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Properties of radonifying operators

Masterscriptie

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Datum Masterexamen: 24 oktober 2014

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Chapter 1

Introduction

In this thesis we study 'radonifying' operators from a separable Hilbert space H into a separable Banach space E. The terminology 'radonification' refers to the property that an operator T maps some cylindrical measure μ on H to a cylindrical Radon measure on E. In this sense the operator T is μ-radonifying. Due to the many nice properties of Gaussian measures, there is a wide theory of γ-radonifying operators ($\gamma = \gamma_H$ is the standard cylindrical Gaussian measure on H). We study this class of γ-radonifying operators in a slightly more general setting by studying them as μ-radonifying operators, where μ is taken from a set that contains the cylindrical Gaussians. The aim of this thesis is to see if some of these results for γ-radonifying operators can be extended to this slightly larger set of cylindrical measures.

A motivation for studying γ-radonifying operators is their central role in stochastic integration theory: In [9], it is shown that a strongly measurable function $\phi : (0,T) \to E$ is stochastically integrable with respect to a standard Brownian motion, if and only if ϕ respresents a γ-radonifying operator $R : L^2(0,T) \to E$. In understanding these radonifying operators, we will only be concerned with them as abstract objects. The thesis is organised as follows: In chapter **2** we give a summary of the basic notions, definitions and a few important results we will need. In chapter **3** we study the space of m-radonifying operators as normed vector space with a suitable norm. In chapter **4** we consider the special case where $m = \gamma_H$, the cylindrical standard Gaussian, and mention some important results for γ-radonifying operators. Important here will be the Reproducing Kernel Hilbert Space (RKHS) in section **4.2**. Section **4.1** will be devoted to a characterization of Gaussian random variables. In section **4.5** we consider some examples of radonifying operators.

Chapter 2

Preliminaries

In this section we give a summary of the basic notions, and definitions. Some familiarity with basic functional analysis is assumed. Unless mentioned otherwise, E will always be a separable real Banach space, and H a separable real Hilbert space.

2.1 Cylinder σ-algebra

In this section we follow [5]. Let E^* be the norm dual of E. For n-tuples $(x_1^*, \cdots, x_n^*) \in E^{*n}$ define the projection

$$\pi_{(x_1^*,\cdots,x_n^*)} : E \to \mathbb{R}^n, \qquad \pi_{(x_1^*,\cdots,x_n^*)}(x) = (\langle x, x_1^* \rangle, \cdots, \langle x, x_n^* \rangle).$$

For $F \in \mathcal{B}(\mathbb{R}^n)$, sets of the form

$$\pi^{-1}_{x_1^*,\cdots,x_n^*}(F) = \{x \in E : (\langle x, x_1^* \rangle, \cdots, \langle x, x_n^* \rangle) \in F\},$$

are called *cylindrical* sets. For each $\Gamma \subseteq E^*$, the set

$$\mathcal{A}(E, \Gamma) := \{\pi^{-1}_{(x_1^*,\cdots,x_n^*)}(F) : x_1^*, \cdots, x_n^* \in \Gamma, F \in \mathcal{B}(\mathbb{R}^n), n \in \mathbb{N}\}$$

is an algebra. The σ-algebra generated by $\mathcal{A}(E, \Gamma)$ is denoted by $\mathcal{C}(E, \Gamma)$, and is called *cylinder σ-algebra* with respect to (E, Γ). For finite $\Gamma \subseteq E^*$ it turns out that $\mathcal{A}(E, \Gamma) = \mathcal{C}(E, \Gamma)$. Easily is verified that $\pi_{x_1^*,\cdots,x_n^*} : E \to \mathbb{R}^n$ is continuous which implies that $\pi^{-1}_{x_1^*,\cdots,x_n^*}(F) \in \mathcal{B}(E)$, and shows shows that $\mathcal{C}(E, E^*) \subseteq \mathcal{B}(E)$. The separability assumption on E implies that even $\mathcal{C}(E, E^*) = \mathcal{B}(E)$. See I.2.2. in [5].

Theorem 2.1. *(E.Moerier). When E is a separable normed space, then $\mathcal{C}(E, E^*) = \mathcal{B}(E)$.*

We postpone the proof of **2.1.**. We begin with a lemma. Recall that the Hahn-Banach theorem implies that for all $x \in E$ we have

$$\sup_{\|x^*\| \leq 1} |\langle x, x^* \rangle| = \|x\| .$$

For a proof of this statement, see for example Theorem **4.3** in Rudin. A sequence of unit vectors $(x_n^*) \subset E^*$ is said to be *norming* if

$$\sup_n |\langle x, x_n^* \rangle| = \|x\| .$$

A generalisation of the following lemma can be found in [8].

Lemma 2.2. *If E is separable, then E^* contains a sequence of unit vectors (x_n^*) that is norming for E.*

Proof. Let $(x_n) \subset E$ be a dense sequence. Choose a sequence of unit vectors $(x_n^*) \subset E^*$ such that $(1 - 1/n) \|x_n\| \leq |\langle x_n, x_n^* \rangle|$ for all $n \geq 1$. The sequence (x_n^*) is norming for E. To see this, fix $x \in E$ and let $\delta > 0$. Pick $N \geq 1$ such that $0 < 1/N \leq \delta$ and $\|x - x_N\| < \delta$. Then,

$$\begin{aligned}
(1 - \delta) \|x\| \leq (1 - 1/N) \|x\| &\leq (1 - 1/N) \|x_N\| + (1 - 1/N) \|x - x_N\| \\
&\leq |\langle x_N, x_N^* \rangle| + \delta \\
&\leq |\langle x_N - x, x_N^* \rangle| + |\langle x, x_N^* \rangle| + \delta \\
&\leq |\langle x, x_N^* \rangle| + 2\delta.
\end{aligned}$$

Since δ was arbitrary it follows that $\|x\| \leq \sup_n |\langle x, x_n^* \rangle|$. The converse inequality is clear. □

With this lemma we can give an easy proof of theorem **2.1.**

Proof. (Theorem **2.1.**)
Since E is separable, by Lemma **2.2.** there exists a norming sequence $(x_n^*) \subset E^*$ with $\|x_n^*\| = 1$. Consequently, the balls $B_y(r)$ can be represented as follows:

$$B_y(r) := \{x \in E : \|x - y\| \leq r\} = \bigcap_{k=1}^{\infty} \{x \in E : |\langle x - y, x_k^* \rangle| \leq r\}.$$

This shows that the balls $B_y(r)$ are contained in $\mathcal{C}(E, E^*)$. Since these balles $B_y(r)$ generate the Borel σ-algebra, it follows that $\mathcal{B}(E) \subseteq \mathcal{C}(E.E^*)$. Consequently we have $\mathcal{B}(E) = \mathcal{C}(E, E^*)$. □

2.2 Cylindrical measures

Definition 2.3. A function

$$\mu : \mathcal{A}(E, E^*) \to [0, \infty)$$

is called a cylindrical probability measure on $\mathcal{A}(E, E^*)$, if for each finite subset $\Gamma \subseteq E^*$ the restriction of μ to the σ-algebra $\mathcal{C}(E, \Gamma)$ is a probability measure.

A σ-additive measure on the algebra of cylindrical sets of E, can by Caratheodory always be extended to the σ-algebra generated by the cylinders. Thus by the separability assumption on E and Theorem **2.1**, a σ-additive cylindrical measure can be extended to a Borel measure. Cylindrical measures are however in general only finitely additive on cylindrical sets. An example is γ_H, the cylindrical standard Gaussian measure on H.

Definition 2.4. The characteristic function of a cylindrical probability measure is defined by

$$\varphi_\mu : E^* \to \mathbb{C}, \qquad \varphi_\mu(x^*) = \int_E \exp\left(i \langle x, x^* \rangle\right) d\mu(x).$$

A cylindrical probability measure is uniqely determined by its characteristic function: See IV Theorem 2.2. in [5].

Theorem 2.5. *For two cylindrical probability measures μ_1, and μ_2 on $\mathcal{A}(E, E^*)$ the following are equivalent:*
(a) $\varphi_{\mu_1} = \varphi_{\mu_2}$,
(b) $\mu_1 = \mu_2$.

The following theorem is due to Bochner; it gives necessary and sufficient conditions for a function to be a characteristic function of a cylindrical probability measure. A complex valued function $\phi : E^* \to \mathbb{C}$ is called *positive definite* if for any N, and $x_1^*, \cdots, x_N^* \in E^*$, and complex numbers $z_1, \cdots, z_N$ we have

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \phi(x_i^* - x_j^*) z_i \overline{z}_j \geq 0.$$

Theorem 2.6. *A function $\phi : E^* \to \mathbb{C}$ is the characteristic function of a cylindrical probability measure on E, if and only if*
(1). $\phi(0) = 1$,

(2). ϕ is positive definite,

(3). The restriction of ϕ to every finite dimensional subset $\Lambda \subseteq E^*$ is continuous with respect to the norm-topology.

Corollary 2.7. *The function*

$$\phi : H \to \mathbb{C}, \qquad \phi(h) = \exp(-\frac{1}{2} \|h\|^2).$$

is the characteristic function of a cylindrical probability measure on H.

Proof. Note that (2) positive definiteness of ϕ is clear from the fact that $\phi(h_i - h_j) = \phi(h_j - h_i)$, and that for any complex numbers z_i, z_j we have $z_i\overline{z}_j + \overline{z}_i z_j \geq 0$. The other two conditions (1), and (3) are clearly satisfied. □

The characteristic function in Corollary **2.7** is the characteristic function of a cylindrical standard Gaussian measure. In Chapter **4** it is shown that every (centred) Gaussian measure μ has a characteristic function of the form $\phi_\mu(x^*) = \exp(-\frac{1}{2} \langle Qx^*, x^* \rangle)$ for a positive symmetric operator $Q : E^* \to E$. In this case $Q = I$ is the identity operator. It is however, as we will see, not the characteristic function of a measure. Hence the cylindrical Gaussian can not be extended to a (Radon) measure.

remark. Unfortunately, there is no analogue for Bochner's theorem for infinite dimensional Banach space. For $\mathbb{R}^n$ Bochner's theorem states that a function $\phi : \mathbb{R}^n \to \mathbb{C}$ satisfying conditions (1),(2), while being continuous at 0, is the characteristic funcion of a measure. For infinite dimensional E such a function is not necessarily the characteristic function of a measure.

2.3 Cylindrical Random variables

Similarly as the correspondence between measures, and random variables there is a random object associated to a cylindrical measure:

Definition 2.8. A *cylindrical random variable* X in E is a linear map

$$X : E^* \to L^0_P(\Omega, \mathbb{R}).$$

Here $L^0_P(\Omega, \mathbb{R})$ denotes the space of all Borel- measurable functions from Ω into $\mathbb{R}$. We may also define its characteristic function:

Definition 2.9. The *characteristic function* of a cylindrical random variable X is defined by

$$\varphi_X : E^* \to \mathbb{C}, \qquad \varphi_X(x^*) = \mathbb{E}[\exp iX(x^*)].$$

The concepts of cylindrical probability measure, and cylindrical random variable match in the sense that given a cylindrical random variable X, the map

$$\mu : \mathcal{A}(E, E^*) \to [0,1], \qquad \mu(\pi^{-1}_{x_1^*,\cdots,x_n^*}(F)) := P((Xx_1^*, \cdots, Xx_n^*) \in F)$$

defines a cylindrical probability measure μ such that $\varphi_X = \varphi_\mu$. The cylindrical measure μ is called *cylindrical probability distribution* of X.

2.4 Radonification

In this section we follow the lecture notes from Onno van Gaans [6].
Let (S, d) be a metric space. A measure μ on the Borel σ-algebra of S, is called *tight* if for all $\varepsilon > 0$, exists a compact $K \subseteq S$ such that $\mu(S \setminus K) < \varepsilon$. One of the main questions when working with a cylindrical measure μ, is whether it can be extended to Radon measure on the borel σ-algebra $\mathcal{B}(S)$. The corresponding question for a cylindrical random variable X is whether it is induced by an S-valued random variable.

Definition 2.10. Let S be a metric space. A finite Borel measure μ that is *tight* is called *Radon* measure on S.

Theorem 2.11. *If (S, d) is a complete separable metric space, then every finite Borel measure on S is tight.*

Consequently, every Borel probability measure on E is a Radon measure. This motivates the following definition.

Definition 2.12. **(a)** A cylindrical measure μ on $\mathcal{A}(E, E^*)$ *extends to a Radon measure* if there exists a Borel-measure ν such that

$$\nu(Z) = \mu(Z) \quad \text{for all } Z \in \mathcal{A}(E, E^*).$$

(b) An E-valued cylindrical random variable X is *induced* by a random variable Y if a.s.

$$\langle Y, x^* \rangle = Xx^* \quad \text{for all } x^* \in E^*.$$

The following theorem taken from [5] reflects how cylindrical random variables correspond with cylindrical measures:

Theorem 2.13. *For a cylindrical random variable $X : E^* \to L^0_P(\Omega, \mathbb{R})$ with cylindrical distribution μ, the following are equivalent:*

(a) X is induced by an E-valued random variable Y;

(b) μ extends to a Radon measure ν on $\mathcal{B}(E)$.

In this situation one has $P_Y = \nu$.

Proof. See [5] Theorem VI.3.1 □

Example 2.14. Let H be a real Hilbert space with orthonormal basis (h_n), let (β_n) be a bounded sequence in $\mathbb{R}$ and (α_n) an orthonormal sequence in $L^2(\Omega)$. Define a cylindrical random variable

$$X(x^*) := \sum_{n=0}^{\infty} \beta_n \alpha_n \langle h_n, x^* \rangle, \qquad x^* \in H,$$

and denote its cylindrical distribution by μ. Then we have by Parseval's identity

$$\mathbb{E}X(x^*)^2 = \sum_n \beta_n^2 \langle h_n, x^* \rangle^2 \leq \sup_n \beta_n^2 \|x^*\|^2 < \infty.$$

Thus $X(x^*)$ converges in $L^2(\Omega)$. The sequence $(\alpha_n h_n)$ is an orthonormal sequence in $L^2(\Omega; H)$ since

$$\langle \alpha_n h_n, \alpha_m h_m \rangle_{L^2(\Omega;H)} = \mathbb{E}\alpha_n \alpha_m \langle h_n, h_m \rangle_H .$$

Thus

$$\sum_n \beta_n^2 < \infty \Longleftrightarrow X := \sum_n \beta_n \alpha_n h_n \quad \text{converges in } L^2(\Omega; H).$$

This means that the cylindrical random variable X is induced by an H-valued random variable Y. Thus by Theorem **2.13** the corresponding cylindrical measure μ extends to a Radon measure ν, if and only if $\sum_n \beta_n^2 < \infty$.

We continue our discussion with the Lebesgue-Bochner space $L^p(\Omega, E)$.

2.5 Bochner Integral

The Bochner integral is a generalisation of the Lebesgue-integral to the Banach-space valued setting. (We follow [4]). Let $(A, \mathcal{F}, \mu)$ be a σ-finite measure space. Thus there exists sets $B_1 \subseteq B_2 \subseteq \cdots$ with $\mu(B_n) < \infty$ for all n, and $A = \bigcup_n B_n$. A μ-simple function with values in E is a function of the form

$$f = \sum_{n=1}^{m} \mathbf{1}_{A_n} x_n,$$

with $x_n \in E$, and $A_n \in \mathcal{F}$ satisfy $\mu(A_n) < \infty$ for all n. The Bochner integral of the simple function f is defined as

$$\int_A f d\mu := \sum_{n=1}^{m} \mu(A_n) x_n.$$

Definition 2.15. A function $f : A \to E$ is μ-*Bochner integrable* if there exists a sequence of μ-simple functions $f_n : A \to E$ such that the following two conditions are met:
(1) $\lim_{n\to\infty} f_n = f \quad \mu$ almost everywhere;
(2) $\lim_{n\to\infty} \int_A \|f_n - f\| \, d\mu = 0$.

If f is μ-Bochner integrable, the limit

$$\int_A f d\mu := \lim_{n\to\infty} \int_A f_n d\mu$$

exists in E, and is called Bochner integral of f with respect to μ. For $1 \leq p < \infty$ the Lebesgue-Bochner space $L^p(A; E, \mu)$ is defined as the linear space of all equivalence classes of μ-measurable $f : A \to E$ for which

$$\int_A \|f\|^p \, d\mu < \infty,$$

identifying functions that are equal μ-almost everywhere. Endowed with the norm

$$\|f\|_{L^p(A;E,\mu)} := \left(\int_A \|f\|^p \, d\mu \right)^{1/p}$$

the space $L^p(A; E, \mu)$ is a Banach space.
Sometimes we abbreviate $L^p(A; E, \mu)$ to $L^p(A; E)$ if it is clear what μ is. Similarly we abbreviate $L^p(A; \mathbb{R})$ to $L^p(A)$.

2.6 Symmetric random variables

Often we will encounter sums of independent, symmetric random variables. Such sums have extra nice convergence properties. Recall that an E-valued random variable is symmetric when $\mathbb{P}(X \in B) = \mathbb{P}(X \in -B)$ for all $B \in \mathcal{B}(E)$. We summarize some important results due to Kahane, Ito and Nisio. Throughout this thesis we assume that all random variables are defined on the probability space $(\Omega, \mathcal{F}, P)$ and the corresponding expectation operator is denoted by $\mathbb{E}$.

2.6.1 Kahane-contraction principle

Theorem 2.16. *(Kahane contraction principle). Let (X_j) be a sequence of independent symmetric E-valued random variables. Then for all $t_1, \cdots, t_n \in \mathbb{R}$, and $1 \leq p < \infty$,*

$$\mathbb{E}\left\|\sum_{j=1}^{n} t_j X_j\right\|^p \leq \left(\max_{1\leq j\leq N} |t_j|\right)^p \mathbb{E}\left\|\sum_{j=1}^{n} X_j\right\|^p .$$

Corollary 2.17. *Let (X_j) be a sequence of symmetric and independent E-valued random variables. Then x_n defined by $\mathbb{E}\left\|\sum_{j=1}^{n} X_j\right\|^p$ is a non-decreasing sequence.*

Proof. Apply the Kahane contraction principle to (X_j) with $t_{n+1} = 0$, and $t_j = 1$ for $j \leq n$ to obtain $x_n \leq x_{n+1}$.

□

Corollary 2.18. *Let (X_j) be a sequence of symmetric and independent E-valued random variables. Let $Y = \sum_j X_j$ a.s. and assume that $Y \in L^p(\Omega, E)$. Then,*

$$\sup_{n\geq 1} \mathbb{E}\left\|\sum_{j=1}^{n} X_j\right\|^p = \mathbb{E}\left\|Y\right\|^p .$$

Proof. Apply the monotone convergence theorem to the non-decreasing sequence (x_n) which is bounded above by $\mathbb{E}\|Y\|^p$ and whose limit is $\sup_n x_n$. Thus $\sup_n x_n \leq \mathbb{E}\|Y\|^p$. The converse inequality follows from Lebesgue's dominated convergence theorem. □

Corollary 2.19. *Let (X_j) be a sequence of symmetric and independent E-valued random variables. Let $Y = \sum_j X_j$ a.s. and assume that $Y \in L^p(\Omega, E)$. Then, for $K \subset \mathbb{N}$,*

$$\mathbb{E}\left\|\sum_{j\in K} X_j\right\|^p \leq \mathbb{E}\,\|Y\|^p .$$

Proof. Let $t_j := 1$ for $j \in K$, and $t_j := 0$ for $j \notin K$. Then apply the Kahane contraction principle to the sequence (X_j) with (t_j) to obtain

$$\mathbb{E}\left\|\sum_{j=1}^{n} t_j X_j\right\|^p \leq \mathbb{E}\left\|\sum_{j=1}^{n} X_j\right\|^p .$$

By Corollary **2.18** and monotone convergence the result follows since

$$\mathbb{E}\left\|\sum_{j\in K} X_j\right\|^p = \mathbb{E}\left\|\sum_{j=1}^{\infty} t_j X_j\right\|^p \leq \sup_{n\geq 1}\mathbb{E}\left\|\sum_{j=1}^{n} t_j X_j\right\|^p \leq \sup_{n\geq 1}\mathbb{E}\left\|\sum_{j=1}^{n} X_j\right\|^p = \mathbb{E}\,\|Y\|^p .$$

□

2.6.2 The Itô-Nisio theorem

The following theorem due to Itô and Nisio [3], is an important tool for studying sums of independent, symmetric E-valued random variables. We will make frequent reference to this theorem.

Theorem 2.20. *(Itô-Nisio) Let (X_j), be a sequence of independent symmetric random variables, put $S_n = \sum_{j=1}^n X_j$. Let μ_n be the probability distribution of S_n, and let S be a random variable. The following assertions are equivalent:*

(a) *we have $\lim_{n\to\infty} S_n = S$ almost surely;*

(b) *we have $\lim_{n\to\infty} S_n = S$ in probability;*

(c) *for all $x^* \in E^*$ we have $\lim_{n\to\infty} \langle S_n, x^*\rangle = \langle S, x^*\rangle$ almost surely;*

(d) *for all $x^* \in E^*$ we have $\lim_{n\to\infty} \langle S_n, x^*\rangle = \langle S, x^*\rangle$ in probability;*

(e) *there exists a probability measure μ on E such that*

$$\mathbb{E}[\exp\left(i\langle S_n, x^*\rangle\right)] = \hat{\mu_n}(x^*) \to \hat{\mu}(x^*)$$

for all $x^* \in E^*$.

If these equivalent conditions hold and $\mathbb{E}\,\|S\|^p < \infty$ for some $p \in [1,\infty)$, then

$$\lim_{n\to\infty} \mathbb{E}\,\|S_n - S\|^p = 0.$$

Chapter 3

Radonifying operators

3.1 m-radonification

As before, let E be a separable Banach space, and H a separable Hilbert space. Let μ be a cylindrical measure on H, and $T : H \to E$ be a bounded linear operator. By the *image measure* $\mu \circ T^{-1}$ we mean the cylindrical measure ν on E defined by

$$\nu(A) := \mu(T^{-1}(A)).$$

The following lemma gives a correspondence between φ_μ, and $\varphi_{\mu\circ T^{-1}}$.

Lemma 3.1. *Let φ_μ be the characteristic function of μ, then*

$$\varphi_{\mu\circ T^{-1}}(x^*) = \varphi_\mu(T^*x^*) \qquad \textit{for all } x^* \in E^*.$$

Proof.

$$\begin{aligned}\varphi_{\mu\circ T^{-1}}(x^*) &= \int_E e^{i\langle x,x^*\rangle} d(\mu \circ T^{-1})(x) = \int_H e^{i\langle Th,x^*\rangle} d\mu(h) \\ &= \int_H e^{i\langle h,T^*x^*\rangle} d\mu(x) = \varphi_\mu(T^*x^*).\end{aligned}$$

□

Definition 3.2. Let μ be a cylindrical probability measure on H. A bounded linear operator $T : H \to E$ is called *μ-radonifying* if $\mu \circ T^{-1}$ extends to a Radon measure on $\mathcal{B}(E)$.

As mentioned in section **2.2**, the cylindrical measure $T(\mu) := \mu \circ T^{-1}$ extends to a Radon measure if it is σ-additive on the algebra of cylindrical sets of E.

We now look at a special class of cylindrical measures. Let (h_n) be an orthonormal basis of H, and (α_n) an orthonormal sequence in $L^2(\Omega, P)$. Define a cylindrical random variable

$$\Lambda : H \to L^2(\Omega, P), \qquad \Lambda h = \sum_{n=1}^{\infty} \alpha_n \langle h_n, h \rangle .$$

Denote by m the cylindrical probability measure of Λ.

Note that Λ is well-defined since by orthonormality of (α_n), and Parseval's identity we have

$$\mathbb{E}(\Lambda h)^2 = \sum_{n=1}^{\infty} \mathbb{E}(\alpha_n)^2 \langle h_n, h \rangle^2 = \sum_{n=1}^{\infty} \langle h_n, h \rangle^2 = \|h\|^2 .$$

Thus indeed $\Lambda h \in L^2(\Omega)$.

Example 3.3. If (α_n) is an i.i.d. sequence standard Gaussians, then $m = \gamma_H$ is the standard cylindrical Gaussian measure. It can be shown that its characteristic function $\phi : H \to \mathbb{C}$ is given by $\phi(h) = \exp(-\frac{1}{2}\|h\|^2)$. Here we enter the theory of γ-radonifying operators; see chapter 4.

Theorem 3.4. *Let $T \in B(H, E)$. Let (h_n) be an orthonormal basis of H, and let (α_n) be an orthonormal sequence in $L^2(\Omega, P)$. The following are equivalent:*

(a) T is m-radonifying;

(b) There exists an E-valued random variable Y such that

$$\langle Y, x^* \rangle = \sum_{n=1}^{\infty} \alpha_n \langle T h_n, x^* \rangle \quad \text{in } L^2(\Omega) \text{ for all } x^* \in E^*.$$

In the case of (a), and (b)

$$\mathbb{E} \langle Y, x^* \rangle \langle Y, y^* \rangle = \langle TT^* x^*, y^* \rangle = \langle T^* x^*, T^* y^* \rangle_H .$$

Proof. Define a cylindrical random variable

$$X : E^* \to L^2(\Omega), \qquad X(x^*) = \sum_{n=1}^{\infty} \alpha_n \langle T h_n, x^* \rangle ,$$

and let μ its cylindrical distribution. Note that the series $X(x^*)$ converges in L^2, since $\mathbb{E}X(x^*)^2 = \|T^*x^*\|^2$ by Parseval's identity. We have

$$X(x^*) = \sum_{n=1}^{\infty} \alpha_n \langle Th_n, x^* \rangle = \sum_{n=1}^{\infty} \alpha_n \langle h_n, Tx^* \rangle = \Lambda T^* x^* \text{ for all } x^* \in E^*,$$

which shows that $\mu = m \circ T^{-1}$ by lemma **3.1** and Theorem **2.5**. It follows from Theorem **2.13** that μ extends to a Radon measure if and only if X is induced by a random variable Y. The last statement is another application of Parseval's identity. □

Corollary 3.5. *If the (α_n) are orthonormal, independent and symmetric, then:*

$$T \text{ is } m\text{-radonifying} \iff \sum_n \alpha_n T h_n \text{ converges almost surely}$$

.

Proof. If T is m-radonifying, then by Theorem **3.4** exists a random variable Y such that $\langle Y, x^* \rangle = \sum_n \alpha_n \langle Th_n, x^* \rangle$ with convergence in $L^2(\Omega)$, and in probability. Since $X_n := \alpha_n Th_n$ are symmetric E-valued random variables, it follows by the Ito Nisio Theorem (**2.5**) that $\sum_n \alpha_n Th_n$ converges almost surely and in probability. The converse is clear. □

3.2 The space of m-radonifying operators.

For a chosen orthonormal sequence (α_n) in $L^2(\Omega)$, denote by $\mathcal{R}_m(H,E)$ the subset of $\mathcal{B}(H,E)$ consisting of m-radonifying operators. By Theorem **3.4** this set is easily seen to be a vector space. We will take a closer look at some of its subspaces, namely the m-radonifying operators of order p.

Definition 3.6. Let $p \in [1,\infty)$. An operator $T \in \mathcal{B}(H,E)$ is called m-radonifying of order p if there exists a $Y_T \in L^p(\Omega, E)$ such that $\langle Y_T, x^* \rangle = \sum_j \alpha_j \langle Th_j, x^* \rangle$ for all $x^* \in E^*$.

Example 3.7. Every finite-rank operator of the form $T \circ \pi_n$, where $T \in B(H,E)$ and $\pi_n : H \to H$ is a orthogonal projection onto the span of $\{h_1, \cdots, h_n\}$, is m-radonifying of order $p = 2$. Indeed, a.s. $Y_{T\circ\pi_n} = \sum_{j=1}^n Th_j \alpha_j \in L^2(\Omega, E)$.

Denote by $\mathcal{R}_m^p(H,E)$ the subset of m-radonifying operators of order p. Clearly for $p < q$ we have inclusions

$$\mathcal{R}_m(H,E) \supset \mathcal{R}_m^p(H,E) \supset \mathcal{R}_m^q(H,E).$$

Notice that for an operator $T \in \mathcal{R}_m^p(H, E)$ the random variable Y_T is uniqely determined by T upto equality almost surely. We proceed with investigating a suitable norm on $\mathcal{R}_m^p(H, E)$. The proof of the following proposition is clear:

Proposition 3.8. *For $T \in \mathcal{R}_m^p(H, E)$, and $p \geq 1$*

$$|||T||| = \|Y_T\|_{L^p} + \|T\|,$$

defines a norm on $\mathcal{R}_m^p(H, E)$.

Remark. For $p \geq 2$ it suffices to take $|||T||| := \|Y_T\|_{L^p}$ since this is in fact a stronger norm then the standard operator norm $\|\cdot\|$. Indeed, observe that

$$\begin{aligned}
\|T\|^2 = \|T^*\|^2 &= \sup_{\|x^*\|\leq 1} \|T^* x^*\|^2 \\
&= \sup_{\|x^*\|\leq 1} \sum_n \langle h_n, T^* x^* \rangle^2 \\
&= \sup_{\|x^*\|\leq 1} \mathbb{E} \left(\sum_n \alpha_n \langle T h_n, x^* \rangle \right)^2 \\
&\leq \mathbb{E} \sup_{\|x^*\|\leq 1} \left(\sum_n \alpha_n \langle T h_n, x^* \rangle \right)^2 = \mathbb{E} \|Y_T\|^2,
\end{aligned}$$

which shows that $\|T\| \leq \|Y_T\|_{L^2}$, and by monotonicity of the L^p-norm we have $\|T\| \leq \|Y_T\|_{L^p}$. Thus for $p \geq 2$ we don't need the extra requirement that T is bounded. Below we give an example to see why this condition is necessary for $1 \leq p < 2$.

Example 3.9. Let $\Omega = \mathbb{N} = \{1, 2, \cdots, \}$, $\mathbb{P}(\{n\}) = 3 \cdot 4^{-n}$, and take $\alpha_n = \frac{1}{\sqrt{3}} \cdot 2^n \mathbf{1}_{\{n\}}$. Then the α_n are orthonormal in $L^2(\Omega, \mathbb{P})$, and for $T \in \mathcal{R}_{m,p}(H, E)$,

$$\mathbb{E} \left\| \sum_{n=1}^{\infty} \alpha_n T h_n \right\|^p = 3 \cdot \sum_{k=1}^{\infty} \left\| \sum_{n=1}^{\infty} \alpha_n(k) T h_n \right\|^p \cdot 4^{-k} = 3^{1-p/2} \cdot \sum_{k=1}^{\infty} \|T h_k\|^p \left(\frac{2^p}{4} \right)^k.$$

Define, for $N \in \mathbb{N}$, bounded linear operators $T_N : H \to \mathbb{R}$ by $T_N(h) = \langle h, h_N \rangle$, and observe that $\|T_N\| = 1$. From the above it follows that for $1 \leq p < 2$

$$\mathbb{E} \left\| \sum_{n=1}^{\infty} \alpha_n T_N h_n \right\|^p = 3^{1-p/2} \left(\frac{2^p}{4} \right)^N \to 0.$$

Hence in contrast with $p \geq 2$, we find that $\mathbb{E} \left\| \sum_n \alpha_n T_N h_n \right\|^p \to 0$ does not in general imply $\|T_N\| \to 0$. Thus for $1 \leq p < 2$ we want to impose the extra condition that T is bounded.

Hence we make the following definition:

Definition 3.10. For T in $\mathcal{R}_m^p(H,E), p \geq 1$ define

$$\|T\|_{m,p} = \begin{cases} \|Y_T\| + \|T\| & \text{if } 1 \leq p < 2 \\ \|Y_T\| & \text{if } p \geq 2 \ . \end{cases}$$

3.3 Separability and completeness.

If H is finite-dimensional, the norms $\|\cdot\|_{m,2}$ and the uniform operator norm are equivalent since

$$\begin{aligned} \|T\| \leq \|Y_T\|_{L^2} = \left\| \sum_{j=1}^{n} \alpha_j T h_j \right\|_{L^2} &= \left(\mathbb{E} \left\| \sum_{j=1}^{n} \alpha_j T h_j \right\|^2 \right)^{1/2} \\ &\leq \left(\mathbb{E} \sum_{i,j=1}^{n} |\alpha_i||\alpha_j| \, \|Th_i\| \, \|Th_j\| \right)^{1/2} \leq n \, \|T\| \, , \end{aligned}$$

where $\dim(H) = n$. Hence for finite-dimensional H we have an isomorphic identification of the spaces $\mathcal{B}(H,E)$, and $\mathcal{R}_m^2(H,E)$. Thus $\mathcal{R}_m^2(H,E)$ is complete since $\mathcal{B}(H,E)$ is complete, and separable if and only $B(H,E)$ is separable. The same is true for $\mathcal{R}_m^p(H,E), p \geq 2$ when additionally the random variables (α_n) are assumed to be normal in $L^p(\Omega)$. In the next lemma we show that $\mathcal{B}(H,E)$ is separable whenever E is separable.

remark. In fact, for finite-dimensional H we have equivalence of the (m,p)-radonifying norm and the uniform operator norm if we merely assume that the random variables (α_n) are contained in $L^p(\Omega)$. A similar calculation as the one above, shows there exists a constant C depending on the choice of $\alpha_1, \cdots, \alpha_n \in L^p(\Omega)$, such that $\|T\| \leq \|Y_T\|_{L^p} \leq C \cdot n \cdot \|T\|$.

Lemma 3.11. *Let E be separable. When H is finite dimensional, the space $\mathcal{R}_m^p(H,E)$ is separable.*

Proof. By the above remarks it suffices show that $\mathcal{B}(H,E)$ is separable in the uniform operator norm. Let $\{h_1,\cdots,h_n\}$ be an orthonormal basis for $H_n := H$. A linear operator $T \in B(H_n,E)$ is completely determined by its values $(T(h_1),\cdots,T(h_n))$. The proof proceeds by induction on $n = \dim(H)$. Let $F \subset E$ be a countable dense subset. The case $H = \mathbb{R}$ is trivial. Assume $B_{n-1} \subset B(H_{n-1},E)$ is a countable dense subset, let $T \in B(H_n,E)$ and let $S_{n-1} \in B_{n-1}$ be such that $\|T \circ \pi_{n-1} - S_{n-1}\| < \varepsilon/2$. Here π_{n-1} is the orthogonal projection onto the span of $\{h_1,\cdots,h_{n-1}\}$. Define $S_n \in B_n$ as $S_n(h_j) = S_{n-1}(h_j)$ for $j < n$, and such that $S_n(h_n) \in F$ with $\|S_n(h_n) - T(h_n)\| < \varepsilon/2$. Thus we have associated to T an operator $S \in B(H_n,E)$. The set B_n of all operators S we obtain this way is countable, since B_{n-1} is countable. Moreover, for $\|h\| \leq 1$ we have

$$\begin{aligned}\|(T-S_n)h\| &= \|(T-S_n)\circ\pi_{n-1}(h) + \langle h,h_n\rangle \cdot (T-S_n)(h_n)\| \\ &\leq \|(T-S_n)\circ\pi_{n-1}(h)\| + |\langle h,h_n\rangle| \cdot \|T(h_n) - S_n(h_n)\| \\ &= \|(T\circ\pi_{n-1} - S_{n-1})h\| + |\langle h,h_n\rangle| \cdot \|T(h_n) - S_n(h_n)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus B_n is dense in H_n, and by induction it follows that $B(H_n,E)$ is separable for all n. □

Suitable conditions on the sequence (α_n) in $L^2(\Omega)$ ensure that m-radonifying operators of order $p \geq 2$ are compact; as we show in Lemma **3.12** .

Lemma 3.12. *Let (α_n) be orthonormal, and $T \in R^p_m(H,E), p \geq 2$. If $Y_T = \sum_n \alpha_n T h_n$ a.s., then T is a uniform limit of finite-rank operators, so T is compact.*

Proof. Let π_N be the orthogonal projection onto the span of $\{h_1,\cdots,h_N\}$. Since the (m,p)-radonifying norm is stronger then the operator norm for $p \geq 2$, we have

$$\|T - T\circ\pi_N\| \leq \|T - T\circ\pi_N\|_{m,p} = \left\|\sum_{n\geq N} \alpha_n T h_n\right\|_{L^p} \to 0.$$

□

remark. If (α_n) are orthonormal, independent and symmetric then $\mathcal{R}^p_m(H,E)$ consists entirely of compact operators. By Corrollary **3.5**, and the Itô-Nisio theorem, T is m-radonifying of order p if and only if $\sum_n \alpha_n T h_n$ converges in $L^p(\Omega,E)$. Thus T is compact by the above lemma. In particular this is true when the α_n are i.i.d. standard Gaussian random variables.

Theorem 3.13. *For $p \geq 2$, let $(\alpha_n) \in L^p(\Omega)$ be independent, symmetric and orthonormal. If E is separable, then $\mathcal{R}_m^p(H, E)$ is separable.*

Proof. Let $T \in \mathcal{R}_m^p(H, E)$. The assumption on (α_n) yields that $Y_T = \sum_n \alpha_n T h_n$ a.s. by the Ito-Nisio Theorem. Let H_n be the closed linear span of $\{h_1, \cdots, h_n\}$, and let

$$B_n := \{T \circ \pi_n | T \in B(H, E)\},$$

which may be identified isometrically with $B(H_n, E)$. By earlier given remarks the space $\mathcal{R}_m^p(H_n, E)$ coincides isometrically with $B(H_n, E)$, and is separable by Lemma **3.11** and the remark above it. Separability of $\mathcal{R}_m^p(H, E)$ follows from the fact that for all $T \in \mathcal{R}_m^p(H, E)$ we have

$$\|T - T \circ \pi_n\| \leq \|T - T \circ \pi_n\|_{m,p} \to 0,$$

as in the proof of Lemma **3.12**. In other words,

$$\mathrm{Cl}\left(\bigcup_{n \geq 1} B_n\right) = \mathcal{R}_m^p(H, E)$$

where the closure is in the (m, p)-radonifying norm. Thus if $K_n \subset B_n = B(H_n, E)$ is a countable dense subset, then $K = \bigcup K_n$ is a countable dense subset of $\mathcal{R}_m^p(H, E)$. Indeed, for all $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $\|T - T \circ \pi_N\|_{m,p} < \varepsilon/2$, and there exists $S \in K_N$ such that $\|T \circ \pi_N - S\|_{m,p} < \varepsilon/2$, showing

$$\|T - S\|_{m,p} \leq \|T - T \circ \pi_N\|_{m,p} + \|T \circ \pi_N - S\|_{m,p} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

It is shown in [8] that the space of γ-radonifying operators is complete in all (m, p)-radonifying norms. Thus if α_n are i.i.d. standard Gaussians, the space $\mathcal{R}_m^p(H, E)$ is complete for all $p \geq 1$. In fact, it can be shown that all (m, p)-norms are equivalent when (α_n) are i.i.d. standard Gaussians, in which case it is sufficient to show completeness $p = 2$. The following theorem is a small extension of the Gaussian case.

Theorem 3.14. *Let the sequence $(\alpha_n) \subset L^2(\Omega)$ be orthonormal, independent and symmetric. Then $(\mathcal{R}_m^p(H, E), \|\cdot\|_m^p)$ is a Banach space.*

Proof. Let (T_j) be a Cauchy sequence in $\mathcal{R}^p_m(H,E)$. Since $\|T_i - T_j\|^p_m < \varepsilon$ implies $\|T_i - T_j\| < \varepsilon$, the sequence (T_j) is Cauchy with respect to the operator norm. By completeness of $B(H,E)$ there exists $T \in B(H,E)$ such that $\|T_j - T\| \to 0$. It remains to show that T is m-radonifying of order p, and $\|T_j - T\|^p_m \to 0$. Since the (α_n) are symmetric and independent we have $Y_{T_i} = \sum_n \alpha_n T_i h_n$ a.s. by Itô-Nisio. By Corollary **2.17**, $\left\|\sum_{n=1}^N \alpha_n (T_i - T_j) h_n\right\|_{L^p}$ is non-decreasing in N, from which it follows that

$$\left\|\sum_{n=1}^{N} \alpha_n (T_i - T_j) h_n\right\|_{L^p} \leq \left\|Y_{T_i} - Y_{T_j}\right\|_{L^p} < \varepsilon.$$

We obtain $\left\|\sum_{n=1}^N \alpha_n (T - T_j) h_n\right\|_{L^p} < \varepsilon$ by letting $i \to \infty$. It follows by Corollary **2.18** that

$$\left\|\sum_{n=1}^{\infty} \alpha_n (T_j - T) h_n\right\|_{L^p} = \sup_{N \geq 1} \left\|\sum_{n=1}^{N} \alpha_n (T_j - T) h_n\right\|_{L^p} < \varepsilon.$$

Notice that the sequence (Y_{T_j}) is Cauchy in $L^p(\Omega, E)$ since for $i, j \in \mathbb{N}$

$$\left\|Y_{T_i} - Y_{T_j}\right\|_{L^p} \leq \|T_i - T_j\|_{m,p} < \varepsilon.$$

Let Y_T be the L^p-limit of the sequence (Y_{T_n}). Then

$$\begin{aligned}\left\|Y_T - \sum_n \alpha_n T h_n\right\|_{L^p} &\leq \left\|Y_T - Y_{T_j}\right\|_{L^p} + \left\|Y_{T_j} - \sum_n \alpha_n T h_n\right\|_{L^p} \\ &= \left\|Y_T - Y_{T_j}\right\|_{L^p} + \left\|\sum_{n=1}^{\infty} \alpha_n (T_j - T) h_n\right\|_{L^p} < 2\varepsilon.\end{aligned}$$

It follows that $Y_T = \sum_n \alpha_n T h_n$ a.s. which shows that T is m-radonifying of order p, and $\|T - T_j\|_{m,p} = \left\|Y_T - Y_{T_j}\right\|_{L^p} + \|T - T_j\| \to 0$. This proves completeness of the space $(\mathcal{R}_{m,p}(H,E), \|\cdot\|_{m,p})$. $\square$

We present the following completeness result as the main theorem of this chapter. Even without the extra independence, and symmetry assumptions on (α_n), the space $\mathcal{R}^2_m(H,E)$ is complete.

Theorem 3.15. *Let the sequence $(\alpha_n) \subset L^2(\Omega)$ be orthonormal. Then $(\mathcal{R}^2_m(H,E), \|\cdot\|^2_m)$ is a Banach space.*

Proof. Replicating the proof of Theorem 21, let $(T_n) \subset (\mathcal{R}^2_m(H,E), \|\cdot\|_{m,2})$ be Cauchy, and let $T \in B(H,E)$ the limit of this sequence in the operator norm. Let Y_T be the L^2-limit of the Cauchy sequence (Y_{T_n}). Then $\left\|Y_{T_j} - Y_T\right\|_{L^2} + \|T_j - T\| \to 0$, and it remains to show that $\langle Y_T, x^*\rangle = \sum_n \alpha_n \langle Th_n, x^*\rangle$ for all $x^* \in E^*$. Suppose $\|T_i - T_j\|_m < \varepsilon$, then

$$\begin{aligned}\sum_{n=1}^{N} \langle (T_i - T_j)h_n, x^*\rangle^2 &\le \sum_{n=1}^{\infty} \langle (T_i - T_j)h_n, x^*\rangle^2 \\ &= \left\| \sum_{n=1}^{\infty} \alpha_n \langle (T_i - T_j)h_n, x^*\rangle \right\|_{L^2}^2 \\ &= \left\| \left\langle Y_{T_i} - Y_{T_j}, x^* \right\rangle \right\|_{L^2}^2 \\ &\le \left\| Y_{T_i} - Y_{T_j} \right\|_{L^2}^2 \|x^*\|^2 \\ &< \varepsilon^2 \|x^*\|^2 .\end{aligned}$$

Let $i \to \infty$ to obtain

$$\sum_{n=1}^{N} \langle (T - T_j)h_n, x^*\rangle^2 < \varepsilon^2 \|x^*\|^2 ,$$

from which it follows

$$\sum_{n=1}^{\infty} \langle (T - T_j)h_n, x^*\rangle^2 \le \sup_N \sum_{n=1}^{N} \langle (T - T_j)h_n, x^*\rangle^2 < \varepsilon^2 \|x^*\|^2 .$$

Consequently

$$\begin{aligned}\left\| \langle Y_T, x^*\rangle - \sum_{n=1}^{\infty} \alpha_n \langle Th_n, x^*\rangle \right\|_{L^2} &= \left\| \langle Y_T, x^*\rangle - \left\langle Y_{T_j}, x^*\right\rangle + \left\langle Y_{T_j}, x^*\right\rangle - \sum_{n=1}^{\infty} \alpha_n \langle Th_n, x^*\rangle \right\|_{L^2} \\ &\le \left\| \left\langle Y_T - Y_{T_j}, x^*\right\rangle \right\|_{L^2} + \left\| \sum_{n=1}^{\infty} \alpha_n \langle (T - T_j)h_n, x^*\rangle \right\|_{L^2} \\ &= \left\| \left\langle Y_T - Y_{T_j}, x^*\right\rangle \right\|_{L^2} + \left(\sum_{n=1}^{\infty} \langle (T - T_j)h_n, x^*\rangle^2 \right)^{1/2} \\ &< 2\varepsilon \|x^*\| .\end{aligned}$$

This proves completeness of the space $(\mathcal{R}_m^2(H,E), \|\cdot\|_{m,2})$. □

Another linear subspace of $R_m(H,E)$ is the following. Let $\mathcal{S}(H,E)$ be the space of operators such that $\sum_{n=1}^{\infty} \|Th_n\| < \infty$. Clearly every $T \in \mathcal{S}(H,E)$ is m-radonifying of order $p = 1$ since

$$\begin{aligned}\mathbb{E} \left\| \sum_n \alpha_n Th_n \right\| &\le \mathbb{E} \sum_n |\alpha_n| \, \|Th_n\| \\ &\le \sum_n \|Th_n\| < \infty.\end{aligned}$$

Thus $\mathcal{R}^1_m(H,E) \supset \mathcal{S}(H,E)$. An example of such operator is the following.

Example 3.16. Define $T_\lambda : l^2 \to l^q$ as the multiplication operator

$$T_\lambda((x(n))_n = (\lambda(n)x(n))_n, \qquad \lambda \in l^p.$$

By an application of Holders inequality, for $\frac{1}{r} + \frac{1}{s} = 1$, we have

$$\sum_n |\lambda(n)x(n)|^q \leq \left(\sum_n |\lambda(n)|^{qr}\right)^{1/r} \left(\sum_n |x(n)|^{qs}\right)^{1/s}$$

where we desire $qs = 2$, and $qr = p$. Solving for q in terms of p yields

$$q = \frac{2}{1 + \frac{2}{p}}.$$

Since we want the range space of T_λ to be a Banach space, this yields the restriction $p \in [2, \infty]$, and for such p we have $1 \leq q \leq 2$. Let (h_n) be the standard basis of l^2, i.e. $h_n(n) = 1$, and $h_n(m) = 0$ for $m \neq n$, then

$$\sum_n \|T_\lambda h_n\|_{l^q} = \sum_n |\lambda(n)|^q = \|\lambda\|^q_{l^q} < \infty.$$

By the above given remarks the operator T_λ is m-radonifying of order 1.

It is well-known that if $T : H \to H'$ where H' is another separable Hilbert space, then T is γ-radonifying if and only if T is Hilbert-Schmidt. This is also known as Sazonov's theorem, which can be found in [4]. (See Theorem **4.19** and proposition **4.20**). The following generalisation is true:

Theorem 3.17. *Let (α_n) be orthonormal, independent and symmetric. When H' is another Hilbert space, the space $\mathcal{R}^2_m(H,H')$ coincides with the space of Hilbert-Schmidt operators $\mathcal{S}^2(H,H')$.*

Proof. First, we show that $\mathcal{R}^2_m(H,H') \subset \mathcal{S}^2(H,H')$. Let $T \in \mathcal{R}^2_m(H,H')$, then there exists $Y_T \in L^2(\Omega, H')$ such that $\langle Y_T, x^* \rangle = \sum_n \alpha_n \langle Th_n, x^* \rangle$ for all $x^* \in H'$. By Ito-Nisio we have that $Y_T = \sum_n \alpha_n Th_n$ with convergence in $L^2(\Omega, H')$. Hence

$$\mathbb{E}\|Y_T\|^2 = \mathbb{E}\left\|\sum_n \alpha_n Th_n\right\|^2_{H'} = \mathbb{E}\left\langle \sum_n \alpha_n Th_n, \sum_n \alpha_n Th_n \right\rangle_{H'} = \sum_n \|Th_n\|^2 < \infty,$$

and it follows that T is Hilbert-Schmidt. The converse is clear; if T is Hilbert-Schmidt, then $\|\sum_{k=m}^n Th_n\|^2_{H'} = \mathbb{E}\|\sum_{k=m}^n \alpha_n Th_k\|^2_{H'}$ which implies that $\sum_n \alpha_n Th_n$ converges in $L^2(\Omega; H')$. Thus T is m-radonifying of order 2. ◻

Chapter 4

Theory of γ-radonifying operators

In this chapter we consider γ-radonifying operators, and summarize some interesting results due to Jan van Neerven, and others. We view γ-radonifying operators as a special case of the m-radonifying operators we considered in Chapter 3. That is, we will assume the sequence (α_n) to be a sequence of i.i.d. standard Gaussian random variables. Gaussian random variables are special in many ways, due to their symmetry and convergence properties. Unlike many other distributions, Gaussians are uniquely determined by their mean and covariance. The main part of this chapter is devoted to Gaussian random variables, and we include some classic (and useful) results by Fernique, and Sazonov.

4.1 Gaussian Random variables

We give a brief treatment of Gaussian random variables. For a more extensive treatment, the reader may consult [1], and the lecture notes from Jan van Neerven. (see [7]).

4.1.1 Real valued Gaussians

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable $X : \Omega \to \mathbb{R}$ is Gaussian if there exists $m \in \mathbb{R}$ and $q \geq 0$ such that for all Borel sets $B \subseteq \mathbb{R}$,

$$\mathbb{P}(X \in B) = \int_B f_{m,q}(x)dx = \int_B \frac{1}{\sqrt{2\pi q}} e^{-(x-m)^2/2q} dx.$$

In the case that $q = 0$, the interpretation of this formula is that $\mathbb{P}(X \in B) = \delta_m(B)$ which means that $X = m$ almost surely. The parameters m and q are called the *mean* and *standard deviation* of X since

$$\mathbb{E}(X) = \int_{\mathbb{R}} x f_{m,q} dx = m,$$

and

$$\mathbb{E}(X-m)^2 = \int_{\mathbb{R}} (x-m)^2 f_{m,q} dx = q.$$

If the random variable X is Gaussian with density $f_{m,q}$ we write $X \sim N(m,q)$. If $m = 0$ and $q = 1$ we call X *standard Gaussian.* From the density function we see that the density is symmetric about m: $f_{m,q}(m-x) = f_{m,q}(m+x)$. By means of complex integration the characteristic function φ_X of a Gaussian X can be shown to be

$$\varphi_X(t) = \mathbb{E}[e^{itX}] = \int_{\mathbb{R}} e^{itx} f_{m,q}(x) dx = e^{imt - \frac{1}{2}qt^2}.$$

The Gaussian property is preserved by linear transformations: If $X \sim N(m,q)$ and $Y = aX + b$ then $Y \sim N(a\mu + b, a^2 q)$. So any Gaussian can be obtained by making a linear transformation on the standard Gaussian. That is, if X is standard Gaussian and $Y = \sqrt{q}X + m$ then $Y \sim N(m,q)$.

The following lemma will be applied many times later on.

Lemma 4.1. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of Gaussian random variables with means m_n and covariances q_n, converging in probability to a random variable X. Then $X \sim N(m,q)$ with $m = \lim_{n\to\infty} m_n$ and $q = \lim_{n\to\infty} q_n$.*

Proof. . Convergence in probability implies the existence of a subsequence (X_{k_n}) that converges almost surely to X. Then, since $|e^{itX_{k_n}}| \leq 1$ for all n we have by the dominated convergence theorem

$$\lim_{n\to\infty} \varphi_{X_{k_n}}(t) = \lim_{n\to\infty} \mathbb{E}[e^{itX_{k_n}}] = \mathbb{E}[e^{itX}] = \varphi_X(t).$$

Since $\varphi_{X_{k_n}}(t) = e^{im_{k_n}t - \frac{1}{2}q_{k_n}t^2}$ converges for all t if and only if $\lim_{n\to\infty} q_{k_n} = q$ and $\lim_{n\to\infty} m_{k_n} = m$, it follows that φ_X is the characteristic function of the Gaussian random variable $X \sim N(m,q)$. □

4.1.2 $\mathbb{R}^n$-valued Gaussians

Definition 4.2. An $\mathbb{R}^n$-valued random variable $X = (X_1, \cdots, X_n)$ is Gaussian if the random variables $\langle X, \xi \rangle$ are Gaussian for all $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$.

Here $\langle X, \xi \rangle = \sum_{k=1}^{n} \xi_k X_k$ denotes the standard inner product in $\mathbb{R}^n$. If the random vector $X = (X_1, \cdots, X_n)$ is Gaussian, the random variables $X_1, \cdots, X_n$ are said to be jointly Gaussian. This definition implies that the coordinate variables X_k are Gaussian (take $\xi = e_k$, the k-th unit vector in $\mathbb{R}^n$). The converse is false: a series of Gaussian random variables $X_1, \cdots, X_n$ may fail to be jointly Gaussian, as is illustrated in the next example.

Example 4.3. Let $X \sim N(0,1)$, and define $Y = WX$ where W is another random variable such that $\mathbb{P}(W = 1) = \mathbb{P}(W = -1) = \frac{1}{2}$, and W is independent of X. Then
(1) X and Y are uncorrelated, (2) X and Y have the same Gaussian distribution, and (3) X and Y are not independent. Moreover, since $\mathbb{P}(X + Y = 0) = \frac{1}{2}$ we see that $X + Y$ is not Gaussian implying that (X, Y) is not Gaussian.

Definition 4.4. The mean of $X = (X_1, \cdots, X_n)$ is defined as $m = (m_1, \cdots, m_n)$ where the m_k are the means of the X_k. The covariance matrix $Q = (q_{jk})_{j,k=1}^{n}$ is defined by

$$q_{jk} = \mathbb{E}(X_j - m_j)(X_k - m_k) \quad (j, k = 1, \cdots, n).$$

We call X centred if $m = 0$. The covariance matrix Q is symmetric, i.e. $q_{j,k} = q_{k,j}$ for all $j, k = 1, \cdots, n$, and positive, i.e. $(Q\xi, \xi) \geq 0$ for all $\xi \in \mathbb{R}^n$. Positivity follows from

$$\langle Q\xi, \xi \rangle = \mathbb{E} \langle X - m, \xi \rangle^2 ,$$

(see below) and symmetry is evident from the definition. We call X standard Gaussian if it is a centred Gaussian and has covariance matrix $Q = I$ (the identity matrix). For every $\xi \in \mathbb{R}^n$ the scalar Gaussian $\langle X, \xi \rangle$ has mean

$$\mathbb{E} \langle X, \xi \rangle = \sum_{k=1} \xi_k \mathbb{E} X_k = \sum_{k} \xi_k m_k = \langle m, \xi \rangle ,$$

and variance

$$\mathbb{E} \langle X - m, \xi \rangle^2 = \mathbb{E} \left(\sum_{k=1}^{n} \xi_k (X_k - m_k) \right)^2 = \mathbb{E} \sum_{j,k=1}^{n} \xi_j \xi_k (X_j - m_j)(X_k - m_k)$$
$$= \sum_{j,k=1}^{n} q_{j,k} \xi_j \xi_k = \langle Q\xi, \xi \rangle .$$

Theorem 4.5. *The random variable $X = (X_1, \cdots, X_n)$ is Gaussian if and only if the joint characteristic function ϕ_X is of the form*

$$\phi_X(\xi) = \mathbb{E}\exp\left(i\langle X, \xi\rangle\right) = \exp\left(i\langle m, \xi\rangle - \frac{1}{2}\langle Q\xi, \xi\rangle\right) \qquad (\xi \in \mathbb{R}^n)$$

for some positive symmetric matrix Q, and some $m \in \mathbb{R}^n$.

Proof. Assume ϕ_X is of the indicated form, and let $\xi \in \mathbb{R}^n$. Then

$$\begin{aligned}\phi_{\langle X,\xi\rangle}(t) &= \mathbb{E}\exp\left(it\langle X, \xi\rangle\right)\\ &= \mathbb{E}\exp\left(i\langle X, t\xi\rangle\right)\\ &= \exp\left(it\langle m, \xi\rangle - \frac{1}{2}t^2\langle Q\xi, \xi\rangle\right).\end{aligned}$$

It follows that $\langle X, \xi\rangle \sim N(\langle m, \xi\rangle, \langle Q\xi, \xi\rangle)$ by the characterization of a Gaussian random variable via its characteristic function. Conversely, if X is Gaussian with covariance operator Q, and mean m, then

$$\phi_X(\xi) = \mathbb{E}\exp(i\langle X, \xi\rangle) = \phi_{\langle X,\xi\rangle}(1) = \exp\left(i\langle m, \xi\rangle - \frac{1}{2}\langle Q\xi, \xi\rangle\right).$$

□

The following proposition is useful, and can be found in many probability theory textbooks. See for example [2].

Proposition 4.6. *Let $X = (X_1, \cdots, X_n)$ be a n-dimensional random vector. Then $X_1, \cdots, X_n$ are independent random variables if and only if*

$$\phi_X(\xi) = \prod_{i=1}^{n}\phi_{X_i}(\xi_i)$$

for all $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$.

Theorem 4.7. *Let $X = (X_1, \cdots, X_n)$ be a Gaussian vector, then $X_1, \cdots, X_n$ are independent if and only if they are uncorrelated.*

Proof. We only prove the if part. Assume that $X_1, \cdots, X_n$ are uncorrelated. Then the covariance matrix Q is diagonal. Therefore, for $(\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$,

$$\begin{aligned}\phi_X(\xi) &= \exp\left(i\langle \xi, m\rangle - \frac{1}{2}\langle Q\xi, \xi\rangle\right)\\ &= \prod_{k=1}^{n} \exp\left(i\xi_k m_k - \frac{1}{2}q_{kk}\xi_k^2\right)\\ &= \prod_{k=1}^{n} \phi_{X_k}(\xi_k).\end{aligned}$$

It follows from proposition **4.6** that $X_1, \cdots, X_n$ are independent. □

The condition that X is Gaussian can not be omitted from the last theorem. Example **4.3** shows that the scalar Gaussians $X_1, \cdots, X_n$ may not be independent, even when they are uncorrelated.

We now give some geometric interpretation of the covariance matrix of a Gaussian $X = (X_1, \cdots, X_n)$ with mean m and covariance matrix Q. From standard linear algebra we know that every symmetric matrix can be orthogonally diagonalized. Choose an orthonormal basis of eigenvectors $\{e_1, \cdots, e_n\}$ of Q. Since Q is positive we have $\lambda_j \geq 0$ for every eigenvalue λ_j corresponding to e_j . With this basis X can be decomposed as

$$X = m + (X - m) = m + \sum_{j=1}^{n} \langle X - m, e_j\rangle e_j$$

Each random variable $\langle X - m, e_j\rangle$ is a centred Gaussian with variance

$$\mathbb{E}\langle X - m, e_j\rangle^2 = \langle Qe_j, e_j\rangle = \lambda_j \langle e_j, e_j\rangle = \lambda_j.$$

We will show that the components $\langle X - m, e_j\rangle e_j$ which take their values in the one-dimensional subspace e_j are independent. This implies that every Gaussian X can be decomposed as a sum of independent one-dimensional Gaussians.

Theorem 4.8. *Let $X = (X_1, \cdots, X_n)$ be a (centrered) Gaussian random variable. Then X can be decomposed as a sum of (centrered) one-dimensional Gaussians, i.e.*

$$X = \sum_{j=1}^{n} \langle X, e_j\rangle e_j,$$

where the e_j are orthonormal eigenvectors of the covariance operator Q.

Proof. Without loss of generality we may assume that X is centred (i.e. $m = 0$). To show independence of the $\langle X, e_j \rangle$ we show they are uncorrelated, which can be done with the following polarization trick. Observe that

$$\begin{aligned}\langle Q(e_j + e_k), (e_j + e_k) \rangle &= \mathbb{E} \langle X, e_j + e_k \rangle^2 \\ &= \mathbb{E} \langle X, e_j \rangle^2 + 2\mathbb{E} \langle X, e_j \rangle \langle X, e_k \rangle + \mathbb{E} \langle X, e_k \rangle^2 . \\ &= \langle Qe_j, e_j \rangle + 2\mathbb{E} \langle X, e_j \rangle \langle X, e_k \rangle + \langle Qe_k, e_k \rangle .\end{aligned}$$

At the same time

$$\begin{aligned}\langle Q(e_j + e_k), (e_j + e_k) \rangle &= \langle Qe_j, e_j \rangle + \langle Qe_j, e_k \rangle + \langle Qe_k, e_j \rangle + \langle Qe_k, e_k \rangle \\ &= \langle Qe_j, e_j \rangle + 2 \langle Qe_j, e_k \rangle + \langle Qe_k, e_k \rangle ,\end{aligned}$$

where symmetry of Q was used in the last step. This implies that

$$\mathbb{E} \langle X, e_j \rangle \langle X, e_k \rangle = \langle Qe_j, e_k \rangle = \lambda_j \langle e_j, e_k \rangle = 0.$$

It now follows from Theorem **4.7** that the $\langle X, e_j \rangle$ are independent. □

Observe that only eigenvectors e_j with strictly positive λ_j contribute in decomposition (2), since if $\lambda_j = 0$ then $\mathbb{E} \langle X - m, e_j \rangle^2 = \langle Qe_j, e_j \rangle = 0$, which means that $\langle X - m, e_j \rangle = 0$ almost surely. Let $J = \{1 \leq j \leq n : \lambda_j > 0\}$, and for $j \in J$ let $x_j = \sqrt{\lambda_j} e_j$, and $g_j = \frac{1}{\sqrt{\lambda_j}} \langle X - m, e_j \rangle$ then each g_j is a $\mathbb{R}$-valued standard Gaussian, and

$$X = m + \sum_{j \in J} g_j x_j .$$

We conclude that every $\mathbb{R}^n$-valued Gaussian has a decomposition of the above form where $\{x_1, \cdots, x_n\}$ is the appropriate orthogonal basis. As a converse we have the following.

Theorem 4.9. *Let $X_1, \cdots, X_N$ be a series of independent $\mathbb{R}^n$-valued Gaussian random variables, then*

$$X = \sum_{k=1}^{N} X_k$$

is Gaussian random variable.

Proof. Let $m_1, \cdots, m_N$, and $Q_1, \cdots, Q_N$ be the means and covariances of the independent Gaussians $X_1, \cdots, X_N$. Then by independence

$$\begin{aligned}\phi_X(\xi) = \mathbb{E}\left(\exp(i\langle X, \xi\rangle)\right) &= \mathbb{E}\left(\prod_{k=1}^{N} \exp(i\langle X_k, \xi\rangle)\right) \\ &= \prod_{k=1}^{N} \mathbb{E}(\exp(i\langle X_k, \xi\rangle)) \\ &= \prod_{k=1}^{N} \exp(i\langle m_k, \xi\rangle - \frac{1}{2}\langle Q_k\xi, \xi\rangle) \\ &= \exp(i\langle m, \xi\rangle - \frac{1}{2}\langle Q\xi, \xi\rangle),\end{aligned}$$

where $m = \sum_{k=1}^{N} m_k$ and $Q = \sum_{k=1}^{N} Q_k$. Since Q_k is symmetric and positive for all k, Q is also symmetric and positive. It follows now by Theorem **4.5** that X is Gaussian. □

4.1.3 E-valued Gaussians

Let E be real Banach space, and denote by E^* its dual. In this section we treat E-valued Gaussians.

Definition 4.10. A random variable $X : \Omega \to E$ is Gaussian if for every $x^* \in E^*$ the random variable $\langle X, x^*\rangle : \Omega \to \mathbb{R}$ defined by $\langle X, x^*\rangle(\omega) = \langle X(\omega), x^*\rangle$ is Gaussian.

Similar to the case of scalar Gaussians, we have the following lemma for E-valued Gaussians.

Lemma 4.11. *Let (X_n) be a sequence of E-valued Gaussians, and let X be an E-valued random variable. If $X_n \to X$ in probability, then X is Gaussian.*

Proof. Convergence $X_n \to X$ in probability implies $\langle X_n, x^*\rangle \to \langle X, x^*\rangle$ in probability for all $x^* \in E^*$. Since for each $x^* \in E^*$, $\langle X_n, x^*\rangle$ is a scalar Gaussian, it follows by lemma **4.1** that $\langle X, x^*\rangle$ is Gaussian. Thus X is an E-valued Gaussian. □

An E-valued random variable X is said to be weakly L^2 if $\mathbb{E}\langle X, x^*\rangle^2 < \infty$ for all $x^* \in E^*$. The following lemma is useful for defining the mean and covariance operator for E-valued Gaussian random variables. We omit the proof which can be found in [7]

Lemma 4.12. *Let X be an E-valued random variable that is weakly L^2. Then for all $Y \in L^2(\Omega, \mathbb{P})$ there exists a unique elment $x_Y \in E$ such that for all $x^* \in E^*$ we have*

$$\langle x_Y, x^* \rangle = \mathbb{E}\ Y \langle X, x^* \rangle .$$

Since any $\mathbb{R}$-valued Gaussian has moments of all orders, any E-valued Gaussian is weakly L^2 and we may apply this lemma to X. We proceed with defining the mean and covariance of X.

Definition 4.13. Let X be an E-valued Gaussian random variable. The *mean* of X is the unique element $m \in E$ satisfying

$$\langle m, x^* \rangle = \mathbb{E} \langle X, x^* \rangle , \qquad x^* \in E^*.$$

The *covariance operator* of X is the operator $Q : E^* \to E$ defined by

$$\begin{aligned} \langle Qx^*, y^* \rangle &= \mathbb{E}[\langle X, x^* \rangle \langle X, y^* \rangle] - \mathbb{E}[\langle X, x^* \rangle] \cdot \mathbb{E}[\langle X, y^* \rangle] \\ &= \mathbb{E} \langle X - m, x^* \rangle \langle X - m, y^* \rangle , \qquad\qquad x^*, y^* \in E^*. \end{aligned}$$

To find the mean m, take $Y \equiv 1$ and apply the above lemma to find $x_Y = m$. As before we call X *centred* if $m = 0$.
For the covariance operator, let $x^* \in E^*$, take $Y = \langle X - m, x^* \rangle$ and apply the above lemma to the centred Gaussian $X - m$ to find a unique $x_Y \in E^*$ such that $\langle x_Y, y^* \rangle = \mathbb{E} \langle X - m, x^* \rangle \langle X - m, y^* \rangle$ for all $y^* \in E^*$. Then define $Qx^* = x_Y$.

Notice that this definition is consistent with the one of covariance matrix. Indeed, when $E = \mathbb{R}^n$, then Q is just the covariance matrix: let $\{e_1, \cdots, e_n\}$, be the standard basis of $\mathbb{R}^n$, $X = (X_1, \cdots, X_n)$, and $m = (m_1, \cdots, m_n)$ then

$$\langle Qe_i, e_j \rangle = \mathbb{E} \langle X - m, e_i \rangle \langle X - m, e_j \rangle = \mathbb{E}(X_i - m_i)(X_j - m_j) = q_{ij}.$$

The covariance operator $Q : E^* \to E$ has similar properties to the covariance matrix; Q is *positive* in the sense that

$$\langle Qx^*, x^* \rangle \geq 0$$

for all $x^* \in E^*$, and *symmetric* in the sense that

$$\langle Qx^*, y^* \rangle = \langle Qy^*, x^* \rangle$$

for all $x^*, y^* \in E^*$.

Theorem 4.14. *An E-valued random variable X is Gaussian if and only if its characteristic function is of the form*

$$\phi_X(x^*) = \exp\left(i\langle m, x^*\rangle - \frac{1}{2}\langle Qx^*, x^*\rangle\right), \qquad x^* \in E^*,$$

for some positive symmetric $Q : E^* \to E$, and $m \in E$.

Proof. If ϕ_X is of the indicated form, then replacing x^* by tx^* we find that the characteristic function of $\langle X, x^*\rangle$ equals

$$\phi_X(tx^*) = \mathbb{E}\exp(it\langle X, x^*\rangle) = \exp(i\langle m, x^*\rangle - \frac{1}{2}t^2\langle Qx^*, x^*\rangle),$$

which shows that $\langle X, x^*\rangle$ is Gaussian with mean $\langle m, x^*\rangle$, and covariance $\langle Qx^*, x^*\rangle$. Conversely, if X is Gaussian then for all $x^* \in E^*$ we have $\mathbb{E}\langle X - m, x^*\rangle^2 = \langle Qx^*, x^*\rangle$ with m, and Q as in definition **4.13.** □

Thus every E-valued Gaussian is uniquely determined by its mean and its covariance operator, similarly as finite dimensional Gaussians. In the next theorem we show that any covariance operator is a bounded linear operator. One can show that a positive linear operator $Q : E^* \to E$ is bounded, but it turns out that symmetry of Q is enough to prove both linearity and boundedness.

Theorem 4.15. *If $Q : E^* \to E$ is symmetric, then Q is a bounded linear operator.*

Proof. Linearity of Q follows from

$$\begin{aligned}\langle Q(\alpha x^* + \beta y^*), z^*\rangle &= \langle Qz^*, \alpha x^* + \beta y^*\rangle \\ &= \langle Qz^*, \alpha x^*\rangle + \langle Qz^*, \beta x^*\rangle \\ &= \langle \alpha Qx^*, z^*\rangle + \langle \beta Qy^*, z^*\rangle \\ &= \langle \alpha Qx^* + \beta Qy^*, z^*\rangle,\end{aligned}$$

which holds for all $x^*, y^*, z^* \in E^*$, and $\alpha, \beta \in \mathbb{R}$, which shows that Q is linear. For the boundedness part, define a set of linear operators $(Q_\alpha)_{\alpha\in I}$ where $I = B_E^*$ is the unit-ball in E^*, and $Q_\alpha : E^* \to \mathbb{R}$ is given by $Q_\alpha x^* = \langle Q\alpha, x^*\rangle$. The linearity of each Q_α is clear from the linearity of Q. Notice that each Q_α is bounded since

$$\|Q_\alpha\| = \sup_{x^* B_E^*} |\langle Q\alpha, x^*\rangle| = \|Q\alpha\|.$$

The set $(Q_\alpha)_\alpha$ is pointwise bounded since for $x^* \in E^*$

$$\sup_{\alpha\in I} |Q_\alpha x^*| = \sup_{\alpha\in I} |\langle Q\alpha, x^*\rangle| = \sup_{\alpha\in I} |\langle Qx^*, \alpha\rangle| = \|Qx^*\|.$$

It now follows from the uniform boundedness principle that

$$\|Q\| = \sup_{\alpha\in I} \|Q\alpha\| = \sup_{\alpha\in I} \|Q_\alpha\| < \infty.$$

□

It is true that every Gaussian with a given mean m is uniquely determined by its covariance operator Q, which is positive and symmetric. It is however not true that every positive symmetric operator is the covariance operator of an E-valued Gaussian: in the following proposition we will show that $Q = I$ on H is not a covariance operator for an H-valued Gaussian, in the case that H is an infinite dimensional Hilbert space. In other words, unlike the finite dimensional case $H = \mathbb{R}^n$, there exists no 'standard Gaussian' in H.

Proposition 4.16. *There exists no Gaussian random variable on an infinite dimensional real Hilbert space, having $Q = I$ as covariance operator.*

Proof. Suppose X is a centred H-valued Gaussian with $Q = I$, and let (e_n) be an orthonormal basis of H. Then $\mathbb{E}\langle X, e_i\rangle \langle X, e_j\rangle = \langle e_i, e_j\rangle$ which shows that the $\langle X, e_i\rangle$ are uncorrelated standard Gaussians. Since the $\langle X, e_i\rangle$ are jointly Gaussian, they are independent by Theorem **4.7**. But since $\sum |\langle X, e_i\rangle| \leq \|X\|^2$ this implies $\langle X(\omega), e_i\rangle \to 0$ almost surely, which is a contradiction since the $\langle X, e_i\rangle$ were i.i.d. standard Gaussians. □

There exists no standard Gaussian on H, but as previously mentioned there exists a cylindrical (finitely additive) standard Gaussian measure γ_H with $Q = I$ as covariance operator. We come back to this later.

In the case that E is separable, the mean and covariance may also have been defined with the help of Fernique's theorem. We state the theorem, and then show how this theorem helps to completely characterize Gaussian covariance operators for a separable real Hilbert space H. Characterizing covariance operators for arbitrary separable Banach spaces E is still an open problem.

Theorem 4.17. *(Fernique). Let X be a centred Gaussian on a separable Banach space E, then there exists $\alpha > 0$ such that*

$$\mathbb{E}\exp(\alpha \|X\|^2) < \infty.$$

Corollary 4.18. *If X is a Gaussian on a separable Banach space, then there exists $\beta > 0$ such that*

$$\mathbb{E}\exp(\beta \|X\|^2) < \infty,$$

in particular $\mathbb{E}\|X\|^p = \int_E \|x\|^p \, d\mu(x) < \infty$ for all $p \in [1, \infty)$.

Proof. (Can be found in [1]). Let X, and Y be be independent Gaussians with the same distribution. Then the random variable $X - Y$ is a centred Gaussian, and by Fernique there exists $\alpha > 0$ such that $\mathbb{E}\exp(\alpha \|X - Y\|^2) < \infty$. Next, by the inequality $xy \leq \frac{1}{2}x^2 + \frac{1}{2}y^2$ we have for any $\gamma > 0$ the inequality $-2xy = -2(\gamma x)(\frac{1}{\gamma}y) \geq -\gamma^2 x^2 - \frac{1}{\gamma^2}y^2$. It follows that

$$\begin{aligned}
\mathbb{E}\exp(\alpha \|X - Y\|^2) &\geq \mathbb{E}\exp(\alpha(\|X\| - \|Y\|)^2) \\
&= \mathbb{E}\exp(\alpha(\|X\|^2 + \|Y\|^2 - 2\|X\|\|Y\|)) \\
&\geq \mathbb{E}\exp(\alpha(1 - \gamma^2)\|X\|^2 + \alpha(1 - \frac{1}{\gamma^2})\|Y\|^2) \\
&= \mathbb{E}\exp(\alpha(1 - \gamma^2)\|X\|^2) \cdot \mathbb{E}\exp(\alpha(1 - \frac{1}{\gamma^2})\|Y\|^2)
\end{aligned}$$

which shows that $\mathbb{E}\exp(\alpha(1 - \gamma^2)\|X\|^2) < \infty$. □

The mean and covariance may now also be defined with Bochner integrals

$$m_X = \mathbb{E}X, \qquad Qx^* = \mathbb{E}\langle X - m_X, x^* \rangle (X - m_X).$$

The following result is by Sazonov and can be found in [4]. The theorem is a complete characterization of covariance operators for Gaussian random variables on a separable real Hilbertspace H. By identifying H^* with H via the Riesz representation theorem, every positive symmetric operator $Q \in L(H^*, H)$ can be identified with positive self-adjoint operator $Q \in L(H)$.

In the next theorem, an operator Q is said to be of *trace class* when the series $\sum_n \langle Qh_n, h_n \rangle$ converges for every orthonormal basis (h_n).

Theorem 4.19. *(Sazonov) A bounded linear operator Q on a separable real Hilbertspace H is a covariance operator of an H-valued Gaussian random variable X if and only if Q is positive, and of trace class. In this situation we have*

$$\operatorname{trace} Q = \mathbb{E}\left\|X\right\|^2.$$

Proof. Let (e_n) be an orthonormal basis of H, and assume Q is positive and of trace class. Since Q is positive, it has a positive square root $Q^{1/2}$ and we may define $e_n := Q^{1/2}h_n$. Let (g_n) be an independent sequence of $\mathbb{R}$-valued standard Gaussians, i.e. $\mathbb{E}g_n^2 = 1$, and $\mathbb{E}g_i g_j = \mathbb{E}g_i\mathbb{E}g_j = 0$ if $i \neq j$. Using independence of the g_n we have

$$\begin{aligned}\mathbb{E}\left\|\sum_{k=N}^{M} g_k e_k\right\|^2 &= \mathbb{E}\left\langle \sum_{k=N}^{M} g_k e_k, \sum_{k=N}^{M} g_k e_k\right\rangle = \sum_{i,j=N}^{M} \langle e_i, e_j\rangle \mathbb{E}g_i g_j \\ &= \sum_{k=N}^{M} \|e_k\|^2 = \sum_{k=N}^{M} \langle Qh_k, h_k\rangle .\end{aligned}$$

Notice that in the last step we used that $Q^{1/2}$ is self-adjoint because of its positiveness. Since Q is of trace class, it follows that the series of partial sums $S_n := \sum_{k=1}^{n} g_k e_k$ is Cauchy, which implies that (S_n) converges in $L^2(\Omega, \mathbb{P}, H)$ to some random variable X. By lemma **4.11** it follows that X is a centred H-valued Gaussian with Q as covariance operator since

$$\begin{aligned}\mathbb{E}\langle X, x\rangle \langle X, y\rangle &= \mathbb{E}\sum_{n,m\geq 1} g_n g_m \langle e_n, x\rangle \langle e_m, y\rangle \\ &= \sum_{k\geq 1} \langle e_k, x\rangle \langle e_k, y\rangle \\ &= \sum_{k\geq 1} \left\langle Q^{1/2}h_k, x\right\rangle \left\langle Q^{1/2}h_k, y\right\rangle \\ &= \sum_{k\geq 1} \left\langle h_k, Q^{1/2}x\right\rangle \left\langle h_k, Q^{1/2}y\right\rangle \\ &= \left\langle Q^{1/2}x, Q^{1/2}y\right\rangle . \\ &= \langle Qx, y\rangle .\end{aligned}$$

For the converse, assume that X is an H-valued Gaussian with covariance operator Q. As a covariance operator, Q is positive and it remains to show that it is of trace class. Notice that by definition of the covariance operator Q we have

$$\sum_n \langle Qh_n, h_n\rangle = \sum_n \mathbb{E}\langle X, h_n\rangle^2 = \mathbb{E}\sum_n \langle X, h_n\rangle^2 = \mathbb{E}\left\|X\right\|^2,$$

where Parseval's identity was used in the last equality. Linearity of the expectation operator was used in the second equality, which is justified since the right-hand side is finite by Ferniques theorem, and it follows that Q is of trace class. □

It is also a direct consequence of Sazonov's theorem that $Q = I$ can not be a covariance operator of a Gaussian random variable on H, as it is clearly not of trace-class. Notice that $Q^{1/2}$ is m-radonifying or order $p = 2$ as in definition **3.6**, in the sense that $Y_{Q^{1/2}} = \sum_{n=1}^{\infty} g_n Q^{1/2} h_n$ converges in $L^2(\Omega, H)$ and (g_n) is an orthonormal sequence in $L^2(\Omega)$. Here $m = \gamma_H$ is the standard Gaussian cylindrical measure on H which means that $Q^{1/2}$ is γ-radonifying. We gather some observations in the following proposition:

Proposition 4.20. *For a bounded linear operator $T : H \to H'$ the following are equivalent:*
(a) T is γ-radonifying,
(b) T is Hilbert-schmidt,
(c) TT^* is a Gaussian covariance operator.

Proof. $(a) \iff (b)$. Similar as in Theorem **3.5**, this follows from $\mathbb{E}\left\|\sum_{n=M}^{N} g_n T h_n\right\|^2 = \sum_{n=M}^{N} \|Th_n\|^2$.
$(b) \iff (c)$. This equivalence follows from $\sum_{n=M}^{N} \|Th_n\|^2 = \sum_{n=M}^{N} \langle h_n, TT^* h_n\rangle$ and Sazonov's theorem.

□

A similar result as proposition **4.20** can be achieved for bounded operators from H into E. It turns out that all γ-radonifying operators from H into E can be classified in terms of Gaussian covariance operators. To this extent we introduce the concept of Reproducing Kernel Hilbert Space (RKHS) in the next section.

4.2 Reproducing Kernel Hilbert space

Let E be a real Banach space and $Q \in L(E^*, E)$ be an arbitrary positive and symmetric operator. In this section we show how to construct the reproducing kernel Hilbert space H_Q, corresponding to the operator Q. The usefulness of this concept will become apparant, as Q enjoys a useful factorisation through this Hilbert space, that is there exist $L_1 \in L(E^*, H_Q)$ and $L_2 \in L(H_Q, E)$ such that $Q = L_2 L_1$. We start by defining an inner product on the range of Q.

Theorem 4.21. *Let $Q \in L(E, E^*)$ be positive and symmetric. The formula*

$$\langle Qx^*, Qy^* \rangle_{H_Q} := \langle Qx^*, y^* \rangle$$

defines an inner product $\langle \cdot, \cdot \rangle_{H_Q}$ on the range of Q.

Proof. To see that $\langle \cdot, \cdot \rangle_{H_Q}$ is well-defined, note that if $Qx_1^* = Qx_2^*$ and $Qy_1^* = Qy_2^*$ then

$$\langle Qx_2^*, y_2^* \rangle = \langle Qx_1^*, y_2^* \rangle = \langle Qy_2^*, x_1^* \rangle = \langle Qy_1^*, x_1^* \rangle = \langle Qx_1^*, y_1^* \rangle .$$

Symmetry of $\langle \cdot, \cdot \rangle_{H_Q}$ follows from the symmetry of Q,

$$\langle Qx^*, Qy^* \rangle_{H_Q} = \langle Qx^*, y^* \rangle = \langle Qy^*, x^* \rangle = \langle Qy^*, Qx^* \rangle_{H_Q} .$$

Bilinearity is obvious from the linearity of Q, and the functionals $y^* \in E^*$. Positive definiteness follows from positivity of Q, since $\langle Qx^*, Qx^* \rangle_{H_Q} = \langle Qx^*, x^* \rangle \geq 0$. If $\langle Qx^*, Qx^* \rangle_{H_Q} = 0$, then $\langle Qx^*, x^* \rangle = 0$, and then it follows by Cauchy-Schwarz applied to the positive semi-definite bilinear map $\langle \cdot, \cdot \rangle_{H_Q}$, for all $y^* \in E^*$, that

$$|\langle Qx^*, y^* \rangle| \leq \langle Qx^*, x^* \rangle^{\frac{1}{2}} \langle Qy^*, y^* \rangle^{\frac{1}{2}} = 0,$$

which shows that $Qx^* = 0$. Conversely, $\langle Qx^*, y^* \rangle = 0$ when $Qx^* = 0$. □

Thus, the range of Q is a pre-Hilbert space and we denote by H_Q the Hilbert space obtained by its completion with respect to $\langle \cdot, \cdot \rangle_{H_Q}$. The Hilbert space $H_Q$is called the Reproducing Kernel Hilbert Space of Q. In the next theorem we collect some properties of Q with respect to its Reproducing Kernel Hilbert Space.

Theorem 4.22. *Let $Q \in L(E^*, E)$ be positive and symmetric, and let H_Q be the corresponding reproducing kernel Hilbert space.*

(a). *The inclusion mapping $i : Q(E^*) \hookrightarrow E$ is continuous with respect to the inner product $\langle \cdot, \cdot \rangle_{H_Q}$, and thus extends to a bounded linear operator $i_Q : H_Q \to E$.*

(b). *The operator Q enjoys the decomposition $Q = i_Q i_Q^*$.*

(c). *The range of i_Q^* is dense in H_Q.*

(d). *The operator $i_Q : H_Q \to E$ is injective.*

(e). *If E is separable then H_Q is also separable.*

Proof. **(a)**. We have

$$\|Qu^*\|^2_{H_Q} = \langle Qu^*, u^*\rangle = |\langle Qu^*, u^*\rangle| \le \|Qu^*\| \, \|u^*\| \le \|Q\|_{L(E^*,E)} \, \|u^*\|^2 .$$

Thus Q is bounded from E^* into H_Q with $\|Q\|_{L(E^*,H_Q)} \le \|Q\|^{1/2}_{L(E^*,E)}$. At the same time, by Cauchy-Schwarz

$$\begin{aligned} |\,\langle Qu^*, v^*\rangle\,| = |\,\langle Qu^*, Qv^*\rangle_{H_Q}\,| &\le \|Qu^*\|_{H_Q} \, \|Qv^*\|_{H_Q} \\ &\le \|Qu^*\|_{H_Q} \, \|Q\|_{L(E^*,H_Q)} \, \|v^*\| , \end{aligned}$$

which shows that

$$\|Qu^*\| = \sup_{\|v^*\|\le 1} |\,\langle Qu^*, v^*\rangle\,| \le \|Qu^*\|_{H_Q} \, \|Q\|_{L(E^*,H_Q)} .$$

Thus the inclusion mapping $i : Q(E^*) \hookrightarrow E$ is continuous with respect to $\langle \cdot, \cdot \rangle_{H_Q}$ and extends to a bounded linear map $i_Q : H_Q \to E$.

(b). For $u^* \in E^*$ denote by h_{u^*} the element of H_Q such that $i_Q(h_{u^*}) = Qu^*$. Then for all $v^* \in E^*$ we have

$$\langle h_{u^*}, h_{v^*}\rangle_{H_Q} = \langle Qu^*, v^*\rangle = \langle i_Q(h_{u^*}), v^*\rangle = \left\langle h_{u^*}, i_Q^* v^*\right\rangle_{H_Q} .$$

Since the elements h_{u^*} form a dense subspace of H_Q this shows that $h_{v^*} = i_Q^* v^*$. Consequently,

$$Qv^* = i_Q h_{v^*} = i_Q i_Q^* v^*$$

for all $v^* \in E^*$.

(c).It follows from $h_{v^*} = i_Q^* v^*$ for all $v^* \in E^*$, that the range of i_Q^* is dense in H_Q.

(d).Assume $i_Q h = 0$ for some $h \in H_Q$. For all $v^* \in E^*$ we have

$$\left\langle h, i_Q^* v^*\right\rangle_{H_Q} = \langle i_Q h, v^*\rangle = 0,$$

which shows $h = 0$ because the range of i_Q^* is dense in H_Q by **(c)**. Thus i_Q is injective.

(e). Let E be separable. Since any separable Banach space can only have separable subspaces, it follows that $i_Q(H_Q)$ must be separable. Take a countable dense subset $F \subset$

$i_Q(H_Q)$. Since i_Q is injective by **(d)**, we have that $i_Q^{-1}(F)$ is countable in H_Q .
Let $y \in H_Q$ and let $y_n \in i_Q^{-1}(F)$ be a sequence such that $\|i_Q(y_n - y)\|_E \to 0$. Thus for all $w^* \in E^*$ we have

$$\langle y_n - y, i_Q^* w^* \rangle = \langle i_Q(y_n - y), w^* \rangle \to 0.$$

Since i_Q^* has dense range in H_Q it follows that $\langle y_n - y, h \rangle_{H_Q} \to 0$ for all $h \in H_Q$. Thus $y_n \to y$ weakly in H_Q. By Mazur's lemma there exists a sequence $\{v_n\}$ in $\text{Conv}\{y_n\}$ such that $\|y - v_n\|_{H_Q} \to 0$. This, with the fact that $i_Q^{-1}(F)$ is countable, implies that $\text{Conv}(i_Q^{-1}(F))$ is dense in H_Q, and shows that H_Q is separable. □

Let X be a Gaussian random variable in E with covariance operator Q. From this point on, as in the previous sections, we will assume that E is separable, and thus H_Q is separable by Theorem **4.22**. Without loss of generality we may assume that the Gaussian X is centred. Let $T_X : E^* \to L^2(\Omega)$ be defined as $T_X x^* := \langle X, x^* \rangle$. The next proposition shows that T_X has a factorisation through H_Q.

Proposition 4.23. *There exists a unique linear isometry $S_X : H_Q \to L^2(\Omega)$ such that the following diagram commutes:*

$$\begin{array}{ccc} E^* & \xrightarrow{T_X} & L^2(\Omega) \\ \downarrow{\scriptstyle i_Q^*} & & \downarrow{\scriptstyle I} \\ H_Q & \xrightarrow{S_X} & L^2(\Omega) \end{array}$$

Proof. For $x^* \in E^*$ define $S_X(i_Q^* x^*) := T_X x^*$. Note that

$$\|T_X x^*\|^2 = \mathbb{E} \langle X, x^* \rangle^2 = \langle Q x^*, x^* \rangle = \left\| i_Q^* x^* \right\|_{H_Q}^2.$$

Since i_Q^* has dense range in H_Q, it follows that S_X can be extended uniquely to an isometry from H_Q into $L^2(\Omega)$. By definition we have $S_X \circ i_Q^* = T_X$. □

Lemma 4.24. *If (h_n) is an orthonormal basis of H_Q, then $g_n := S_X h_n$ defines a sequence $(g_n)_{n \geq 1}$ of independent standard Gaussian random variables.*

Proof. Let $h \in H_Q$. We start by showing that $S_X h$ is a centred Gaussian. First assume that h is of the form $h = i_Q^* x^*$ for $x^* \in E^*$. Since

$$S_X h = S_X i_Q^* x^* = T_X x^* = \langle X, x^* \rangle,$$

it follows that $S_X h$ is a centred Gaussian with variance

$$\|h\|_{H_Q} = \left\langle i_Q^* x^*, i_Q^* x^* \right\rangle_{H_Q} = \langle Qx^*, x^* \rangle = \mathbb{E} \langle X, x^* \rangle^2 .$$

The general case now follows since i_Q^* has dense range in H_Q. Indeed, take a sequence $h_k = i_Q^* x_k^*$, $x_k^* \in E^*$, such that $h_k \to h$ in H_Q. Since $S_X h_k \to S_X h$ in $L^2(\Omega)$, it follows from lemma **4.11** that $S_X h$ is a centred Gaussian. In particalar $S_X h$ is standard Gaussian when $\|h\|_{H_Q} = 1$. Next, observe that the R^N-valued random variable $(g_1, \cdots, g_N)$ is Gaussian since any linear combination

$$\sum_{i=1}^{N} \xi_i g_i = \sum_{i=1}^{N} \xi_i S_X h_i = S_X \sum_{i=1}^{N} \xi_i h_i$$

is Gaussian for all $\xi = (\xi_1, \cdots, \xi_N) \in \mathbb{R}^N$. Since S_X is an isometry it preserves the inner product between H_Q and $L^2(\Omega)$. Hence it follows that

$$\mathbb{E}(g_m g_n) = \mathbb{E}(S_X h_m \cdot S_X h_n) = \langle h_m, h_n \rangle_{H_Q} = \delta_{mn}.$$

Thus the standard Gaussians $g_1, \cdots, g_N$ are uncorrelated, and it now follows from Theorem **4.7** that $g_1, \cdots, g_N$ are independent for all $N \geq 1$. □

4.3 Sum decomposition of E-valued Gaussians

Similarly as for finite dimensional Gaussians, it turns out that every (centred) E-valued has a sum-decomposition in (centred) one-dimensional Gaussians. Recall that we assume that E is separable. The following theorem we refer to as the Karhunen-Loève theorem.

Theorem 4.25. *Let X be an E-valued Gaussian with covariance operator Q. Let (h_n) be an orthonormal basis of H_Q. Put $g_n := S_X h_n$, then*

$$X = \sum_n g_n i_Q h_n,$$

with a.s. convergence in E, and in $L^p(\Omega, E)$ for all $p \in [1, \infty)$.

Proof. Let $X = \sum_n g_n i_Q h_n$, and apply Itô-Nisio to the symmetric random variables $X_n :=$

$g_n i_Q h_n$. Let $S_n := \sum_{i=1}^n X_i$, and let Q_n be the covariance operator of S_n. Then

$$\begin{aligned}
\lim_{n\to\infty} \langle Q_n x^*, y^* \rangle &= \lim_{n\to\infty} \mathbb{E} \langle S_n, x^* \rangle \langle S_n, y^* \rangle \\
&= \lim_{n\to\infty} \mathbb{E} \sum_{i,j=1}^n g_i g_j \langle i_Q h_i, x^* \rangle \langle i_Q h_j, y^* \rangle \\
&= \lim_{n\to\infty} \sum_{j=1}^n \langle h_j, i_Q^* x^* \rangle_H \langle h_j, i_Q^* y^* \rangle_H . \\
&= \langle i_Q^* x^*, i_Q^* y^* \rangle_H \\
&= \langle Q x^*, y^* \rangle .
\end{aligned}$$

Denote by $\hat{\mu}_n$ the characteristic function of S_n. Then it follows that for all $x^* \in E^*$ that $\hat{\mu}_n(x^*) = \mathbb{E} e^{i\langle S_n, x^* \rangle} = e^{-\frac{1}{2}\langle Q_n x^*, x^* \rangle} \to e^{-\frac{1}{2}\langle Q x^*, x^* \rangle}$. Now it follows from the Itô-Nisio theorem that S_n converges almost surely to a Gaussian random variable Y with distribution μ. By Fernique's theorem, we have $\mathbb{E} \|Y\|^p < \infty$ for all $p \in [1, \infty)$, and agian by the Itô-Nisio theorem, the convergence is now also in $L^p(\Omega; E)$. It remains to show that $Y = X$ almost surely: Since $i_Q^* x^* \in H_Q$ for all $x^* \in E^*$ and (h_n) is an orthonormal basis for H_Q, we have that $S_X(i_Q x^*)$ is contained in the closed linear span of $(S_X h_n)_n$ in $L^2(\Omega)$. It follows that for all $x^* \in E^*$ we have

$$\begin{aligned}
\langle Y, x^* \rangle &= \sum_n g_n \langle i_Q h_n, x^* \rangle_{H_Q} = \sum_n S_X h_n \langle h_n, i_Q^* x^* \rangle_{H_Q} \\
&= \sum_n S_X h_n \langle S_X h_n, S_X i_Q^* x^* \rangle_{L^2(\Omega)} = S_X i_Q^* x^* \\
&= T_X x^* = \langle X, x^* \rangle ,
\end{aligned}$$

where for the two equalities in the middle we used that S_X is an isometry and that $i_Q x^*$ is contained in the closed linear span of $\{(S_X h_n)_n\}$. □

4.4 γ-radonifying operators

In Section **3.1** we defined a cylindrical random variable $\Lambda : H \to L^2(\Omega)$ by $\sum_n \alpha_n \langle h_n, h \rangle$ where $\alpha_n \in L^2(\Omega)$ are such that (α_n) is an orthonormal sequence. We have shown that the space $\mathcal{R}_m^2(H, E)$ is a Banach space, where m is the distribution of Λ. We consider in this section the special case where (α_n) is chosen to be an i.i.d. sequence of $N(0,1)$ Gaussians.

Definition 4.26. Let (g_n) be an i.i.d. sequence of $N(0,1)$-Gaussians. Let γ_H be the cylindrical measure corresponding to the cylindrical random variable

$$\Lambda : H \to L^2(\Omega), \quad \Lambda(h) = \sum_n g_n \langle h_n, h \rangle .$$

Then γ_H is called the *cylindrical standard Gaussian* on H.

remark. We mentioned in example **3.3** that the cylindrical standard Gaussian has characteristic function $e^{-\frac{1}{2}\|h\|^2}$. Indeed, we have

$$\begin{aligned}
\phi_\Lambda(h) &= \mathbb{E}e^{i\Lambda h} = \mathbb{E}e^{i\sum_n g_n \langle h_n, h\rangle} \\
&= \lim_{N\to\infty} \mathbb{E}e^{i\sum_{n=1}^N g_n \langle h_n, h\rangle} = \lim_{N\to\infty} \mathbb{E} \prod_{n=1}^N e^{i g_n \langle h_n, h\rangle} \\
&= \lim_{N\to\infty} \prod_{n=1}^N \mathbb{E}e^{i g_n \langle h_n, h\rangle} = \lim_{N\to\infty} \prod_{n=1}^N e^{-\frac{1}{2}\langle h_n, h\rangle^2} \\
&= \lim_{N\to\infty} e^{-\frac{1}{2}\sum_{n=1}^N \langle h_n, h\rangle^2} = e^{-\frac{1}{2}\|h\|^2}.
\end{aligned}$$

An operator $T : H \to E$ is γ-radonifying if $\gamma_H \circ T^{-1}$ extends to a Radon measure which according to Theorem **3.4** happens if and only if there exists a random variable Y_T such that

$$\langle Y_T, x^* \rangle = \sum_n g_n \langle T h_n, x^* \rangle , \qquad x^* \in E^*.$$

Theorem 4.27. *For a bounded linear operator $T : H \to E$ the following are equivalent:*
(a) T is γ-radonifying;
(b) $\sum_n g_n T h_n$ converges almost surely in E;
(c) $\sum_n g_n T h_n$ converges in $L^p(\Omega, E)$ for some $p \in [1, \infty)$;
(d) $\sum_n g_n T h_n$ converges in $L^p(\Omega, E)$ for all $p \in [1, \infty)$;
(e) TT^* is the covariance operator of a Gaussian measure μ on $\mathcal{B}(E)$.

Proof. Let T be γ-radonifying, and let $\langle Y_T, x^* \rangle = \sum_n g_n \langle T h_n, x^* \rangle$ a.s. Put $X_n := g_n T h_n$, and $S_n := \sum_{k=1}^n X_n$, then a.s. $\langle S_n, x^* \rangle \to \langle Y_T, x^* \rangle$. Since the X_n are symmetric, and independent it follows that a.s. $S_n \to Y_T$ by Ito-Nisio, and the convergence is in $L^p(\Omega, E)$ if Y_T is in $L^p(\Omega, E)$. Moreover, since S_n is Gaussian for all n, it follows that a.s. $\lim_n \langle S_n, x^* \rangle = \langle Y_T, x^* \rangle$ is Gaussian. Thus Y_T is an E-valued Gaussian which belongs to L^p for all p by Fernique (Theorem **4.17**). Here we established the equivalences between $(a), (b), (c), (d)$.

$(b) \implies (e)$. Since $S = \sum_n g_n T h_n$ is the a.s. limit of the Gaussians (S_n) it follows that S is Gaussian by lemma **4.11** with as distribution a Gaussian measure μ. To see that its covariance operator is TT^*,note that for $x^*, y^* \in E^*$ we have

$$\begin{aligned}
\mathbb{E}\left\langle \sum_{n=1}^{\infty} g_n T h_n, x^* \right\rangle \left\langle \sum_{n=1}^{\infty} g_n T h_n, y^* \right\rangle &= \mathbb{E} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g_n g_m \langle T h_n, x^* \rangle \langle T h_n, y^* \rangle \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \delta_{mn} \langle T h_n, x^* \rangle \langle T h_n, y^* \rangle \\
&= \sum_{n=1}^{\infty} \langle h_n, T^* x^* \rangle_H \langle h_n, T^* y^* \rangle_H \\
&= \langle T^* x^*, T^* y^* \rangle_H \\
&= \langle T T^* x^*, y^* \rangle .
\end{aligned}$$

$(e) \implies (b)$. This implication follows from the Karhunen-Loeve Theorem (see the proof of **4.25**): If Q_n is the covariance operator of $S_n = \sum_{k=1}^n g_k T h_k$, one can show that $\langle Q_n x^*, y^* \rangle \to \langle TT^* x^*, y^* \rangle$. It then follows that $\hat{\mu}_n(x^*) \to \hat{\mu}(x^*)$ which implies that S_n converges almost surely to a random variable Y with distribution μ, by Ito-Nisio. □

4.5 Examples of radonifying operators

We continu example **3.16**.

Example 4.28. Let $H = l^2$, and $E = l^p$, and let $\alpha \in l^p$. Consider the linear operator $T_\alpha \in B(H, E)$ given by

$$T_\alpha (x(n))_n = (\alpha(n) x(n))_n.$$

We have seen that T_α is m-radonifying, and thus also γ-radonifying. We calculate $T_\alpha^* : E^* \to H^*$ where $H^* = H = l^2$, and $E^* = l^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. By definition of the adjoint we have $\langle T_\alpha x^*, y^* \rangle = \langle x^*, T_\alpha^* y^* \rangle$ for $x^* \in l^2, y^* \in l^q$, i.e.

$$\sum_n \alpha(n) x^*(n) y^*(n) = \sum_n x^*(n) (T_\alpha^* y^*)(n).$$

Take $x^* = e_n$ to see that $(T_\alpha^* y^*)(n) = \alpha(n) y^*(n)$ for all n. Thus

$$T_\alpha T_\alpha^* : l^q \to l^p, \qquad (x(n))_n \mapsto (\alpha^2(n) x(n))_n,$$

is the covariance operator of a Gaussian measure μ_α on l^p.

Example 4.29. Let $E = L^p(\mu)$, where (S, μ) is a measure space. Consider the map

$$i : L^p(\mu, H) \to \mathcal{R}^p_m(H, L^p(\mu)), \qquad \phi \mapsto T_\phi := [h \mapsto \langle \phi, h \rangle_H].$$

Notice that for $1 \leq p < 2$ the function $x \mapsto x^{p/2}$ is concave on $(0, \infty)$, and thus by Jensen's inequality

$$\begin{aligned}
\mathbb{E}\left\|\sum_n \alpha_n T_\phi h_n\right\|^p &= \mathbb{E}\left\|\sum_{n=1}^{\infty} \alpha_n f_n\right\|^p = \int_S \mathbb{E}\left|\sum_{n=1}^{\infty} \alpha_n f_n(s)\right|^{2\cdot\frac{p}{2}} ds \\
&\leq \int_S \left(\mathbb{E}\left|\sum_{n=1}^{\infty} \alpha_n f_n(s)\right|^2\right)^{p/2} ds \\
&= \int_S \left(\sum_{n=1}^{\infty} f_n(s)^2\right)^{p/2} ds \\
&= \int_S \left(\sum_n \langle \phi, h_n \rangle^2\right)^{p/2} ds \\
&= \|\phi\|^p_{L^p(\mu)}.
\end{aligned}$$

Note that this inequality holds for $1 \leq p < 2$, thus it remains to check that T_ϕ is bounded in the operator norm. We have

$$\|T_\phi\|_{m,p} \leq \|\phi\|_{L^p(\mu)} + \|T_\phi\|,$$

and by Cauchy-Schwarz

$$\|T_\phi\| = \sup_{\|h\| \leq 1} \|\langle \phi, h \rangle\|_{L^p(\mu)} = \sup_{\|h\| \leq 1} \left(\int_S |\langle \phi(s), h \rangle|^p d\mu(s)\right)^{1/p} \leq \|\phi\|_{L^p(\mu)}.$$

Thus for $1 \leq p < 2$ the map i is a continuous linear injection, and for $1 \leq q \leq p < 2$

$$i : L^p(\mu, H) \hookrightarrow \mathcal{R}^p_m(H, L^p(\mu)) \subset \mathcal{R}^q_m(H, L^p(\mu))$$

with $\|i(\phi)\|_{m,p} \leq 2\|\phi\|_{L^p(\mu)}$.
If additionally (α_n) are assumed to be independent and symmetric, the map i is an isomorphism between $L^p(\mu, H)$ and a subspace of $\mathcal{R}^p_m(H, E)$.

Of course, as we have seen before, for $p = 2$ we have $\mathcal{S}^2(H, L^2(\mu)) = \mathcal{R}^2_m(H, L^2(\mu))$ by Theorem **3.17** , and

$$\sum_{n=1}^{\infty} \|T_\phi h_n\|^2_{L^2(\mu)} = \mathbb{E}\left\|\sum_n \alpha_n T_\phi h_n\right\|^2_{L^2(\mu)} = \|\phi\|^2_{L^2(\mu)},$$

and i is an isometric isomorphism

$$i : L^2(\mu, H) \overset{\sim}{\to} \mathcal{R}^2_m(H, L^2(\mu)) = \mathcal{S}^2(H, L^2(\mu)).$$

Note that this isomorphism establishes a well-known fact, namely that Hilbert-Schmidt operators between L^p-spaces are precisely the integral-operators with square integrable kernel. Thus for measure spaces (S, μ), and (Ω, σ), let $H = L^2(\Omega, \sigma)$, then every $K \in L^2(\mu, L^2(\Omega, \sigma)) = L^2(S \times \Omega, \mu \times \sigma)$ corresponds to a Hilbert-schmidt operator
$T_K \in \mathcal{S}^2(L^2(\Omega), L^2(S))$ given by

$$g \mapsto \left(s \mapsto \int_\Omega K(s, \omega) g(\omega) d\sigma(\omega) \right).$$

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