1, \qquad y_1(t) = e^{-\frac{t}{\tau_1}}.$$

The water level in the next vessels depends on the vessel that is in front of it, as in how much water flows into the vessel. Looking at the second vessel we can say that the outflow of that vessel is given by

$$\begin{aligned} Q_2(t) &= \frac{\pi r_2^4 \rho g}{8\mu L_2} \cdot h_2(t) = \frac{A_2}{\tau_2} h_2(t), \\ A_2 \frac{dh_2(t)}{dt} &= -Q_2(t) + Q_1(t), \\ \frac{dh_2(t)}{dt} &= -\frac{1}{\tau_2} h_2(t) + \frac{Q_1(t)}{A_2}. \end{aligned}$$

So,

$$\frac{dy_2(t)}{dt} = -\frac{1}{\tau_2} y_2(t) + \frac{Q_1(t)}{A_2 h_{2,\max}}, \qquad y_2(0) = 1.$$

All together, for the polyvascular model we have

$$\begin{aligned}
\frac{dy_1(t)}{dt} &= -\frac{1}{\tau_1}y_1(t), \qquad y_1(0)=1 \\
\frac{dy_2(t)}{dt} &= -\frac{1}{\tau_2}y_2(t) + \frac{Q_1(t)}{A_2 h_{2,\max}}, \qquad y_2(0)=1 \\
&\vdots \\
\frac{dy_n(t)}{dt} &= -\frac{1}{\tau_n}y_n(t) + \frac{Q_{n-1}(t)}{A_n h_{n,\max}}, \qquad y_n(0)=1.
\end{aligned}$$

Let us take

$$Q_i(t) = q_i h_i(t), \qquad q_i = \frac{\pi r_i^4 \rho g}{8\mu L_i}, \qquad i = 1, \ldots n-1.$$

This means that we can write the polyvascular model as

$$\begin{aligned}
\frac{dy_1(t)}{dt} &= -\frac{1}{\tau_1}y_1(t), \qquad y_1(0)=1 \\
\frac{dy_2(t)}{dt} &= -\frac{1}{\tau_2}y_2(t) + \frac{q_1 h_{1,\max}}{A_2 h_{2,\max}}y_1(t) \\
&= -\frac{1}{\tau_2}y_2(t) + c_1 e^{-\frac{t}{\tau_1}}, \qquad y_2(0)=1 \\
&\vdots \\
\frac{dy_n(t)}{dt} &= -\frac{1}{\tau_n}y_n(t) + c_{n-1}y_{n-1}(t), \qquad y_n(0)=1.
\end{aligned}$$

Here

$$c_i = \frac{q_i h_{i,\max}}{A_{i+1} h_{i+1,\max}} = \frac{1}{\tau_i} \cdot \frac{h_{i,\max} A_i}{h_{i+1,\max} A_{i+1}} \qquad \text{for } i = 1, 2, 3, \ldots, n$$

Now we can solve $\frac{dy_2(t)}{dt}$, because

$$\begin{aligned}
\frac{dy_2(t)}{dt} &= -\frac{1}{\tau_2}y_2(t) + c_1 e^{-\frac{t}{\tau_1}} \\
y_2(t) &= e^{-\frac{t}{\tau_2}} \cdot y_2(0) + \int_0^t e^{-\frac{t-s}{\tau_2}} \cdot c_1 e^{-\frac{s}{\tau_1}} ds \\
&= e^{-\frac{t}{\tau_2}} + e^{-\frac{t}{\tau_2}} \cdot c_1 \cdot \int_0^t e^{\frac{s}{\tau_2} - \frac{s}{\tau_1}} ds \\
&= e^{-\frac{t}{\tau_2}} + c_1 e^{-\frac{t}{\tau_2}} \left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right)^{-1} \left(e^{t\left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right)} - 1\right) \\
&= e^{-\frac{t}{\tau_2}} + c_1 \left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right)^{-1} \left(e^{-\frac{t}{\tau_1}} - e^{-\frac{t}{\tau_2}}\right). \qquad (12)
\end{aligned}$$

Note that in (12) we can have that $\tau_2 = \tau_1$ for example when all vessels and outflow nozzles are the same. If that is the case we get

$$y_2(t) = (1 + c_1 t)e^{-\frac{t}{\tau_1}}, \tag{13}$$

Proposition 2.1. *If all $\tau_i = \tau$, then*

$$y_n(t) = e^{-\frac{t}{\tau}} \left(1 + c_{n-1}t + \frac{1}{2}c_{n-2}c_{n-1}t^2 + \ldots + \frac{1}{(n-1)!}c_1c_2\ldots c_{n-1}t^{n-1}\right) \tag{14}$$

for $n \geq 1$.

Proof. This will be proven by mathematical induction. It is already proven for $n = 1$ and $n = 2$ (see (13)). Assume that this statement holds for n. Let us prove the statement for $n + 1$: The differential equation for y_{n+1} yields

$$\frac{dy_{n+1}}{dt}(t) + \frac{1}{\tau}y_{n+1}(t) = c_n y_n(t)$$

so,

$$\left[e^{\frac{t}{\tau}}y_{n+1}(t)\right] = c_n e^{\frac{t}{\tau}}y_n(t).$$

Consequently,

$$\begin{aligned} y_{n+1}(t) &= e^{-\frac{t}{\tau}}y_{n+1}(0) + \int_0^t e^{-\frac{t-s}{\tau}}c_n y_n(s)ds \\ &= e^{-\frac{t}{\tau}} + e^{-\frac{t}{\tau}}\int_0^t c_n y_n(s)e^{\frac{s}{\tau}}ds \\ &= e^{-\frac{t}{\tau}} + e^{-\frac{t}{\tau}}\left(c_n t + \frac{1}{2}c_{n-1}c_n t^2 + \ldots + \frac{1}{n!}c_1c_2\ldots c_n t^n\right) \qquad (15) \\ &= e^{-\frac{t}{\tau}}\left(1 + c_n t + \frac{1}{2}c_{n-1}c_n t^2 + \ldots + \frac{1}{n!}c_1c_2\ldots c_n t^n\right). \end{aligned}$$

In (15) the induction step is used. So (14) holds for all n. □

In Proposition 2.1 we recognise the Taylor polynomial for e^t in $y_n(t)$ in case we choose $c_n = \frac{1}{\tau}$. The authors of [2] have chosen for this option, so they made all the vessels identical and all dimensions were set for the vessels.
Recall that the main goal of the polyvascular model is to get a waterflow that goes (almost) linear in time, because otherwise it would not be an accurate way of time-measurement. This means that the level of the last vessel should be as steady as possible $y_n(t) \approx 1$ for

all time. In case we choose all $c_n = \frac{1}{\tau}$, ie all the vessels are the same, we obtain (see [2])

$$\begin{aligned} y_n(t) &= e^{-\frac{t}{\tau}} \left(1 + \frac{t}{\tau} + \frac{1}{2} \left(\frac{t}{\tau} \right)^2 + \ldots + \frac{1}{(n-1)!} \left(\frac{t}{\tau} \right)^{n-1} \right) \\ &= \left(e^{\frac{t}{\tau}} - \sum_{k=n}^{\infty} \frac{1}{k!} \left(\frac{t}{\tau} \right)^k \right) e^{-\frac{t}{\tau}} \\ &= 1 - e^{-\frac{t}{\tau}} \sum_{k=n}^{\infty} \frac{1}{k!} \left(\frac{t}{\tau} \right)^k . \end{aligned}$$

We want the water flow to be (almost) linear in time. Let us have a look at the estimate for the error after n vessels.

$$1 - y_n(t) = 1 - \left(1 - e^{-\frac{t}{\tau}} \sum_{k=n}^{\infty} \frac{1}{k!} \left(\frac{t}{\tau} \right)^k \right) = e^{-\frac{t}{\tau}} \sum_{k=n}^{\infty} \frac{1}{k!} \left(\frac{t}{\tau} \right)^k$$

Using an expression for the error term for the Taylor expansion to order $n-1$ for some θ_x between 0 and x (see [3],p.259),

$$R_{n-1}(x) = \frac{f^{(n)}(\theta_x)}{n!} x^n = \frac{e^{\theta_x}}{n!} x^n \qquad x = \frac{t}{\tau}.$$

This gives us

$$0 \leq e^{-\frac{t}{\tau}} \sum_{k=n}^{\infty} \frac{1}{k!} \left(\frac{t}{\tau} \right)^k \leq \frac{1}{n!} \left(\frac{t}{\tau} \right)^n .$$

Using Stirling's formula for $n!$, we get

$$\frac{1}{n!} \left(\frac{t}{\tau} \right)^n \approx \frac{1}{\sqrt{2\pi n} \left(\frac{n}{e} \right)^n} \left(\frac{t}{\tau} \right)^n = \frac{1}{\sqrt{2\pi n}} \cdot \left(\frac{e}{n} \right)^n \cdot \left(\frac{t}{\tau} \right)^n =: E_n \tag{16}$$

We get the following inequality to assume that $1 - y_n(t) \leq \epsilon$ for a given fixed error ϵ

$$\begin{aligned} E_n &\leq \epsilon \\ \left(\frac{t}{\tau} \right) \cdot \frac{e}{n} \left(\frac{1}{\sqrt{2\pi n}} \right)^{\frac{1}{n}} &\leq \epsilon^{\frac{1}{n}} \\ \frac{t}{\tau} &\leq \epsilon^{\frac{1}{n}} \cdot \frac{n}{e} \cdot (\sqrt{2\pi n})^{\frac{1}{n}} \end{aligned} \tag{17}$$

In the table below we listed the duration (in terms of $\frac{t}{\tau}$) of the period in which $1 - y_n(t) \leq 0.05$ for different number of n vessels in the polyvascular clepsydra using formula (17).

n	1	2	3	4	5
$\frac{t}{\tau}$	0.0461	0.3098	0.6633	1.0412	1.4262

Recall that τ is the characteristic outflow rate of the (identical) vessels. Note that the exact solutions t to $1 - y_n(t) = 0.05$ is $t = -\ln(0.95) \approx 0.0513$ for $n = 1$.

2.3.2 Overflow model

We are now going to look at the overflow model.

The equation for the first vessel is equal to the one of the polyvascular (and the simple

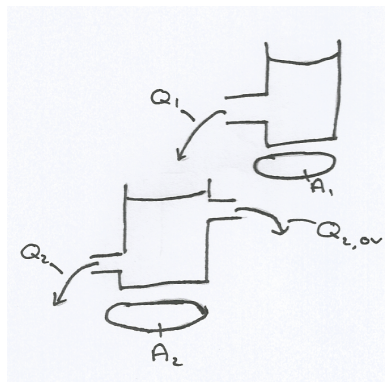


Figure 9: Overflow

outflow) model. The second equation gets an extra term, since there is a second outflow. This outflow depends on the water level, because when the water level reaches below the nozzle for the overflow, we have the same equation as in the polyvascular model. So, for the overflow nozzle we get

$$Q_{2,ov}(t) = \frac{\pi r_{ov}^4 \rho g}{8\mu L_{ov}} H(h_2 - h_{ov}) \cdot (h_2 - h_{ov}).$$

Here $H(x)$ is the Heaviside function, which is 1 if $x \geq 0$ and 0 if $x < 0$. For the overflow vessel we get

$$\begin{aligned}
A_2 \frac{dh_2(t)}{dt} &= -Q_2(t) - Q_{2,ov}(t) + Q_1(t) \\
\frac{dh_2(t)}{dt} &= -\frac{1}{\tau_2} h_2 - \frac{1}{\tau_{ov}} H(h_2 - h_{ov}) + \frac{Q_1(t)}{A_2} \\
\frac{dy_2(t)}{dt} &= -\frac{1}{\tau_2} y_2(t) - \frac{1}{\tau_{ov}} \cdot \left(y_2(t) - \frac{h_{ov}}{h_{2,ov}} \right) H \left(y_2(t) - \frac{h_{ov}}{h_{2,ov}} \right) + \frac{Q_1(t)}{A_2 h_{2,ov}} \\
\frac{dy_2(t)}{dt} &= -\frac{1}{\tau_2} y_2(t) - \frac{1}{\tau_{ov}} (y_2(t) - 1) H(y_2(t) - 1) + \frac{A_1 h_{1,ov}}{A_2 h_{2,ov}} \cdot \frac{1}{\tau_1} e^{-\frac{t}{\tau_1}} \\
&= \left(\frac{\alpha}{\tau_1} e^{-\frac{t}{\tau_1}} - \frac{1}{\tau_2} y_2(t) \right) - \frac{1}{\tau_{ov}} (y_2(t) - 1) H(y_2(t) - 1)
\end{aligned}$$

In the final step we have replaced $\frac{A_1 h_{1,ov}}{A_2 h_{2,ov}}$ by α and $h_{2,ov} = h_{ov}$.

There are now two possibilities: $\frac{\alpha}{\tau_1} < \frac{1}{\tau_2}$ and $\frac{\alpha}{\tau_1} > \frac{1}{\tau_2}$.

- $\frac{\alpha}{\tau_1} < \frac{1}{\tau_2}$
  In this case we can deduce that the outflow of vessel 2 is quicker than the inflow from

vessel 1. The water level in vessel 2 drops. This means that the Heaviside function becomes 0. This also means that we have the same situation as in (12), only here we have $\frac{\alpha}{\tau_1}$ instead of c_1. If vessel 1 and 2 are identical, we can see here that $y_2(t) = e^{-\frac{t}{\tau_1}}$ because $\tau_2 = \tau_1$ and the second term becomes 0.

- $\frac{\alpha}{\tau_1} > \frac{1}{\tau_2}$
  In this case the water level in vessel 2 rises and we have to deal with the overflow term, because at a point the Heaviside function becomes 1.

$$\frac{dy_2(t)}{dt} = \frac{\alpha}{\tau_1}e^{-\frac{t}{\tau_1}} - \frac{1}{\tau_2}y_2(t) - \frac{1}{\tau_{ov}}(y_2(t) - 1)$$

2.4 Conclusion

It may be clear that keeping track of time by using a water clock is not the most accurate way. There are too many external elements that play a role in the accuracy, such as temperature and viscosity.
In Section 2.2.3 we have seen (see Figure 7) that the inviscid functions get closer to a straight line, resulting in better time-measurement. However, the physical relevance of this model is limited because eg. friction, temperature, etc. are no longer taken into account, which makes it less close to reality. For the viscosity there are too many elements that play a role.
In the polyvascular model we have taken the option for $c_n = \frac{1}{\tau}$ for all n. One can discuss if this is the best option as a choice for c_n in order to minimize the deviation for constant outflow.
In the case of the overflow model, we have chosen for every vessel to be different, which makes it very difficult to find explicit functions in case the Heaviside function equals 1. Too many factors are involved.
We have not discussed the problem of a constant water flow. The water clock needs a 'refill' in the first vessel to be able to be a clock. It seems that a water clock is not a convenient way to keep track of time.

3 Sundials

There are several locations in Leiden with sundials: 'Molen de Valk', which has a horizontal sundial, the 'Hooglandse kerk', which has a vertical sundial. Astronomy deals with the position of all stars, such as the Sun, so also the 'Observatory' is a location that is related to this subject, see Figure 1. For long the sundial yielded the most exact method for measuring time.

3.1 Introduction

The sundial was already mentioned in the introduction of the water clock. It is not clear when the sundial was invented and by whom. Several different sundials have appeared in history. Sundials are quite known, but we will still need to introduce some terms connected to sundials in order to explain this subject.
Most sundials have a *gnomon*. A gnomon is the object that casts the shadow. Gnomon is the Greek word for 'one who knows.' Besides the gnomon hour lines need to be visible on the sundial. That is, marks that indicate the time of day. There are two different kinds of hour lines, namely *temporary hour lines* and *equal hours lines*. A temporary hour is different every day: it is $\frac{1}{12}$ of the time that the Sun is above the horizon. Equal hours are just $\frac{1}{24}$ of a day. That is, the time between the moments when the Sun reaches its highest location in the sky. Or the time between the moments that a point of Earth faces the Sun. Note that due to the movement of the Earth around the Sun a day as this defined is slightly shorter than the time needed by the Earth to make a full rotation around its axis. Note also that a temporary hour is longer during Summer and shorter during Winter. Until the 14th century temporary hour lines were used.
At the end of the 16th century everyone was interested in telling the time by a sundial 'all over Europe', which can be referred to as 'dialing with a sundial'. It was even placed as a teaching subject in the school curriculum, since mathematics books had chapters on this subject. Around 1700 the sundial became the standard for setting the pendulum clocks, which were inaccurate at that time. It is interesting that sundials are believed not to be accurate. This does not have to do with the time the sundial indicates, but it has to do with the translation from the solar time (given by the sundial) to the standard time (GMT). This can even reach a difference of fifteen minutes. The different time settings and the time translation will be discussed later [4].
Since this thesis is written at University Leiden, we will use the latitude and longitude of Leiden, The Netherlands to describe the set-up of a sundial.

Leiden	actual	approximately
latitude	52.1603° north	52° north
longitude	4.4939° east	4.5° east

3.2 Different time settings

The reader has to realise that there are several time settings. While discussing sundials we have *solar time*, *mean time*, as in Greenwich Mean Time (GMT) and *standard time*. In these definitions the term *hour angle* will be mentioned. This describes the angular distance along the celestial equator from the meridian of the observer to the *hour circle* of a particular celestial body. The hour circle is analogous to longitudes since it is a great circle through the object and its celestial poles, in the case of the Earth the North and South pole. Actually the hour angle is the angle between two planes: the plane containing the Earth's axis and the zenith, the highest point in the sky that is reached by the Sun, and the plane containing the Earth's axis and the point where the observer is.

- *Solar time*; time based on the rotation of the Earth around its axis and around the Sun. Solar time units are slightly longer than sidereal (related to stars) units due to the continuous movement of the Earth along its orbital path [13]. This is also the time that is measured by sundials.

- *Mean time*; this is the hour angle of the Sun if the Earth is situated in such a way that the Sun is at the equator. The mean time considers a mean Sun, which is a Sun placed along the equator of the Earth and its orbit is circular since we take the time that the Earth takes to circle this mean Sun at a uniform rate.

- *Standard time*; this is the time that is indicated by our clocks nowadays. In general our clocks are synchronized to atomic clocks. In Europe there are two atomic clocks, in Braunschweig, Germany and Teddington, England.

The translation between solar time and standard time, also called '*Equation of Time*', is influenced by two factors:

- *Eccentricity.* The Earth's orbit around the Sun is not circular but an ellipse. As a result the Earth does not travel at a constant speed around the Sun. The eccentricity of the Earth's orbit is 0.0167 [5], p.47-56. See Appendix G for its definition.

- *Obliquity.* This is the fact that the Earth's rotation axis is not at a 90° angle to the orbital plane of the Earth around the Sun. Instead the axis is tilted approximately 23 degrees (23.44 degrees). The tilt changes slowly over time [14].

The Equation of Time is set up as a translation between the solar time and the standard time. The main reason why there had to be a translation between those two time settings was the railroad. Due to the position of the Sun, time could be different in one city compared to another city. This is why a standard time was introduced.
The Earth has been divided into zones over 15 degrees of longitude. Still, the lines are not regular, due to geographical and population clusters. The Sun travels westwards by 1 degree every 4 minutes:

$$\begin{aligned} 24 \cdot 60 &= 1440 \text{ minutes per day} \\ 1440 : 360 &= 4 \text{ minutes per degree} \end{aligned}$$

This part of the difference between standard time and solar time caused by longitude differences is called the longitude correction. In order to find your longitude correction you have to calculate the difference between your longitude and the central meridian that is appropriate with respect to standard time. Multiply this difference by 4. When being east of this meridian, solar time will be earlier than standard time, while being west of a meridian means that solar time will be later than solar time on the meridian.
We know that the longitude for Leiden is approximately 4.5 east from Greenwich, which is on the zero meridian. This tells us that the sundial in Leiden will be $4.5 \cdot 4 = 18$ minutes ahead of the solar time at Greenwich.

3.3 Celestial motion of the Sun

The celestial motion of the Sun determines the read-out of a sundial. In this section we consider this motion in detail. The motion of the Sun has two parts. The Sun rises and sets daily and through the year, the Earth travels around the Sun which makes the daily rise and set a little different to the day before.

3.3.1 Daily movement

The position of the Sun in the sky is given by two angles [5]; see Figure 10:

- *Altitude*; the angle made between the Sun and the horizon.
- *Azimuth*; the angle made between the Sun and the true north.

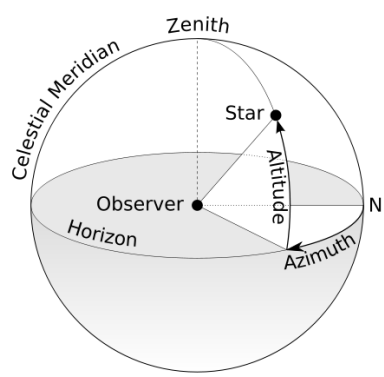


Figure 10: Graphic view of altitude and azimuth.

A question rises while dealing with the motion of the Sun. What kind of figure does the Sun's motion describe along the sky during a day? It seems to be a part of a circle, but as mentioned, the sunrise and sunset 'move' along the horizon and the highest point of the Sun is also different for every day. It depends on the day that the Sun is observed and so the shape of the daily trajectory is a little different everyday.

3.3.2 Daily movement: using polar coordinates

$\mathbf{a}$	Earth's axis, vector from the centre of the Earth to the North Pole
ρ	radius of the Earth
ρ_a	radius of the Earth, in the direction of the Earth's axis
ϕ	latitude of observer
$\omega_{\mathbf{a}}$	angular velocity of the rotation of the Earth
ω_0	mean angular velocity of the rotation of the Earth around the Sun
p	position of the observer on Earth

Besides these symbols we also will use

$$r = \rho \sin \phi, \qquad z_\phi = \rho \cos \phi$$

We introduce three coordinate systems:

1. *The* $(\hat{x}, \hat{y}, \hat{z})$*-system*: In this coordinate system the Sun is in the origin. The $\hat{x}$-axis and $\hat{y}$-axis lie in the orbital plane of the Earth around the Sun. They are fixed with respect to the 'fixed stars' (far away). The $\hat{z}$-axis is orthogonal to the orbital plane. The Earth's North pole lies in $\{\hat{z} > 0\}$.

2. *The* (x', y', z')*-system*: Here the origin is the centre of the Earth, while x', y', z' are parallel to $\hat{x}, \hat{y}, \hat{z}$.

3. *The* (x, y, z)*-system*: This is a rotation of the (x', y', z')-system around the origin such that the positive z'-axis is moved to the direction of the Earth's rotation axis ($\mathbf{a}$). So $(x, y, z)^T = R_{\mathbf{a}}(x', y', z')$, where $R_{\mathbf{a}}$ is the corresponding rotation matrix.

First we start with the (x', y', z')- system. Actually (x', y', z') is a 'moving' coordinate system, since the Earth changes position throughout the year.

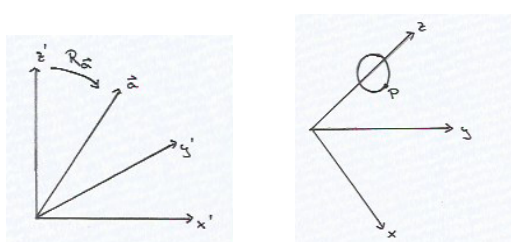


Figure 11: Moving the (x', y', z')-system into the (x, y, z)-system (left). The movement of the observer (P) in the (x, y, z)-system (right).

We want to describe the movement of the Sun in the sky as observed by an observer at a fixed position on Earth. To place the coordinate system in such a way that we can reason from the Earth we rotate z' to the position of $\mathbf{a}$ by $R_{\mathbf{a}}$. This also means that the (x', y', z')-coordinate system changes to a (x, y, z)-coordinate system and z being the position of $\mathbf{a}$

as mentioned. Note that the direction of the Earth's axis does change really slowly. This will be discussed in Section 3.3.7. We will ignore this small change in this exposition. We can describe the orbit of the observer relative to the centre of the Earth and the equatorial plane (x, y) by

$$\mathbf{x}(t) = (r\cos(\omega_{\mathbf{a}}t + \theta_0), r\sin(\omega_{\mathbf{a}}t + \theta_0), z_\phi)^T$$

The position of the observer in the (x', y', z')-coordinate system is now given by

$$\mathbf{x}'(t) = R_{\mathbf{a}}^{-1}\mathbf{x}(t).$$

$\mathbf{x}'(t)$ is a normal vector to the tangent plane V to the Earth at position P of the observer, pointing to the sky, right above the observer (the *zenith*, see Figure 10). Using the $(\hat{x}, \hat{y}, \hat{z})$-

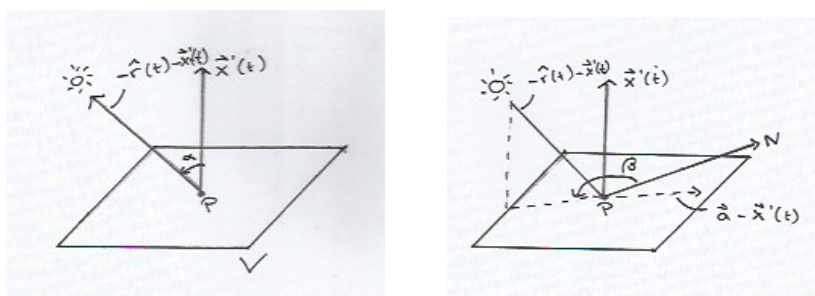


Figure 12: Plane V with the altitude (left) and azimuth (right).

system we have the position of the centre of the Earth given by

$$\hat{r}(t) = (\hat{x}(t), \hat{y}(t), 0)^T = (\rho_{\mathbf{a}}\cos(\omega_0 t + \theta_{\mathbf{a},0}), \rho_{\mathbf{a}}\sin(\omega_0 t + \theta_{\mathbf{a},0}), 0)^T.$$

Note that we have removed the eccentricity. This means in the (x', y', z') moving frame that the Sun is at position of $-\hat{r}(t) - \mathbf{x}'(t)$ from the point of view of the observer. In case $-\hat{r}(t) - \mathbf{x}'(t)$ is above the tangent plane V, it is day. When it is below the tangent plane V it is night.
Let α be the angle between $\mathbf{x}'(t)$ and $-\hat{r}(t) - \mathbf{x}'(t)$. The altitude is then given by $\frac{\pi}{2} - \alpha$. Let $||x||$ denote the length of a vector x. Then (omitting time dependence)

$$\begin{aligned}
\sin(\text{altitude}) &= \sin(\frac{\pi}{2} - \alpha) = \cos(\alpha) && (18)\\
&= \frac{\mathbf{x}' \bullet (-\hat{r} - \mathbf{x}')}{||\mathbf{x}'||||\hat{r} + \mathbf{x}'||} = -\frac{\mathbf{x}' \bullet \hat{r}}{||\mathbf{x}'||||\hat{r} + \mathbf{x}'||} - \frac{||\mathbf{x}'||}{||\hat{r} + \mathbf{x}'||} && (19)\\
&= -\frac{\mathbf{x}' \bullet \hat{r}}{\rho \cdot \rho_{\mathbf{a}}} \cdot \frac{||\hat{r}||}{||\hat{r} + \mathbf{x}'||} - \frac{||\mathbf{x}'||}{||\hat{r}||} \cdot \frac{||\hat{r}||}{||\hat{r} + \mathbf{x}'||}\\
&= -\frac{\mathbf{x}' \bullet \hat{r}}{\rho \cdot \rho_{\mathbf{a}}} \cdot \frac{1}{||\frac{\hat{r}}{||\hat{r}||} + \frac{\mathbf{x}'}{||\hat{r}||}||} - \frac{||\mathbf{x}'||}{||\hat{r}||} \cdot \frac{||\hat{r}||}{||\hat{r} + \mathbf{x}'||}\\
&\approx -\frac{\mathbf{x}' \bullet \hat{r}}{\rho \cdot \rho_{\mathbf{a}}} = \frac{\mathbf{x}' \bullet R_{\mathbf{a}}\hat{r}}{\rho \cdot \rho_\alpha} && (20)
\end{aligned}$$

since $||\mathbf{x}'|| \ll ||\hat{r}||$. Thus the altitude is given approximately by

$$\arcsin\left[-\frac{(R_{\mathbf{a}}^{-1}\mathbf{x}(t)) \bullet (\hat{r}(t))}{\rho \cdot \rho_{\mathbf{a}}}\right] = \arcsin\left[-\frac{\mathbf{x}(t) \bullet R_{\mathbf{a}}\hat{r}(t)}{\rho \cdot \rho_{\mathbf{a}}}\right].$$

For the position of the Sun seen from the Earth we not only need the altitude, but we also need the azimuth. In Figure 12 we see again the tangent plane V. $\mathbf{n}$ is the direction of the true north in the observer's plane V, given by the projection onto V of $\mathbf{a} - \mathbf{x}'(t)$. Let us take an arbitrary vector $\mathbf{v}'(t)$. Let $\mathbf{v}'_{\perp}(t)$ be the projection of $\mathbf{v}'(t)$ onto the line through $\mathbf{x}'(t)$. This projection is given by

$$\mathbf{v}'_{\perp}(t) = \left(\mathbf{v}'(t) \bullet \frac{\mathbf{x}'(t)}{||\mathbf{x}'(t)||}\right) \frac{\mathbf{x}'(t)}{||\mathbf{x}'(t)||}$$

So, the projection of $\mathbf{v}'(t)$ onto V is given by

$$\mathbf{v}'(t)_V = \mathbf{v}'(t) - \mathbf{v}'_{\perp}(t).$$

Omitting again the time dependence we get

$$\begin{aligned}
\mathbf{n} &= (\mathbf{a} - \mathbf{x}')_V = (\mathbf{a} - \mathbf{x}') - (\mathbf{a} - \mathbf{x}')_{\perp} \\
&= (\mathbf{a} - \mathbf{x}') - \left[(\mathbf{a} - \mathbf{x}') \bullet \frac{\mathbf{x}'}{||\mathbf{x}'||}\right] \frac{\mathbf{x}'}{||\mathbf{x}'||} \\
&= \mathbf{a} - \frac{\mathbf{a} \bullet \mathbf{x}'}{||\mathbf{x}'||} \cdot \frac{\mathbf{x}'}{||\mathbf{x}'||} - \mathbf{x}' + \mathbf{x}' \\
&= \mathbf{a} - \frac{\mathbf{a} \bullet \mathbf{x}'}{||\mathbf{x}'||} \cdot \frac{\mathbf{x}'}{||\mathbf{x}'||}.
\end{aligned}$$

Note that in case of the North or South Pole we have $\mathbf{x}' \equiv \pm\mathbf{a}$:

$$\mathbf{n} = \mathbf{a} - \frac{\pm||\mathbf{a}||^2}{||\mathbf{a}||} \cdot \frac{\pm\mathbf{a}}{||\mathbf{a}||} = 0.$$

Recall that the direction of the Sun from the point of view of the observer is $-\hat{r}(t) - \mathbf{x}'(t)$ in the (x', y', z')-system.

$$\cos(\text{azimuth}) = \frac{\mathbf{n} \bullet (-\hat{r} - \mathbf{x}')_V}{||\mathbf{n}|| \cdot ||(\hat{r} + \mathbf{x}')_V||} \tag{21}$$

We have

$$-\mathbf{n} \bullet (\hat{r} + \mathbf{x}')_V = -\left(\mathbf{a} - \frac{\mathbf{a} \bullet \mathbf{x}'}{||\mathbf{x}'||} \cdot \frac{\mathbf{x}'}{||\mathbf{x}'||}\right) \bullet (\hat{r} + \mathbf{x}')_V$$

We can write $(\hat{r} + \mathbf{x}')_V$ as

$$\begin{aligned}
(\hat{r} + \mathbf{x}')_V &= (\hat{r} + \mathbf{x}') - \left[(\hat{r} + \mathbf{x}') \bullet \frac{\mathbf{x}'}{||\mathbf{x}'||}\right] \cdot \frac{\mathbf{x}'}{||\mathbf{x}'||} \\
&= \hat{r} + \mathbf{x}' - \frac{\hat{r} \bullet \mathbf{x}'}{||\mathbf{x}'||} \cdot \frac{\mathbf{x}'}{||\mathbf{x}'||} - \mathbf{x}' \\
&= \hat{r} - \frac{\hat{r} \bullet \mathbf{x}'}{||\mathbf{x}'||} \cdot \frac{\mathbf{x}'}{||\mathbf{x}'||}.
\end{aligned}$$

Note:

$$\begin{aligned} ||(\hat{r}+\mathbf{x}')_V||^2 &= ||\hat{r}||^2 - \left(\frac{\hat{r}\cdot\mathbf{x}'}{||\mathbf{x}'||}\right)^2 \\ &= ||\hat{r}||^2 \cdot \left[1-\cos^2\angle(\hat{r},\mathbf{x}')\right] = ||\hat{r}||^2 \sin^2\angle(\hat{r},\mathbf{x}'). \end{aligned}$$

This all together yields

$$\begin{aligned} -\mathbf{n}\bullet(\hat{r}+\mathbf{x}')_V &= -\left[\mathbf{a}\bullet\hat{r} - 2\frac{\hat{r}\bullet\mathbf{x}'}{||\mathbf{x}'||}\cdot\frac{\mathbf{a}\bullet\mathbf{x}'}{||\mathbf{x}'||} + \frac{\hat{r}\bullet\mathbf{x}'}{||\mathbf{x}'||}\cdot\frac{\mathbf{a}\bullet\mathbf{x}'}{||\mathbf{x}'||}\cdot\frac{\mathbf{x}'\bullet\mathbf{x}'}{||\mathbf{x}'||^2}\right] \\ &= -\mathbf{a}\bullet\hat{r} + \frac{\hat{r}\bullet\mathbf{x}'}{||\mathbf{x}'||}\cdot\frac{\mathbf{a}\bullet\mathbf{x}'}{||\mathbf{x}'||}. \end{aligned}$$

At the equator we have $\mathbf{a}\bullet\mathbf{x}' \equiv 0$ and $\mathbf{n}=\mathbf{a}$. So, for the azimuth at the equator we get

$$\cos(\text{azimuth}) = -\frac{\mathbf{a}\bullet\hat{r}(t)}{||\mathbf{a}||\cdot||\hat{r}||}\cdot\frac{1}{\sin\angle(\hat{r},\mathbf{x}')} = -\frac{\cos\angle(\hat{r},\mathbf{a})}{\sin\angle(\hat{r},\mathbf{x}')}.$$

Note that the Sun on the equator is in the south precisely when $\mathbf{x}'$ lies in the plane spanned by $\hat{r}$ and $\mathbf{a}$.

We started this section by the raising the question what kind of figure the Sun's motion describes during a day. We have expressions for the altitude and the azimuth so let us have a look at this question. Note that we have been dealing with a sphere to describe the azimuth and altitude, but we will look at the Sun's motion in a plane, at distance d from the observer and orthogonal to the south direction. The x-coordinate of the Sun in this plane will be given by

$$x = d\tan(\pi - \text{azimuth}) = d\cdot\frac{\sin(\text{azimuth})}{-\cos(\text{azimuth})} = -d\tan(\text{azimuth}(t)). \tag{22}$$

The y-coordinate will be given by

$$y = d\tan(\text{altitude}(t)) \tag{23}$$

Note that d is just a scale factor. Note that in a right-angled triangle with shorter sides γ (opposite side) and $\sqrt{1-\gamma^2}$ (adjacent side) and the hypothenuse being 1 we have $\tan(\arcsin(\gamma)) = \frac{\gamma}{\sqrt{1-\gamma^2}}$. Using this information we can write (22) and (23) as

$$\begin{aligned} x &= d\cdot\frac{\sqrt{1-\beta^2}}{\beta}, \qquad \beta = \frac{\mathbf{n}\bullet(-\hat{r}-\mathbf{x}')_V}{||\mathbf{n}||\cdot||(\hat{r}+\mathbf{x}')_V||}, \\ y &= d\cdot\frac{\gamma}{\sqrt{1-\gamma^2}}, \qquad \gamma = -\frac{\mathbf{x}'\bullet\hat{r}}{||\mathbf{x}'||||\hat{r}+\mathbf{x}'||} - \frac{||\mathbf{x}'||}{||\hat{r}+\mathbf{x}'||}. \end{aligned}$$

3.3.3 Yearly movement: the analemma

If you would take a picture of the Sun every day of the year at the same standard time you would get an eight-shaped figure, called an analemma, see Figure 13. This shape is formed in the following way:

- The north-south change has to do with the Sun's declination. When you take the lines on a map of Earth that run east-west parallel to the equator (the latitude) and project these onto the sky, the lines become lines of declination. It measures the angular distance of a celestial object north/south of the equator. The declination is measured by the angle between the celestial equator and the position of the body.
- In the east-west change the major role is for the Equation of Time.

In Figure 13 two analemmas are shown. Although it is stated that one is from the northern hemisphere and the other one from the southern hemisphere, this depends on the time of the day. The left picture actually belongs to a moment in which the Sun is rising on the northern hemisphere. The shape of the right picture could also belong to the northern hemisphere, but that would belong to a setting Sun. For the southern hemisphere it is exactly the other way around, since the dates of the summer and winter solstice are reversed in comparison to the northern hemisphere [6].

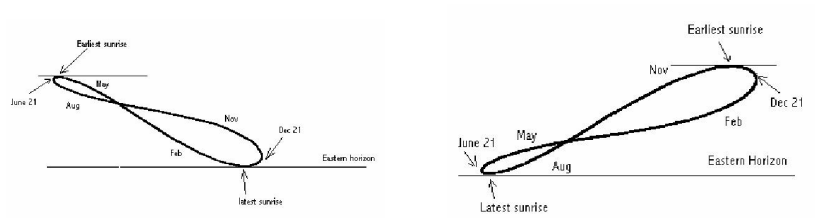


Figure 13: Analemma. Left: Northern hemisphere. Right: Southern hemisphere

3.3.4 Approximation of the analemma

According to [5] components of the analemma, both the altitude and the azimuth, can be analysed using Fourier analysis. In Figure 14 the Fourier components are shown for Leiden at noon. It can be seen that for the altitude only the first Fourier component has influence. This is actually the yearly movement of the Sun between the tropics of Capricorn and Cancer. For the azimuth we see a large impact for the first two Fourier components and number three is substantially smaller. This indicates a change every half a year.
Figure 14 has been created by the use of 365 datapoints, the number of days in a year. This dataset, for Leiden at noon, can be found in Appendix A. We have used Discrete Fourier Transform with $T = N = 365$. Definition and more explanation concerning Discrete Fourier Transform can be found in Appendix B as well for the proof why Figure 14 is symmetrical. One remark should be made. In Figure 14 it is shown that the second

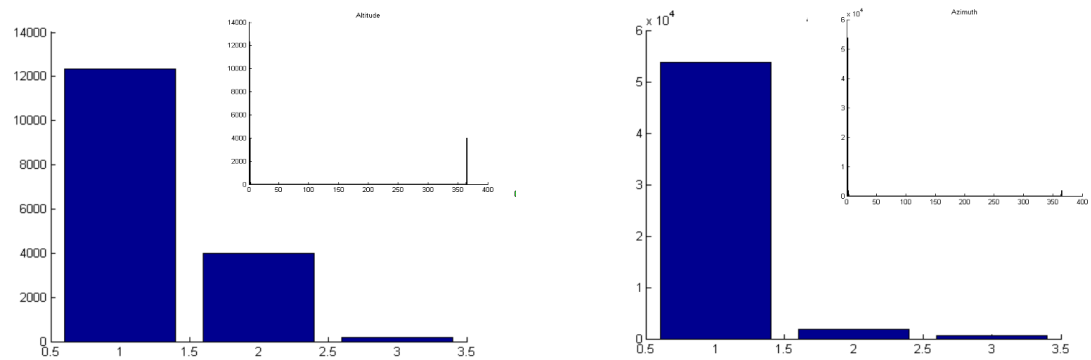


Figure 14: First three components in the Discrete Fourier Transform of the 365 day time series of the Sun for altitude (left) and azimuth (right) at the same standard time (12:00h) in Leiden. Insets: plots of modulus of all components. Note the symmetry (see Proposition B.1).

component is similar to the last component. This has to do with the following. The dataset is formed as a 365-vector $x[1, 2, \ldots, 365]$ with Fast Fourier Transform $y = \text{fft}(x)$, again a 365-vector $y[1, 2, \ldots, 365]$. We have $y(i) \longleftrightarrow \overline{X_{i-1}}$. Proposition B.1 tells us $\overline{X_k} = X_{N-k}$ with $N = 365$ while $k = 0, 1, \ldots, 364$. This gives us $y(2) = X_1 = \overline{X_{364}} = \overline{y(365)}$. For $y(1) = X_0 = \overline{X_N} = \overline{y(366)} = y(1)$. The code for transforming the dataset into Figure 14 and the analemmas can be found in Appendix C.

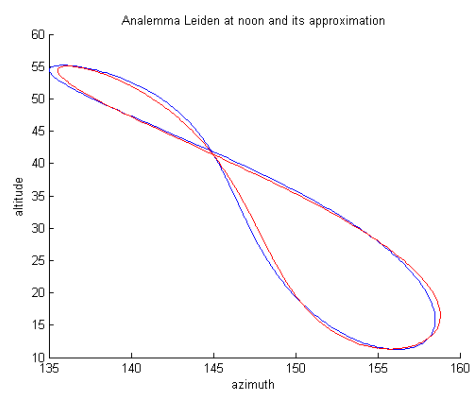


Figure 15: Analemma for Leiden at noon. The blue line are the datapoints, the red line the reconstruction of the dataset using the first two components of the DFT.

In Figure 15 we have the analemma for the dataset given in Appendix A, the blue graph, as well as the approximation of the analemma, the red graph, by using the truncated vectors for the altitude and azimuth after the use of Discrete Fourier Transform.
The truncated vectors are the vectors that are made by using the symmetry of Proposition B.1. By using this symmetry we can transform the altitude and the azimuth back in order to get an approximation of these truncated vectors. For the altitude we have seen that

only the first Fourier component plays a role, although three components are shown. By using the inverse formula we get

$$x_k = \frac{1}{N}\left(\hat{x_0} + \hat{x_N} e^{2\pi i k}\right) = \frac{1}{365}\left(\hat{x_0} + \hat{x_N}\right)$$

k is the number of day in the year. For the azimuth we get a similar equation, but in this case the first two Fourier components play a role, as well as the altitude three components are shown in Figure 14

$$\hat{x_k} = \frac{1}{365}\left(\hat{x_0} + \hat{x_1} e^{2\pi i \frac{k}{365}} + \hat{x_{n-1}} e^{2\pi i \frac{k}{365}} + \hat{x_n}\right).$$

3.3.5 Yearly movement: the equinox

The word equinox means the time of year when night and day have equal length. '*Equi*' means equal and '*nox*' means night. Twice a year this phenomenon takes place. Interesting to know is that day and night are equal for the whole year at the equator. In general we can say that the Vernal equinox (Spring equinox) takes place around the 21st March and the Autumnal equinox around the 22nd September. The Earth makes an elliptic orbit around the Sun. From the Earth it seems as if the Sun is moving. During Summer at the northern hemisphere, the Sun is above the equator and during Winter at the northern hemisphere below the equator. This means that the Sun 'crosses' the equator twice a year. This can be seen in Figure 16. As mentioned, the dates of the equinoxes are not the same every

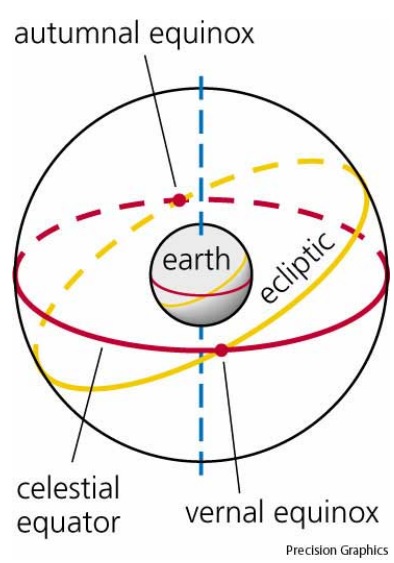


Figure 16: Equinox

year. This has to do with the fact that the Earth takes 365.25 days to go around the Sun. In general it differs 6 hours every year, except for a leap year when is moves back one day. One remark we can make is the fact that the dates of the equinoxes do not coincide with

the exactly equal day-night division, considering the sunrise and sunset of these dates. By Kepler's Law we know that the Earth moves faster when closer to the Sun. More about this will be discussed in the next part. The Sun also does not cross the meridian at noon each day. Here the Equation of Time shows up again. This equation will be discussed later [15].

3.3.6 Yearly movement: the aphelion and the perihelion

As mentioned in the part covering the equinox, the Earth does not travel at a constant speed around the Sun. Since the Earth's orbit is elliptic the distance between the Earth and the Sun differs. See Figure 17.

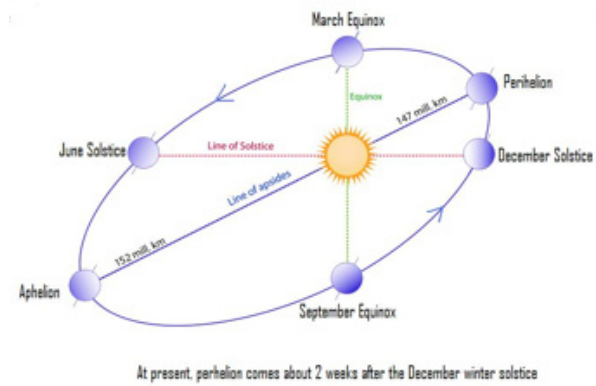


Figure 17: Aphelion and perihelion [16]

The place where the Earth is closest to the Sun is called *perihelion* and this happens in the beginning of January. For 2014 it was the 4th that month [17]. The place where the Earth is furthest away is called *aphelion* and this happens in the beginning of July. For 2014 this was the 3rd. These two words come from the Greek language. 'Helios' means Sun, while 'peri' means near and 'apo' means away from. As you can see in Figure 17 at perihelion the distance to the Sun is 147 million km while at aphelion it is 152 million km. According to Kepler's Second Law a line joining the Earth and the Sun sweeps out equal areas during equal intervals of time. Physically this means that the Earth moves faster while closer to the Sun. Let us have a look at the Earth's speed at perihelion and aphelion. The velocity of an object at distance r from the Sun is given by

$$v = \sqrt{GM \cdot \left(\frac{2}{r} - \frac{1}{a}\right)}$$

G is the universal gravitational constant, being

$$G = 6.673 \cdot 10^{-11} \mathrm{Nm}^2/\mathrm{kg}^2.$$

M is the mass of the Sun, being

$$M = 1.989 \cdot 10^{30}\text{kg}.$$

a is the Earth's semi-major axis, being

$$a = 149\ 597\ 887\text{km}.$$

This means that at perihelion the distance equals

$$r = a(1 - e)$$

and at aphelion

$$r = a(1 + e).$$

e is the eccentricity of the Earth

$$e = 0.0167.$$

Using this all we find that the Earth's speed at perihelion is 30.287 km/s relative to the Sun which equals about 109 km/h. At aphelion the speed equals 29.291 km/s which is about 105.4 km/h.

3.3.7 Longterm movement: precession of the equinoxes

Besides the daily and yearly movement there is one more change in the position of the Earth. This is the change of orientation of the Earth's rotational axis. This change takes about 26000 years to be back again at its starting point. This period of time is called a Great or Platonic year. This section is called the precession of the equinoxes which is actually a historical name. Nowadays this term is used when the mathematical details are unknown, in non-technical discussions.
The precession is caused by gravitational forces. Since the Earth is not actually a sphere but an oblate spheroid and the Earth is tilted, the gravitational forces pull differently on the Earth. The part that is closer to the Sun has a stronger force working on it than the part that is in the opposite position.
The seasons are influenced by the precession since the cycle of seasons is 20.4 minutes less than the period for the the Earth to return to the same position in comparison to the previous year. This rotation of the projection slowly takes place one degree every 71 years [14].
In the left-hand top corner of Figure 18 is actually meant; the projection of the south-north earth rotation axis onto the orbital plane (normalised).

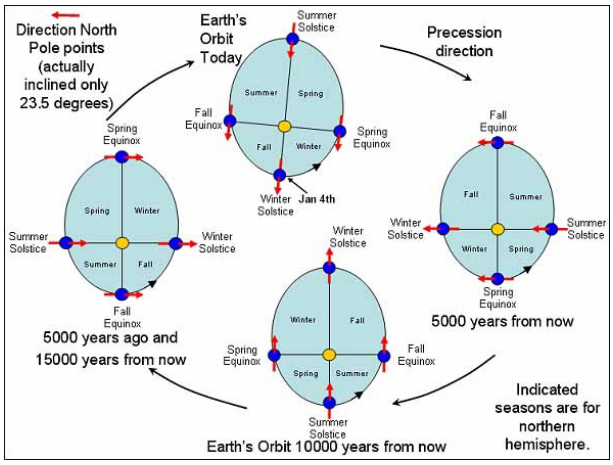


Figure 18: Precession [14]

3.4 Construction of sundials

On a sundial you need hour lines that indicate the time. In the early days of the sundial the day was divided into three parts, the morning, afternoon and night. The shadows of the gnomon are large during the morning, grow smaller towards noon and get larger again passing noon. During the night there is no shadow. During the morning the shadows are stretched to the west and in the afternoon to the east. The division of the three parts, the morning, afternoon and night was done by focusing on dawn, noon and sunset. Noon was done by the fact that the shadow had to be the shortest during the day. This way the temporary hours were created [7], p1-2.

3.4.1 Different kinds of sundials

There are several kinds of sundials [19]. We will focus on three types of sundials, namely:

- *Equatorial sundial*; This is a sundial of which the plane of the dial is equal to the plane of the equator. The gnomon of this sundial is vertical and all the hour lines on the plane are equally spaced by 15 degrees.

- *Horizontal sundial*; This sundial is similar to the equatorial sundial, except this sundial is not placed at the equator but its plane is parallel to the equator. The gnomon of this dial is placed in an angle that is equal to the local latitude of the place the sundial is situated.

- *Vertical sundial*; The plane of this type of sundial is vertical, placed on walls for example. This type of sundial works best when facing south. If that is the case, the gnomon is placed in an angle equal to the co-latitude (the complementary angle of the latitude) of the location. In case the sundial is not facing south, the gnomon will be placed in an angle less than the co-latitude.

3.4.2 Hour line configurations

The location of hour lines on the sundial depends on the type of sundial.
As mentioned, the equatorial sundial has hour lines that are equally divided by 15 degrees. This division can be done by construction (see Appendix E). This division can be seen in Figure 19.

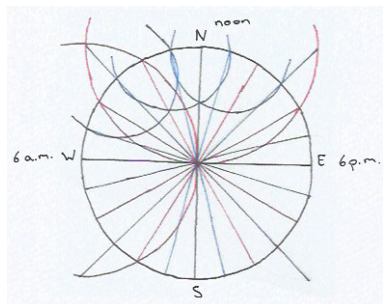


Figure 19: Construction of dividing a circle in 24 pieces

From this equatorial dial we can also make the hour lines for any horizontal or vertical sundial, for any latitude. For this we first need a right-angled triangle in order to know how to draw the circles for these dials. The constructions for the triangle and the hour lines can be found in the Appendix F. In Figure 20 the triangle for Leiden's latitude can be viewed as well as the elliptic form which the hour lines will give. Do notice that the lengths of the radii of the circles have been doubled in comparison to the construction in order to get a better picture.

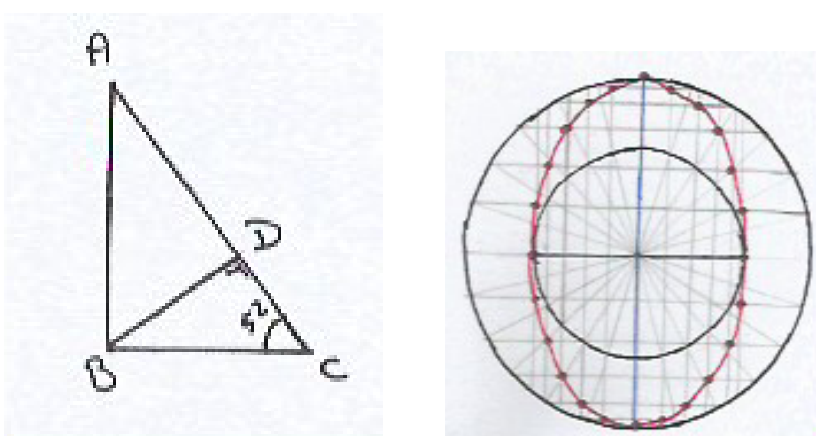


Figure 20: Left: triangle for Leiden's latitude. Right: hour lines in ellipse for Leiden for a horizontal dial.

One geometric way of drawing sundial hour lines for any place is given on page 16 of [4]. In Figure 21 the construction for Leiden has been drawn. This construction, described in Section 3.5.1, is based on the sundial formula (24) that will be presented in Section 3.4.3.

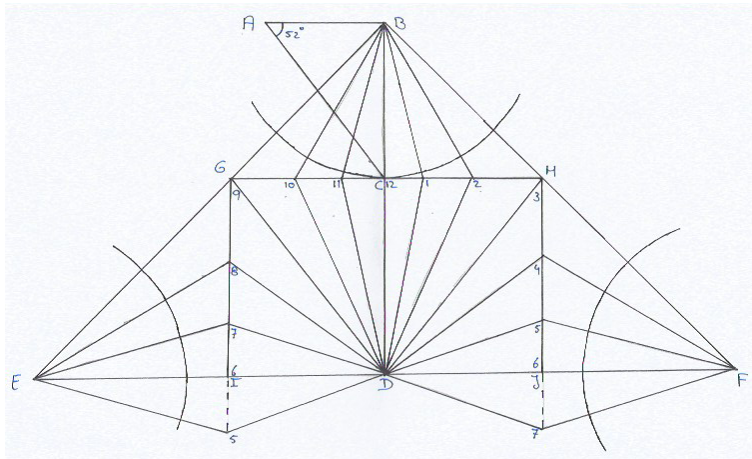


Figure 21: Hour lines for Leiden.(a large picture is available in Appendix D.) The numbers at the points on rectangle GHIJ indicate the hours of the day.

3.4.3 The sundial formula

The sundial formula describes the angle α_n between the noon hour line (which runs from the base of the gnomon to the true north) and the hour line that corresponds to hour $n \in \{-6, -5, \ldots, 0, \ldots, 5, 6\}$ where negative values correspond to morning (a.m.) and positive to afternoon (p.m.). This formula is used for a horizontal sundial, an example in Leiden is the one near Molen de Valk, seen in Figure 22. One can see the 52 ^{o} angle of the gnomon with respect to the horizontal plane, indicating it is a horizontal sundial.

Figure 22: Sundial near Molen de Valk, Leiden; the gnomon has recently been removed.

The construction of Figure 21, is based on this sundial formula. Note that β is the latitude of the location. The sundial formula is given by

$$\alpha_n = \arctan(\sin\beta \tan(15^o n))$$

or

$$\tan\alpha_n = \sin\beta \tan(15^o n). \tag{24}$$

In order to derive (24), let us have a look at the equatorial sundial as explained above. This dial can be placed anywhere on Earth as long as the dial plane stays parallel to the equator. The Earth is tilted by 23.44 degrees. It is easier to have the face of the sundial parallel to the local horizon instead of parallel to the equator. This brings us to the translation from the hour lines on an equator dial to a horizontal dial. Do notify that the angle between the gnomon and the horizontal dial should be equal to the latitude [4]. Figure 23 shows this graphically.

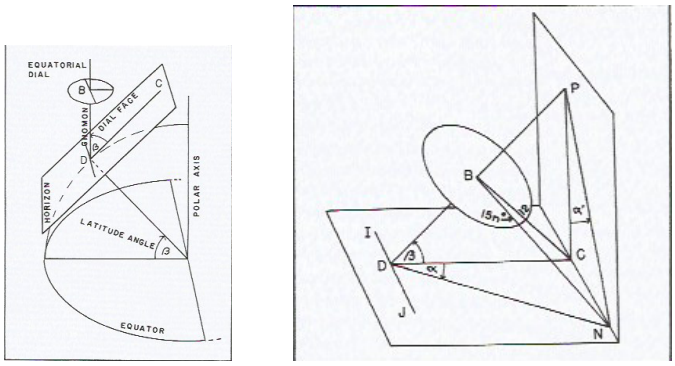


Figure 23: Left: position of a horizontal dial. Right: Zoomed in and levelled.

Looking at Figure 23 we can derive the sundial formula.

$$\angle DCP = \angle BCN = \angle DCN = \angle DBC = 90^o.$$

This is easier to see in the left picture of Figure 23.

$$\angle CDN = \alpha, \qquad \angle CDB = \beta,$$

respectively the hour angle and the latitude. BN is the extension of an hour line on the equatorial dial. We already know that on an equatorial dial all hour lines are spaced by 15 degrees, so for any angle such as $\angle CBN$ we have that it equals $15n$ for n hours from noon.

$$\begin{aligned}
\Delta DBC &: \quad BC = DC \cdot \sin\beta \\
\Delta CBN &: \quad \angle CBN = 15^o n \\
& \quad\quad CN = BC \cdot \tan 15n \Rightarrow CN = DC \cdot \sin\beta \cdot \tan(15^o n) \\
\Delta DCN &: \quad \tan\alpha_n = \frac{CN}{DC} = \sin\beta \tan(15^o n)
\end{aligned}$$

The degrees for the hour lines on a horizontal dial for Leiden are given in the table below (see Appendix D).

Time	n	α
1 p.m.	1	11.9
2 p.m	2	24.5
3 p.m	3	38.2
4 p.m	4	53.8
5 p.m.	5	71.2
6 p.m.	6	90

3.5 Construction for hour line configurations

The hour lines according to the sundial formula can be constructed geometrically. There are two constructions.

3.5.1 Construction geometric method hour lines

This construction has been taken from p.16 of [4] and has been clarified, see Figure 21.

1. Draw a line AB.
2. At A, draw AC, making angle BAC equal to your latitude.
3. Measure CD, which is the extension of BC, equal to CA.
4. Draw EF perpendicular to D to BD such that DE and DF are equal to DB.
5. Connect B with E and B with F.
6. Make GH perpendicular through C, perpendicular to BD, stopping at BE and BF.
7. Drop GI and HJ perpendicular to EF.
8. At B, E and F, draw arcs of radius BC.
9. Divide those arcs into 15-degree sectors, and extend each radius until it meets a side of the rectangle GHIJ.
10. Draw lines from D to the points of intersection with the rectangle. These are the hour lines for a sundial at your latitude, with gnomon placed at D pointing in the direction of C, see Figure 23.

3.5.2 Using the ellipse

Another way of constructing hour lines according to the sundial formula is by using an ellipse [8], see Figure 20. Let a be the semi-minor axis of the ellipse equal to the radius of the equatorial dial. Let b be the semi-major axis. β is still the angle that the gnomon makes, which is the latitude. b is related to a by

$$\sin\beta = \frac{a}{b}, \text{hence } b = \frac{a}{\sin\beta}. \tag{25}$$

The equation for the ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(See Appendix G). By substituting (25) we get

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2 \cdot \sin^2\beta}{a^2} &= 1 \\ x^2 + y^2 \sin^2\beta &= a^2 \end{aligned}$$

We could write x as $x = a\sin(15^o n)$, see Figure 24. Substituting this gives

$$\begin{aligned} a^2 \cdot \sin^2(15^o n) + y^2 \sin^2\beta &= a^2 \\ y^2 &= \frac{a^2 - a^2\sin^2(15^o n)}{\sin^2\beta} \\ &= \frac{a^2(1 - \sin^2(15^o n))}{\sin^2\beta} \\ &= \frac{a^2\cos^2(15^o n)}{\sin^2\beta} \\ y &= \frac{a\cos(15^o n)}{\sin\beta} \end{aligned}$$

Looking again at Figure 24 we get

$$\begin{aligned} \tan\alpha_n &= \frac{x}{y} \\ &= \frac{a\sin(15^o n)}{\frac{a\cos(15^o n)}{\sin\beta}} \\ &= \frac{a\sin(15^o n)\sin\beta}{a\cos(15^o n)} \\ &= \tan(15^o n) \cdot \sin\beta \end{aligned}$$

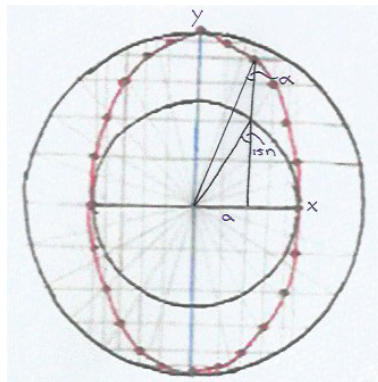


Figure 24: Ellipse for the derivation.

3.6 Equation of Time

With the Equation of Time is meant the difference between the solar time and the standard time as mentioned before. As mentioned, eccentricity and obliquity play the most important roles in the difference of time. The Earth's orbit is a Keplerian ellipse, which means that the motion of the Earth forms a two-dimensional orbit plane in a three-dimensional space. The ratio between the mean angular velocity of revolution, ω, and the rotation Ω is $\frac{1}{365}$, since the time between perihelion and aphelion is half a year and it takes the Earth 365 days to go around the Sun. In [5] a derivation is done for eccentricity part of the time equation. Do notify that the numbers could be more precise, but the focus in [5] is on the derivation itself. The rate of change in the difference per day due to eccentricity is

$$\Delta_E = \frac{2e\omega}{\Omega} \cos\left(\frac{2\pi(D - D_p)}{365}\right). \tag{26}$$

e is the constant for the eccentricity, which is approximately 0.0167 [5], D is the day number, while D_p is the date of perihelion, which was 4th January 2014 for this year [17]. Do note that

$$\frac{2e\omega}{\Omega} \approx 9.15 \times 10^{-5}\mathrm{ss}^{-1},$$

which is 7.9 seconds a day. For the obliquity another derivation is done in [5], namely the rate of change in the difference per day to obliquity is

$$\Delta_O = \frac{\epsilon^2\omega}{2\Omega} \cos 2\left(\frac{2\pi(D - D_w)}{365}\right) \tag{27}$$

ϵ is the angle by which the Earth is tilted, so 23.44 degrees, D_w is the day of the winter solstice, which is the 21st December 2014 for this year [21]. Notice that

$$\frac{\epsilon^2\omega}{2\Omega} \approx 2.29 \times 10^{-4}\mathrm{ss}^{-1}$$

which is 19.8 seconds a day. The accumulated difference from starting day D_0 to day D is then given by

$$\begin{aligned} E &= \int_{D_0}^{D} \Delta_E(D') + \Delta_O(D') dD' \\ &= \left[\frac{365}{2\pi} \cdot \left[\frac{2e\omega}{\Omega} \sin\left(\frac{2\pi(D' - D_p)}{365} \right) + \frac{\epsilon^2 \omega}{4\Omega} \sin 2 \left(\frac{2\pi(D' - D_w)}{365} \right) \right] \right]_{D'=D_0}^{D} \\ &= \frac{365}{2\pi} \left[\frac{2e\omega}{\Omega} \sin\left(\frac{2\pi(D' - D_p)}{365} \right) + \frac{\epsilon^2 \omega}{4\Omega} \sin 2 \left(\frac{2\pi(D' - D_w)}{365} \right) \right] - E_0 \end{aligned}$$

where E_0 depends on the starting day. Since $\frac{365}{2\pi}$ day $\approx 58,1$ day, we find (in minutes)

$$E = 7.6 \sin\left(\frac{2\pi(D-4)}{365} \right) + 9.6 \sin 2 \left(\frac{2\pi(D-355)}{365} \right) - E_0 \tag{28}$$

Note that in [5], formula (28) the prefactor $\frac{365}{2\pi}$ was omitted leading to an slightly different equation with prefactors 7.9 and 9.9 instead of 7.6 and 9.6 respectively. If D_0 is taken on a day where solar time and standard time are equal, E_0 can be computed and E yields the required correction each day.

In Figure 25 we can see that the eccentricity gives a sine wave in a period of a year, while the obliquity gives a sine wave in a period of half a year. This was also shown by the Fourier analysis of the altitude and the azimuth. When putting the eccentricity and obliquity together we get the equation of time.

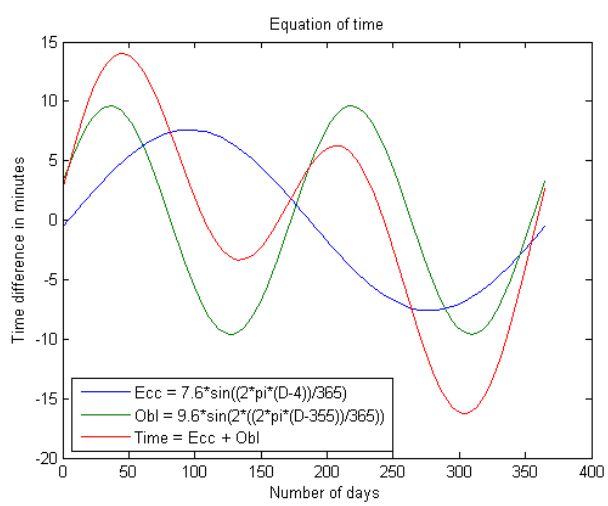


Figure 25: Equation of time

3.7 Conclusion

Keeping track of time using the Sun is an old method and a lot is known. Difficulties can be found in the different time settings when discussing sundials. Although there are all kinds of sundials, one important feature for them to work is the Sun itself. When there is no Sun, there is no clear shadow and finding out the time can become very difficult. Interesting is that a sundial is only exact on time when it is an equinox day. All other days during the year the sundial is a little ahead or behind.
As seen in the construction, constructing the hour lines using the ellipse is quite difficult. Especially when getting a correct ellipse with $b = a \sin \beta$.

4 A pendulum clock on a ship

4.1 Introduction

For navigation at sea time is essential. The connection to Leiden can be made by the nautical college. We have seen water clocks not being accurate, sundials depending on longitude and latitude, so how to measure time on a ship? At the end of the 16th century clocks only worked because of a weight falling down. In 1657 the Dutch scientist Christiaan Huygens applied for a patent for a pendulum clock. Huygens used the information that he got from Galileo Galilei, who discovered that not the weight of the pendulum but the length between the pivot and the centre of mass played the most important role in keeping track of time [23].
When the clock is on a stable surface, there is not much of a problem. But time also played an important role at sea in finding the position of a ship. In order to determine the position of the ship you needed the ship's longitude. One rotation of the Earth is 24 hours, so a full turn (360 degrees) equals 24 hours, which means 15 degrees every hour,as already mentioned in the section about sundials. But how could they keep track of time on a ship that is not stable at all? In 1714 this became a serious question. The British government set up the Board of Longitude in the early 1700s to ensure more safety for British sailors. The longitude prize, organized by the Board of Longitude, was offered to the person who could produce an accurate way of determining the longitude of a ship. Huygens came up with a helical spring in order to keep the balance for a clock at sea. Huygens did not succeed because of the movement of the ship and differences in temperature [24].
Father and son Harrison worked on the longitude problem for almost their entire lives. This is an interesting story, but we want to look at what the influence of a ship is on the motion of a pendulum clock. Besides motion, temperature and humidity play a role in the accurately working of a clock, since humidity can make the clock rust and difference in temperature can make the clock go faster or slower. Since we can make sure that a clock can be placed under such conditions that temperature and humidity can be controlled, we are going to focus on the motion part [25].

4.2 Pendulum equation

Our goal in this chapter is to figure out the behaviour of a pendulum on a moving ship. As mentioned in the introduction, temperature and humidity do have their influences but we will ignore these factors.

g	gravitational constant, ~ 9.81 m$^2/s$
l	length between the pivot and the centre of mass
h	height of the mast
F_g	force of gravity
F_c	force of constraint applied by the pivot

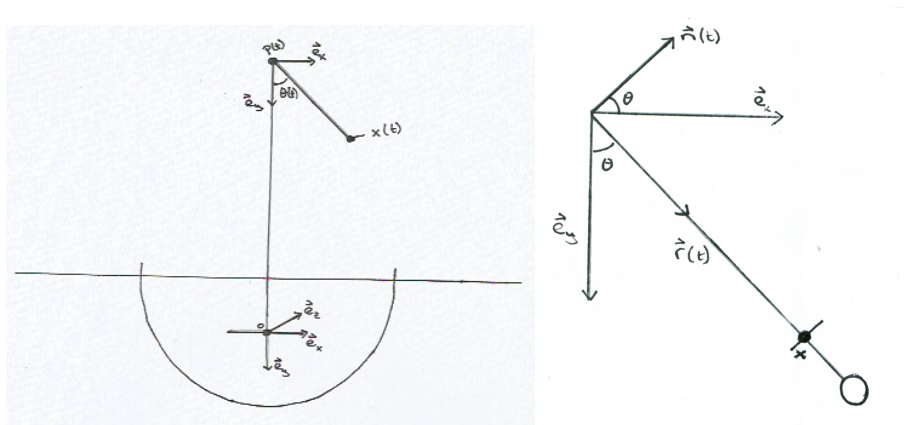


Figure 26: Initial situation of a pendulum attached to the mast of a ship (left) and detailed situation near the pendulum (right).

In [9] a derivation of the pendulum equation is done. The advantage of this derivation is the fact that it is done for a pendulum on a moving support. Do notice that the following assumptions are made:

- The pendulum is a rigid body.
- The position $\mathbf{p}(t)$ of the pivot is prescribed *a priori*.
- The forces working on $\mathbf{p}(t)$ will not be mentioned explicitly, only the acceleration of $\mathbf{p}(t)$, which is described by $\alpha(t)$.

In Figure 26 we see the initial situation of the pendulum. $\mathbf{p}(t)$ is the position of the pivot with respect to the 'fixed' point O around which the ship will roll:

$$\mathbf{p}(t) = p_x(t)\mathbf{e_x} + p_y(t)\mathbf{e_y}.$$

Here $\mathbf{e_x}$ and $\mathbf{e_z}$ lie in the horizontal plane, $\mathbf{e_y}$ is pointing downwards orthogonal to this plane (to a right-handed coordinate system). $\theta(t)$ is the angle between $\mathbf{e_y}$ and the vector from the pivot to the pendulum, and let $\mathbf{k_t}(\theta(t))$ be the vector of unit length for the pivot to the end of the pendulum. The t stands for tangent (to the pendulum). Then

$$\mathbf{r}(t) := \mathbf{x}(t) - \mathbf{p}(t) = l\mathbf{k_t}(\theta(t)), \tag{29}$$

where $\mathbf{x}(t)$ is the position of centre of mass (recall that we assume that the pendulum is a rigid body).

$$\mathbf{x}(t) = x(t)\mathbf{e_x} + y(t)\mathbf{e_y}.$$

Besides the tangent component $\mathbf{k_t}(\theta(t))$ we also need a normal component, which will be represented by $\mathbf{k_n}(\theta(t))$:

$$\begin{aligned} \mathbf{k_t}(\theta(t)) &= \sin\theta(t)\mathbf{e_x} + \cos\theta(t)\mathbf{e_y}, \\ \mathbf{k_n}(\theta(t)) &= \cos\theta(t)\mathbf{e_x} - \sin\theta(t)\mathbf{e_y}. \end{aligned}$$

We will need the derivatives of $\mathbf{k_t}(\theta(t))$ and $\mathbf{k_n}(\theta(t))$.

$$\begin{aligned}
\frac{d}{dt}\mathbf{k_t}(\theta(t)) &= \cos\theta(t)\cdot\theta'(t)\cdot\mathbf{e_x} - \sin\theta(t)\cdot\theta'(t)\cdot\mathbf{e_y} \\
&= \theta'(t)\cdot\mathbf{k_n}(\theta(t)), \qquad (30)\\
\frac{d}{dt}\mathbf{k_n}(\theta(t)) &= -\sin\theta(t)\cdot\theta'(t)\cdot\mathbf{e_x} - \cos\theta(t)\cdot\theta'(t)\mathbf{e_y} \\
&= -\theta'(t)\mathbf{k_t}(\theta(t)). \qquad (31)
\end{aligned}$$

We could also express $\mathbf{e_x}$ and $\mathbf{e_y}$ in $\mathbf{k_t}(\theta(t))$ and $\mathbf{k_n}(\theta(t))$, namely

$$\begin{aligned}
\mathbf{e_x} &= \mathbf{k_t}\sin\theta(t) + \mathbf{k_n}\cos\theta(t), \\
\mathbf{e_y} &= \mathbf{k_t}\cos\theta(t) - \mathbf{k_n}\sin\theta(t).
\end{aligned}$$

In [9] the author derives the pendulum equation from Newton's Second Law:

$$m\mathbf{x}''(t) = \mathbf{F}(t).$$

Where $\mathbf{F}(t)$ is the net force which is applied to the rigid body, given by

$$\mathbf{F}(t) = F_t(t)\mathbf{k_t}(\theta(t)) + F_n(t)\mathbf{k_n}(\theta(t)). \qquad (32)$$

By using (29), (30) and (31) we get

$$\mathbf{x}''(t) = \mathbf{p}''(t) + l\theta''(t)\mathbf{k_n}(\theta(t)) - l(\theta'(t))^2\mathbf{k_t}(\theta(t)).$$

If we write

$$\mathbf{p}''(t) = \alpha_t(t)\mathbf{k_t}(\theta(t)) + \alpha_n(t)\mathbf{k_n}(\theta(t))$$

then we can rewrite Newton's Second Law as the following system

$$\begin{aligned}
-ml(\theta'(t))^2 + m\alpha_t(t) &= F_t(t) \qquad (33)\\
ml\theta''(t) + m\alpha_n(t) &= F_n(t) \qquad (34)
\end{aligned}$$

In (33) and (34) we have F_t and α_t being the components of force on the centre of mass at $\mathbf{x}(t)$ and acceleration of the pivot at $\mathbf{p}(t)$ in the direction of the pendulum. The force in the direction perpendicular to the pendulum and the acceleration of $\mathbf{p}(t)$ perpendicular to the pendulum are given by F_n and α_n.

In Figure 27 all forces on the pendulum are shown. We see that we can write (34) as

$$ml\theta''(t) = F_{gn}(t) - F_{pn}(t). \qquad (35)$$

Let us have a look at the different kinds of forces. The gravitational force is given by

$$F_g(t) = mg\mathbf{e_y} = mg(\cos\theta(t)\mathbf{k_t}(\theta(t)) - \sin\theta(t)\mathbf{k_n}(\theta(t))). \qquad (36)$$

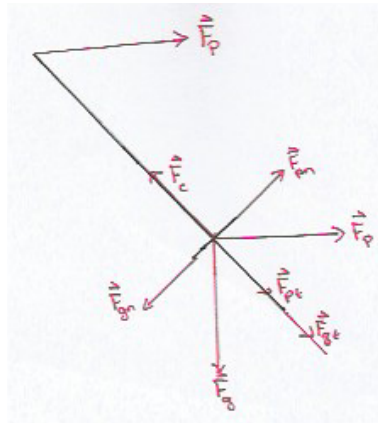


Figure 27: Forces working on the pendulum

There is the force of constraint, given by

$$F_c(t) = m\alpha_t(t) - ml(\theta'(t))^2 - mg\cos\theta(t).$$

Since the pendulum is a rigid body, the force of constraint can also be given by

$$F_c(t) = -(F_{pt}(t) + F_{gt}(t))$$

so it cancels F_t in (33). We will ignore the frictional forces. This all together brings us to the pendulum equation

$$l\theta''(t) = -g\sin\theta(t) - \alpha_n(t). \tag{37}$$

This second order ODE can also be written as two-dimensional system

$$\begin{aligned} \theta'(t) &= W(t) && (38)\\ W'(t) &= -\frac{g}{l}\sin\theta(t) - \frac{1}{ml}\cdot m\alpha_n(t) && (39)\\ &= -\frac{g}{l}\sin\theta(t) - \frac{1}{l}\cdot\alpha_n(t) && (40) \end{aligned}$$

Due to the lack of friction, mass does not play a role anymore, only the length of the pendulum. In case θ is small we get in (37) $\sin\theta(t) \approx \theta(t)$ and then equation (37) reduces to the harmonic oscillator with forcing. This could be solved explicitly, but we leave this to the reader. In case $\theta(t)$ is larger, the solution to the pendulum equation cannot be given explicitly.

4.3 Pivot equation

Besides the pendulum we also have to deal with the pivot, $\mathbf{p}(t)$, that is moving, due to the movement of the ship. Let us consider rolling (the rotation of a ship about its longitudinal axis). As pictured in Figure 28, $\phi(t)$ is the angle between O and the mast and h is the height of the mast at which the pivot of the pendulum is fixed to the mast.

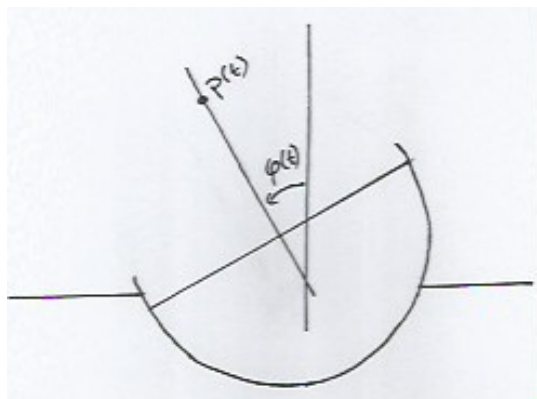


Figure 28: Movement of the position $\mathbf{p}(t)$ of the pivot

The formula for $\mathbf{p}(t)$ is given by

$$\mathbf{p}(t) = h\left[\sin(\phi(t))\mathbf{e_x} - \cos(\phi(t))\mathbf{e_y}\right]. \tag{41}$$

The acceleration of $\mathbf{p}(t)$ will be given by the second derivative.

$$\begin{aligned}
\mathbf{p}'(t) &= h\left[\cos(\phi(t))\cdot\phi'(t)\cdot\mathbf{e_x} + \sin(\phi(t))\cdot\phi'(t)\cdot\mathbf{e_y}\right] \\
\mathbf{p}''(t) &= h\left[\left\{-\sin(\phi(t))\cdot(\phi'(t))^2 + \cos(\phi(t))\cdot\phi''(t)\right\}\mathbf{e_x}\right. \\
&\quad \left. + \left\{\cos(\phi(t))\cdot(\phi'(t))^2 + \sin(\phi(t))\cdot\phi''(t)\right\}\mathbf{e_y}\right]
\end{aligned} \tag{42}$$

We can find the expression for $\phi(t)$ by using the pendulum equation.

$$\begin{aligned}
mh\phi''(t) &= -mg\sin(\phi(t)) \\
\phi''(t) &= -\frac{g}{h}\sin(\phi(t)) \\
\phi''(t) &\approx -\frac{g}{h}\phi(t) \qquad \phi(0) = 0 \\
\phi(t) &= \sin\left(\sqrt{\frac{g}{h}}t\right) =: \sin(\omega t)
\end{aligned}$$

Let us consider the harmonic oscillator in this situation, $\sin(\phi(t)) \approx \phi(t)$. We can rewrite (42) by using that

$$\begin{aligned}
\phi''(t) &= -\omega^2\sin(\phi(t)) \approx -\omega^2\phi(t) = -\omega^2\sin(\omega t) \\
\phi'(t) &= \omega\cos(\omega t), \qquad \cos(\phi(t)) \approx 1
\end{aligned}$$

$$
\begin{aligned}
\mathbf{p}''(t) &\approx h\left[\left\{-\omega^2\sin(\omega t)\cdot(\cos^2(\omega t)+1)\right\}\mathbf{e_x}+\left\{\omega^2\cos(2\omega t)\right\}\mathbf{e_y}\right] \\
&= h\left[\left\{-\omega^2\sin(\omega t)\cdot(\cos^2(\omega t)+1)\right\}(\mathbf{k_t}(\theta(t))\sin(\theta(t))+\mathbf{k_n}(\theta(t))\cos(\theta(t)))\right. \\
&\quad \left.\left\{\omega^2\cos(2\omega t)\right\}(\mathbf{k_t}(\theta(t))\cos(\theta(t))-\mathbf{k_n}(\theta(t))\sin(\theta(t)))\right] \qquad (43)
\end{aligned}
$$

4.4 Simulating the pendulum equation

Looking at (37) we still have the $\alpha_n(t)$ term that is not expressed. By using (43) we can express $\alpha_n(t)$ since that will be given by the coefficients of $\mathbf{k_n}(\theta(t))$.

$$
\begin{aligned}
\alpha_n(t) &= \cos\theta(t)\cdot h\cdot\left\{-\omega^2\sin(\omega t)\cdot(\cos^2(\omega t)+1)\right\}-\sin(\theta(t))\cdot h\cdot\left\{\omega^2\cos(2\omega t)\right\} \\
&= -h\omega^2\left[\cos(\theta(t))\sin(\omega t)(\cos^2(\omega t)+1)+\sin(\theta(t))\cos(2\omega t)\right] \qquad (44)
\end{aligned}
$$

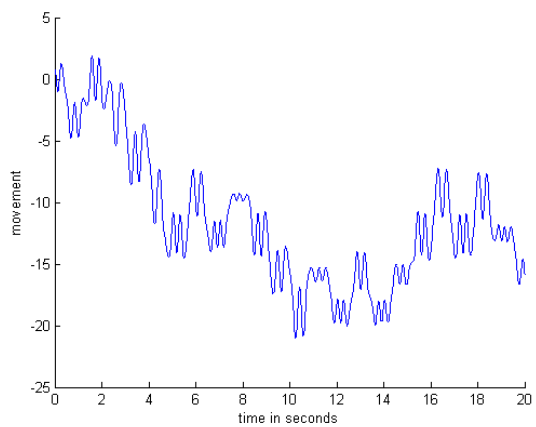


Figure 29: Simulation of the two dimensional system with parameter setting $h = 3$ and $l = 0.15$, both in metres. Vertically, $\theta(t)$ is plotted. Initial conditions: $\theta(0) = 1, \theta'(0) = 0$.

4.5 Conclusion

By simulating the pendulum equation it became clear why keeping track of time on a ship was a major problem back in those days. A ship is not a stable base for a pendulum clock. One could even try to figure out the working of a pendulum on a ship during a storm. We expect nothing else than 'chaotic' behaviour.

5 A brief comment on modern time-measurement

We have discussed 'natural' ways of measuring time, by water clocks and sundials. In history, these ways of measuring time were followed by the pendulum clock, also mentioned in a special way in this thesis. This all was followed in 1928 by the mechanism that nowadays still makes our wristwatches work, namely the quartz clock. A quartz clock has a quartz crystal inside in order to keep the electronic oscillator working. An advantage of quartz is the fact that temperature does not effect the crystal [26].
Today's world is a high-tech world. Everything depends on an accurate timing. Think of the GPS-system, especially the satellites involved, but also in radio-telescopes and synchronising computers. The most reliable source for keeping track of time is the atomic clock. At first GMT, Greenwich Mean Time, was the standard global time setting. This was introduced in 1884 and it was followed by the atomic version in 1972. An atomic clock works on a cesium-133 atom. In general an atomic clock uses a very stable atom. An atom has its own frequency, which is actually the movement of the electrodes in the atom. For the working of an atom clock a radio wave is sent through the atom. Since the atom has a specific frequency, the radio wave should be very close to this frequency in order to keep the atom in a constant and very precize oscillation. The results of oscillations of the atom are the new measure for time. This gives an accuracy of one second in 15 million years. Scientists now are working on an accuracy of one second in 10 billion years. This new atomic clock would work on a mercury ion [27].

A Dataset Azimuth and Altitude

From [22] we have take the following altitude and azimuth for the sun at noon in Leiden over the year. The data has last been updated at the 17th April 2014.

date	altitude	azimuth		date	altitude	azimuth		date	altitude	azimuth
1 jan	11.42	155.36		1 feb	16.41	151.27		1 mar	25.53	148.18
2 jan	11.47	155.24		2 feb	16.68	151.14		2 mar	25.91	148.09
3 jan	11.53	155.11		3 feb	16.95	151.01		3 mar	26.29	148.0
4 jan	11.6	154.99		4 feb	17.22	150.88		4 mar	26.67	147.91
5 jan	11.68	154.86		5 feb	17.5	150.76		5 mar	27.05	147.83
6 jan	11.77	154.73		6 feb	17.79	150.64		6 mar	27.44	147.74
7 jan	11.86	154.6		7 feb	18.08	150.51		7 mar	27.82	147.66
8 jan	11.96	154.47		8 feb	18.37	150.39		8 mar	28.21	147.58
9 jan	12.07	154.34		9 feb	18.67	150.27		9 mar	28.6	147.5
10 jan	12.18	154.2		10 feb	18.98	150.15		10 mar	28.99	147.42
11 jan	12.3	154.07		11 feb	19.29	150.04		11 mar	29.38	147.34
12 jan	12.43	153.94		12 feb	19.6	149.92		12 mar	29.78	147.26
13 jan	12.57	153.8		13 feb	19.92	149.81		13 mar	30.17	147.19
14 jan	12.71	153.67		14 feb	20.24	149.69		14 mar	30.57	147.11
15 jan	12.86	153.53		15 feb	20.57	149.58		15 mar	30.96	147.04
16 jan	13.02	153.4		16 feb	20.9	149.47		16 mar	31.36	146.96
17 jan	13.18	153.26		17 feb	21.24	149.36		17 mar	31.76	146.89
18 jan	13.35	153.13		18 feb	21.58	149.26		18 mar	32.15	146.81
19 jan	13.53	152.99		19 feb	21.92	149.15		19 mar	32.55	146.74
20 jan	13.71	152.86		20 feb	22.27	149.05		20 mar	32.95	146.67
21 jan	13.9	152.72		21 feb	22.62	148.94		21 mar	33.35	146.6
22 jan	14.1	152.59		22 feb	22.97	148.84		22 mar	33.74	146.52
23 jan	14.3	152.45		23 feb	23.33	148.74		23 mar	34.14	146.45
24 jan	14.51	152.32		24 feb	23.69	148.64		24 mar	34.54	146.38
25 jan	14.73	152.18		25 feb	24.05	148.55		25 mar	34.94	146.31
26 jan	14.95	152.05		26 feb	24.42	148.45		26 mar	35.33	146.24
27 jan	15.18	151.92		27 feb	24.79	148.36		27 mar	35.73	146.16
28 jan	15.42	151.79		28 feb	25.16	148.27		28 mar	36.12	146.09
29 jan	15.66	151.65						29 mar	36.51	146.02
30 jan	15.9	151.52						30 mar	36.9	145.95
31 jan	16.16	151.39						31 mar	37.29	145.87

date	altitude	azimuth		date	altitude	azimuth		date	altitude	azimuth
1 apr	37.68	145.8		1 may	48.11	142.82		1 june	54.36	137.98
2 apr	38.07	145.72		2 may	48.39	142.69		2 june	54.46	137.82
3 apr	38.46	145.64		3 may	48.68	142.55		3 june	54.55	137.67
4 apr	38.84	145.57		4 may	48.95	142.41		4 june	54.64	137.52
5 apr	39.22	145.49		5 may	49.29	142.27		5 june	54.72	137.37
6 apr	39.6	145.41		6 may	49.49	142.13		6 june	54.8	137.22
7 apr	39.98	145.33		7 may	49.75	141.99		7 june	54.87	137.07
8 apr	40.36	145.25		8 may	50.01	141.84		8 june	54.93	136.93
9 apr	40.73	145.16		9 may	50.26	141.69		9 june	54.98	136.8
10 apr	41.1	145.08		10 may	50.5	141.54		10 june	55.03	136.66
11 apr	41.47	144.99		11 may	50.74	141.39		11 june	55.07	136.53
12 apr	41.84	144.9		12 may	50.98	141.23		12 june	55.11	136.41
13 apr	42.2	144.81		13 may	51.2	141.08		13 june	55.14	136.29
14 apr	42.56	144.72		14 may	51.43	140.92		14 june	55.16	136.17
15 apr	42.91	144.63		15 may	51.64	140.76		15 june	55.18	136.06
16 apr	43.27	144.53		16 may	51.85	140.6		16 june	55.19	135.95
17 apr	43.62	144.43		17 may	52.06	140.43		17 june	55.19	135.85
18 apr	43.96	144.33		18 may	52.25	140.27		18 june	55.19	135.75
19 apr	44.31	144.23		19 may	52.45	140.11		19 june	55.18	135.66
20 apr	44.65	144.13		20 may	52.63	139.94		20 june	55.17	135.58
21 apr	44.98	144.02		21 may	52.81	139.78		21 june	55.15	135.5
22 apr	45.31	143.91		22 may	52.98	139.61		22 june	55.12	135.42
23 apr	45.64	143.8		23 may	53.15	139.45		23 june	55.09	135.35
24 apr	45.96	143.68		24 may	53.31	139.28		24 june	55.05	135.29
25 apr	46.28	143.57		25 may	53.46	139.12		25 june	55.0	135.23
26 apr	46.6	143.45		26 may	53.61	138.95		26 june	54.95	135.19
27 apr	46.91	143.33		27 may	53.75	138.79		27 june	54.89	135.14
28 apr	47.22	143.21		28 may	53.89	138.62		28 june	54.83	135.11
29 apr	47.52	143.08		29 may	54.01	138.46		29 june	54.76	135.08
30 apr	47.81	142.95		30 may	54.13	138.3		30 june	54.69	135.05
				31 may	54.25	138.14				

date	altitude	azimuth		date	altitude	azimuth		date	altitude	azimuth
1 july	54.61	135.04		1 aug	49.72	137.89		1 sept	41.33	145.42
2 july	54.53	135.03		2 aug	49.49	138.08		2 sept	41.02	145.69
3 july	54.43	135.02		3 aug	49.26	138.27		3 sept	40.71	145.97
4 july	54.34	135.03		4 aug	49.03	138.47		4 sept	40.39	146.24
5 july	54.24	135.04		5 aug	48.8	138.68		5 sept	40.07	146.51
6 july	54.13	135.06		6 aug	48.56	138.89		6 sept	39.75	146.79
7 july	54.02	135.08		7 aug	48.32	139.1		7 sept	39.43	147.06
8 july	53.9	135.12		8 aug	48.07	139.32		8 sept	39.11	147.33
9 july	53.78	135.16		9 aug	47.82	139.54		9 sept	38.78	147.6
10 july	53.65	135.2		10 aug	47.57	139.77		10 sept	38.46	147.87
11 july	53.52	135.25		11 aug	47.32	140.0		11 sept	38.13	148.14
12 july	53.38	135.32		12 aug	47.06	140.23		12 sept	37.8	148.41
13 july	53.24	135.38		13 aug	46.8	140.47		13 sept	37.47	148.68
14 july	53.1	135.46		14 aug	46.53	140.71		14 sept	37.13	148.94
15 july	52.94	135.54		15 aug	46.27	140.95		15 sept	36.8	149.2
16 july	52.79	135.63		16 aug	46.0	141.2		16 sept	36.46	149.47
17 july	52.63	135.72		17 aug	45.73	141.44		17 sept	36.12	149.73
18 july	52.46	135.82		18 aug	45.45	141.7		18 sept	35.78	149.98
19 july	52.29	135.93		19 aug	45.17	141.95		19 sept	35.44	150.24
20 july	52.12	136.04		20 aug	44.89	142.21		20 sept	35.1	150.49
21 july	51.94	136.16		21 aug	44.61	142.47		21 sept	34.76	150.74
22 july	51.76	136.29		22 aug	44.32	142.73		22 sept	34.42	150.99
23 july	51.57	136.42		23 aug	44.03	142.99		23 sept	34.07	151.24
24 july	51.38	136.56		24 aug	43.74	143.26		24 sept	33.73	151.48
25 july	51.19	136.71		25 aug	43.45	143.52		25 sept	33.38	151.72
26 july	50.99	136.86		26 aug	43.15	143.79		26 sept	33.03	151.96
27 july	50.79	137.02		27 aug	42.86	144.06		27 sept	32.69	152.19
28 july	50.58	137.18		28 aug	42.56	144.33		28 sept	32.34	152.42
29 july	50.37	137.35		29 aug	42.25	144.6		29 sept	31.99	152.65
30 july	50.16	137.52		30 aug	41.95	144.87		30 sept	31.64	152.87
31 july	49.94	137.7		31 aug	41.64	145.15				

date	altitude	azimuth		date	altitude	azimuth		date	altitude	azimuth
1 oct	31.29	153.09		1 nov	20.77	157.8		1 dec	13.32	158.19
2 oct	30.94	153.31		2 nov	20.46	157.88		2 dec	13.15	158.15
3 oct	30.59	153.52		3 nov	20.15	157.95		3 dec	12.99	158.09
4 oct	30.24	153.73		4 nov	19.85	158.02		4 dec	12.84	158.04
5 oct	29.89	153.93		5 nov	19.55	158.08		5 dec	12.69	157.98
6 oct	29.54	154.13		6 nov	19.25	158.14		6 dec	12.55	157.91
7 oct	29.19	154.33		7 nov	18.96	158.2		7 dec	12.42	157.85
8 oct	28.84	154.52		8 nov	18.67	158.25		8 dec	12.29	157.78
9 oct	28.49	154.71		9 nov	18.39	158.29		9 dec	12.17	157.71
10 oct	28.14	154.9		10 nov	18.1	158.33		10 dec	12.06	157.63
11 oct	27.79	155.08		11 nov	17.83	158.37		11 dec	11.96	157.56
12 oct	27.44	155.25		12 nov	17.55	158.4		12 dec	11.86	157.47
13 oct	27.09	155.42		13 nov	17.29	158.42		13 dec	11.77	157.39
14 oct	26.75	155.59		14 nov	17.02	158.45		14 dec	11.68	157.3
15 oct	26.4	155.75		15 nov	16.76	158.46		15 dec	11.6	157.22
16 oct	26.05	155.91		16 nov	16.51	158.48		16 dec	11.53	157.12
17 oct	25.71	156.07		17 nov	16.26	158.49		17 dec	11.47	157.03
18 oct	25.37	156.21		18 nov	16.01	158.49		18 dec	11.42	156.93
19 oct	25.03	156.36		19 nov	15.77	158.49		19 dec	11.37	156.83
20 oct	24.69	156.5		20 nov	15.54	158.49		20 dec	11.33	156.73
21 oct	24.35	156.63		21 nov	15.31	158.48		21 dec	11.29	156.63
22 oct	24.01	156.76		22 nov	15.08	158.47		22 dec	11.27	156.52
23 oct	23.68	156.89		23 nov	14.86	158.46		23 dec	11.25	156.42
24 oct	23.35	157.01		24 nov	14.65	158.44		24 dec	11.24	156.31
25 oct	23.02	157.12		25 nov	14.44	158.41		25 dec	11.23	156.19
26 oct	22.69	157.23		26 nov	14.24	158.39		26 dec	11.24	156.06
27 oct	22.36	157.34		27 nov	14.04	158.36		27 dec	11.25	155.97
28 oct	22.04	157.44		28 nov	13.85	158.32		28 dec	11.27	155.85
29 oct	21.72	157.54		29 nov	13.67	158.28		29 dec	11.29	155.73
30 oct	21.4	157.63		30 nov	13.49	158.24		30 dec	11.33	155.61
31 oct	21.08	157.72						31 dec	11.37	155.49

B Discrete Fourier Transform

A way to analyse time series for periodicty in the data is by use of Discrete Fourier Transform (DFT). Let us take $\mathbf{x} = (x_0, x_1, \ldots, x_{N-1}) \in \mathbf{C}^N$, representing an N-dimensional data vector. For sample times we take

$$t_j = j \cdot \frac{T}{N}, \qquad j = 0, 1, 2, \ldots, N-1,$$

where T is the length of the sampling interval and N the number of samples. We define

$$\omega_k = k \cdot \frac{2\pi}{T}.$$

Take the matrix

$$U = \begin{pmatrix} e^{-i\omega_0 t_0} & e^{-i\omega_0 t_1} & \ldots & e^{-i\omega_0 t_{N-1}} \\ e^{-i\omega_1 t_0} & e^{-i\omega_1 t_1} & \ldots & e^{-i\omega_1 t_{N-1}} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-i\omega_{N-1} t_0} & \ldots & \ldots & e^{-i\omega_{N-1} t_{N-1}} \end{pmatrix}$$

The Discrete Fourier Transform of $\mathbf{x}$ is $X := Ux$. That is $\left[x_n = \sum_{k=0}^{N-1} e^{-i\omega_n t_k} x_k\right]$ Let us have a look at U^*U.

$$U^*U = \begin{pmatrix} e^{i\omega_0 t_0} & e^{i\omega_0 t_1} & \ldots & e^{i\omega_0 t_{N-1}} \\ e^{i\omega_1 t_0} & e^{i\omega_1 t_1} & \ldots & e^{i\omega_1 t_{N-1}} \\ \vdots & \vdots & \vdots & \vdots \\ e^{i\omega_{N-1} t_0} & \ldots & \ldots & e^{i\omega_{N-1} t_{N-1}} \end{pmatrix} \cdot U$$

For the diagonal of U^*U we will have

$$(U^*U)_{jj} = \sum_{k=0}^{N-1} e^{i\omega_k t_j} \cdot e^{-i\omega_k t_j} = N$$

For all $l \neq j$ we have

$$\begin{aligned} (U^*U)_{lj} &= \sum_{k=0}^{N-1} e^{i\omega_k t_l} \cdot e^{-i\omega_k t_j} = \sum_{k=0}^{N-1} e^{i\omega_k (t_l - t_j)} \\ &= \sum_{k=0}^{N-1} \left[e^{i \cdot \frac{2\pi}{T}(t_l - t_j)}\right]^k \\ &= \frac{e^{\frac{2\pi i}{T}(t_l - t_j)N} - 1}{e^{\frac{2\pi i}{T}(t_l - t_j)} - 1} = \frac{e^{2\pi i (l-j)} - 1}{e^{\frac{2\pi i}{T}(t_l - t_j)} - 1} = 0 \end{aligned}$$

Thus, the inverse formula for the Discrete Fourier Transform is given by

$$x = \frac{1}{N} U^* X = \left(\frac{1}{N} \sum_{k=0}^{N-1} X_k e^{i\omega_k t_n}\right)_{n=0}^{N-1}$$

Proposition B.1. *Using Discrete Fourier Transform and all $x_k \in \mathbf{R}, k = 0, 1, \ldots, N-1$, we have $X_{N-k} = \overline{X_k}$.*

Proof. Let $k \in \{0, 1, \ldots N-1\}$ Then

$$\begin{aligned} X_{N-k} &= \sum_{n=0}^{N-1} x_n e^{-i\omega_{N-k} t_n} = \sum_{n=0}^{N-1} x_n e^{\frac{-2\pi i(N-k)n}{N}} \\ &= \left(\sum_{n=0}^{N-1} x_n e^{\frac{2\pi i}{N} kn} \right) e^{-2\pi i} \\ &= \overline{\left(\sum_{n=0}^{N-1} x_n e^{-i\omega_k t_n} \right)} = \overline{X_k} \end{aligned}$$

The for last step can be done because $x_n = \overline{x_n}$. □

In Matlab not discrete Fourier transforms (DFT) are used, but the fast Fourier transforms (FFT). This is done since the computational complexity of straightforward implementation of DFT is $O(n^2)$ while FFT is $O(n \log n)$ [18]. A discussion of FFT and computational complexity of both is beyond the scope of this thesis.

C Matlab code Discrete Fourier Transform

C.1 Datasets in vectors

For altyear and aziyear this function is exaclty the same, only dataA changes into dataB and altyear into aziyear. DataA and dataB are just all components shown in Appendix A put into months of 31 days. When a month does not consists of 31 days it is extended by zeros.

```
function altyear = dataA(A)
b=zeros(1,372);
for i=0:11
for j=1:31
b(31*i+j)=A(i+1,j);
end
end

for j=1:7
for i=1:371
if b(i)==0
b(i) = b(i+1);
b(i+1) = 0;
```

```
end
end
j = j+1;
end

altyear = b([1:365]);
return
```

C.2 Main program

```
clear all
clc

A = dataalt;
B = dataazi;
altyear = dataA(A);
aziyear = dataB(B);

y = fft(altyear);
r=abs(y);

p = zeros(1,3);
for i = 1:3
p(i) = r(i);
end

z = fft(aziyear);
s= abs(z);

t = zeros(1,3);
for i = 1:3
t(i) = s(i);
end

figure(1)
hold on
bar(y)
hold off

figure(2)
hold on
bar(r)
hold off
```

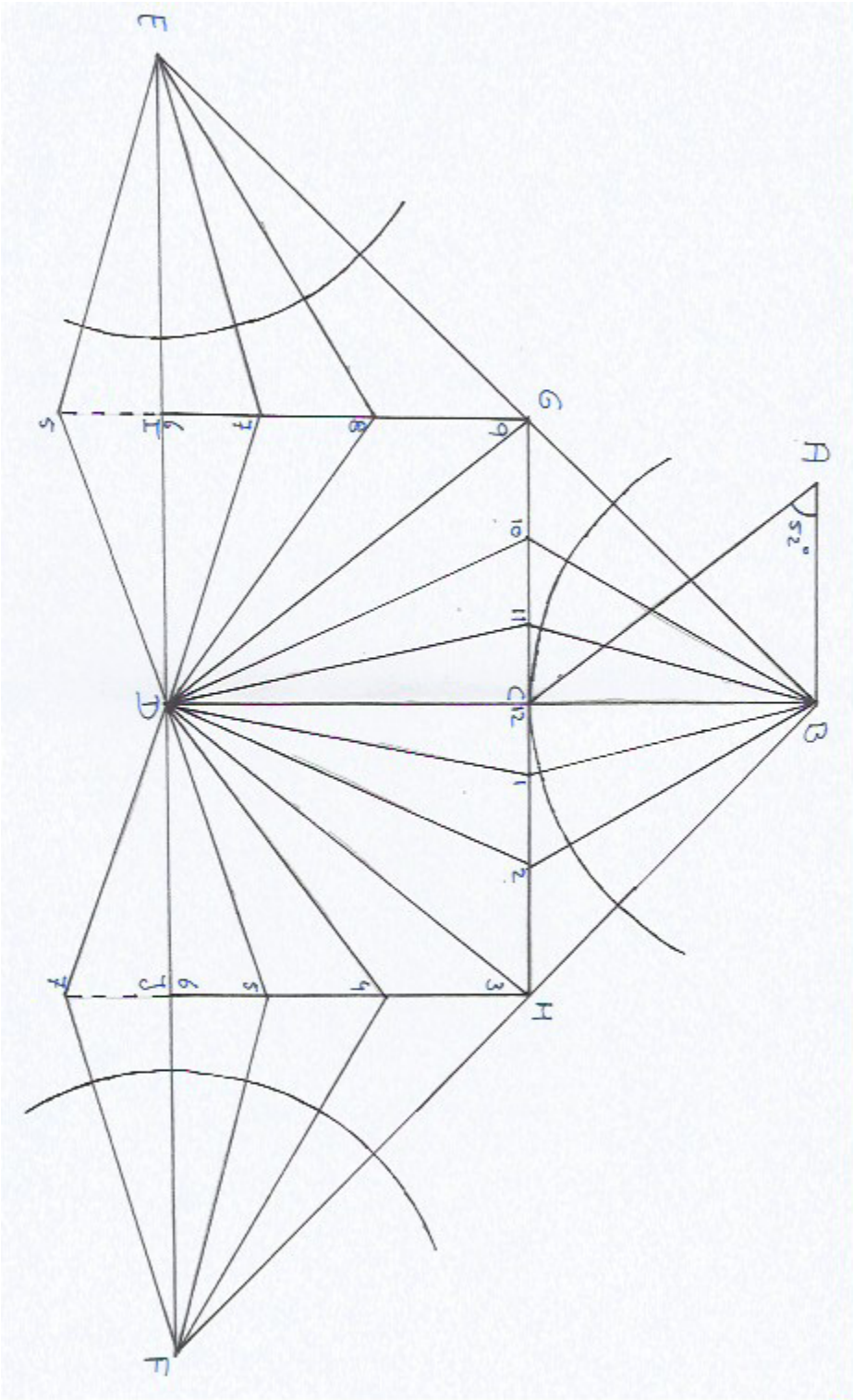
```
figure(3)
hold on
bar(s)
hold off

figure(1)
hold on
bar(t)
hold off
```

C.3 Analemma

```
clear all
clc

A = dataalt;
B = dataazi;
altyear = dataA(A);
aziyear = dataB(B);

y = fft(altyear);

p = zeros(1,365);
for i = 1:3
p(i)=y(i);
end
p(365)=conj(y(2));
p(364)=conj(y(3));

r = ifft(p);

z = fft(aziyear);

t = zeros(1,365);
t(1)=z(1);
t(2)=z(2);
t(3)=z(3);
t(365)=conj(z(2));
t(364)=conj(z(3));

s = ifft(t);
```

```
figure(1)
hold on
title('Analemma Leiden at noon and its approximation')
xlabel('azimuth');
ylabel('altitude');
plot(aziyear,altyear)
plot(s,r, 'r')
hold off
```

D Hour lines for Leiden



E Construction hour lines on an equatorial sundial

This construction has been taken from [20]. The colours are seen in Figure 19.

1. Draw a circle (black).
2. Divide the circle into four segments by drawing two perpendicular diameters in the circle (black).
3. Set the pair of compasses at the endpoints of each diameters and draw the arcs intersecting the perimeter with an equal radius to the circle (red).
4. Connect all the intersection points to the midpoint (red).
5. Use the bisector of one of the segments to divide the circle even further. Take the length of a cord segment and from the bisector on draw arcs such that you divide all segments into two (blue).

F Construction hour lines on a horizontal/vertical dial

This construction has been taken from [20].

1. Draw a right-angled triangle ABC with $\angle BCA$ equal to the latitude.
2. Draw line BD perpendicular to AC.
3. Assign BC the value of 1.
4. Calculate the lengths of BD and AB for your latitude.
5. Draw two concentric circles such that the inner circle has radius BD and the outer circle has radius BC. This is for the horizontal dial. For the vertical version, the outer circle has radius AB.
6. Draw 2 diameters, one for each circle, perpendicular to each other. For convenience, draw the longest diameter vertically.
7. Divide the circle in 24 pieces. (see Appendix E).
8. Choose a radial and draw a line through its intersection with the outer circle and perpendicular to the longest diameter. Do the same for the intersection of that radial with the inner circle and a line perpendicular to the shortest diameter.
9. The intersection of the two lines in step 8 is the coordinate for that hour on an elliptical radiation.
10. Repeat step 8 and 9 in order to generate the ellipse.

G Ellipse, derivation of the equation

The derivation of the equation of the ellipse can be found in [3], p.436-437.
We take an ellipse of which the x-axis is the the semi-major axis and the y-axis is the semi-minor axis. The ellipse passes through the points $(\pm a, 0)$ and $(0, \pm b)$. Take two focus points at $(\pm c, 0)$. The ellipse is the set of points (x, y) for which the sum of the distance to the two focus points remain constant. Thus, because $(a, 0)$ is on the ellipse

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a$$

Substituting the point $(0, b)$ into this equation yields

$$2\sqrt{b^2+c^2}=2a$$

Hence

$$b^2+c^2=a^2 \tag{45}$$

and

$$\sqrt{(x-c)^2+y^2}+\sqrt{(x+c)^2+y^2}=2a.$$

The derivation is continued as follows:

$$\begin{aligned}
\sqrt{(x-c)^2+y^2} &= 2a-\sqrt{(x+c)^2+y^2} \\
(x-c)^2+y^2 &= 4a^2-4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2 \\
4a\sqrt{(x+c)^2+y^2} &= 4a^2+4xc \\
a\sqrt{(x+c)^2+y^2} &= a^2+xc \\
a^2(x^2+2xc+c^2+y^2) &= a^4+2a^2cx+c^2x^2 \\
(a^2-c^2)x^2+a^2y^2 &= a^2(a^2-c^2)
\end{aligned}$$

Using (45) we find

$$b^2x^2+a^2y^2=a^2b^2,$$

which leads to the algebraic equation of an ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The *eccentricity* of an ellipse is the ratio $\frac{c}{a}$. Thus, a circle has eccentricity 0.

H Mathlab code 'Pendulum on a ship'

g = 9.81; *gravitational constant*
h = 3; *height of the mast in m*
l = 0.15; *length in m*

```
om = sqrt(g/h);

f=@(t,y)[y(2);-g/l*sin(y(1))-1/l*(-h*om*om*(cos(y(1))*sin(om*t)*(cos(om*t)*cos(om*t)+1)-
h*om*om*sin(y(1))*cos(2*om*t)))];
figure(1)
hold on
xlabel ('time in seconds')
ylabel ('movement')
[ts, ys] = ode45(f,[0,20],[1;0]);
y = ys(:,1); plot(ts,y)
hold off
```

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