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Algebraic and Analytic Multiplier Ideal Sheaves and Their Applications

Master thesis, defended on June 27, 2008

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Introduction

We show how the theory of multiplier ideals can be developed and discuss several applications of this theory. In the second section the same theory in the analytic setting is developed and several applications are given. Let X be a smooth algebraic variety and D an effective $\mathbb{Q}$-divisor. We associate to D (or to the pair (X, D)) an ideal sheaf $I(D)$ which controls the behavior of the fractional part of D and determines how close it is to have an simple normal crossing support. Other applications can be treated such as singularities of projective hypersurfaces and characterization of divisors. In the former case a result of Esnault-Viehweg concerning the least degree of hypersurfaces with multiplicity greater than or equal to a given positive integer at each point of a finite set is explained and proved in two different ways. A slight generalization is also given. Several vanishing and non-vanishing results including a global generation theorem are treated which will be used to prove the results about singularities. In the second section the analytic analogues of the materials in section one are given and the characterization of analytic nef and good divisors are explained.

1 ALGEBRAIC MULTIPLIER IDEALS

1.Definitions and basic theorems

Throughout these notes X is a smooth complex algebraic variety and D is a $\mathbb{Q}$-Cartier divisor on X. We can associate to this divisor an ideal sheaf which will turn out to give several valuable informations about D such as the extent to which the fractional part of D fails to have normal crossing support or informations about its singularities.

Notation. Let $D = \sum a_j D_j$ is a $\mathbb{Q}$-divisor on X. We denote by $\lfloor D \rfloor$ the integer part of D.i.e. $\lfloor D \rfloor = \sum \lfloor a_j \rfloor D_j$.

Definition 1.1. *Let X be as above and $D = \Sigma a_i D_i$ a $\mathbb{Q}$-divisor on it. A* log resolution *of D is a projective birational mapping $\mu : Y \to X$ where Y is non-singular and $\mu^* D + excep(\mu)$ has simple normal crossing support (SNC). Here $excep(\mu)$ denotes the exceptional divisor of μ.*

Now we are ready to define the multiplier ideal sheaf associated to D:

Definition 1.2. *Let D be an effective $\mathbb{Q}$-divisor on X (a smooth complex projective variety) , fix a log resolution of $\mu : Y \to X$ of D .Then the* multiplier ideal sheaf *$I(D)$ is defined to be*

$$I(D) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor),$$

where $K_{Y/X} = K_Y - \mu^* K_X$ is the relative canonical divisor of Y over X

Remark. *The fact that this is really an ideal sheaf follows from the fact that $\mu_* \mathcal{O}_Y(K_{Y/X}) = \mathcal{O}_X$ (or $\mu_* \omega_Y = \omega_X$)*

For the future needs, we have also to define the multiplier ideal associated to a linear series. Of course first we would have to define the notion of log resolution in these two cases;

Definition 1.3. *Let L be an integral divisor on X and let $V \subseteq H^0(X, \mathcal{O}_X(L))$ be a non-zero finite-dimensional space of sections of the line bundle determined by L.As usual write $|V|$ for the corresponding linear series of divisors in the linear series $|L|$. A log resolution of $|V|$ is a projective birational map $\mu : Y \to X$ again with Y non-singular, such that $\mu^*|V| = |W| + F$, where $F + excep(\mu)$ is a divisor with SNC support, and $W \subseteq H^0(Y, \mathcal{O}_Y(\mu^* L - F))$ defines a free linear series.*

The following lemma showes that in general, a log resolution of $|V|$ gives a log resolution of a general divisor $D \in |V|$.

Lemma 1.4. *Let X be a smooth variety and let $|V|$ be a finite dimensional free linear series on X. Suppose we are given a simple normal crossing divisor ΣE_i on X, with the E_i distinct components of E . If $A \in |V|$ is a general divisor, $A + \Sigma E_i$ again has simple normal crossings.*

Proof. We show that a general divisor $A \in |V|$ is smooth, and that it meets each of the intersections $E_{i_1} \bigcap ..... \bigcap E_{i_k}$ either transversally or not at all. Since ΣE_i has normal crossings it suffices to prove that given any finite collection of smooth subvarieties $Z_1, ..., Z_s \subseteq X$, a general divisor $A \in |V|$ meets each Z_j transversally or not at all. But the Bertini theorem implies that general members of a free linear series (and in particular their restrictions to each Z_j) is non-singular. This implies the assertion.

Using the above definitions we can define the multiplier ideal associated to a linear series :

Definition 1.5. *(Multiplier ideal associated to a linear series) Let $|V| \subseteq |L|$ and let $\mu : Y \to X$ be a log resolution of $|V|$ so that*

$$\mu^*|V| = |W| + F,$$

with W a free linear series and F the fixed part. Given a rational number $c > 0$ the multiplier ideal $I(c.|V|)$ corresponding to c and $|V|$ is

$$I(c.|V|) = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor c.F \rfloor).$$

According to a theorem of Esnault and Viehweg [EV] one should not be worried about the fixed log resolution in the definition of multiplier ideals:

theorem 1.6. *The multiplier ideal sheaves $I(D)$, $I(c.|V|)$ described above are all independent of the log resolutions used to construct them.*

Proof. [EV, 7.3]. The proof is based on the fact that any two resolutions can be dominated by a third one and also theorem 1.9 below to guarntee that if the underlying divisor -$\mu^* D$- has SNC, then nothing changes by passing to a further resolution.

Before giving the definition of an ideal sheaf associated to a $\mathbb{Q}$-divisor we mentioned that it measures how far is the fractional part of the divisor of having normal crossing support. Now we can elucidate this remark by the following theorems :

theorem 1.7. *Let D be an integral divisor on X then $I(D) = \mathcal{O}_X(-D)$.*

Proof. The proof is just by observing that if D is an integral divisor $\lfloor \mu^* D \rfloor = \mu^* D$, and then applying the projection formula.

theorem 1.8. *Let A be an integral divisor and D an effective $\mathbb{Q}$-divisor on a smooth variety X. Then:*

$$I(D + A) = I(D) \otimes \mathcal{O}_X(-A).$$

Proof. Let $\mu : Y \to X$ be a log resolution of D . If A is integral, then

$$\lfloor \mu^* D + \mu^* A \rfloor = \lfloor \mu^* D \rfloor + \mu^* A.$$

Therefore

$\mu_*(K_{Y/X} - \lfloor \mu^*(D+A) \rfloor) = \mu_*(\mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor) \otimes \mathcal{O}_Y(-\mu^* A)) = \mu_*(\mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor)) \otimes \mathcal{O}_X(-A) = I(D) \otimes \mathcal{O}_X(-A),$

where in the second to last equality again we have used the projection formula.

theorem 1.9. *If D is a $\mathbb{Q}$-divisor on X with normal crossing support, then $I(D) = \mathcal{O}_X(-\lfloor D \rfloor)$*

Proof. [LAZ, Lemma 9.2.19].

Relations between different kinds of multiplier ideals

We study the relation between the multiplier ideals associated to a linear series and to a divisor.

Proposition 1.10. *Let X be a smooth variety and $|V| \subseteq |L|$ a non-empty linear series on X .Fix a rational number $c > 0$ and choose $k > c$ general divisors A_1 ,....., $A_k \in |V|$, and set*

$$D = \frac{1}{k}.(A_1 + ... + A_k)$$

Then $I(c.|V|) = I(c.D)$

Proof. The proof uses Lemma 1.4 to guarantee the SNC property. Explicitly, We have $\mu^*|V| = |W| + F$ for a log resolution μ. Then for general $A'_i \in |W|$ we have $\mu^* A_i = A'_i + F$. But $|W|$ is free by definition so by lemma 1.4 , $F + \sum A'_i + excep(\mu)$ has SNC support. Therefore Y is in fact a log resolution for D and since by assumption $k > c$ we have $K_{Y/X} - \lfloor \mu^*(cD) \rfloor = K_{Y/X} - \lfloor k.\frac{c}{k} F + \sum \frac{c}{k} A'_i \rfloor = K_{Y/X} - \lfloor cF \rfloor$. Which completes the proof.

2.Some geometric properties of the multiplier ideals

Let H be a smooth irreducible hypersurface in X.Let D be a $\mathbb{Q}$-divisor and let α be the coefficent of H in D . We assume that $0 \leq \alpha < 1$. Since the construction of $I(X, D)$ does not depend on the log resolution, we might assume that μ^*H, the exceptional divisors of μ, and μ^*D are all in normal crossing in X'. (μ as always represents the corresponding log resolution). Let F be the proper transform of D in X' . We set $D' = D - \alpha H$ and $D'_H = D'|_H$. We see that $K_H + D'|_H$ is a $\mathbb{Q}-divisor$ on H. Then we can consider the multiplier ideal determined by D'_H in $\mathcal{O}_H$. The following comparison of I(X,D) and I(H,D'_H) is due to Esnault and Viehweg([EV]).

Proposition 1.11. $I(H, D'_H) \subset Im(I(X, D) \to \mathcal{O}_H)$

Proof. [EV, Prop 7.5].

In particular from this proposition we conclude that if $I(H, D'_H)$ is trivial at a point p in H , then $I(X, D)$ is also trivial at p.

Let us denote by Z_D the zero scheme associated to the ideal sheaf $I(D)$.The next proposition due to Anghern and Siu [AS], was originally proved using analytic methods.

Proposition 1.12. *Let X be a smooth variety and let T be a smooth curve.We consider the product variety $X \times T$. Let D be an effective $\mathbb{Q}$-divisor on $X \times T$.We assume that the support of D does not contain any fiber $X_t = X \times \{t\}$ for $t \in T$. Denote by D_t the restriction of D to X_t.Let $s : T \to X \times T$ be a section .Assume that the multiplier ideal of D_t is non-trivial at the point s(t) for a general t.If $X_0 = X \times \{0\}$ is a special fiber, then the multiplier ideal of D_0 is also non-trivial at s(0).*

Proof. [LAZ, Thm 9.5.35].

The behavior of a multiplier ideal when the $\mathbb{Q}$-divisor is pulled back by a generically finite map is studied in the following proposition:

Proposition 1.13. *Let X and M be two smooth irreducible varieties.Let D be an effective $\mathbb{Q}$-divisor on X . Let $\phi : M \to X$ be a proper generically finite map.Let q be a point in M. Then the multiplier ideal of the $\mathbb{Q}$-divisor $\phi^*D - K_{M/X}$(assuming that it is effective) is nontrivial at q iff the multiplier ideal of D is nontrivial at $\phi(q)$.*

Proof. Consider the log resolutions of D and $\phi^*D - K_{M/X}$ respectively as $f : Y \to X$ and $g : W \to M.\phi$ extends to proper generically finite map $\phi' : W \to Y$. Let $R = K_Y - f^*(K_X + D)$ and $R_1 = K_W - \phi^* f^*(K_X + D)$.Then the multiplier ideal of D is given by $f_*(\mathcal{O}_Y(\lceil R \rceil)$. Similarly the multiplier ideal for $\phi^*D - K_{M/X}$ is given by $g_*\mathcal{O}_W(\lceil R_1 \rceil)$. Let F be an irreducible component of R with coefficents a and let E be an irredcible divisor in W which maps onto F. Denote by m the coefficents of E in $K_{W/Y}$ and b the coefficient of E in R_1. Then $b = a(m + 1) + m$. Note that $a \leq -1$ iff $b \leq -1$.Conversely if E is an irreducible component of R_1, after further blowing up of Y and W , we may assume that $\phi'(E)$ is a divisor in Y. It follows that the multiplier ideal of D is nontrivial at $\phi(q)$ iff the multiplier ideal of $\phi^*D - K_{M/X}$ is nontrivial at q.

3. Vanishing and non-vanishing theorems involving multiplier ideals

Before proving vanishing theorems involving multiplier ideals we need to state some preparatory Theorems and Lemmas. The first theorem is the Kawamata-Viehweg vanishing theorem which gives the vanishing of higher cohomology of some adjoint type divisors with certain conditions on their fractional parts.

theorem 1.14. *(Kawamata-Viehweg) Let X be a non-singular projective variety of dimension n, and let N be an integral divisor on X, Assume that*

$$N \equiv_{num} B + \Delta$$

where B is a nef and big $\mathbb{Q}$-divisor, and $\Delta = \Sigma a_i \Delta_i$ is a $\mathbb{Q}$-divisor with simple normal crossing support and fractional coefficients :

$$0 \leq a_i < 1$$

Then

$$H^i(X, \mathcal{O}_X(K_X + N)) = 0$$

for every $i > 0$

Proof. [M, 5.2]

Another useful result of this theorem is as follows:

Lemma 1.15. *In the setting of theorem , let L be an integral divisor and D a $\mathbb{Q}$-divisor. Assume that $L - D$ is big and nef and that D has SNC support.Then*

$$H^i(X, \mathcal{O}_X(K_X + L - \lfloor D \rfloor)) = 0$$

for all $i > 0$.

One of the properties of multiplier ideals is that they are *acyclic* in the sense that higher direct images vanish. Other vanishing results mostly reduce to this result:

theorem 1.16. *(Local vanishing theorem)Let D be any divisor on a smooth variety X . If $\mu : Y \to X$ is a log resolution of D, then*

$$R^j \mu_*(K_{Y/X} - \lfloor \mu^* D \rfloor) = 0$$

for $j > 0$.

Proof (When X and Y are both projective) choose A be any ample divisor on X such that $A - D$ becomes itself ample. Then $\mu^*(A - D)$ is nef and big because μ is perojective (and hence proper) then [KL] proves the claim. Now Kawamata-Viehweg vanishing implies that

$$H^j(Y, \mathcal{O}_Y(K_Y + \mu^* A - \lfloor \mu^* D \rfloor)) = 0$$

for every $j > 0$.

As this is true for sufficiently positive divisor A, it follows from [LAZ, Lemma 4.3.10], that

$$R^j \mu_* \mathcal{O}_Y(K_Y + \mu^* A - \lfloor \mu^* D \rfloor = R^j \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor) \otimes \mathcal{O}_X(K_X + A) = 0$$

for every $j > 0$.

The same result holds for multiplier ideals of linear series :

theorem 1.17. *Let $\mu : Y \to X$ be a log resolution of a linear series $|V| \subseteq |D|$, with $\mu^*|V| = |W| + F$ and $|W|$ free and let $c > 0$ be any rational number. Then*

$$R^j \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor c.F \rfloor) = 0$$

for every $j > 0$.

Proof. [LAZ, 9.4.5]. There are also global vanishing results for multiplier ideal sheaves

theorem 1.18. *(Nadel vanishing theorem) Let X be a smooth complex projective variety, let D be any $\mathbb{Q}$-divisor on X, and let L be any integral divisor such that $L - D$ is nef and big. Then*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes I(D)) = 0$$

for every $i > 0$.

Proof. The proof is based on Kawamata-Viehweg vanishing theorem and the Leray spectral sequence ([H,Ex8.1]).Specifically, let $\mu : Y \to X$ be a log resolution of D. Then it follows from Kawamat-Viehweg vanishing that

$$H^i(Y, \mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor) \otimes \mu^* \mathcal{O}_X(K_X + L)) = H^i(Y, \mathcal{O}_Y(K_Y + \mu^* L - \lfloor \mu^* D \rfloor)) = 0$$

for $i > 0$.

Now by the local vanishing theorem

$R^j \mu_*(\mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor) \otimes \mu^* \mathcal{O}_X(K_X + L)) = R^j \mu_*(\mathcal{O}_Y(K_{Y/X} - \lfloor \mu^* D \rfloor)) \otimes \mathcal{O}_X(K_X + L) = 0$ for every $j > 0$.

The assertion then follows from the Leray spectral sequence.

theorem 1.19. *(Nadel vanishing for linear series).Let X be a smooth projective variety, let $c > 0$ be rational, and let L and A be integral divisors on X such that $L - c.A$ is big and nef if $|V| \subseteq |A|$ is any linear series, then*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes I(c.|V|)) = 0$$

for $i > 0$.

Proof. [LAZ, Cor 9.4.15].

Later on, we will discuss some global generation results for which we need to mention a result known as "Castelnuovo-Mumford regularity".

Proposition 1.20. *If $\mathcal{F}$ is an $m-$regular sheaf i.e a coherent sheaf on X with $H^i(X, \mathcal{F}(m-i)) = 0$ for $i \geq 1$ then we have:*

(i) We have $H^i(X, \mathcal{F}(r)) = 0$ whenever $i \geq 1$ and $r \geq m - i$.In other words, if $\mathcal{F}$ is $m-regular$, then it is $n-regular$ for $n \geq m$.

(ii) All the higher cohomology of $\mathcal{F}(r)$ vanish and this sheaf is generated by its global sections for $r \geq m$.

Proof. [LAZ, Thm 1.8.5].

Using this result along with some non-vanishing theorems of the multiplier ideals we can prove useful results concerning the global generation of multiplier ideal sheaves.

theorem 1.21. *Let X be a smooth projective variety of dimension n. Let D be an effective $\mathbb{Q}$-divisor on X and L an integral divisor such that $L - D$ is nef and big (section 6), and let be H ample (or nef and big) then there is a $0 \leq t_0 \leq n$ such that*

$$H^0(X, \mathcal{O}_X(K_X + L + t_0H) \otimes I(D)) \neq 0$$

Proof. Using the Nadel vanishing we know that

$$H^i(X, \mathcal{O}_X(K_X + L + tH)) \otimes I(D)) = 0$$

for $i > 0$ and $t \geq 0$. Therefore

$$h^0(X, \mathcal{O}_X(K_X + L + tH) \otimes I(D) = \chi(X, \mathcal{O}_X(K_X + L + tH) \otimes I(D))$$

.The assertion then follows from the fact that the above Euler characteristic is a polynomial of degree n and therefore must be non-zero at some $0 \leq t_0 \leq n$.

Proposition 1.22. *Let L be a divisor with non-negative Iitaka dimension (Definition 1.37) and H any ample divisor on X. Then there exists an integer $1 \leq t \leq n+1$ such that $H^0(X, \mathcal{O}_X(K_X + L + tH)) \neq 0$*

Proof. By the assumption we may choose $A \in |mL|$. Let $D = \frac{1}{m}A$, then $D \equiv_{num} L$ and the result follows from the previous theorem applied to $L + H$.

Using Nadel vanishing theorem and the Castelnuovo-Mumford regularity we can also assert the following result regarding the global generation of the multilier ideals:

Proposition 1.23. *With the assumptions of the last theorem with the additional hypothesis that H is very ample. Then*

$$\mathcal{O}_X(K_X + L + mH) \otimes I(D)$$

is generated by global sections for $m \geq n = dimX$.

4. Multiplier Ideals and singularities of hypersurfaces

As we mentioned before, multiplier ideals reveal valuable information about the singularities of divisors. In the following section we study some applications of the theory of multiplier ideals to understand the singularities of hypersurfaces in a projective space.

theorem 1.24. *Assume that X has dimension n, and let D be an effective $\mathbb{Q}$-divisor on X. If $mult_x D \geq n$ at some point $x \in X$, then $I(D)$ is non-trivial at x .i.e. $I(D) \subseteq m_x$, where m_x is the maximal ideal of x. More generally, if $mult_x D \geq n + s$ for s an integer with $s \geq 0$ then $I(D) \subseteq m_x^{s+1}$.*

Proof. Let $\mu : Y \to X$ be a log resolution provided by the first blowing up of X at x. We have

$$ord_E(K_{Y/X}) = n - 1$$

and

$$ord_E(\mu^*(D)) = mult_x D$$

Now since we have :$\mu_* \mathcal{O}_Y(-pE) = m_x^p$, a simple calculation implies that :

$$ord_E(K_{Y/X} - \lfloor \mu^* D \rfloor) \leq -s - 1$$

i.e. $I(D) = \mu_*(K_{Y/X} - \lfloor \mu^* D \rfloor) \subseteq \mu_* \mathcal{O}_Y(-(s+1)E) = m_x^{s+1}$.

Remark. *This theorem is not valid for $1 \leq mult_x D < n$. Indeed one can construct divisors with non-trivial multiplier ideals whose multiplicity are arbitrary close to 1 $[LAZ, Ex9.5.14]$. But we have the following theorem:*

theorem 1.25. *With the above assumptions if $mult_x D < 1$ then the multiplier ideal of D is trivial at x , i.e. $I(D)_x = \mathcal{O}_{X,x}$.*

Proof. We proceed by induction on $n = dimX$. By theorem 1.9 the theorem holds if $dimX = 1$ i.e. when X is a smooth curve. If $n > 1$ we might choose a smooth divisor $H \subseteq X$ which contains x and is not contained in the support of D . Let $D_H = D|H$. It follows by induction that we have $I(H, D_H) = \mathcal{O}_x H$, and the following lemma completes the proof.

Lemma 1.26. *With the above notation, if $I(H, D_H)$ is trivial at x then $I(D)$ is trivial at x.*

Proof. [LAZ, Cor.9.5.11].

One of the most interesting results due to Esnault and Viehweg [EV2], which uses the method of multiplier ideals in the proof is the relation between singular hypersurfaces and postulation of a finite set in a complex projective space.

theorem 1.27. *Let $S \subset \mathbb{P}^n$ be a finite set of points. Assume that there is a hypersurface A of degree d in $\mathbb{P}^n$, such that $mult_p A \geq k$ for each $p \in S$. Then there is a hypersurface of degree $[\frac{nd}{k}]$ that contains S.*

Proof. Let $D = \frac{n}{k}A$. We have $mult_p D \geq n$ for all $p \in S$. Let $I = I(\mathbb{P}^n, D)$ be the multiplier ideal of D . Then $S \subset Z = Z_D$ by theorem 1.24. Let H be the hypersurface class of $\mathbb{P}^n$. Let $m = [\frac{nd}{k}]$. Then $(m+1)H - D$ is ample . Since $K_{\mathbb{P}^n} \sim (-n-1)H$, we conclude that $H^i(I \otimes \mathcal{O}_{\mathbb{P}^n}(t)) = 0$ for $t \geq m - n$ and $i > 0$. As the Hilbert polynomial of I is a polynomial in t of degree less than or equal to n, we can find an integer t_0 where $m - n \leq t_0 \leq m$, such that $h^0(I \otimes \mathcal{O}_{\mathbb{P}^n}(t_0))$ is nonzero. Since $t_0 \leq m$ and $S \subset Z$, this implies the theorem.

This theorem can also be proved as a result of proposition 1.23 and theorem 1.24. Let $D = \frac{n}{k}A$ be as before. Then $mult_x D \geq n$ and hence by the theorem, $I(D) \subseteq I_S$. Now let H be the hyperplane divisor on $\mathbb{P}^n$, for $r > \lfloor \frac{dn}{k} \rfloor$ we have that $rH - D = (r - \frac{dn}{k})H$ is (very) ample. Since $K_{\mathbb{P}^n} = -(n+1)H$, proposition 1.23 implies that for $r \geq \lfloor \frac{dn}{k} \rfloor$,$\mathcal{O}_{\mathbb{P}^n}(r) \otimes I(D)$ is generated by global sections. But $I(D) \subseteq I_S$. This gives the desired result.

Definition 1.28. *Let $S \subseteq \mathbb{P}^n$ be a finite set of points.We denote by $\omega_t(S)$, the least degree of hypersurfaces with multiplicity $\geq t$ at each point of S.*

Definition 1.29. *Let us make the following definition*

$$\mathcal{D}_t = \{D : \text{hypersurface} | degD = \omega_t(S), mult_x D \geq t, \forall x \in S\}$$

Proposition 1.30. *Let S and $\omega_t(S)$ be as above. then :*

$$\frac{\omega_{t_1}(S)}{t_1+n-1} \leq \frac{\omega_{t_2}(S)}{t_2}$$

for all t_1 and t_2 $in\mathbb{N}$ with $t_1 \leq t_2$.

Proof. Let $A \in \mathcal{D}_{t_2}$. Put $D = \frac{t_1+n-1}{t_2}A$ then D has multiplicity $\geq t_1+n-1$ at each point of S. Now applying theorem gives us that $I(D) \subseteq I_S^{t_1}$. Continuing as above we see that for $r \geq \lfloor \frac{t_1+n-1}{t_2}\omega_{t_2}(S)\rfloor$ (and so in particular for $r = \lfloor \frac{t_1+n-1}{t_2}\omega_{t_2}(S)\rfloor$) $\mathcal{O}_{\mathbb{P}^n}(r) \otimes I(D)$ is generated by global sections. Therefore by the fact that $I(D) \subseteq I_S^{t_1}$ we conclude that there exists a hypersurface of degree greater than or equal to $\lfloor \omega_{t_2}(S)\frac{t_1+n-1}{t_2}\rfloor$ having multiplicity $\geq t_1$ at each point of S . Which completes the proof of the proposition.

5. The Asymptotic constructions

The importance of the asymptotic construction of the multiplier ideals was first realized by Siu in [S] where he tries to prove a conjecture on the invariance of the plurigenera . More explicitly he proves that if $\pi : X \to \Delta$ is a family of compact complex manifolds parametrized by the open unit 1-disk Δ and if for every $t \in \Delta$ $\pi^{-1}(t) = X_t$ is of general type, then for every $m \in \mathbb{N}$ the plurigenus $dim_{\mathbb{C}}\Gamma(X_t, m_tK_X)$ is independent of $t \in \Delta$. This in turn means that for every $t \in \Delta$ and every integer m every element of $\Gamma(X_t, mK_{X_t})$ can be extended to an element of $\Gamma(X, mK_X)$.

The asymptotic construction enables us to study the geometry of the asymptotic linear series instead of using the log resolutions which may not resolve all the $|mL|$ for different m. The main property of the multiplier ideals which makes us to use them instead of the log resolutions is the finiteness property which says that for a given c the family $\{I(\frac{c}{p}|pL|)\}_{p\geq 0}$ has a *unique* maximal element. This means that this family stabilizes for $p \gg 0$.

theorem 1.31. *If $k \geq 1$*

$$I(\frac{c}{p}|pL|) \subseteq I(\frac{c}{pk}|pkL|).$$

Proof. We first make the convention that for a line bundle L with negative Iitaka dimension(i.e.$|pL| = \varnothing$ for $p > 0$) we let $I(\frac{c}{p}|pL|) = 0$. With this convention we might assume that the Iitaka dimension is greater than or equal to zero . Then one can choose a log resolution for both $|pL|$ and $|pkL|$:

$$\mu : Y \to X$$

$$\mu^*(|pL|) = |W_p| + F_p$$

and

$$\mu^*(|pkL|) = |W_{pk}| + F_{pk}$$

The image of the natural map $W_p \to W_{pk}$ is a free linear subseries of $\mu^*(|pkL|)$ whose fixed divisor is naturally kF_p . Therefore we must have $F_{pk} \leq kF_p$ (due to the freeness of the linear series $|W_{pk}|$) and hence :

$$I(\frac{c}{p}|pL|) = \mu_*\mathcal{O}_Y(K_{Y/X} - \lfloor \frac{c}{p}F_p \rfloor) \subseteq \mu_*\mathcal{O}_Y(K_{Y/X} - \lfloor \frac{c}{pk}F_{pk} \rfloor) = I(\frac{c}{pk}|pkL|)$$

The above theorem enables us to define an *asymptotic* notion of multiplier ideals.

theorem 1.32. *The family of ideals $I(\frac{c}{p}|pL|)_{(p\geq 0)}$ has a unique maximal element which we denote by $I(c.\|L\|)$*

Proof. It follows from the ascending chain condition that at least one such maximal element exists. If $I(\frac{c}{p}|pL|)$ and $I(\frac{c}{q}|qL|)$ are both maximal elements, by the above theorem they both must coincide with $I(\frac{c}{pq}|pqL|)$ and hence are equal. Therefore the maximal element exists and is unique.

It follows from the way that we constructed the asymptotic multiplier ideal sheaves that there exists $p_0 \in \mathbb{N}$ such that

$$I(c.\|L\|) = I(\frac{c}{p}.|pL|)$$

for $p \geq p_0$.

6. Nef and Big line bundles and divisors

We now turn to studying some properties of divisors which will turn out can be characterized by the constructions we described above i.e. multiplier and asymptotic multiplier ideal sheaves.

Definition 1.33. *Let X be a complete variety or scheme. A line bundle on X is said to be* nef, *if $\int_C c_1(L) \geq 0$ for every irreducible curve $C \subseteq X$. Similarly, a Cartier divisor D on X is* nef *if $(D.C) \geq 0$ for all irreducible curves $C \subset X$.*

The most important result about nef divisors is that of Kleiman [KL].

theorem 1.34. *If D is a nef ($\mathbb{Q}-$)divisor on the complete variety X, then $(D^k.V) \geq 0$, for every irreducible subvariety $V \subseteq X$ of dimension k.Or, in the language of line bundles*

$$\int_V c_1(L)^{dimV} \geq 0$$

for every nef line bundle L on X.
Proof. [K].

Actually, nef divisors can be thought of as limits of ample divisors:

Proposition 1.35. *Let X be a projective variety or scheme, and D a $\mathbb{Q}$-divisor on X . If H is any ample $\mathbb{Q}$-divisor on X, and $D + \epsilon H$ is ample for every $\epsilon > 0$ then D is a nef divisor.*

Proof. In fact if $D + \epsilon H$ is ample then for each irreducible curve C we have

$$(D + \epsilon H).C = D.C + \epsilon(H.C) > 0,$$

letting $\epsilon \to 0$ we get that $D.C \geq 0$. Whence the claim.

There are other types of divisors, namely *big* divisors which as we will observe can be characterized by means of multiplier ideal sheaves. To define them we first need to define another important concept, namely *Iitaka dimension*:

Definition 1.36. *Let L be a line bundle on X. For each $m \in \mathbb{N}$ with $H^0(X, L^m) \neq 0$ we have a natural rational mapping $\phi_m : X \to \mathbb{P}H^0(X, L^m)$ associated to the complete linear series $|L^m|$. Let $Y_m = \phi_m(X)$. Then the Itaka dimension of L is defined to be*

$$\kappa(L) = max\{dimY_m\}$$

with the maximum being taken over all $m \in \mathbb{N}$ such that $H^0(X, L^m) \neq 0$ we put $\kappa(L) = -\infty$ if there is no such m. For a Cartier divisor D we define $\kappa(D) = \kappa(\mathcal{O}_X(D))$.

Definition 1.37. *A line bundle on an irreducible projective variety X is said to be* big *if $\kappa(X, L) = dimX$. In terms of divisors a (Cartier) divisor on X is* big *if $\mathcal{O}_X(D)$ is big as a line bundle.*

When X is normal one can observe by making use of the *Iitaka fibration* that a line bundle L is big iff the so called Iitaka fibration $\phi_m : X \to \mathbb{P}H^0(X, L^m)$ is birational for some $m > 0$.

The following theorem is useful to understand the nature of big divisors:

theorem 1.38. *On a projective variety X of dimension n, a divisor D is big iff there is a constant $C > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geq C.m^n$ for $m \gg 0$.*

Proof. [LAZ, Lemma 2.2.3].

Another very useful characterization of big divisors is that big divisors can be thought of divisors such that a positive multiple of them is the sum of an ample divisor and an effective one. More explicitly‘:

theorem 1.39. *The divisor D on a projective variety is big if and only if there is am $m \in \mathbb{N}$ and an effective divisor N such that for some ample divisor A:*

$$mD \equiv_{lin} A + N.$$

Proof. [LAZ, Cor 2.2.7].

This in turn implies that every sufficiently large multiple of a big divisor is effective.

To define another important class of divisors let us define a necessary concept.

Definition 1.40. *We define the* numerical dimension *of the divisor D as:*

$$\nu(X, D) = sup\{\nu \in \mathbb{N} : D^\nu \not\equiv 0\}$$

theorem 1.41. $dim(X) \geq \nu(X, D) \geq \kappa(X, D)$.

Proof. If $\nu(X, D) = dimX$ then the statement is clear. If $\kappa(X, D) = dimX$ then we have [Laz,Thm 2.2.16] that $D^n > 0$ for $n = dimX$ which again proves the assertion . Hence we can assume that both dimensions are strictly less than $dimX = n$. In this case we observe that $\kappa(D_H) \geq \kappa(D)$ ([EV,5.4])for H an ample divisor on X. But clearly $\nu(D_H) = \nu(D)$ which gives the assertion by using induction.

Now we can introduce the third important class of divisors:

Definition 1.42. *A nef divisor is said to be* good *if $\nu(X, D) = \kappa(X, D)$.*

It follows from the definitions that a nef and big divisor is good.

Examples 1. *We construct a divisor on a surface which is nef but not good. Let S be the blowing-up of $\mathbb{P}^2$ at d^2 points which are general on a smooth curve H of degree d and take D to be the strict transform of H. Then $D^2 = 0$ but $D.C > 0$ for any irreducible curve $C \neq D$ on S. To show this let μ be the blowing-up morphism then $\mu^* H = D + E$ where E is the exceptional divisor and hence equal to ΣE_i. Since $E_i^2 = -1$ and $E_i E_j = 0$ for $i \neq j$ and also $DE_i = mult_{x_i} H = 1$ we get that $d^2 = H^2 = \mu^* H.\mu^* H = D^2 + 2d^2 - d^2$ which clearly proves that $D^2 = 0$. Now if C is an irreducible curve not equal to D we know that $\mu(C) = C_0$ an irreducible curve and C is the strict transform of C_0.*

Note that we can move C_0 so that the obtained curve doesn't contain any of the d^2 points above . Therefore we have

$$C_0.H = \mu^* C_0.\mu^* H = C.D.$$

Now the assertion follows by Bezout theorem, because $C_0.H > 0$.

By making use of asymptotic construction of multiplier ideals one can characterize nef and big divisors:

theorem 1.43. *Let L be a big line bundle on a smooth projective variety X of dimension n. Then L is nef iff*

$$I(X, \|mL\|) = \mathcal{O}_X$$

for every $m \geq 1$.
Proof. [LAZ, 11.1.20].
In [G] Goodman introduced the following notion:

Definition 1.44. *A divisor D on X is said to be* almost base point free *if $\forall x \in X$ and $\forall \epsilon > 0$ there exists $n = n(\epsilon, x)$ and $D_n \in |nD|$ such that $mult_x D_n < n\epsilon$.*

The connection between almost base point freeness and nef and good divisors is given in the following theorem:

theorem 1.45. *([MR]) Let D be a divisor on a complete normal complex variety X. Then D is almost base point free iff it is nef and good.*

Proof. (Of the only if part) We may check the theorem at the generic points of the irreducible components D_i of *supp*(D). Now a result of Kawamata ([K, prop2.1]), states that a divisor D is nef and good iff for a birational map $f : Z \to X$, with Z a non-singular algebraic variety there exists a positive integer n_0 and an effective divisor N on Z such that $nf^*D - N$ is semiample (some positive power of it is free) for each n divisible by n_0. Therefore we may assume that there exists a positive integer n_0 and an effective divisor E on X such that $nD - E$ is semiample for every n divisible by n_0, then for such an n there exists a positive integer r such that $|rnD - E|$ is base point free. Therefore

$$Fix|rnD| \leq rE + Fix|rnD - rE| = rE.$$

Now for fixed ϵ and i, if $m_i = ord_{D_i}(E)$ we can choose a divisible n such that $m_i < n\epsilon$. Therefore there exists $E_n \in |rnD|$ with $ord_{D_i}(E_n) \leq rm_i < rn\epsilon$. Which completes the proof.

theorem 1.46. *Let D be a divisor on a smooth proper complete variety with non-negative Iitaka dimension and let $e(D)$ be the exponent of D i.e. the gcd of those integers m, for which $H^0(X, mD) \neq 0$. Then D is nef and good iff $I(n\|e(D)D\|) = \mathcal{O}_X$ for $n \gg 0$.*

Proof. We might assume that $e(D) = 1$. If D is not nef and good then by the theorem it can not be almost base point free. Therefore there exists $\epsilon > 0$ and a (closed)point $x \in X$, such that for all $k > 0$ and all $D_k \in |kD|$ we have $mult_x D_k \geq k\epsilon$. We may choose n with $\lfloor n\epsilon \rfloor \geq dimX$. By the properties of the asymptotic multiplier ideal sheaves there is an r suficiently large such that $I(\|nD\|) = I(\frac{1}{r}|rnD|)$. Let $\mu : Y \to X$ be a log resolution of the linear series $|rnD|$ provided by the first blowing up of X at x. It follows that $mult_x D_{rn} = ord_E(\mu^* D_{rn})$ where E is the exceptional divisor of the blowing up and $D_{rn} \in |rnD|$. In the same manner as in the remark to theorem 1.25 we know that $ord_E(K_{Y/X}) = n - 1$ where $n = dimX$. By the definition of the log resolution of linear series, we have $\mu^*(|rnD|) = |W| + F_{rn}$. Therefore $rn\epsilon \leq ord_E(F_{rn})$ thanks to the freeness of the linear series $|W|$. It follows that :

$$ord_E(K_{Y/X} - \lfloor \frac{1}{r} F_{rn} \rfloor) \leq n - 1 - \lfloor n\epsilon \rfloor \leq -1$$

. This together with the fact that $\mu^* \mathcal{O}_Y(-pE)$ for $p > 0$ (Example) gives us :

$$I(\|nD\|)_x = \mu_* \mathcal{O}_Y(K_{Y/X} - \lfloor \frac{1}{r} F_{rn} \rfloor))_x \subseteq m_x$$

which contradicts our assumption. For the reverse conclusion again choose r large enough so that $I(\|nD\|) = I(\frac{1}{r}|rnD|)$. Assume on the contrary that there exists a point $x \in X$ with $I(\|nD\|) \subseteq m_x$ for some $n \geq 1$. Now for a general divisor $D_{rn} \in |rnD|$ we must have $mult_x D_{rn} \geq r$ for otherwise by theorem 1.25 it follows that $I(\|nD\|)$ is trivial at x which is absurd. Therefore D is not almost base point free and by the theorem is not nef and good as well. The proof is complete.

As we have remarked earlier, the asymptotic multiplier ideals are interesting just in the case of infinite generation of $R = R(X, D) = \oplus H^0(X, nD)$. It is well known (due to Zariski in [Z]) that if in addition to the above hypothesis , X is normal then a nef and good divisor is semiample iff $R(X, D)$ is finitely generated . This means that the triviality of the asymptotic multiplier ideals can not characterize semiampleness. Moreover there are examples of divisors which are nef and good but not semiample (i.e. none of their multiples are generated by global sections)

2 Analytic Constructions

In this section we are going to study the analytic analogue of the concepts which were introduced in section 1.

Definition 2.1. *Let Ω be an open set in $\mathbb{C}^n$. A function $\phi : \Omega \to [-\infty, \infty)$ is said to be plurisubharomonic if it is upper semicontinuous and for every complex line $L \in \mathbb{C}^n$ it is subharmonic on $\Omega \cap L$ i.e. for every $a \in \Omega$ and $|\xi| < d(a, \partial\Omega)$ we have*

$$\phi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(a + e^{i\theta}\xi) d\theta.$$

The definition shows that plurisubharmonicity is the complex analogue of convexity, in fact we have

Lemma 2.2. *A function $\phi \in C^2(\Omega, \mathbb{R})$ is plurisubharmonic iff the associated Hermitian form $H\phi(a) = \sum_{1 \leq j.k \leq n} \frac{\partial^2 \phi}{\partial z_j \partial \overline{z}_k}(a)$ is semipositive at every point $a \in \Omega$.*

Proof. Letting $d\lambda$ be the Lebesgue measure on $\mathbb{C}$ we actually have

$$\frac{1}{2\pi} \int_0^{2\pi} \phi(a + e^{i\theta}\xi) d\theta - \phi(a) = \frac{2}{\pi} \int_0^1 \frac{dt}{t} \int_{|l|<t} H\phi(a + l\xi)(\xi) d\lambda(l).$$

We denote the set of plurisubharmonic functions on an open set $\Omega \subseteq \mathbb{C}^n$ by $Psh(\Omega)$.

Examples 2. *For every holomorphic function ϕ the function $log|\phi|$ is Psh.*

So far we have defined plurisubharmonic functions only on open sets in $\mathbb{C}^n$, but actually, we can observe that this definition also makes sense for any complex manifold. In fact let X be any complex analytic manifold of dimension n and u a differential form of bidegree (p, q). We can write u as

$$u = \sum_{|I|=p, |J|=q} u_{IJ} dz_I \wedge d\overline{z}_J$$

with multiindices $I = (i_1, ...i_p)$ and $J = (j_1, ...j_q)$ and $dz_I = dz_{i_1} \wedge ... \wedge dz_{i_p}$ and $d\overline{z}_J = d\overline{z}_{j_1} \wedge .... \wedge d\overline{z}_{j_q}$. Note that the exterior derivative splites as $d = d' + d''$, where

$$d'u = \sum_{|I|=p, |J|=q, 1 \leq k \leq n} \frac{\partial v_{I,J}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J$$

and

$$d''u = \sum_{|I|=p, |J|=q, 1 \leq k \leq n} \frac{\partial v_{I,J}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J$$

Moreover, if $\Phi : X \to Y$ is a holomorphic mapping of manifolds and if $v \in C^2(Y, \mathbb{R})$, we have $d'd''(v \circ \Phi) = \Phi^* d'd''v$, and hence

$$H(v\Phi)(a, \xi) = Hv(\Phi(a), \Phi'(a).\xi)$$

Therefore Hv as a Hermitian form does not depend on the choice of coordinates.Which shows that the notion of Psh function can be defined on any complex manifold. More generally we can assert that

Proposition 2.3. *If $\Phi : X \to Y$ is a holomorphic map of manifolds and $v \in Psh(Y)$ then $v \circ \Phi \in Psh(X)$.*

Proof. [D, 1.8].

Now let X be a complex analytic manifold as above and $\mathcal{L}$ a line bundle on X. The notions of nef and good divisors can be defined in the analytic setting .

Definition 2.4. *A holomorphic line bundle $\mathcal{L}$ over a compact complex manifold X with Hermitian metric ω is nef if for every $\epsilon > 0$, there is a smooth hermitian metric h_ϵ on $\mathcal{L}$ such that $i\Theta_{h_\epsilon} \geq -\epsilon\omega$ where Θ stands for the curvature form.*

Remark. *One can observe that this definition is really the analytic analogue of nef divisors introduced in the last section. In fact if X is assumed to be a complex projective manifold then we can be sure that there exist some irreducible curves on X then under the conditions of definition , we get $\mathcal{L}.C = \int_C \frac{i}{2\pi}\Theta_{h_\epsilon}(\mathcal{L}) \geq -\frac{\epsilon}{2\pi}\int_C \omega$ for every curve C and every $\epsilon > 0$, hence $\mathcal{L}.C \geq 0$ which is the definition that we gave for nefness in the last section.*

Likewise we can define the notion of Iitaka dimension of a line bundle.

Definition 2.5. *For a line bundle $\mathcal{L}$, the Iitaka dimension $\kappa(\mathcal{L})$ is defined as follows. Let B_m be the base locus of the line bundle $\mathcal{L}^m$ and let $\Phi_m : X - B_m \to \mathbb{P}H^0(\mathcal{L}^m)$ be the canonical maps. The Iitaka dimension of $\mathcal{L}$ is defined to be the supremum of the dimensions of the images of the Φ_m. If for all $m \geq 1$ we have $H^0(X, m\mathcal{L}) = 0$ then we set $\kappa(\mathcal{L}) = -\infty$. We clearly have $\kappa(\mathcal{L}) \leq dimX$ A line bundle $\mathcal{L}$ is said to be big if $\kappa(\mathcal{L}) = dimX$.*

A well-known lemma due to Serre in [Se] asserts that

Lemma 2.6. *Let $\mathcal{L}$ and X be as above. Then $h^0(X, m\mathcal{L}) = O(m^{\kappa(\mathcal{L})})$ for $m \to \infty$.*

There is also an analogue of the numerical dimension of $\mathcal{L}$ in the analytic setting, which uses the notion of Chern classes on Kähler manifolds

Definition 2.7. *Let $\mathcal{L}$ be a nef line bundle on a compact Kähler manifold X. Then the numerical dimension of $\mathcal{L}$ is defined to be*

$$\nu(\mathcal{L}) = max\{0 \leq k \leq n | 0 \neq c_1(\mathcal{L})^k \in H^{2k}(X, \mathbb{R})\}.$$

Likewise we can prove that

Proposition 2.8. *If $\mathcal{L}$ is a nef line bundle on a compact Kähler manifold, then $\kappa(\mathcal{L}) \leq \nu(\mathcal{L})$.*

Proof. [D, Prop 6.10].

Therefore it makes sense to to discuss the notion of plurisubharmonicity on any complex manifold.

Definition 2.9. *A line bundle $\mathcal{L}$ is called good if $\kappa(\mathcal{L}) = \nu(\mathcal{L})$.*

The analogue of multiplier ideals can be defined as follows:

Definition 2.10. *Let X be a complex manifold and ϕ a Psh function on X. The analytic multiplier ideal sheaf associated to ϕ is defined to be the sheaf of germs of holomorphic functions f such that $|f|^2e^{-2\phi}$ is locally integrable with respect to Lebesgue measure in a local coordinate system.We denote this ideal sheaf with $\mathcal{I}(\phi)$.*

The relation between algebraic and analytic multiplier ideal sheaves can be described in the following way:

Let X be a smooth algebraic variety viewed as a complex manifold and let D be a given effective $\mathbb{Q}$-divisor $D = \sum_{i=1}^{n} D_i$ on X. Let U be an arbitrary open set and let g_i be a holomorphic function locally defining D_i on U. Then the function $\phi_D = \sum_{i=1}^{n} a_i log|g_i|$ is plurisubharmonic on U thanks to example 2. The multiplier ideal sheaf of ϕ is given by the definition as

$$\mathcal{I}(\phi_D) = \{f \in H^0(X, \mathcal{O}_X) | \frac{|f|^2}{\Pi |g_i|^{2a_i}} \in L^1_{loc}\}$$

theorem 2.11. *The analytic multiplier ideal sheaf $\mathcal{I}(\phi_D)$ is the analytic sheaf associated to the algebraic multiplier ideal sheaf I(D).i.e $\mathcal{I}(\phi_D) = I(D)^{an}$.*

There is an analogue of the notion of multiplicity in the analytic setting,namely Lelong numbers:

Definition 2.12. *Let ϕ be a Psh function on a coordinate open set Ω of a compact complex manifold X. The Lelong number of ϕ at $x \in \Omega$ is defined to be*

$$\mu(\phi, x) = \liminf_{z \to x} \frac{\phi(z)}{log(|z - x|)}.$$

Note that we are considering $\Omega \subseteq \mathbb{C}^n$. By [D, Thm 2.10] however, Lelong numbers are independent of changes of local coordinates.

We have the following theorem :

theorem 2.13. *Let ϕ and Ω be as in the definition then we have*

(i)If $\mu(\phi, x) < 1$, then $e^{-2\phi}$ is integrable in a neighborhood of x, therefore using the definition of analytic multiplier ideals it follows that in particular the analytic multiplier ideal of ϕ is trivial at x i.e. $\mathcal{I}(\phi)_x = \mathcal{O}_{\Omega,x}$.

(ii)If $\mu(\phi, x) \geq n + s$ for some s non-negative integer, then we have the inequality $e^{-2\phi} \geq A|z - x|^{-2n-2s}$ in a neighborhood of x and $\mathcal{I}(\phi)_x \subseteq m^{s+1}$

Proof. [D, Lemma 5.6].

This theorem was originally proved by Skoda in [Sk]. Comparing this theorem with theorem 1.24 of section 1 exactly shows that the Lelong numbers work as multiplicity in the analytic setting.

The definition of almost base point free divisor in the algebraic setting suggests the following definition

Definition 2.14. *A line bundle $\mathcal{L}$ on X is said to be almost base point free if $\forall \epsilon > 0$and $\forall x \in X$ there exists a possibly singular Hermitian metric $h = e^{-2\phi}$ on $\mathcal{L}$, which is positive in the sense of currents and for which $\mu(\phi, x) < \epsilon$.*

As in the algebraic case the almost base point freeness property can be characterized by means of vanishing of asymptotic multiplier ideals . For this aim we first define the notion of asymptotic multiplier ideal sheaves in the analytic setting.which uses the notion of Pseudoeffective divisors i.e. divisors whose first chern class lies in the closure of the cone of effective divisors.

Definition 2.15. *Let $\mathcal{L}$ be a pseudoeffective line bundle on X, let h_∞ be any smooth Hermitian metric on $\mathcal{L}$ and $u = i\Theta_{h_\infty}(\mathcal{L})$. Now let $h_{min} = h_\infty e^{-\psi_{max}}$ where*

$$\psi_{max} = sup\{\psi(x) | \psi usc, \psi \leq 0, i\partial\overline{\partial} log(\psi) + u \geq 0\}$$

where by usc we mean upper semicontinuous.

h_{min} is called the singular Hermitian metric with minimal singularities.

Definition 2.16. *The analytic asymptotic multiplier ideal sheaf associated to the line bundle $\mathcal{L}$ is defined to be $\mathcal{I}(\psi_{min})$ where $\psi_{min} = -\psi_{max}$ and [D, 13] shows that this function is Psh.*

Proposition 2.17. *Let $\mathcal{L}$ be a line bundle on a compact complex manifold X. Then $\mathcal{L}$ is analytic almost base point free iff $\mathcal{I}(\psi_{min}) = \mathcal{O}_X$ iff for every $x \in X$ the Lelong numbers of h_{min} are all zero.*

Proof. [R]. The proof uses the same method as in theorem 1.47 along with Theorem 2.13.

Therefore we have the following:

Proposition 2.18. *Let $\mathcal{L}$ be a nef and good line bundle on a compact complex projective manifold X. Then for m sufficiently large we have $I(\|\mathcal{L}^{e(\mathcal{L})m}\|) = \mathcal{I}(\psi_{min}) = \mathcal{O}_X$.*

Proof. As mentioned in the remark to definition 2.4, $\mathcal{L}$ will be nef and good in the setting of section 1 and therefore it is almost base point free by theorem 1.47 which implies that it is analytic almost base point free and the previous proposition proves the claim.

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