

**Crossed product algebras associated with  
topological dynamical systems**

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THOMAS STIELTJES INSTITUTE  
FOR MATHEMATICS



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# Preface

This thesis consists of an introduction, acknowledgements, a summary (in Dutch), a curriculum vitae (in Dutch) and the following five papers.

- Svensson, C., Silvestrov, S., de Jeu, M., *Dynamical systems and commutants in crossed products*, Internat. J. Math. **18** (2007), 455-471.
- Svensson, C., Silvestrov S., de Jeu M., *Connections between dynamical systems and crossed products of Banach algebras by  $\mathbb{Z}$* , in “Methods of Spectral Analysis in Mathematical Physics”, Janas, J., Kurasov, P., Laptev, A., Naboko, S., Stolz, G. (Eds.), Operator Theory: Advances and Applications **186**, Birkhäuser, Basel, 2009, 391-401.
- Svensson, C., Silvestrov, S., de Jeu, M., *Dynamical systems associated with crossed products*, to appear in Acta Applicandae Mathematicae (Preprints in Mathematical Sciences 2007:22, LUTFMA-5088-2007; Leiden Mathematical Institute report 2007-30; arXiv:0707.1881).
- Svensson, C., Tomiyama, J., *On the commutant of  $C(X)$  in  $C^*$ -crossed products by  $\mathbb{Z}$  and their representations*, to appear in Journal of Functional Analysis (Leiden Mathematical Institute report 2008-13; arXiv:0807.2940).
- Svensson, C., Tomiyama, J., *On the Banach  $*$ -algebra crossed product associated with a topological dynamical system*, submitted (Leiden Mathematical Institute report 2009-03; arXiv:0902.0690).





# Chapter 1

## Introduction

Consider a pair  $\Sigma = (X, \sigma)$  consisting of a compact Hausdorff space  $X$  and a homeomorphism  $\sigma$  of  $X$ . We shall understand  $\Sigma$  as a topological dynamical system by letting the integers act on  $X$  via iterations of  $\sigma$ . Denote by  $C(X)$  the algebra of continuous complex-valued functions on  $X$  endowed with the supremum norm and the natural pointwise operations. The map  $\alpha : C(X) \rightarrow C(X)$  defined, for  $f \in C(X)$ , by  $\alpha(f) = f \circ \sigma^{-1}$  is then easily seen to be an automorphism of  $C(X)$ . Conversely, given a pair  $(C(X), \alpha)$ , where  $X$  is compact Hausdorff and  $\alpha$  is an automorphism of  $C(X)$ , there exists a unique homeomorphism  $\sigma$  of  $X$  such that for all  $f \in C(X)$  we have  $\alpha(f) = f \circ \sigma^{-1}$ . To realize this, denote by  $\Delta(C(X))$  the set of all characters, i.e. all non-zero multiplicative linear functionals, of  $C(X)$  and note firstly that  $\alpha$  permutes this set by composition. Namely, denoting the permutation by  $\hat{\alpha}$ , we have, for  $\zeta \in \Delta(C(X))$ , that  $\zeta \xrightarrow{\hat{\alpha}} \zeta \circ \alpha$ . Secondly, recall that  $\Delta(C(X))$  can be shown to coincide with the set of all point evaluations of  $C(X)$ . Denoting, for  $x \in X$ , such an evaluation by  $\mu_x$ , the permutation  $\hat{\alpha}$  above induces a bijection,  $\sigma$ , of  $X$  by  $\mu_x \xrightarrow{\hat{\alpha}} \mu_x \circ \alpha = \mu_{\sigma^{-1}(x)}$ . Thus, for  $f \in C(X)$ , we have indeed that  $\alpha(f)(x) = f \circ \sigma^{-1}(x)$  for all  $x \in X$  and furthermore one can show that  $\sigma$  is a homeomorphism and unique, as desired. Hence, studying the dynamical system  $\Sigma$  is equivalent to studying the pair  $(C(X), \alpha)$ , where the integers act on  $C(X)$  via iterations of  $\alpha$ . Given another compact Hausdorff space,  $Y$ , one can use an argument similar to the above to conclude that there exists a homeomorphism between  $X$  and  $Y$  if and only if  $C(X)$  is isomorphic, as an algebra, to  $C(Y)$ , whence to study the space  $X$  is equivalent to study the algebra  $C(X)$ . Having this appealing correspondence in mind, it is quite natural to try to transplant the pair  $(C(X), \alpha)$  above into some algebraic object such that its structure reflects topological dynamical properties of the system  $\Sigma = (X, \sigma)$ . Since  $C(X)$  is a typical commutative unital  $C^*$ -algebra, one natural choice of category for the object associated with  $(C(X), \alpha)$  would be that of unital  $C^*$ -algebras. Given such a pair, one can indeed construct a certain  $C^*$ -algebra, a so called  $C^*$ -crossed product, which is generated by a copy of  $C(X)$  and a unitary element,  $\delta$ , that implements the action of the integers on  $C(X)$  via  $\alpha$ . We denote this  $C^*$ -algebra by  $C^*(\Sigma)$  to indicate that it is associated with the dynamical system  $\Sigma$ . In the literature it is also commonly denoted by  $C(X) \rtimes_{\alpha} \mathbb{Z}$ . One way of obtaining  $C^*(\Sigma)$  is as the completion of a  $*$ -algebra,  $k(\Sigma)$ , in a certain norm.

We shall go through the construction of  $C^*(\Sigma)$  in detail in the following section, but we now introduce  $k(\Sigma)$  to give the reader a basic idea of how the pair  $(C(X), \alpha)$  can be transplanted into an algebraic structure.

Hence, we shall endow the set

$$k(\Sigma) = \{a : \mathbb{Z} \rightarrow C(X) : \text{only finitely many } a(n) \text{ are non-zero}\}$$

with the structure a  $*$ -algebra. We define scalar multiplication and addition on  $k(\Sigma)$  as the natural pointwise operations. Multiplication is defined by convolution twisted by the automorphism  $\alpha$  as follows:

$$(ab)(n) = \sum_{k \in \mathbb{Z}} a(k) \cdot \alpha^k(b(n-k)),$$

for  $a, b \in k(\Sigma)$  and  $n \in \mathbb{Z}$ . The involution,  $*$ , is defined by

$$a^*(n) = \overline{\alpha^n(a(-n))},$$

for  $a \in k(\Sigma)$  and  $n \in \mathbb{Z}$ . The bar denotes the usual pointwise complex conjugation. One can then view  $C(X)$  as a  $*$ -subalgebra of  $k(\Sigma)$ , namely as

$$\{a : \mathbb{Z} \rightarrow C(X) : a(n) = 0 \text{ if } n \neq 0\}.$$

A useful way of working with  $k(\Sigma)$  is to write an element  $a \in k(\Sigma)$  in the form  $a = \sum_{k \in \mathbb{Z}} a_k \delta^k$ , for  $a_k = a(k)$  and  $\delta = \chi_{\{1\}}$  where, for  $n, m \in \mathbb{Z}$ ,

$$\chi_{\{n\}}(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

It is then readily checked that  $\delta^* = \delta^{-1}$  and that  $\delta^n = \chi_{\{n\}}$ , for  $n \in \mathbb{Z}$ . As promised, the unitary element  $\delta$  implements the action of the integers on  $C(X)$  via  $\alpha$ . Namely, for  $f \in C(X)$ , we have

$$\delta f \delta^* = \alpha(f) = f \circ \sigma^{-1}$$

and this clearly implies that, for  $n \in \mathbb{Z}$ , the relation

$$\delta^n f \delta^{*n} = \alpha^n(f) = f \circ \sigma^{-n}$$

holds. Note that  $k(\Sigma)$ , and hence  $C^*(\Sigma)$ , is commutative precisely when the system  $\Sigma$  is trivial in the sense that  $\sigma$  is the identity map of  $X$ .

The type of construction that yields  $C^*(\Sigma)$  was first used in a systematic way in [2]. Since then, the connections between topological dynamical properties of  $\Sigma$  and the structure of  $C^*(\Sigma)$  have been intensively studied. To give the reader an idea of what the nature of such connections can be like, we shall now state three known theorems on this so-called interplay between  $\Sigma$  and  $C^*(\Sigma)$  which will play a central role in this thesis.

For  $\Sigma = (X, \sigma)$ , a point  $x \in X$  is called *aperiodic* if for every non-zero  $n \in \mathbb{Z}$  we have  $\sigma^n(x) \neq x$ . The system  $\Sigma$  is called *topologically free* if the set of its aperiodic points is dense in  $X$ . An equivalent statement of the following theorem appeared for the first time as [21, Theorem 4.3.5]. It is also to be found as [22, Theorem 5.4].

**Theorem 1.** *The following three properties are equivalent.*

- $\Sigma$  is topologically free;
- Every non-zero closed ideal  $I$  of  $C^*(\Sigma)$  is such that  $I \cap C(X) \neq \{0\}$ ;
- $C(X)$  is a maximal abelian  $C^*$ -subalgebra of  $C^*(\Sigma)$ .

A system  $\Sigma = (X, \sigma)$  is said to be *minimal* if there are no non-empty proper closed subsets  $C$  of  $X$  such that  $\sigma(C) \subseteq C$ . Equivalently, for every  $x \in X$  the orbit of  $x$  under  $\sigma$ ,  $\mathcal{O}_\sigma(x) = \{\sigma^n(x) : n \in \mathbb{Z}\}$ , is dense in  $X$ . A  $C^*$ -algebra is called *simple* if it lacks non-zero proper closed ideals (in  $C^*(\Sigma)$ , which is unital, this is equivalent to lacking arbitrary non-zero proper ideals). The following classical result follows from the main result of [12]. It is also proved in [22, Theorem 5.3].

**Theorem 2.** *If  $X$  consists of infinitely many points, then  $\Sigma = (X, \sigma)$  is minimal if and only if  $C^*(\Sigma)$  is simple.*

A system  $\Sigma = (X, \sigma)$  is called *topologically transitive* if for every pair  $U, V$  of non-empty open subsets of  $X$ , there exists an integer  $n$  such that  $\sigma^n(U) \cap V \neq \emptyset$ . A  $C^*$ -algebra is *prime* if every pair of non-zero closed ideals have non-zero intersection. For a proof of the following result we refer to [22, Theorem 5.5].

**Theorem 3.** *If  $X$  consists of infinitely many points, then  $\Sigma = (X, \sigma)$  is topologically transitive if and only if  $C^*(\Sigma)$  is prime.*

Note that all these theorems are concerned with ideals of  $C^*(\Sigma)$ . This is not a coincidence. Understanding the ideal structure of an algebra of crossed product type is crucial to several major directions of investigation of it, e.g. its representation theory and examinations of the relations between its ideals and its “building block” algebra (or “coefficient algebra”), which for  $C^*(\Sigma)$  is  $C(X)$ . Ideals will play a prominent role in large parts of this work as well.

Although it is beyond the scope of this thesis, the aforementioned equivalence of homeomorphism of two compact Hausdorff spaces,  $X_1$  and  $X_2$ , and isomorphism of  $C(X_1)$  and  $C(X_2)$  naturally raises the question whether there is a known relation between two arbitrary topological dynamical systems necessary and sufficient for the existence of an isomorphism between their associated  $C^*$ -crossed products. The answer is in the negative. A natural candidate would, for example, be that of topological conjugacy or flip conjugacy since there is an obvious isomorphism between the  $C^*$ -crossed products of two systems that are topologically conjugate or flip conjugate. However, one can show that there exist systems  $\Sigma_1 = (X_1, \sigma_1)$  and  $\Sigma_2 = (X_2, \sigma_2)$  such that the spaces  $X_1$  and  $X_2$  are not even homeomorphic, but where  $C^*(\Sigma_1)$  is isomorphic to  $C^*(\Sigma_2)$ , as is mentioned in [23]. Nevertheless, there are some results available in this direction. For example, let  $\theta_1, \theta_2$  be two irrational real numbers and denote, for  $i = 1, 2$ , by  $\Sigma_i$  the dynamical system defined by rotation of the unit circle by the angle  $2\pi\theta_i$ . Then it follows from the work in [11] and [14] that  $C^*(\Sigma_1)$  is isomorphic to  $C^*(\Sigma_2)$  if and only if  $\theta_1 \equiv \pm\theta_2 \pmod{\mathbb{Z}}$ . Furthermore, a well-known result being proved in [3] states that so-called strong orbit equivalence of minimal systems on the Cantor set is equivalent to isomorphism of their associated  $C^*$ -crossed products. In

relation to this it is interesting to mention the main result of [13], from which it follows that topological conjugacy of two systems,  $\Sigma_1$  and  $\Sigma_2$ , is equivalent to isomorphism of their associated so-called analytic crossed products. The analytic crossed product associated with a system  $\Sigma$  is a certain natural closed non self-adjoint subalgebra of  $C^*(\Sigma)$ ; it is generated, as a Banach algebra, by  $C(X)$  and the unitary element  $\delta$  as introduced above.

In this thesis we focus not only on the  $C^*$ -algebra  $C^*(\Sigma)$  associated with an arbitrary topological dynamical system  $\Sigma = (X, \sigma)$ , but also on a Banach  $*$ -algebra,  $\ell^1(\Sigma)$ , and the non-complete  $*$ -algebra  $k(\Sigma)$  as introduced above, of both of which  $C^*(\Sigma)$  contains a dense  $*$ -isomorphic copy;  $C^*(\Sigma)$  is the so-called enveloping  $C^*$ -algebra of  $\ell^1(\Sigma)$ . The algebras  $\ell^1(\Sigma)$  and  $C^*(\Sigma)$  can be obtained as completions of  $k(\Sigma)$  in two different norms, which we define in the following section. We investigate the interplay between  $\Sigma$  and, respectively,  $k(\Sigma)$  and  $\ell^1(\Sigma)$ , none of which is a  $C^*$ -algebra. While studies of connections between  $\Sigma$  and  $C^*(\Sigma)$  have an extensive history, considerations of  $k(\Sigma)$  and  $\ell^1(\Sigma)$  are new. The algebras  $C^*(\Sigma)$ ,  $\ell^1(\Sigma)$  and  $k(\Sigma)$  all contain a copy of  $C(X)$ , whose commutant,  $C(X)'$ , is being investigated in detail in all three algebras. In particular, we are concerned with its intersection properties for ideals of these algebras. We also consider the interplay between algebras generalizing  $k(\Sigma)$  and corresponding dynamical systems.

It is worth mentioning that there has been a long-standing strong link between ergodic theory and the theory of von Neumann algebras dating back to the seminal work of Murray and von Neumann (cf. [6], [7], [24]), which appeared before the counterpart for topological dynamical systems and  $C^*$ -algebras that serves as departure point for the work in this thesis. There, one associates a crossed product von Neumann algebra with the action of a countable group of non-singular transformations on a standard Borel space equipped with a  $\sigma$ -finite measure. One of the most famous results on this interplay states, under the condition that the action is free, that the associated crossed product is a factor if and only if the action is ergodic, and furthermore gives precise conditions on the measure-theoretic side under which it is a factor of certain types. Another well-known result is the theorem of Krieger ([4],[5]) saying that two such ergodic group actions are orbit equivalent if and only if their associated crossed product von Neumann algebras are isomorphic. This strong ergodic interplay has stimulated studies of the topological case as introduced above.

There is a general theory of  $C^*$ -crossed products of which  $C^*(\Sigma)$  is a special case. There, one starts with a triple  $(G, A, \beta)$  consisting of a locally compact group  $G$ , a  $C^*$ -algebra  $A$  and a homomorphism  $\beta : G \rightarrow \text{Aut}(A)$  such that, for every  $a \in A$ , the map  $g \mapsto (\beta(g))(a)$ , from  $G$  to  $A$ , is norm continuous. With such a triple a  $C^*$ -crossed product,  $C^*(G, A, \beta)$ , is then associated. In our case,  $G$  is the group of integers, with the discrete topology,  $A = C(X)$  and  $\beta$  is the homomorphism mapping an integer  $n$  to  $\alpha^n$ . By the Gelfand-Naimark theorem for commutative  $C^*$ -algebras, the commutative unital  $C^*$ -algebras are precisely the algebras  $C(X)$ , with  $X$  compact Hausdorff, whence the studies of  $C^*(\Sigma)$  amount precisely to the special case when  $A$  is commutative and unital, and the integers act on  $A$  via iterations of a single automorphism,  $\alpha$ , of  $A$ . Some results holding for the aforementioned general  $C^*$ -crossed products can be obtained by simpler means in this particular situation. There are also results on  $C^*(\Sigma)$  that have no known analogues in the general context. When  $A$  is commutative, the  $C^*$ -crossed product is sometimes referred to as a transformation group  $C^*$ -algebra in the literature. For the interested reader, we mention [10] and [25] as standard references for the theory of general  $C^*$ -crossed products. We

also refer to the work in [8] and [9] for results in the same vein as ours, e.g. on intersection properties for ideals of certain maximal abelian subalgebras, in the context of various purely algebraic crossed product structures, of some of which the algebra  $k(\Sigma)$  mentioned above is a special case.

We shall devote Section 1.1 to going through the remaining details of the constructions of the aforementioned algebras to be able to state, in Section 1.2, the questions investigated in this thesis in a clear fashion. We then conclude this chapter by summarizing, in Section 1.3, the contents and some of the main results of the papers corresponding to the remaining chapters.

## 1.1. Three crossed product algebras associated with a dynamical system

Consider a topological dynamical system  $\Sigma = (X, \sigma)$ . As usual,  $X$  is a compact Hausdorff space,  $\sigma$  a homeomorphism of  $X$  and the integers act on  $X$  via iterations of  $\sigma$ . Again, we denote by  $\alpha$  the automorphism of  $C(X)$  defined, for  $f \in C(X)$ , by  $\alpha(f) = f \circ \sigma^{-1}$ . Above we introduced the  $*$ -algebra  $k(\Sigma)$ , which has its multiplication defined in terms of  $\alpha$ . Although the algebras  $\ell^1(\Sigma)$  and  $C^*(\Sigma)$  can both be obtained directly as completions of  $k(\Sigma)$  in different norms, we shall, to be consistent with the presentation in the following chapters, first define  $\ell^1(\Sigma)$  as a completion of  $k(\Sigma)$  and then regard  $C^*(\Sigma)$  as the enveloping  $C^*$ -algebra of  $\ell^1(\Sigma)$ .

Recall that an arbitrary element  $a$  of  $k(\Sigma)$  can be written, in a unique way, as a finite sum,

$$a = \sum_k f_k \delta^k,$$

where the  $f_k$  are in  $C(X)$  and  $\delta$  is unitary, meaning that  $\delta^* = \delta^{-1}$ . We endow  $k(\Sigma)$  with a norm as follows:

$$\|a\| = \sum_k \|f_k\|_\infty.$$

Completing  $k(\Sigma)$  in this norm then yields the Banach  $*$ -algebra  $\ell^1(\Sigma)$ . As in [20], we understand a Banach  $*$ -algebra (or involutive Banach algebra) to be a complex Banach algebra with an *isometric* involution. The algebra  $\ell^1(\Sigma)$  can be concretely realized as

$$\{a = \sum_k f_k \delta^k : \sum_k \|f_k\|_\infty < \infty\},$$

with the operations of  $k(\Sigma)$  extended by continuity. Note that the representation of an element of  $\ell^1(\Sigma)$  as such an infinite sum is unique, and that the closed  $*$ -subalgebra  $\{a : a = f_0 \delta^0 \text{ for some } f_0 \in C(X)\} \subseteq \ell^1(\Sigma)$  constitutes an isometrically  $*$ -isomorphic copy of  $C(X)$ . The  $C^*$ -crossed product,  $C^*(\Sigma)$ , associated with  $\Sigma$  is the enveloping  $C^*$ -algebra of  $\ell^1(\Sigma)$ . This is defined as the completion of  $\ell^1(\Sigma)$  in a different norm. This new norm is defined, for  $a \in \ell^1(\Sigma)$ , by

$$\|a\|_{C^*} = \sup \{\|\tilde{\pi}(a)\| : \tilde{\pi} \text{ is a Hilbert space representation of } \ell^1(\Sigma)\}.$$

While it is readily checked that  $\|\cdot\|_{C^*}$  is a  $C^*$ -seminorm, it is not obvious that it is actually a norm. One can show, however, that  $\ell^1(\Sigma)$  has sufficiently many Hilbert space representations, meaning that for every  $a \in \ell^1(\Sigma)$  there is such a representation  $\tilde{\pi}$  such that  $\tilde{\pi}(a) \neq 0$ . This can be done as follows. Consider a Hilbert space representation on  $\mathcal{H}$ , say, of  $C(X)$ . We shall construct a representation,  $\tilde{\pi}$ , of  $\ell^1(\Sigma)$  on the Hilbert space  $\ell^2(\mathbb{Z}, \mathcal{H})$  of all square summable functions  $x$  of  $\mathbb{Z}$  into  $\mathcal{H}$  endowed with the norm

$$\|x\|_2^2 = \sum_{k \in \mathbb{Z}} \|x(k)\|^2.$$

We first define  $\tilde{\pi}$  on the generating set  $C(X) \cup \{\delta\} \subseteq k(\Sigma)$  by

$$\begin{aligned} (\tilde{\pi}(f)x)(n) &= \pi(\alpha^{-n}(f))(x(n)), \\ (\tilde{\pi}(\delta)x)(n) &= x(n-1), \end{aligned}$$

for all  $f \in C(X)$ ,  $x \in \ell^2(\mathbb{Z}, \mathcal{H})$  and  $n \in \mathbb{Z}$ . One can then check that setting, for  $a = \sum_k f_k \delta^k \in k(\Sigma)$ ,  $\tilde{\pi}(a) = \sum_k \tilde{\pi}(f_k) \tilde{\pi}(\delta)^k$  yields a well-defined representation of  $k(\Sigma)$  on  $\ell^2(\mathbb{Z}, \mathcal{H})$ , that extends by continuity to  $\ell^1(\Sigma)$ . By the Gelfand-Naimark theorem there exist faithful, hence isometric, Hilbert space representations of  $C(X)$ . Knowing this, it is not difficult to choose, for a given arbitrary non-zero element  $a \in \ell^1(\Sigma)$ , a suitable Hilbert space representation  $\pi$  of  $C(X)$  and an element  $x \in \ell^2(\mathbb{Z}, \mathcal{H})$  such that  $\tilde{\pi}(a)x \neq 0$ . Similarly one shows that the embedded copy of  $C(X)$  in  $C^*(\Sigma)$  is isometric with  $C(X)$ .

To sum up we have that, up to  $*$ -isomorphisms,

$$C(X) \subsetneq k(\Sigma) \subsetneq \ell^1(\Sigma) \subsetneq C^*(\Sigma),$$

where the last two inclusions are dense.

## 1.2. Overview of the main directions of investigation

The research carried out in this thesis can be roughly divided into three parts, all intimately related, the questions of which we now outline briefly.

### 1.2.1. The algebras $k(\Sigma)$ and $\ell^1(\Sigma)$

Let  $\Sigma = (X, \sigma)$  be a topological dynamical system. As stated in Section 1.1 its associated  $C^*$ -algebra,  $C^*(\Sigma)$ , contains a dense  $*$ -isomorphic copy of the  $*$ -algebra  $k(\Sigma)$  and of the Banach  $*$ -algebra  $\ell^1(\Sigma)$ . One of our main directions of investigation is the study of the interplay between  $\Sigma$  and the algebras  $k(\Sigma)$  and  $\ell^1(\Sigma)$ , respectively. We consider analogues of results from the  $C^*$ -algebra context for these structures, e.g. Theorems 1-3 above, and also investigate links between their structure and  $\Sigma$  lacking counterparts for  $C^*(\Sigma)$ . While  $C^*$ -algebras have several attractive properties that fail for general Banach  $*$ -algebras,  $\ell^1(\Sigma)$  has an obvious advantage to  $C^*(\Sigma)$  in that its norm is defined such that each of its elements can be written, in a unique way, as an infinite sum of the form  $\sum_n f_n \delta^n$ . Hence this allows one to approximate it by elements of  $k(\Sigma)$  in an obvious manner. This should be compared

to [23, Proposition 1] which provides a more complicated formula for approximating an arbitrary element  $a \in C^*(\Sigma)$  by a sequence of elements in  $k(\Sigma)$ , defined in terms of the so-called generalized Fourier coefficients of  $a$ . Naturally, the fact that  $\ell^1(\Sigma)$  has  $C^*(\Sigma)$  as its enveloping  $C^*$ -algebra furthermore allows us to make use of some of the known facts concerning the latter to derive new results on the former. In the case of  $k(\Sigma)$  it turns out that, although it lacks the obvious advantage of being a complete normed algebra, the fact that its elements can be written in a unique way as *finite* sums of the form  $\sum_n f_n \delta^n$  makes it a very computable object, which enables us to give considerably simpler proofs of theorems on this algebra than of their counterparts for  $\ell^1(\Sigma)$  and  $C^*(\Sigma)$ .

### 1.2.2. Varying the “coefficient algebra” in $k(\Sigma)$

Denote, as above, by  $\alpha$  the automorphism of  $C(X)$  induced by  $\sigma$  and suppose that  $A \subseteq C(X)$  is an arbitrary subalgebra that is invariant under  $\alpha$  and its inverse. To the then naturally defined action of the integers on  $A$  one can associate a crossed product type algebra constructed in the same way as  $k(\Sigma)$ , in which  $A$  plays the role of “coefficient algebra” as  $C(X)$  does in  $k(\Sigma)$ . As we shall see later, it turns out that many results on the interplay between  $\Sigma$  and  $C^*(\Sigma)$  survive if we replace the latter by  $k(\Sigma)$ , while if one chooses the  $A$  above to be the complex numbers, the associated crossed product will be canonically isomorphic to the Laurent polynomial algebra in one variable regardless of the homeomorphism  $\sigma$ . Hence in the latter case the choice of  $A$  yields a crossed product independent of the nature of the dynamical system  $\Sigma = (X, \sigma)$ : all dynamical information is lost. Inspired by these facts, we investigate which choices of  $\alpha$ - and  $\alpha^{-1}$ -invariant subalgebras  $A$  of  $C(X)$  that yield interesting connections between  $\Sigma$  and the associated analogue of  $k(\Sigma)$ . Furthermore, given a pair  $(B, \Psi)$ , where  $B$  is a commutative Banach algebra and  $\Psi$  an automorphism of  $B$ , we consider again an associated crossed product type algebra whose construction is analogous to that of  $k(\Sigma)$ . The automorphism naturally induces a dynamical system on the character space of  $B$ , and we investigate connections between the crossed product associated with  $(B, \Psi)$  and this system. When  $B$  is a commutative unital  $C^*$ -algebra this crossed product is precisely  $k(\Sigma)$ , where  $\Sigma$  is the induced system on the character space of  $B$ . Hence this generalizes the situation where the interplay between  $\Sigma$  and  $k(\Sigma)$  is considered.

### 1.2.3. The commutant of $C(X)$

Recall from Section 1.1 that for a topological dynamical system  $\Sigma = (X, \sigma)$ ,  $C(X)$  can be naturally embedded into the algebras  $k(\Sigma)$ ,  $\ell^1(\Sigma)$  and  $C^*(\Sigma)$ , respectively, by a  $*$ -isomorphism which for the last two algebras is also an isometry (although  $k(\Sigma)$  is  $*$ -isomorphically embedded in both  $\ell^1(\Sigma)$  and  $C^*(\Sigma)$ , we always regard it as a mere algebra and hence do not make any norm considerations when working with it). An object being analyzed in detail in this thesis is the commutant of  $C(X)$  in these algebras, and of the “coefficient algebras” in the crossed product type algebras generalizing  $k(\Sigma)$  as discussed in the previous subsection. While Theorem 1 gives statements equivalent to maximal commutativity of  $C(X)$  in  $C^*(\Sigma)$ , we show that the commutant of  $C(X)$ , which we denote by  $C(X)'$ , is always commutative in all crossed product structures under consideration here.

Hence  $C(X)'$  is always the unique maximal commutative subalgebra that contains  $C(X)$ . Inspired by this fact together with Theorem 1, from which it follows that  $C(X)$  has a certain intersection property for closed ideals of  $C^*(\Sigma)$  precisely when it coincides with  $C(X)'$ , we investigate, for arbitrary  $\Sigma$ , intersection properties of  $C(X)'$  for ideals of the various crossed products. Our conclusions turn out to serve as key results when proving e.g. analogues of Theorems 1-3 for other crossed products than  $C^*(\Sigma)$ . Furthermore, we investigate ideal intersection properties of algebras  $B$  such that  $C(X) \subseteq B \subseteq C(X)'$  and, for  $C^*(\Sigma)$ , of  $\tilde{\pi}(C(X))'$ , where  $\tilde{\pi}$  is a Hilbert space representation of the former. Many of our results related to  $C(X)'$  rely on the crucial fact that we can describe it explicitly, as well as its character space in the cases of  $\ell^1(\Sigma)$  and  $C^*(\Sigma)$ .

### 1.3. Brief summary of the included papers

Chapter 2 to Chapter 6 of this thesis correspond to, respectively, [15], [16], [17], [18] and [19]. We shall now summarize their contents briefly.

#### Chapter 2: Dynamical systems and commutants in crossed products

To a pair  $(A, \Psi)$  of an arbitrary associative commutative complex algebra  $A$  and an automorphism  $\Psi$  of  $A$ , we associate a purely algebraic crossed product containing an isomorphic copy of  $A$ . Namely, we endow the set

$$\{f : \mathbb{Z} \rightarrow A : f(n) = 0 \text{ for all but finitely many } n \in \mathbb{Z}\}$$

with operations making it an associative, in general non-commutative, complex algebra, which we denote by  $A \rtimes_{\Psi} \mathbb{Z}$ . Its multiplication is defined in terms of  $\Psi$  in a way analogous to the case of the algebra  $k(\Sigma)$ , as introduced above, which is the special case here obtained as the crossed product associated with the pair  $(C(X), \alpha)$ . We show that the commutant,  $A'$ , of  $A$  is commutative and describe it explicitly in the case when  $A$  is a function algebra and  $\Psi$  a composition automorphism defined via a bijection of the domain of  $A$ . Commutativity of  $A'$  implies that it is the unique maximal abelian subalgebra of  $A \rtimes_{\Psi} \mathbb{Z}$  that contains  $A$ . Given various classes of pairs  $(X, \sigma)$  of a topological space  $X$  and a homeomorphism  $\sigma$  of  $X$ , which we consider as dynamical systems by letting the integers act on  $X$  via iterations of  $\sigma$  as above, we prove that suitable subalgebras  $A$  of  $C(X)$ , invariant under the automorphism  $\alpha : C(X) \rightarrow C(X)$  induced by  $\sigma$  and under  $\alpha^{-1}$ , constitute maximal abelian subalgebras of  $A \rtimes_{\alpha} \mathbb{Z}$  if and only if the set of aperiodic points of  $(X, \sigma)$  is dense in  $X$ . We show that a specific class of such  $A$  are maximal abelian in  $A \rtimes_{\alpha} \mathbb{Z}$  precisely when  $\sigma$  is not of finite order. An example of this is the case when  $A$  is the algebra of all holomorphic functions on a connected complex manifold,  $M$ , and  $\sigma$  is a biholomorphic function on  $M$ .

Furthermore, for pairs  $(A, \Psi)$ , where  $A$  is a semi-simple commutative Banach algebra and  $\Psi$  is an automorphism of  $A$ , we introduce a topological dynamical system on the character space,  $\Delta(A)$ , of  $A$  naturally induced by  $\Psi$  and prove e.g. that when  $A$  is also completely regular we have equivalence between maximal commutativity of  $A$  in  $A \rtimes_{\Psi} \mathbb{Z}$  and density of the set of aperiodic points of the associated system on  $\Delta(A)$ . When  $A$  is the algebra  $L_1(G)$  of integrable functions on a locally compact abelian group  $G$  with connected dual



group, we use the above results to conclude that  $L_1(G)$  is maximal abelian in the crossed product precisely when the automorphism  $\Psi$  of  $L_1(G)$  is not of finite order.

All these results should be compared to the first and third statement of Theorem 1.

### Chapter 3: Connections between dynamical systems and crossed products of Banach algebras by $\mathbb{Z}$

Here we start the investigation of the ideal structure of algebraic crossed products as defined in Chapter 2. We mainly focus on crossed products associated with pairs  $(A, \Psi)$  where  $A$  is a commutative semi-simple completely regular Banach algebra and  $\Psi$  is an automorphism of  $A$ . We prove that  $A$  has non-zero intersection with every non-zero ideal precisely when  $A$  is a maximal abelian subalgebra of  $A \rtimes_{\Psi} \mathbb{Z}$ , which is in turn equivalent to density of the aperiodic points of the associated dynamical system on  $\Delta(A)$  as introduced in Chapter 2. Thus we obtain a result analogous to Theorem 1. When  $A$  is the algebra  $L_1(G)$  of integrable functions on a locally compact abelian group  $G$  with connected dual group, the set of aperiodic points of  $\Delta(A)$  is dense precisely when the automorphism  $\Psi$  is not of finite order. We prove equivalence between simplicity of  $A \rtimes_{\Psi} \mathbb{Z}$  and minimality of the system on  $\Delta(A)$ , provided that  $A$  is unital and  $\Delta(A)$  is infinite. This is analogous to Theorem 2 in the  $C^*$ -algebra context. We also show that every non-zero ideal of  $A \rtimes_{\Psi} \mathbb{Z}$  always has non-zero intersection with  $A'$ . Finally, we show that for unital  $A$  such that  $\Delta(A)$  is infinite, topological transitivity of the system on  $\Delta(A)$  is equivalent to primeness of  $A \rtimes_{\Psi} \mathbb{Z}$  and hence find the analogue of Theorem 3.

### Chapter 4: Dynamical systems associated with crossed products

We give simplified proofs of generalizations of some results from Chapter 3, which we thereby show to hold in a broader context than when  $A$  is a certain kind of Banach algebra. For example, we give an elementary proof of the fact that if  $(A, \Psi)$  is a pair consisting of an arbitrary commutative associative complex algebra  $A$  and an automorphism  $\Psi$  of  $A$ , the associated crossed product  $A \rtimes_{\Psi} \mathbb{Z}$  is such that  $A'$  has non-zero intersection with every non-zero ideal. This allows us to prove the analogue of Theorem 1 in a greater generality than in Chapter 3. We also investigate subalgebras properly between  $A$  and its commutant  $A'$  and show that for suitable  $(A, \Psi)$ , one may find two such subalgebras,  $B_1$  and  $B_2$ , of  $A \rtimes_{\Psi} \mathbb{Z}$  where  $B_1$  has non-zero intersection with every non-zero ideal and where  $B_2$  does not have this property. Finally, we start the investigation of the Banach algebra crossed product,  $I_1^{\sigma}(\mathbb{Z}, C_0(X))$ , associated with a pair  $(X, \sigma)$  of a locally compact Hausdorff space  $X$  and a homeomorphism  $\sigma$  of  $X$  where, as usual, the integers act on  $X$  via iterations of  $\sigma$ . When  $X$  is compact, this is precisely the algebra  $\ell^1(\Sigma)$  as introduced in Section 1.1. We determine the closed commutator ideal of  $I_1^{\sigma}(\mathbb{Z}, C_0(X))$  in terms of the set of fixed points,  $\text{Per}^1(X)$ , of  $(X, \sigma)$  and furthermore find a bijection between the characters of  $I_1^{\sigma}(\mathbb{Z}, C_0(X))$  and  $\text{Per}^1(X) \times \mathbb{T}$ .

## Chapter 5: On the commutant of $C(X)$ in $C^*$ -crossed products by $\mathbb{Z}$ and their representations

In this chapter, which is focused entirely on  $C^*(\Sigma)$  for a topological dynamical system  $\Sigma = (X, \sigma)$ , we analyze the commutant of  $C(X)$ , denoted as usual by  $C(X)'$ , in detail. We describe its elements explicitly in terms of their generalized Fourier coefficients and conclude that it is commutative. More generally, we analyze  $\tilde{\pi}(C(X))'$ , where  $\tilde{\pi}$  is an arbitrary Hilbert space representation of  $C^*(\Sigma)$ . We describe the spectrum of  $\tilde{\pi}(C^*(\Sigma))'$  and consider a topological dynamical system on it which is derived from the action of the integers on  $C(X)$  via iterations of  $\alpha$ , the automorphism induced by  $\sigma$ . Inspired by the results on  $C(X)'$  in  $k(\Sigma)$  and its generalizations as obtained in Chapter 3 and 4, we prove that  $C(X)'$  always has non-zero intersection with every non-zero (not necessarily closed or self-adjoint) ideal of  $C^*(\Sigma)$ , and that  $\tilde{\pi}(C(X))'$  has the corresponding property if a certain condition on  $\tilde{\pi}(C^*(\Sigma))$  holds. This enables us to provide a sharpened version of Theorem 1 above that holds for arbitrary ideals of  $C^*(\Sigma)$  rather than just for closed ones. Furthermore, we consider  $C^*$ -subalgebras properly between  $C(X)$  and  $C(X)'$  (cf. Chapter 4) and conclude that, as soon as  $C(X) \neq C(X)'$ , one may find two such subalgebras,  $B_1$  and  $B_2$ , of  $C^*(\Sigma)$  where  $B_1$  has non-zero intersection with every non-zero ideal and where  $B_2$  does not have this property. Finally, we discuss existence of norm one projections of  $C^*(\Sigma)$  onto  $C(X)'$ .

## Chapter 6: On the Banach $*$ -algebra crossed product associated with a topological dynamical system

Having focused mainly on  $C^*(\Sigma)$  and  $k(\Sigma)$ , and its generalizations, in the previous chapters, we now consider the Banach  $*$ -algebra  $\ell^1(\Sigma)$  associated with a topological dynamical system  $\Sigma = (X, \sigma)$ . As in the other algebras, we describe  $C(X)'$  explicitly and conclude that it is commutative. By invoking the theory of ordered linear spaces, we also determine its character space and furthermore prove that it has non-zero intersection with every non-zero closed ideal of  $\ell^1(\Sigma)$ . Using this result, we deduce analogues of Theorems 1-3 for  $\ell^1(\Sigma)$ . We conclude by proving a result whose analogue in the  $C^*$ -algebra context is clearly false:  $\ell^1(\Sigma)$  has non self-adjoint closed ideals if and only if  $\Sigma$  has periodic points.

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## Chapter 2

# Dynamical systems and commutants in crossed products

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**Abstract.** In this paper we describe the commutant of an arbitrary subalgebra  $A$  of the algebra of functions on a set  $X$  in a crossed product of  $A$  with the integers, where the latter act on  $A$  by a composition automorphism defined via a bijection of  $X$ . The resulting conditions which are necessary and sufficient for  $A$  to be maximal abelian in the crossed product are subsequently applied to situations where these conditions can be shown to be equivalent to a condition in topological dynamics. As a further step, using the Gelfand transform we obtain for a commutative completely regular semi-simple Banach algebra a topological dynamical condition on its character space which is equivalent to the algebra being maximal abelian in a crossed product with the integers.

### 2.1. Introduction

The description of commutative subalgebras of non-commutative algebras and their properties is an important direction of investigation for any class of non-commutative algebras and rings, because it allows one to relate representation theory, non-commutative properties, graded structures, ideals and subalgebras, homological and other properties of non-commutative algebras to spectral theory, duality, algebraic geometry and topology naturally associated with the commutative subalgebras. In representation theory, for example, one of the keys for the construction and classification of representations is the method of induced representations. The underlying structures behind this method are the semi-direct products or crossed products of rings and algebras by various actions. When a non-commutative algebra is given, one looks for a subalgebra such that its representations can be studied and classified more easily, and such that the whole algebra can be decomposed as a crossed

product of this subalgebra by a suitable action. Then the representations for the subalgebra are extended to representations of the whole algebra using the action and its properties. A description of representations is most tractable for commutative subalgebras as being, via the spectral theory and duality, directly connected to algebraic geometry, topology or measure theory.

If one has found a way to present a non-commutative algebra as a crossed product of a commutative subalgebra by some action on it of the elements from outside the subalgebra, then it is important to know whether this subalgebra is maximal abelian or, if not, to find a maximal abelian subalgebra containing the given subalgebra, since if the selected subalgebra is not maximal abelian, then the action will not be entirely responsible for the non-commutative part as one would hope, but will also have the commutative trivial part taking care of the elements commuting with everything in the chosen commutative subalgebra. This maximality of a commutative subalgebra and related properties of the action are intimately related to the description and classifications of representations of the non-commutative algebra.

Little is known in general about connections between properties of the commutative subalgebras of crossed product algebras and properties of dynamical systems that are in many situations naturally associated with the construction. A remarkable result in this direction is known, however, in the context of crossed product  $C^*$ -algebras. When the algebra is described as the crossed product  $C^*$ -algebra  $C(X) \rtimes_{\alpha} \mathbb{Z}$  of the algebra of continuous functions on a compact Hausdorff space  $X$  by an action of  $\mathbb{Z}$  via the composition automorphism associated with a homeomorphism  $\sigma$  of  $X$ , it is known that  $C(X)$  sits inside the  $C^*$ -crossed product as a maximal abelian subalgebra if and only if for every positive integer  $n$  the set of points in  $X$  having period  $n$  under iterations of  $\sigma$  has no interior points [4], [7], [8], [9], [10]. By the category theorem, this condition is equivalent to the action of  $\mathbb{Z}$  on  $X$  being topologically free in the sense that the aperiodic points of  $\sigma$  are dense in  $X$ . This result on the interplay between the topological dynamics of the action on the one hand, and the algebraic property of the commutative subalgebra in the crossed product being maximal abelian on the other hand, provided the main motivation and starting point for our work.

In this article, we bring such interplay results into a more general algebraic and set theoretical context of a crossed product  $A \rtimes_{\alpha} \mathbb{Z}$  of an arbitrary subalgebra  $A$  of the algebra  $\mathbb{C}^X$  of functions on a set  $X$  (under the usual pointwise operations) by  $\mathbb{Z}$ , where the latter acts on  $A$  by a composition automorphism defined via a bijection of  $X$ . In this general algebraic set theoretical framework the essence of the matter is revealed. Topological notions are not available here and thus the condition of freeness of the dynamics as described above is not applicable, so that it has to be generalized in a proper way in order to be equivalent to the maximal commutativity of  $A$ . We provide such a generalization. In fact, we describe explicitly the (unique) maximal abelian subalgebra containing  $A$  (Theorem 2.3.3), and then the general result, stating equivalence of maximal commutativity of  $A$  in the crossed product and the desired generalization of topological freeness of the action, follows immediately (Theorem 2.3.5). It involves separation properties of  $A$  with respect to the space  $X$  and the action.

The general set theoretical framework allows one to investigate the relation between the maximality of the commutative subalgebra in the crossed product on the one hand, and the properties of the action on the space on the other hand, for arbitrary choices of

the set  $X$ , the subalgebra  $A$  and the action, different from the previously cited classical choice of continuous functions  $C(X)$  on a compact Hausdorff topological space  $X$ . As a rather general application of our results we obtain that, for a Baire topological space  $X$  and a subalgebra  $A$  of  $C(X)$  satisfying a mild separation condition, the property of  $A$  being maximal abelian in the crossed product is equivalent to the action being topologically free in the sense above, i.e., to the set of aperiodic points being dense in  $X$  (Theorem 2.3.7). This applies in particular when  $X$  is a compact Hausdorff space, so that a result analogous to that for the crossed product  $C^*$ -algebra  $C(X) \rtimes_{\alpha} \mathbb{Z}$  is obtained. We also demonstrate that, for a general topological space  $X$  and a subalgebra  $A$  of  $C(X)$  satisfying a less common separation condition, the subalgebra  $A$  is maximal abelian if and only if  $\sigma$  is not of finite order, which is a much less restrictive condition than topological freeness (Theorem 2.3.11). Examples of this situation are provided by crossed products of subalgebras of holomorphic functions on connected complex manifolds by biholomorphic actions (Corollary 2.3.12). It is interesting to note that these two results, Theorem 2.3.7 and Theorem 2.3.11, have no non-trivial situations in common (Remark 2.3.13).

In the motivating Section 2.4.1, we illustrate by several examples that, generally speaking, topological freeness and the property of the subalgebra  $A$  being maximal abelian in the crossed product are unrelated. In Section 2.4.2, we consider the crossed product of a commutative semi-simple Banach algebra  $A$  by the action of an automorphism. Since the automorphism of  $A$  induces a homeomorphism of  $\Delta(A)$ , the set of all characters on  $A$  endowed with the Gelfand topology, such a crossed product is via the Gelfand transform canonically isomorphic (by semi-simplicity) to the crossed product of the subalgebra  $\hat{A}$  of the algebra  $C_0(\Delta(A))$  by the induced homeomorphism, where  $C_0(\Delta(A))$  denotes the algebra of continuous functions on  $\Delta(A)$  that vanish at infinity. When  $A$  is also assumed to be completely regular, the general result in Theorem 2.3.7 can be applied to this isomorphic crossed product. We thus prove that, when  $A$  is a commutative completely regular semi-simple Banach algebra, it is maximal abelian in the crossed product if and only if the associated dynamical system on the character space  $\Delta(A)$  is topologically free in the sense that its aperiodic points are dense in the topological space  $\Delta(A)$  (Theorem 2.4.8). It is then possible to understand the algebraic properties in the examples in Section 2.4.1 as dynamical properties after all, by now looking at the “correct” dynamical system on  $\Delta(A)$  instead of the originally given dynamical system. In these examples, this corresponds to removing parts of the space or enlarging it.

In Section 2.4.3, we apply Theorem 2.4.8 in the case of an automorphism of  $L_1(G)$ , the commutative completely regular semi-simple Banach algebra consisting of equivalence classes of complex-valued Borel measurable functions on a locally compact abelian group  $G$  that are integrable with respect to a Haar measure, with multiplication given by convolution. We use a result saying that every automorphism on  $L_1(G)$  is induced by a piecewise affine homeomorphism of the dual group of  $G$  (Theorem 2.4.16) to prove that, for  $G$  with connected dual,  $L_1(G)$  is maximal abelian in the crossed product if and only if the given automorphism of  $L_1(G)$  is not of finite order (Theorem 2.4.17). We also provide an example showing that this equivalence may fail when the dual of  $G$  is not connected.

Finally, in Section 2.4.4, we provide another application of the commutant description in Theorem 2.3.3 and of the isomorphic crossed product on the character space, showing that, for a semi-simple commutative Banach algebra  $A$ , the commutant of  $A$  in the crossed

product is finitely generated as an algebra over  $\mathbb{C}$  if and only if  $A$  has finite dimension as a vector space over  $\mathbb{C}$ .

## 2.2. Crossed products associated with automorphisms

### 2.2.1. Definition

Let  $A$  be an associative commutative  $\mathbb{C}$ -algebra and let  $\Psi : A \rightarrow A$  be an automorphism. Consider the set

$$A \rtimes_{\Psi} \mathbb{Z} = \{f : \mathbb{Z} \rightarrow A \mid f(n) = 0 \text{ except for a finite number of } n\}.$$

We endow it with the structure of an associative  $\mathbb{C}$ -algebra by defining scalar multiplication and addition as the usual pointwise operations. Multiplication is defined by *twisted convolution*,  $*$ , as follows;

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k) \cdot \Psi^k(g(n-k)),$$

where  $\Psi^k$  denotes the  $k$ -fold composition of  $\Psi$  with itself. It is trivially verified that  $A \rtimes_{\Psi} \mathbb{Z}$  is an associative  $\mathbb{C}$ -algebra under these operations. We call it the *crossed product of  $A$  and  $\mathbb{Z}$  under  $\Psi$* .

A useful way of working with  $A \rtimes_{\Psi} \mathbb{Z}$  is to write elements  $f, g \in A \rtimes_{\Psi} \mathbb{Z}$  in the form  $f = \sum_{n \in \mathbb{Z}} f_n \delta^n$ ,  $g = \sum_{m \in \mathbb{Z}} g_m \delta^m$ , where  $f_n = f(n)$ ,  $g_m = g(m)$ , addition and scalar multiplication are canonically defined, and multiplication is determined by  $(f_n \delta^n) * (g_m \delta^m) = f_n \cdot \Psi^n(g_m) \delta^{n+m}$ , where  $n, m \in \mathbb{Z}$  and  $f_n, g_m \in A$  are arbitrary. Using this notation, we may think of the crossed product as a complex Laurent polynomial algebra in one variable (having  $\delta$  as its indeterminate) over  $A$  with twisted multiplication.

### 2.2.2. A maximal abelian subalgebra of $A \rtimes_{\Psi} \mathbb{Z}$

Clearly one may canonically view  $A$  as an abelian subalgebra of  $A \rtimes_{\Psi} \mathbb{Z}$ , namely as  $\{f_0 \delta^0 \mid f_0 \in A\}$ . Here we prove that the commutant of  $A$  in  $A \rtimes_{\Psi} \mathbb{Z}$ , denoted by  $A'$ , is commutative, and thus there exists a *unique* maximal abelian subalgebra of  $A \rtimes_{\Psi} \mathbb{Z}$  containing  $A$ , namely  $A'$ .

Note that we can write the product of two elements in  $A \rtimes_{\Psi} \mathbb{Z}$  as follows;

$$f * g = \left( \sum_{n \in \mathbb{Z}} f_n \delta^n \right) * \left( \sum_{m \in \mathbb{Z}} g_m \delta^m \right) = \sum_{r, n \in \mathbb{Z}} f_n \cdot \Psi^n(g_{r-n}) \delta^r.$$

We see that the precise condition for commutation of two such elements  $f$  and  $g$  is

$$\forall r : \sum_{n \in \mathbb{Z}} f_n \cdot \Psi^n(g_{r-n}) = \sum_{m \in \mathbb{Z}} g_m \cdot \Psi^m(f_{r-m}). \quad (2.1)$$

**Proposition 2.2.1.** *The commutant  $A'$  is abelian, and thus it is the unique maximal abelian subalgebra containing  $A$ .*



*Proof.* Let  $f = \sum_{n \in \mathbb{Z}} f_n \delta^n$ ,  $g = \sum_{m \in \mathbb{Z}} g_m \delta^m \in A \rtimes_{\Psi} \mathbb{Z}$ . We shall verify that if  $f, g \in A'$ , then  $f$  and  $g$  commute. Using (2.1) we see that membership of  $f$  and  $g$  respectively in  $A'$  is equivalent to the equalities

$$\forall h \in A, \forall n \in \mathbb{Z} : f_n \cdot \Psi^n(h) = f_n \cdot h, \quad (2.2)$$

$$\forall h \in A, \forall m \in \mathbb{Z} : g_m \cdot \Psi^m(h) = g_m \cdot h. \quad (2.3)$$

Insertion of (2.2) and (2.3) into (2.1) we realize that the precise condition for commutation of such  $f$  and  $g$  can be rewritten as

$$\forall r : \sum_{n \in \mathbb{Z}} f_n \cdot g_{r-n} = \sum_{m \in \mathbb{Z}} g_m \cdot f_{r-m}.$$

This clearly holds, and thus  $f$  and  $g$  commute. From this it follows immediately that  $A'$  is the unique maximal abelian subalgebra containing  $A$ .  $\square$

## 2.3. Automorphisms induced by bijections

Fix a non-empty set  $X$ , a bijection  $\sigma : X \rightarrow X$ , and an algebra of functions  $A \subseteq \mathbb{C}^X$  that is invariant under  $\sigma$  and  $\sigma^{-1}$ , i.e., such that if  $h \in A$ , then  $h \circ \sigma \in A$  and  $h \circ \sigma^{-1} \in A$ . Then  $(X, \sigma)$  is a discrete dynamical system (the action of  $n \in \mathbb{Z}$  on  $x \in X$  is given by  $n : x \mapsto \sigma^n(x)$ ) and  $\sigma$  induces an automorphism  $\tilde{\sigma} : A \rightarrow A$  defined by  $\tilde{\sigma}(f) = f \circ \sigma^{-1}$  by which  $\mathbb{Z}$  acts on  $A$  via iterations.

In this section we will consider the crossed product  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  for the above setup, and explicitly describe the commutant,  $A'$ , of  $A$  and the center,  $Z(A \rtimes_{\tilde{\sigma}} \mathbb{Z})$ . Furthermore, we will investigate equivalences between properties of aperiodic points of the system  $(X, \sigma)$ , and properties of  $A'$ . First we make a few definitions.

**Definition 2.3.1.** For any nonzero  $n \in \mathbb{Z}$  we set

$$\text{Sep}_A^n(X) = \{x \in X \mid \exists h \in A : h(x) \neq \tilde{\sigma}^n(h)(x)\},$$

$$\text{Per}_A^n(X) = \{x \in X \mid \forall h \in A : h(x) = \tilde{\sigma}^n(h)(x)\},$$

$$\text{Sep}^n(X) = \{x \in X \mid x \neq \sigma^n(x)\},$$

$$\text{Per}^n(X) = \{x \in X \mid x = \sigma^n(x)\}.$$

Furthermore, let

$$\text{Per}_A^\infty(X) = \bigcap_{n \in \mathbb{Z} \setminus \{0\}} \text{Sep}_A^n(X),$$

$$\text{Per}^\infty(X) = \bigcap_{n \in \mathbb{Z} \setminus \{0\}} \text{Per}^n(X).$$

Finally, for  $f \in A$ , put

$$\text{supp}(f) = \{x \in X \mid f(x) \neq 0\}.$$

It is easy to check that all these sets, except for  $\text{supp}(f)$ , are  $\mathbb{Z}$ -invariant and that if  $A$  separates the points of  $X$ , then  $\text{Sep}_A^n(X) = \text{Sep}^n(X)$  and  $\text{Per}_A^n(X) = \text{Per}^n(X)$ . Note also that  $X \setminus \text{Per}_A^n(X) = \text{Sep}_A^n(X)$ , and  $X \setminus \text{Per}^n(X) = \text{Sep}^n(X)$ . Furthermore  $\text{Sep}_A^n(X) = \text{Sep}_A^{-n}(X)$  with similar equalities for  $n$  and  $-n$  (where  $n \in \mathbb{Z}$ ) holding for  $\text{Per}_A^n(X)$ ,  $\text{Sep}^n(X)$  and  $\text{Per}^n(X)$  as well.

**Definition 2.3.2.** We say that a non-empty subset of  $X$  is a *domain of uniqueness* for  $A$  if every function in  $A$  that vanishes on it, vanishes on the whole of  $X$ .

For example, using results from elementary topology one easily shows that for a completely regular topological space  $X$ , a subset of  $X$  is a domain of uniqueness for  $C(X)$  if and only if it is dense in  $X$ .

**Theorem 2.3.3.** *The unique maximal abelian subalgebra of  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  that contains  $A$  is precisely the set of elements*

$$A' = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} : f_n|_{\text{Sep}_A^n(X)} \equiv 0 \right\}.$$

*Proof.* Quoting a part of the proof of Proposition 2.2.1, we have that

$$\sum_{n \in \mathbb{Z}} f_n \delta^n \in A'$$

if and only if

$$\forall h \in A, \forall n \in \mathbb{Z} : f_n \cdot \tilde{\sigma}^n(h) = f_n \cdot h.$$

Clearly this is equivalent to

$$\forall n \in \mathbb{Z} : f_n|_{\text{Sep}_A^n(X)} \equiv 0.$$

The result now follows from Proposition 2.2.1. □

Note that for any non-zero integer  $n$ , the set  $\{f_n \in A : f_n|_{\text{Sep}_A^n(X)} \equiv 0\}$  is a  $\mathbb{Z}$ -invariant ideal in  $A$ . Note also that if  $\sigma$  has finite order, then by Theorem 2.3.3,  $A$  is not maximal abelian. The following corollary follows directly from Theorem 2.3.3.

**Corollary 2.3.4.** *If  $A$  separates the points of  $X$ , then  $A'$  is precisely the set of elements*

$$A' = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} : \text{supp}(f_n) \subseteq \text{Per}^n(X) \right\}.$$

*Proof.* Immediate from Theorem 2.3.3 and the remarks following Definition 2.3.1. □

Applying Definition 2.3.2 yields the following direct consequence of Theorem 2.3.3.

**Theorem 2.3.5.** *The subalgebra  $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  if and only if, for every  $n \in \mathbb{Z} \setminus \{0\}$ ,  $\text{Sep}_A^n(X)$  is a domain of uniqueness for  $A$ .*

In what follows we shall mainly focus on cases where  $X$  is a topological space. Before we turn to such contexts, however, we use Theorem 2.3.3 to give a description of the center of the crossed product,  $Z(A \rtimes_{\tilde{\sigma}} \mathbb{Z})$ , for the general setup described in the beginning of this section.

**Theorem 2.3.6.** *An element  $g = \sum_{m \in \mathbb{Z}} g_m \delta^m$  is in  $Z(A \rtimes_{\tilde{\sigma}} \mathbb{Z})$  if and only if both of the following conditions are satisfied:*

- (i) *for all  $m \in \mathbb{Z}$ ,  $g_m$  is  $\mathbb{Z}$ -invariant, and*
- (ii) *for all  $m \in \mathbb{Z}$ ,  $g_m \upharpoonright_{\text{Sep}_A^m(X)} \equiv 0$ .*

*Proof.* If  $g$  is in  $Z(A \rtimes_{\tilde{\sigma}} \mathbb{Z})$  then certainly  $g \in A'$ , and hence condition (ii) follows from Theorem 2.3.3. For condition (i), note that  $g$  is in the center if and only if  $g$  commutes with every element on the form  $f_n \delta^n$ . Multiplying out, or looking at (2.1), we see that this means that

$$g = \sum_{m \in \mathbb{Z}} g_m \delta^m \text{ is in } Z(A \rtimes_{\tilde{\sigma}} \mathbb{Z}) \iff \\ \forall n, m \in \mathbb{Z}, \forall f \in A : f \cdot \tilde{\sigma}^n(g_m) = g_m \cdot \tilde{\sigma}^m(f).$$

We fix  $m \in \mathbb{Z}$  and an  $x \in \text{Per}_A^m(X)$ . Then for all  $f \in A : f(x) = \tilde{\sigma}^m(f)(x)$ . If there is a function  $f \in A$  that does not vanish in  $x$ , then for  $g$  to be in the center we clearly must have that for all  $n \in \mathbb{Z} : g_m(x) = \tilde{\sigma}^n(g_m)(x)$ . If all  $f \in A$  vanish in  $x$ , then in particular both  $g_m$  and  $\tilde{\sigma}^n(g_m)$  do. Thus for all points  $x \in \text{Per}_A^m(X)$  we have that  $g_m$  is constant along the orbit of  $x$  (i.e., for all  $n \in \mathbb{Z} : g_m(x) = \tilde{\sigma}^n(g_m)(x)$ ) for all  $m \in \mathbb{Z}$ , since  $m$  was arbitrary in our above discussion. It remains to consider  $x \in \text{Sep}_A^m(X)$ . For such  $x$ , we have concluded that  $g_m(x) = 0$ . If there exists  $f \in A$  that does not vanish in  $x$ , we see that in order for the equality above to be satisfied we must have  $\tilde{\sigma}^n(g_m)(x) = 0$  for all  $n$ , and if all  $f \in A$  vanish in  $x$ , then in particular  $\tilde{\sigma}^n(g_m)$  does for all  $n$  and the result follows.  $\square$

We now focus solely on topological contexts. The following theorem makes use of Corollary 2.3.4.

**Theorem 2.3.7.** *Let  $X$  be a Baire space, and let  $\sigma : X \rightarrow X$  be a homeomorphism inducing, as usual, an automorphism  $\tilde{\sigma}$  of  $C(X)$ . Suppose  $A$  is a subalgebra of  $C(X)$  that is invariant under  $\tilde{\sigma}$  and its inverse, separates the points of  $X$  and is such that for every non-empty open set  $U \subseteq X$ , there is a non-zero  $f \in A$  that vanishes on the complement of  $U$ . Then  $A$  is a maximal abelian subalgebra of  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  if and only if  $\text{Per}^\infty(X)$  is dense in  $X$ .*

*Proof.* Assume first that  $\text{Per}^\infty(X)$  is dense in  $X$ . This means in particular that any continuous function that vanishes on  $\text{Per}^\infty(X)$  vanishes on the whole of  $X$ . Thus Corollary 2.3.4 tells us that  $A$  is a maximal abelian subalgebra of  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$ . Now assume that  $\text{Per}^\infty(X)$  is not dense in  $X$ . This means that  $\bigcap_{n \in \mathbb{Z}_{>0}} (X \setminus \text{Per}^n(X))$  is not dense. Note that the sets  $X \setminus \text{Per}^n(X)$ ,  $n \in \mathbb{Z}_{>0}$  are all open. Since  $X$  is a Baire space there exists an  $n_0 \in \mathbb{Z}_{>0}$  such that  $\text{Per}^{n_0}(X)$  has non-empty interior, say  $U \subseteq \text{Per}^{n_0}(X)$ . By the assumption on  $A$ , there is a nonzero function  $f_{n_0} \in A$  that vanishes outside  $U$ . Hence Corollary 2.3.4 shows that  $A$  is not maximal abelian in the crossed product.  $\square$

**Example 2.3.8.** Let  $X$  be a locally compact Hausdorff space, and  $\sigma : X \rightarrow X$  a homeomorphism. Then  $X$  is a Baire space, and  $C_c(X)$ ,  $C_0(X)$ ,  $C_b(X)$  and  $C(X)$  all satisfy the required conditions for  $A$  in Theorem 2.3.7. For details, see for example [5]. Hence these function algebras are maximal abelian in their respective crossed products with  $\mathbb{Z}$  under  $\sigma$  if and only if  $\text{Per}^\infty(X)$  is dense in  $X$ .

**Example 2.3.9.** Let  $X = \mathbb{T}$  be the unit circle in the complex plane, and let  $\sigma$  be counterclockwise rotation by an angle which is an irrational multiple of  $2\pi$ . Then every point is aperiodic and thus, by Theorem 2.3.7,  $C(\mathbb{T})$  is maximal abelian in the associated crossed product.

**Example 2.3.10.** Let  $X = \mathbb{T}$  and  $\sigma$  counterclockwise rotation by an angle which is a rational multiple of  $2\pi$ , say  $2\pi p/q$  (where  $p, q$  are relatively prime positive integers). Then every point on the circle has period precisely  $q$ , and the aperiodic points are certainly not dense. Using Corollary 2.3.4 we see that

$$C(\mathbb{T})' = \left\{ \sum_{n \in \mathbb{I}} f_{nq} \delta^{nq} \mid f_{nq} \in C(\mathbb{T}) \right\}.$$

We will use the following theorem to display an example different in nature from the ones already considered.

**Theorem 2.3.11.** *Let  $X$  be a topological space,  $\sigma : X \rightarrow X$  a homeomorphism, and  $A$  a non-zero subalgebra of  $C(X)$ , invariant both under the usual induced automorphism  $\tilde{\sigma} : C(X) \rightarrow C(X)$  and under its inverse. Assume that  $A$  separates the points of  $X$  and is such that every non-empty open set  $U \subseteq X$  is a domain of uniqueness for  $A$ . Then  $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  if and only if  $\sigma$  is not of finite order (that is,  $\sigma^n \neq \text{id}_X$  for any non-zero integer  $n$ ).*

*Proof.* By Corollary 2.3.4,  $A$  being maximal abelian implies that  $\sigma$  is not of finite order. Indeed, if  $\sigma^p = \text{id}_X$ , where  $p$  is the smallest such positive integer, then  $X = \text{Per}^p(X)$  and hence  $f \delta^p \in A'$  for any  $f \in A$ . For the converse, assume that  $\sigma$  does not have finite order. The sets  $\text{Sep}^n(X)$  are non-empty open subsets of  $X$  for all  $n \neq 0$  and thus domains of uniqueness for  $A$  by assumption of the theorem. The implication now follows directly from Corollary 2.3.4.  $\square$

**Corollary 2.3.12.** *Let  $M$  be a connected complex manifold and suppose the function  $\sigma : M \rightarrow M$  is biholomorphic. If  $A \subseteq H(M)$  is a subalgebra of the algebra of holomorphic functions of  $M$  which separates the points of  $M$  and is invariant under the induced automorphism  $\tilde{\sigma}$  of  $C(M)$  and its inverse, then  $A \subseteq A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is maximal abelian if and only if  $\sigma$  is not of finite order.*

*Proof.* On connected complex manifolds, open sets are domains of uniqueness for  $H(M)$ . See for example [2].  $\square$

**Remark 2.3.13.** It is important to point out that the required conditions in Theorem 2.3.7 and Theorem 2.3.11 can only be simultaneously satisfied in case  $X$  consists of a single point and  $A = \mathbb{C}$ . To see this we assume that both conditions are satisfied. Note first of all that this implies that every non-empty open subset of  $X$  is dense. Assume to the contrary that there is a non-empty open subset  $U \subseteq X$  that is not dense in  $X$ . We may then choose a non-zero  $f \in A$  that vanishes on  $X \setminus U$ . Certainly,  $f$  must then vanish on  $V = X \setminus \bar{U}$ . As  $U$  is not dense, however, this implies that  $f$  is identically zero since the non-empty open set  $V$  is a domain of uniqueness by assumption. Hence we have a contradiction and conclude that every non-empty open subset of  $X$  is dense. Secondly, we note that since  $\mathbb{C}$  is Hausdorff

and  $A \subseteq C(X)$  separates the points of  $X$ ,  $X$  must be Hausdorff. Assume now that there are two distinct points  $p, q \in X$ . As  $X$  is Hausdorff we can separate them by two disjoint open sets. Since every non-empty open subset is dense, however, this is not possible and hence  $X$  consists of one point. If  $X = \{p\}$ , the only possibilities for  $A$  are  $\{0\}$  and  $\mathbb{C}$ . As the conditions in Theorem 2.3.7 imply existence of non-zero functions in  $A$  we conclude that  $A = \mathbb{C}$ . Conversely, if  $X = \{p\}$  and  $A = \mathbb{C}$ , the conditions in both Theorem 2.3.7 and Theorem 2.3.11 are satisfied.

## 2.4. Automorphisms of commutative semi-simple Banach algebras

### 2.4.1. Motivation

In the setup in Theorem 2.3.7 we concluded that we have an appealing equivalence between density of the aperiodic points of  $X$  and  $A$  being maximal abelian in the associated crossed product, under a certain condition on  $A$ . Example 2.3.9 and Example 2.3.10, respectively, show an instance where both these equivalent statements are true, and where they are false. Generally speaking, however, the density of the aperiodic points and  $A$  being maximal abelian are unrelated properties; all four logical possibilities can occur. We show this by giving two additional examples.

**Example 2.4.1.** As in Example 2.3.9, let  $X = \mathbb{T}$  be the unit circle and  $\sigma$  counterclockwise rotation by an angle that is an irrational multiple of  $2\pi$ . If we use  $A = \mathbb{C}$  instead of  $C(\mathbb{T})$ , the whole crossed product is commutative and thus  $A$  is clearly *not* maximal abelian in it. The aperiodic points, however, are of course still dense. Here we simply chose a subalgebra of  $C(\mathbb{T})$  so small that the homeomorphism  $\sigma$  was no longer visible in the crossed product.

**Example 2.4.2.** Let  $X = \mathbb{T} \cup \{0\}$  with the usual subspace topology from  $\mathbb{C}$ , and let  $\sigma'$  be such that it fixes the origin and rotates points on the circle counterclockwise with an angle that is an irrational multiple of  $2\pi$ . As function algebra  $A$ , we take  $C(\mathbb{T})$  and extend every function in it to  $X$  so that it vanishes in the origin. This is obviously an algebra of functions being continuous on  $X$ , which is invariant under  $\sigma'$  and its inverse. Since  $A$  separates points in  $X$ , Corollary 2.3.4 assures us that  $A$  is maximal abelian in the crossed product, even though the aperiodic points are not dense in  $X$ .

In the following example, the equivalence in Theorem 2.3.7 fails in the same fashion as in Example 2.4.2. It is included, however, since it will be illuminating to refer to it in what follows.

**Example 2.4.3.** As in Example 2.4.2, let  $X = \mathbb{T} \cup \{0\}$  and  $\sigma'$  the map defined as counterclockwise rotation by an angle that is an irrational multiple of  $2\pi$  on  $\mathbb{T}$  and  $\sigma'(0) = 0$ . Let  $A$  be the restriction to  $X$  of all continuous functions on the closed unit disc that are holomorphic on the open unit disc. By the maximum modulus theorem, none of these functions are non-zero solely in the origin. Thus, by Corollary 2.3.4, we again obtain a case where the aperiodic points are not dense, but  $A$  is a maximal abelian subalgebra in the crossed product.

In summary, we have now displayed three examples where we do not have an equivalence between algebra and topological dynamics as in Theorem 2.3.7. In the following subsection we prove a general result - in the context of automorphisms of Banach algebras - that in particular shows that for a certain class of pairs of discrete dynamical systems and  $\mathbb{Z}$ -invariant function algebras on it,  $((X, \sigma), A)$ , yielding the associated crossed product  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  as usual, one can always find another such pair  $((Y, \phi), B)$  with associated crossed product  $B \rtimes_{\tilde{\phi}} \mathbb{Z}$  *canonically isomorphic* to  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$ , where the equivalence *does* hold: the aperiodic points of  $Y$  are dense if and only if  $B$  is maximal abelian in  $B \rtimes_{\tilde{\phi}} \mathbb{Z}$  (which it is, by the canonical isomorphism, if and only if  $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$ ). In this way, the equivalence of an algebraic property with a topological dynamical property is restored. Examples 2.4.1 through 2.4.3 all fall into this mentioned class of pairs, as we will see in Example 2.4.10 - 2.4.12.

### 2.4.2. A system on the character space

We will now focus solely on Banach algebras, and start by recalling a number of basic results concerning them. We refer to [3] for details. All Banach algebras under consideration will be *complex* and *commutative*.

**Definition 2.4.4.** Let  $A$  be a complex commutative Banach algebra. The set of all non-zero multiplicative linear functionals on  $A$  is denoted by  $\Delta(A)$  and called the *character space* of  $A$ .

**Definition 2.4.5.** Given any  $a \in A$ , we define a function  $\hat{a} : \Delta(A) \rightarrow \mathbb{C}$  by  $\hat{a}(\mu) = \mu(a)$  ( $\mu \in \Delta(A)$ ). The function  $\hat{a}$  is called the *Gelfand transform* of  $a$ . Let  $\hat{A} = \{\hat{a} \mid a \in A\}$ . The character space  $\Delta(A)$  is endowed with the topology generated by  $\hat{A}$ , which is called the *Gelfand topology* on  $\Delta(A)$ . The Gelfand topology is locally compact and Hausdorff. A commutative Banach algebra  $A$  for which the Gelfand transform, i.e., the map sending  $a$  to  $\hat{a}$ , is injective, is called *semi-simple*.

Let  $A$  be a commutative semi-simple complex Banach algebra and  $\tilde{\sigma} : A \rightarrow A$  an algebra automorphism. Then  $\tilde{\sigma}$  induces a bijection  $\sigma : \Delta(A) \rightarrow \Delta(A)$  defined by  $\sigma(\mu) = \mu \circ \tilde{\sigma}^{-1}$ , ( $\mu \in \Delta(A)$ ), which is automatically a homeomorphism when  $\Delta(A)$  has the Gelfand topology. Note that by semi-simplicity of  $A$ , the map

$$\phi : \text{Aut}(A) \rightarrow \{\sigma \in \text{Homeo}(\Delta(A)) \mid \hat{a} \circ \sigma, \hat{a} \circ \sigma^{-1} \in \hat{A} \text{ for all } a \in A\}$$

defined by

$$\phi(\tilde{\sigma})(\mu) = \mu \circ \tilde{\sigma}^{-1}$$

is an isomorphism of groups. In turn,  $\sigma$  induces an automorphism  $\hat{\sigma}$  on  $\hat{A}$  as in Section 2.3, namely  $\hat{\sigma}(\hat{a}) = \hat{a} \circ \sigma^{-1} = \widehat{\tilde{\sigma}(a)}$ .

The following result shows that in the context of a semi-simple Banach algebra one may pass to an isomorphic crossed product, but now with an algebra of continuous functions on a topological space. It is here that topological dynamics can be brought into play again. The proof consists of a trivial direct verification.

**Theorem 2.4.6.** *Let  $A$  be a commutative semi-simple Banach algebra and  $\tilde{\sigma}$  an automorphism, inducing an automorphism  $\hat{\sigma} : \hat{A} \rightarrow \hat{A}$  as above. Then the map  $\Phi : A \rtimes_{\tilde{\sigma}} \mathbb{Z} \rightarrow \hat{A} \rtimes_{\hat{\sigma}} \mathbb{Z}$  defined by  $\sum_{n \in \mathbb{Z}} a_n \delta^n \mapsto \sum_{n \in \mathbb{Z}} \hat{a}_n \delta^n$  is an isomorphism of algebras mapping  $A$  onto  $\hat{A}$ .*

**Definition 2.4.7.** A commutative Banach algebra  $A$  is said to be *completely regular* if for every subset  $F \subseteq \Delta(A)$  that is closed in the Gelfand-topology and for every  $\phi_0 \in \Delta(A) \setminus F$  there exists an  $a \in A$  such that  $\hat{a}(\phi) = 0$  for all  $\phi \in F$  and  $\hat{a}(\phi_0) \neq 0$ . In Banach algebra theory it is proved that  $A$  is completely regular if and only if the hull-kernel topology on  $\Delta(A)$  coincides with the Gelfand topology, see for example [1].

**Theorem 2.4.8.** *Let  $A$  be a commutative completely regular semi-simple Banach algebra,  $\tilde{\sigma} : A \rightarrow A$  an algebra automorphism and  $\sigma$  the homeomorphism of  $\Delta(A)$  in the Gelfand topology induced by  $\tilde{\sigma}$  as described above. Then the aperiodic points of  $(\Delta(A), \sigma)$  are dense if and only if  $\hat{A}$  is a maximal abelian subalgebra of  $\hat{A} \rtimes_{\hat{\sigma}} \mathbb{Z}$ . In particular,  $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  if and only if the aperiodic points of  $(\Delta(A), \sigma)$  are dense.*

*Proof.* As mentioned in Definition 2.4.5,  $\Delta(A)$  is locally compact Hausdorff in the Gelfand topology, and clearly  $\hat{A}$  is by definition a separating function algebra on it. Since we assumed  $A$  to be completely regular, we see that all the conditions assumed in Theorem 2.3.7 are satisfied, and thus this theorem yields the equivalence. Furthermore, by Theorem 2.4.6,  $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  if and only if  $\hat{A}$  is maximal abelian in  $\hat{A} \rtimes_{\hat{\sigma}} \mathbb{Z}$ .  $\square$

**Theorem 2.4.9.** *Let  $A$  be a commutative semi-simple Banach algebra such that  $\hat{A}$  separates the points of  $\Delta(A)$  and every open set of  $\Delta(A)$  is a domain of uniqueness for  $\hat{A}$ . Let  $\tilde{\sigma} : A \rightarrow A$  be an algebra automorphism and  $\sigma$  the homeomorphism of  $\Delta(A)$  in the Gelfand topology induced by  $\tilde{\sigma}$  as described above. Then  $\hat{A}$  is a maximal abelian subalgebra of  $\hat{A} \rtimes_{\hat{\sigma}} \mathbb{Z}$  precisely when  $\hat{\sigma}$  is not of finite order. In particular,  $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  precisely when  $\tilde{\sigma}$  is not of finite order.*

*Proof.* By semi-simplicity of  $A$  it is clear that the induced homeomorphism  $\sigma$  of  $\Delta(A)$  is of finite order precisely when  $\tilde{\sigma}$  is of finite order. By Theorem 2.3.11 the equivalence follows and it is easy to see that  $\hat{\sigma}$  is, by construction, of finite order precisely when  $\tilde{\sigma}$  is. And, by Theorem 2.4.6,  $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  if and only if  $\hat{A}$  is maximal abelian in  $\hat{A} \rtimes_{\hat{\sigma}} \mathbb{Z}$ .  $\square$

We shall now revisit Examples 2.4.1 through 2.4.3 and use Theorem 2.4.8 and Theorem 2.4.9 to conclude algebraic properties from topological dynamics after all.

**Example 2.4.10.** Consider again Example 2.4.1. Obviously  $\Delta(\mathbb{C}) = \{\text{id}_{\mathbb{C}}\}$ . Thus  $\mathbb{C}$  is a commutative semi-simple completely regular Banach algebra. Trivially, a set with only one element has no aperiodic point, so that  $A = \mathbb{C}$  is not maximal abelian by Theorem 2.4.8.

**Example 2.4.11.** Consider again Example 2.4.2. Clearly the function algebra  $A$  on  $X$  is isometrically isomorphic to  $C(\mathbb{T})$  (when both algebras are endowed with the sup-norm) and thus a commutative completely regular semi-simple Banach algebra. It is furthermore a well known result from the theory of Banach algebras that  $\Delta(C(X)) = \{\mu_x \mid x \in X\}$  for any compact Hausdorff space  $X$ , where  $\mu_x$  denotes the point evaluation in  $x$ , and that

$\Delta(C(X))$  equipped with the Gelfand topology is homeomorphic to  $X$  (see for example [3]). Thus clearly here  $\Delta(A) = \Delta(C(\mathbb{T})) = \mathbb{T}$ . Of course the induced map  $\sigma$  defined on  $\Delta(A)$  by  $\sigma(\mu_x) = \mu_{\sigma(x)}$  can simply be identified with rotation  $\sigma'$  of the circle by an angle that is an irrational multiple of  $2\pi$ . So on  $\Delta(A)$  the aperiodic points are dense, and hence  $A$  is maximal abelian by Theorem 2.4.8. Here passing to the system on the character space corresponded to deleting the origin from  $X$ , thus recovering  $\mathbb{T}$ , and restricting  $A$  to  $\mathbb{T}$ , hence recovering  $C(\mathbb{T})$  so that in the end we recovered the setup of Example 2.3.9.

**Example 2.4.12.** Consider again Example 2.4.3. Using the maximum modulus theorem one sees that  $A$  is isometrically isomorphic to  $\mathcal{A}(\overline{\mathbb{D}})$ , the algebra of all functions that are continuous on the unit circle and holomorphic on the open unit disc (denoted by  $\mathbb{D}$ ), as  $A$  is the restrictions of such functions to  $\mathbb{T} \cup \{0\}$ . It is a standard result from Banach algebra theory that  $\Delta(\mathcal{A}(\overline{\mathbb{D}}))$  (endowed with the Gelfand topology) is canonically homeomorphic to  $\overline{\mathbb{D}}$ ; the elements in  $\Delta(\mathcal{A}(\overline{\mathbb{D}}))$  are precisely the point evaluations in  $\overline{\mathbb{D}}$ , and that  $\mathcal{A}(\overline{\mathbb{D}})$  may thus be identified with its own Gelfand transform (see [3]). Furthermore open subsets of  $\overline{\mathbb{D}}$  are domains of uniqueness for  $\mathcal{A}(\overline{\mathbb{D}})$  (see for example [2]). So we conclude that  $\Delta(A)$  is also equal to  $\overline{\mathbb{D}}$  and that  $A$  is a commutative semi-simple Banach algebra whose Gelfand transform separates the points of  $\Delta(A)$  and open subsets of  $\Delta(A)$  are domains of uniqueness for  $\widehat{A}$ . The induced homeomorphism  $\sigma$  on  $\Delta(A) = \overline{\mathbb{D}}$  is rotation by the same angle as for  $\sigma'$ . It is now clear that  $\sigma$  is not of finite order, so that  $A$  is maximal abelian by Theorem 2.4.9.

Note the difference in nature between Examples 2.4.11 and 2.4.12. In the former, passing to the system on the character space corresponds to deleting a point from the original system and restricting the function algebra and homeomorphism, while in the latter it corresponds to adding (a lot of) points and extending.

We conclude this subsection by giving yet another example of an application of Theorem 2.4.8, recovering one of the results we obtained in Example 2.3.8 by using Theorem 2.3.7.

**Example 2.4.13.** Let  $X$  be a locally compact Hausdorff space, and  $\sigma : X \rightarrow X$  a homeomorphism. Let  $A = C_0(X)$ ; then  $\sigma$  induces an automorphism  $\tilde{\sigma}$  on  $A$  as in Section 2.3. Here  $\Delta(A)$  is canonically homeomorphic to  $X$  and  $A$  is canonically isomorphic (with respect to the homeomorphism between  $\Delta(A)$  and  $X$ ) to  $\widehat{A}$  (see for example [3]). Hence by Theorem 2.4.8  $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  if and only if the aperiodic points of  $X$  are dense, as already mentioned in Example 2.3.8.

### 2.4.3. Integrable functions on locally compact abelian groups

In this subsection we consider the crossed product  $L_1(G) \rtimes_{\Psi} \mathbb{Z}$ , where  $G$  is a locally compact abelian group and  $\Psi : L_1(G) \rightarrow L_1(G)$  an automorphism. We will show that under an additional condition on  $G$ , a stronger result than Theorem 2.4.8 is true (cf. Theorem 2.4.17).

We start by recalling a number of standard results from the theory of Fourier analysis on groups, and refer to [3] and [6] for details. Let  $G$  be a locally compact abelian group. Recall that  $L_1(G)$  consists of equivalence classes of complex-valued Borel measurable functions of  $G$  that are integrable with respect to a Haar measure on  $G$ , and that  $L_1(G)$  equipped with convolution product is a commutative completely regular semi-simple Banach algebra. A



group homomorphism  $\gamma : G \rightarrow \mathbb{T}$  from a locally compact abelian group  $G$  to the unit circle is called a *character* of  $G$ . The set of all *continuous* characters of  $G$  forms a group  $\Gamma$ , the *dual group* of  $G$ , if the group operation is defined by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \quad (x \in G; \gamma_1, \gamma_2 \in \Gamma).$$

If  $\gamma \in \Gamma$  and if

$$\widehat{f}(\gamma) = \int_G f(x)\gamma(-x)dx \quad (f \in L_1(G)),$$

then the map  $f \mapsto \widehat{f}(\gamma)$  is a non-zero complex homomorphism of  $L_1(G)$ . Conversely, every non-zero complex homomorphism of  $L_1(G)$  is obtained in this way, and distinct characters induce distinct homomorphisms. Thus we may identify  $\Gamma$  with  $\Delta(L_1(G))$ . The function  $\widehat{f} : \Gamma \rightarrow \mathbb{C}$  defined as above is called the *Fourier transform* of  $f$ ,  $f \in L_1(G)$ , and is hence precisely the Gelfand transform of  $f$ . We denote the set of all such  $\widehat{f}$  by  $A(\Gamma)$ . Furthermore,  $\Gamma$  is a locally compact abelian group in the Gelfand topology.

**Definition 2.4.14.** Given a set  $X$ , a *ring of subsets* of  $X$  is a collection of subsets of  $X$  which is closed under the formation of finite unions, finite intersections, and complements (in  $X$ ). Note that any intersection of rings is again a ring. The *coset-ring* of  $\Gamma$  is defined to be the smallest ring of subsets of  $\Gamma$  which contains all *open* cosets, i.e., all subsets of  $\Gamma$  of the form  $a + U$ , where  $a \in \Gamma$  and  $U$  is an open subgroup of  $\Gamma$ .

We are now ready to define a particular type of map on the coset ring of  $\Gamma$  (cf. [6]).

**Definition 2.4.15.** Let  $E$  be a coset in  $\Gamma$ . A continuous map  $\sigma : E \rightarrow \Gamma$  which satisfies the identity

$$\sigma(\gamma + \gamma' - \gamma'') = \sigma(\gamma) + \sigma(\gamma') - \sigma(\gamma'') \quad (\gamma, \gamma', \gamma'' \in E)$$

is called *affine*. Suppose that

- (i)  $S_1, \dots, S_n$  are pairwise disjoint sets belonging to the coset-ring of  $\Gamma$ ;
- (ii) each  $S_i$  is contained in an open coset  $K_i$  in  $\Gamma$ ;
- (iii) for each  $i$ ,  $\sigma_i$  is an affine map of  $K_i$  into  $\Gamma$ ;
- (iv)  $\sigma$  is the map of  $Y = S_1 \cup \dots \cup S_n$  into  $\Gamma$  which coincides on  $S_i$  with  $\sigma_i$ .

Then  $\sigma$  is said to be a *piecewise affine* map from  $Y$  to  $\Gamma$ .

The following theorem is a key result for what follows. It states that every automorphism of  $L_1(G)$  is induced by a piecewise affine homeomorphism, and that a piecewise affine homeomorphism induces an injective homomorphism from  $L_1(G)$  to itself.

**Theorem 2.4.16.** *Let  $\tilde{\sigma} : L_1(G) \rightarrow L_1(G)$  be an automorphism. Then for every  $f \in L_1(G)$  we have that  $\widehat{\tilde{\sigma}(f)} = \widehat{f} \circ \sigma$ , where  $\sigma : \Gamma \rightarrow \Gamma$  is a fixed piecewise affine homeomorphism. Also, if  $\sigma : \Gamma \rightarrow \Gamma$  is a piecewise affine homeomorphism, then  $\widehat{f} \circ \sigma \in A(\Gamma)$  for every  $\widehat{f} \in A(\Gamma)$ .*

*Proof.* Follows from the more general Theorems 4.1.3 and 4.6.2 in [6].  $\square$

Now let  $\tilde{\sigma} : L_1(G) \rightarrow L_1(G)$  be an automorphism and consider the crossed product  $L_1(G) \rtimes_{\tilde{\sigma}} \mathbb{Z}$ . Letting  $\tilde{\sigma}$  induce a homeomorphism as described for arbitrary commutative completely regular semi-simple Banach algebras in the paragraph following Definition 2.4.5, we obtain  $\sigma^{-1} : \Gamma \rightarrow \Gamma$ , where  $\sigma$  is the piecewise affine homeomorphism inducing  $\tilde{\sigma}$  in accordance with Theorem 2.4.16.

**Theorem 2.4.17.** *Let  $G$  be a locally compact abelian group with connected dual group and let  $\tilde{\sigma} : L_1(G) \rightarrow L_1(G)$  be an automorphism. Then  $L_1(G)$  is maximal abelian in  $L_1(G) \rtimes_{\tilde{\sigma}} \mathbb{Z}$  if and only if  $\tilde{\sigma}$  is not of finite order.*

*Proof.* Denote by  $\Gamma$  the dual group of  $G$ . By Theorem 2.4.8,  $\sigma$ , the homeomorphism induced by  $\tilde{\sigma}$  in accordance with the discussion following Definition 2.4.5, is not of finite order if  $L_1(G)$  is maximal abelian. Assume now that  $L_1(G)$  is not maximal abelian. By Theorem 2.4.8, this implies that  $\text{Per}^\infty(\Gamma)$  is not dense in  $\Gamma$ . As argued in the proof of Theorem 2.3.7, there must then exist  $n_0 \in \mathbb{N}$  such that  $\text{Per}^{n_0}(\Gamma)$  has non-empty interior. Namely, since in this case  $\bigcap_{n \in \mathbb{Z}_{>0}} (\Gamma \setminus \text{Per}^n(\Gamma))$  is not dense and that the sets  $\Gamma \setminus \text{Per}^n(\Gamma)$ ,  $n \in \mathbb{Z}_{>0}$  are all open, the fact that  $\Gamma$  is a Baire space (being locally compact and Hausdorff) implies existence of an  $n_0 \in \mathbb{Z}_{>0}$  such that  $\text{Per}^{n_0}(\Gamma)$  has non-empty interior. Note that  $\Gamma$  being connected implies that  $\sigma^{-1}$ , the piecewise affine homeomorphism of  $\Gamma$  inducing  $\tilde{\sigma}$  in accordance with Theorem 2.4.16, must be affine by connectedness of  $\Gamma$  (the coset-ring is trivially  $\{\emptyset, \Gamma\}$ ) and hence so is  $\sigma$ . It is readily verified that the map  $\sigma^{n_0} - I$  is then also affine. Now clearly  $\text{Per}^{n_0}(\Gamma) = (\sigma^{n_0} - I)^{-1}(\{0\})$ . The affine nature of  $\sigma^{n_0} - I$  assures us that  $\text{Per}^{n_0}(\Gamma)$  is a coset. In a topological group, however, continuity of the group operations implies that cosets with non-empty interior are open, hence also closed. We conclude that  $\text{Per}^{n_0}(\Gamma)$  is a non-empty closed and open set. Connectedness of  $\Gamma$  now implies that every point in  $\Gamma$  is  $n_0$ -periodic under  $\sigma$ . Hence, by the discussion following Definition 2.4.5,  $\tilde{\sigma}^{n_0}$  is the identity map on  $L_1(G)$ .  $\square$

The following example shows that if the dual of  $G$  is not connected, the equivalence in Theorem 2.4.17 need not hold.

**Example 2.4.18.** Let  $G = \mathbb{T}$  be the circle group. Here  $\Gamma = \mathbb{Z}$  (see [3] for details), which is not connected. Define  $\sigma : \mathbb{Z} \rightarrow \mathbb{Z}$  by  $\sigma(n) = n$  ( $n \in 2\mathbb{Z}$ ) and  $\sigma(m) = m + 2$  ( $m \in 1 + 2\mathbb{Z}$ ). Obviously  $\sigma$  and  $\sigma^{-1}$  are then piecewise affine homeomorphisms that are not of finite order. By Theorem 2.4.16,  $\sigma$  induces an automorphism  $\tilde{\sigma}^{-1} : L_1(\mathbb{T}) \rightarrow L_1(\mathbb{T})$ , which in turn induces the homeomorphism  $\sigma^{-1} : \mathbb{Z} \rightarrow \mathbb{Z}$ . Now  $A(\mathbb{Z})$  is not maximal abelian in  $A(\mathbb{Z}) \rtimes_{\tilde{\sigma}^{-1}} \mathbb{Z}$  since by Corollary 2.3.4 we have

$$A(\mathbb{Z})' = \left\{ \sum_{n \in \mathbb{Z}} \widehat{f}_n \delta^n \mid \text{for all } n \in \mathbb{Z} \setminus \{0\} : \text{supp}(\widehat{f}_n) \subseteq 2\mathbb{Z} \right\},$$

and hence by Theorem 2.4.6

$$L_1(\mathbb{T})' = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} \setminus \{0\} : \text{supp}(\widehat{f}_n) \subseteq 2\mathbb{Z} \right\}.$$

Note that

$$\left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid \text{for all } n \in \mathbb{Z} \setminus \{0\} : f_n \in \mathbb{C}[z^2, z^{-2}] \right\} \subseteq L_1(\mathbb{T})',$$

and thus we conclude that  $L_1(\mathbb{T})$  is not maximal abelian.

### 2.4.4. A theorem on generators for the commutant

A natural question to ask is whether or not the commutant  $A' \subseteq A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is finitely generated as an algebra over  $\mathbb{C}$  or not. Here we give an answer in the case when  $A$  is a semi-simple commutative Banach algebra.

**Theorem 2.4.19.** *Let  $A$  be a semi-simple commutative Banach algebra and let  $\tilde{\sigma} : A \rightarrow A$  be an automorphism. Then  $A'$  is finitely generated as an algebra over  $\mathbb{C}$  if and only if  $A$  has finite dimension as a vector space over  $\mathbb{C}$ .*

*Proof.* Let the induced homeomorphism  $\sigma : \Delta(A) \rightarrow \Delta(A)$  be as usual. Assume first that  $A$  has infinite dimension. By basic theory of Banach spaces,  $A$  must then have uncountable dimension. If  $A'$  were generated by finitely many elements, then  $A'$ , and in particular  $A$ , would have countable dimension, which is a contradiction. Hence  $A'$  is not finitely generated as an algebra over  $\mathbb{C}$ . For the converse, we need two results from Banach algebra theory. Suppose that  $A$  has finite dimension. By Proposition 26.7 in [1],  $\Delta(A)$  is then a finite set, and thus there exists a positive integer  $n_0$  such that  $\sigma^{n_0} = \text{id}$ . Furthermore, by Corollary 21.6 in [1]  $A$  must then also be unital. We pass now to the crossed product  $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ , which is isomorphic to  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  by Theorem 2.4.6. Clearly  $\widehat{A}$  is unital and has finite linear dimension since  $A$  does. By Corollary 2.3.4, for a general element  $\sum_{n \in \mathbb{Z}} \widehat{a}_n \delta^n \in (\widehat{A})'$  the set of possible coefficients of  $\delta^n$  is a vector subspace (and even an ideal) of  $\widehat{A}$ ,  $K_n$  say, and hence of finite dimension. Since  $\sigma^{n_0} = \text{id}$ , Corollary 2.3.4 also tells us that  $K_{r+l \cdot n_0} = K_r$  for all  $r, l \in \mathbb{Z}$ . Now note that since  $\widehat{A}$  is unital,  $\delta^{n_0}, \delta^{-n_0} \in (\widehat{A})'$ . Thus, denoting a basis for a  $K_l$  by  $\{e_{(l,1)}, \dots, e_{(l,l_r)}\}$  (where  $l_r \leq s$ ), the above reasoning assures us that  $\bigcup_{l=1}^{n_0} \bigcup_{j=1}^{l_r} \{e_{(l,j)} \delta^j\}$  generates  $(\widehat{A})'$  as an algebra over  $\mathbb{C}$ . By Theorem 2.4.6 this implies that also  $A' \subseteq A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is finitely generated as an algebra over  $\mathbb{C}$ .  $\square$

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## Chapter 3

# Connections between dynamical systems and crossed products of Banach algebras by $\mathbb{Z}$

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**Abstract.** Starting with a complex commutative semi-simple completely regular Banach algebra  $A$  and an automorphism  $\sigma$  of  $A$ , we form the crossed product of  $A$  by the integers, where the latter act on  $A$  via iterations of  $\sigma$ . The automorphism induces a topological dynamical system on the character space  $\Delta(A)$  of  $A$  in a natural way. We prove equivalence between the property that every non-zero ideal of the crossed product has non-zero intersection with the subalgebra  $A$ , maximal commutativity of  $A$  in the crossed product, and density of the aperiodic points of the induced system on the character space. We also prove that every non-trivial ideal of the crossed product always intersects the commutant of  $A$  non-trivially. Furthermore, under the assumption that  $A$  is unital and such that  $\Delta(A)$  consists of infinitely many points, we show equivalence between simplicity of the crossed product and minimality of the induced system, and between primeness of the crossed product and topological transitivity of the system.

### 3.1. Introduction

A lot of work has been done in the direction of connections between certain topological dynamical systems and crossed product  $C^*$ -algebras. In [6] and [7], for example, one starts with a homeomorphism  $\sigma$  of a compact Hausdorff space  $X$  and constructs the crossed prod-

uct  $C^*$ -algebra  $C(X) \rtimes_{\alpha} \mathbb{Z}$ , where  $C(X)$  is the algebra of continuous complex-valued functions on  $X$  and  $\alpha$  is the automorphism of  $C(X)$  naturally induced by  $\sigma$ . One of many results obtained is equivalence between simplicity of the algebra and minimality of the system, provided that  $X$  consists of infinitely many points, see [1], [3], [6], [7] or, for a more general result, [8]. In [5], a purely algebraic variant of the crossed product is considered, with more general classes of algebras than merely continuous functions on compact Hausdorff spaces serving as “coefficient algebras”. For example, it was proved there that, for such crossed products, the analogue of the equivalence between density of aperiodic points of a dynamical system and maximal commutativity of the “coefficient algebra” in the associated crossed product  $C^*$ -algebra is true for significantly larger classes of coefficient algebras and associated dynamical systems. In this paper, we go beyond these results and investigate the ideal structure of some of the crossed products considered in [5]. More specifically, we consider crossed products of complex commutative semi-simple completely regular Banach algebras  $A$  by the integers under an automorphism  $\sigma : A \rightarrow A$ .

In Section 3.2 we give the most general definition of the kind of crossed product that we will use throughout this paper. We also mention the elementary result that the commutant of the coefficient algebra is automatically a commutative subalgebra of the crossed product. The more specific setup that we will be working in is introduced in Section 3.3. There we also introduce some notation and mention two basic results concerning a canonical isomorphism between certain crossed products, and an explicit description of the commutant of the coefficient algebra in one of them.

According to [7, Theorem 5.4], the following three properties are equivalent:

- The aperiodic points of  $(X, \sigma)$  are dense in  $X$ ;
- Every non-zero closed ideal  $I$  of the crossed product  $C^*$ -algebra  $C(X) \rtimes_{\alpha} \mathbb{Z}$  is such that  $I \cap C(X) \neq \{0\}$ ;
- $C(X)$  is a maximal abelian  $C^*$ -subalgebra of  $C(X) \rtimes_{\alpha} \mathbb{Z}$ .

In Section 3.4, an analogue of this result is proved for our setup. A reader familiar with the theory of crossed product  $C^*$ -algebras will easily recognize that if one chooses  $A = C(X)$  for  $X$  a compact Hausdorff space in Corollary 3.4.5 below, then the crossed product is canonically isomorphic to a norm-dense subalgebra of the crossed product  $C^*$ -algebra coming from the considered induced dynamical system. We also combine this with a theorem from [5] to conclude a stronger result for the Banach algebra  $L_1(G)$ , where  $G$  is a locally compact abelian group with connected dual group.

In Section 3.5, we prove the equivalence between algebraic simplicity of the crossed product and minimality of the induced dynamical system in the case when  $A$  is unital with its character space consisting of infinitely many points. This is analogous to [7, Theorem 5.3], [1, Theorem VIII 3.9], the main result in [3] and, as a special case of a more general result, [8, Corollary 8.22] for the crossed product  $C^*$ -algebra.

In Section 3.6, the fact that the commutant of  $A$  always has non-zero intersection with every non-zero ideal of the crossed product is shown. This should be compared with the fact that  $A$  itself may well have zero intersection with such ideals, as Corollary 3.4.5 shows. The analogue of this result in the context of crossed product  $C^*$ -algebras was open at the time this paper was submitted.

Finally, in Section 3.7 we show equivalence between primeness of the crossed product and topological transitivity of the induced system, in the case when  $A$  is unital and has an infinite character space. The analogue of this in the context of crossed product  $C^*$ -algebras is [7, Theorem 5.5].

## 3.2. Definition and a basic result

Let  $A$  be an associative commutative complex algebra and let  $\Psi : A \rightarrow A$  be an algebra automorphism. Consider the set

$$A \rtimes_{\Psi} \mathbb{Z} = \{f : \mathbb{Z} \rightarrow A \mid f(n) = 0 \text{ except for a finite number of } n\}.$$

We endow it with the structure of an associative complex algebra by defining scalar multiplication and addition as the usual pointwise operations. Multiplication is defined by *twisted convolution*,  $*$ , as follows;

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k) \cdot \Psi^k(g(n - k)),$$

where  $\Psi^k$  denotes the  $k$ -fold composition of  $\Psi$  with itself. It is trivially verified that  $A \rtimes_{\Psi} \mathbb{Z}$  is an associative  $\mathbb{C}$ -algebra under these operations. We call it the *crossed product of  $A$  by  $\mathbb{Z}$  under  $\Psi$* .

A useful way of working with  $A \rtimes_{\Psi} \mathbb{Z}$  is to write elements  $f, g \in A \rtimes_{\Psi} \mathbb{Z}$  in the form  $f = \sum_{n \in \mathbb{Z}} f_n \delta^n$ ,  $g = \sum_{m \in \mathbb{Z}} g_m \delta^m$ , where  $f_n = f(n)$ ,  $g_m = g(m)$ , addition and scalar multiplication are canonically defined, and multiplication is determined by  $(f_n \delta^n) * (g_m \delta^m) = f_n \cdot \Psi^n(g_m) \delta^{n+m}$ , where  $n, m \in \mathbb{Z}$  and  $f_n, g_m \in A$  are arbitrary.

Clearly one may canonically view  $A$  as an abelian subalgebra of  $A \rtimes_{\Psi} \mathbb{Z}$ , namely as  $\{f_0 \delta^0 \mid f_0 \in A\}$ . The following elementary result is proved in [5, Proposition 2.1].

**Proposition 3.2.1.** *The commutant  $A'$  of  $A$  is abelian, and thus it is the unique maximal abelian subalgebra containing  $A$ .*

## 3.3. Setup and two basic results

In what follows, we shall focus on cases when  $A$  is a commutative complex Banach algebra, and freely make use of the basic theory for such  $A$ , see e.g. [2]. As conventions tend to differ slightly in the literature, however, we mention that we call a commutative Banach algebra  $A$  *semi-simple* if the Gelfand transform on  $A$  is injective, and that we call it *completely regular* (the term *regular* is also frequently used in the literature) if, for every subset  $F \subseteq \Delta(A)$ , where  $\Delta(A)$  is the character space of  $A$ , that is closed in the Gelfand topology and for every  $\phi_0 \in \Delta(A) \setminus F$ , there exists an  $a \in A$  such that  $\widehat{a}(\phi) = 0$  for all  $\phi \in F$  and  $\widehat{a}(\phi_0) \neq 0$ . All topological considerations of  $\Delta(A)$  will be done with respect to its Gelfand topology (the weakest topology making all elements in the image of the Gelfand transform of  $A$  continuous on  $\Delta(A)$ ).

Now let  $A$  be a complex commutative semi-simple completely regular Banach algebra, and let  $\sigma : A \rightarrow A$  be an algebra automorphism. As in [5],  $\sigma$  induces a map  $\tilde{\sigma} : \Delta(A) \rightarrow \Delta(A)$  defined by  $\tilde{\sigma}(\mu) = \mu \circ \sigma^{-1}$ ,  $\mu \in \Delta(A)$ , which is automatically a homeomorphism when  $\Delta(A)$  is endowed with the Gelfand topology. Hence we obtain a topological dynamical system  $(\Delta(A), \tilde{\sigma})$ . In turn,  $\tilde{\sigma}$  induces an automorphism  $\widehat{\sigma} : \widehat{A} \rightarrow \widehat{A}$  (where  $\widehat{A}$  denotes the algebra of Gelfand transforms of all elements of  $A$ ) defined by  $\widehat{\sigma}(\widehat{a}) = \widehat{a} \circ \tilde{\sigma}^{-1} = \widehat{\sigma(a)}$ . Therefore we can form the crossed product  $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$ . We also mention that when speaking of ideals, we will always mean two-sided ideals.

In what follows, we shall make frequent use of the following fact. Its proof consists of a trivial direct verification.

**Theorem 3.3.1.** *Let  $A$  be a commutative semi-simple Banach algebra and  $\sigma$  an automorphism, inducing an automorphism  $\widehat{\sigma} : \widehat{A} \rightarrow \widehat{A}$  as above. Then the map  $\Phi : A \rtimes_{\sigma} \mathbb{Z} \rightarrow \widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$  defined by  $\sum_{n \in \mathbb{Z}} a_n \delta^n \mapsto \sum_{n \in \mathbb{Z}} \widehat{a}_n \delta^n$  is an isomorphism of algebras mapping  $A$  onto  $\widehat{A}$ .*

Before stating the next result, we make the following basic definitions.

**Definition 3.3.2.** For any nonzero  $n \in \mathbb{Z}$  we set

$$\text{Per}^n(\Delta(A)) = \{\mu \in \Delta(A) \mid \mu = \tilde{\sigma}^n(\mu)\}.$$

Furthermore, we denote the aperiodic points by

$$\text{Per}^{\infty}(\Delta(A)) = \bigcap_{n \in \mathbb{Z} \setminus \{0\}} (\Delta(A) \setminus \text{Per}^n(\Delta(A))).$$

Finally, for  $f \in \widehat{A}$ , put

$$\text{supp}(f) = \overline{\{\mu \in \Delta(A) \mid f(\mu) \neq 0\}}.$$

We can now give the following explicit description of  $\widehat{A}'$  in  $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$ .

**Theorem 3.3.3.**

$$\widehat{A}' = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid f_n \in \widehat{A}, \text{ and for all } n \in \mathbb{Z} : \text{supp}(f_n) \subseteq \text{Per}^n(\Delta(A)) \right\}.$$

*Proof.* This follows from [5, Corollary 3.4], as  $\widehat{A}$  trivially separates the points of  $\Delta(A)$  and  $\text{Per}^n(\Delta(A))$  is a closed set.  $\square$

### 3.4. Three equivalent properties

In this section we shall conclude that, for certain  $A$ , two different algebraic properties of  $A \rtimes_{\sigma} \mathbb{Z}$  are equivalent to density of the aperiodic points of the naturally associated dynamical system on the character space  $\Delta(A)$ , and hence obtain equivalence of three different



properties. The analogue of this result in the context of crossed product  $C^*$ -algebras is [7, Theorem 5.4]. We shall also combine this with a theorem from [5] to conclude a stronger result for the Banach algebra  $L_1(G)$ , where  $G$  is a locally compact abelian group with connected dual group.

In [5, Theorem 4.8], the following result is proved.

**Theorem 3.4.1.** *Let  $A$  be a complex commutative completely regular semi-simple Banach algebra,  $\sigma : A \rightarrow A$  an automorphism and  $\tilde{\sigma}$  the homeomorphism of  $\Delta(A)$  in the Gelfand topology induced by  $\sigma$  as described above. Then the aperiodic points are dense in  $\Delta(A)$  if and only if  $\hat{A}$  is a maximal abelian subalgebra of  $\hat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ . In particular,  $A$  is maximal abelian in  $A \rtimes_{\sigma} \mathbb{Z}$  if and only if the aperiodic points are dense in  $\Delta(A)$ .*

We shall soon prove another algebraic property of the crossed product equivalent to density of the aperiodic points of the induced system on the character space. First, however, we need two easy topological lemmas.

**Lemma 3.4.2.** *Let  $x \in \Delta(A)$  be such that the points  $\tilde{\sigma}^i(x)$  are distinct for all  $i$  such that  $-m \leq i \leq n$ , where  $n$  and  $m$  are positive integers. Then there exist an open set  $U_x$  containing  $x$  such that the sets  $\tilde{\sigma}^i(U_x)$  are pairwise disjoint for all  $i$  such that  $-m \leq i \leq n$ .*

*Proof.* It is easily checked that any finite set of points in a Hausdorff space can be separated by pairwise disjoint open sets. Separate the points  $\tilde{\sigma}^i(x)$  with disjoint open sets  $V_i$ . Then it is readily verified that the set

$$U_x := \tilde{\sigma}^m(V_{-m}) \cap \tilde{\sigma}^{m-1}(V_{-m+1}) \cap \dots \cap V_0 \cap \tilde{\sigma}^{-1}(V_1) \cap \dots \cap \tilde{\sigma}^{-n}(V_n)$$

is an open neighbourhood of  $x$  with the required property.  $\square$

**Lemma 3.4.3.** *The aperiodic points of  $(\Delta(A), \tilde{\sigma})$  are dense if and only if  $\text{Per}^n(\Delta(A))$  has empty interior for all positive integers  $n$ .*

*Proof.* Clearly, if there is a positive integer  $n_0$  such that  $\text{Per}^{n_0}(\Delta(A))$  has non-empty interior, the aperiodic points are not dense. For the converse, we recall that  $\Delta(A)$  is a Baire space since it is locally compact and Hausdorff, and note that we may write

$$\Delta(A) \setminus \text{Per}^\infty(\Delta(A)) = \bigcup_{n>0} \text{Per}^n(\Delta(A)).$$

If the set of aperiodic points is not dense, its complement has non-empty interior, and as the sets  $\text{Per}^n(\Delta(A))$  are clearly all closed, there must exist an integer  $n_0 > 0$  such that  $\text{Per}^{n_0}(\Delta(A))$  has non-empty interior since  $\Delta(A)$  is a Baire space.  $\square$

We are now ready to prove the promised result.

**Theorem 3.4.4.** *Let  $A$  be a complex commutative semi-simple completely regular Banach algebra,  $\sigma : A \rightarrow A$  an automorphism and  $\tilde{\sigma}$  the homeomorphism of  $\Delta(A)$  in the Gelfand topology induced by  $\sigma$  as described above. Then the aperiodic points are dense in  $\Delta(A)$  if and only if every non-zero ideal  $I \subseteq A \rtimes_{\sigma} \mathbb{Z}$  is such that  $I \cap A \neq \{0\}$ .*

*Proof.* We first assume that  $\overline{\text{Per}^\infty(\Delta(A))} = \Delta(A)$ , and work initially in  $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ . Assume that  $I \subseteq \widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is a non-zero ideal, and that  $f = \sum_{n \in \mathbb{Z}} f_n \delta^n \in I$ . By definition, only finitely many  $f_n$  are non-zero. Denote the set of integers  $n$  for which  $f_n \neq 0$  by  $S = \{n_1, \dots, n_r\}$ . Pick an aperiodic point  $x \in \Delta(A)$  such that  $f_{n_1}(x) \neq 0$  (by density of  $\text{Per}^\infty(\Delta(A))$  such  $x$  exists). Using the fact that  $x$  is not periodic we may, by Lemma 3.4.2, choose an open neighbourhood  $U_x$  of  $x$  such that  $\tilde{\sigma}^{-n_i}(U_x) \cap \tilde{\sigma}^{-n_j}(U_x) = \emptyset$  for  $n_i \neq n_j$ ,  $n_i, n_j \in S$ . Now by complete regularity of  $A$  we can find a function  $g \in \widehat{A}$  that is non-zero in  $\tilde{\sigma}^{-n_1}(x)$ , and vanishes outside  $\tilde{\sigma}^{-n_1}(U_x)$ . Consider  $f * g = \sum_{n \in \mathbb{Z}} f_n \cdot (g \circ \tilde{\sigma}^{-n}) \delta^n$ . This is an element in  $I$  and clearly the coefficient of  $\delta^{n_1}$  is the only one that does not vanish on the open set  $U_x$ . Again by complete regularity of  $A$ , there is an  $h \in \widehat{A}$  that is non-zero in  $x$  and vanishes outside  $U_x$ . Clearly  $h * f * g = [h \cdot (g \circ \tilde{\sigma}^{-n_1}) f_{n_1}] \delta^{n_1}$  is a non-zero monomial belonging to  $I$ . Now any ideal that contains a non-zero monomial automatically contains a non-zero element of  $\widehat{A}$ . Namely, if  $a_i \delta^i \in I$  then  $[a_i \delta^i] * [(a_i \circ \tilde{\sigma}^i) \delta^{-i}] = a_i^2 \in \widehat{A}$ . By the canonical isomorphism in Theorem 3.3.1, the result holds for  $A \rtimes_{\sigma} \mathbb{Z}$  as well.

For the converse, assume that  $\overline{\text{Per}^\infty(\Delta(A))} \neq \Delta(A)$ . Again we work in  $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ . It follows from Lemma 3.4.3 that since  $\text{Per}^\infty(\Delta(A)) \neq \Delta(A)$ , there exists an integer  $n > 0$  such that  $\text{Per}^n(\Delta(A))$  has non-empty interior. As  $A$  is assumed to be completely regular, there exists  $f \in \widehat{A}$  such that  $\text{supp}(f) \subseteq \text{Per}^n(\Delta(A))$ . Consider now the ideal  $I = (f + f \delta^n)$ . Using that  $f$  vanishes outside  $\text{Per}^n(\Delta(A))$ , we may rewrite this as follows

$$\begin{aligned} a_i \delta^i (f + f \delta^n) a_j \delta^j &= [a_i \cdot (a_j \circ \tilde{\sigma}^{-i}) \delta^i] * [f \delta^j + f \delta^{n+j}] \\ &= [a_i \cdot (a_j \circ \tilde{\sigma}^{-i}) \cdot (f \circ \tilde{\sigma}^{-i})] \delta^{i+j} + [a_i \cdot (a_j \circ \tilde{\sigma}^{-i}) \cdot (f \circ \tilde{\sigma}^{-i})] \delta^{i+j+n}. \end{aligned}$$

This means that any element in  $I$  may be written in the form  $\sum_i (b_i \delta^i + b_i \delta^{n+i})$ . As  $i$  runs only through a finite subset of  $\mathbb{Z}$ , this is not a non-zero monomial. In particular, it is not a non-zero element in  $\widehat{A}$ . Hence  $I$  intersects  $\widehat{A}$  trivially. By the canonical isomorphism in Theorem 3.3.1, the result carries over to  $A \rtimes_{\sigma} \mathbb{Z}$ .  $\square$

Combining Theorem 3.4.1 and Theorem 3.4.4, we now have the following result.

**Corollary 3.4.5.** *Let  $A$  be a complex commutative semi-simple completely regular Banach algebra,  $\sigma : A \rightarrow A$  an automorphism and  $\tilde{\sigma}$  the homeomorphism of  $\Delta(A)$  in the Gelfand topology induced by  $\sigma$  as described above. Then the following three properties are equivalent:*

- *The aperiodic points  $\text{Per}^\infty(\Delta(A))$  of  $(\Delta(A), \tilde{\sigma})$  are dense in  $\Delta(A)$ ;*
- *Every non-zero ideal  $I \subseteq A \rtimes_{\sigma} \mathbb{Z}$  is such that  $I \cap A \neq \{0\}$ ;*
- *$A$  is a maximal abelian subalgebra of  $A \rtimes_{\sigma} \mathbb{Z}$ .*

We shall make use of Corollary 3.4.5 to conclude a result for a more specific class of Banach algebras. We start by recalling a number of standard results from the theory of Fourier analysis on groups, and refer to [2] and [4] for details. Let  $G$  be a locally compact abelian group. Recall that  $L_1(G)$  consists of equivalence classes of complex-valued Borel measurable functions of  $G$  that are integrable with respect to a Haar measure on  $G$ , and that

$L_1(G)$  equipped with convolution product is a commutative completely regular semi-simple Banach algebra. A group homomorphism  $\gamma : G \rightarrow \mathbb{T}$  from a locally compact abelian group to the unit circle is called a *character* of  $G$ . The set of all *continuous* characters of  $G$  forms a group  $\Gamma$ , the *dual group* of  $G$ , if the group operation is defined by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x)\gamma_2(x) \quad (x \in G; \gamma_1, \gamma_2 \in \Gamma).$$

If  $\gamma \in \Gamma$  and if

$$\widehat{f}(\gamma) = \int_G f(x)\gamma(-x)dx \quad (f \in L_1(G)),$$

then the map  $f \mapsto \widehat{f}(\gamma)$  is a non-zero complex homomorphism of  $L_1(G)$ . Conversely, every non-zero complex homomorphism of  $L_1(G)$  is obtained in this way, and distinct characters induce distinct homomorphisms. Thus we may identify  $\Gamma$  with  $\Delta(L_1(G))$ . The function  $\widehat{f} : \Gamma \rightarrow \mathbb{C}$  defined as above is called the *Fourier transform* of  $f \in L_1(G)$ , and is precisely the Gelfand transform of  $f$ . We denote the set of all such  $\widehat{f}$  by  $A(\Gamma)$ . Furthermore,  $\Gamma$  is a locally compact abelian group in the Gelfand topology.

In [5, Theorem 4.16], the following result is proved.

**Theorem 3.4.6.** *Let  $G$  be a locally compact abelian group with connected dual group and let  $\sigma : L_1(G) \rightarrow L_1(G)$  be an automorphism. Then  $L_1(G)$  is maximal abelian in  $L_1(G) \rtimes_{\sigma} \mathbb{Z}$  if and only if  $\sigma$  is not of finite order.*

Combining Corollary 3.4.5 and Theorem 3.4.6 the following result is immediate.

**Corollary 3.4.7.** *Let  $G$  be a locally compact abelian group with connected dual group and let  $\sigma : L_1(G) \rightarrow L_1(G)$  be an automorphism. Then the following three statements are equivalent.*

- $\sigma$  is not of finite order;
- Every non-zero ideal  $I \subseteq L_1(G) \rtimes_{\sigma} \mathbb{Z}$  is such that  $I \cap L_1(G) \neq \{0\}$ ;
- $L_1(G)$  is a maximal abelian subalgebra of  $L_1(G) \rtimes_{\sigma} \mathbb{Z}$ .

### 3.5. Minimality versus simplicity

Recall that a topological dynamical system is said to be *minimal* if all of its orbits are dense, and that an algebra is called *simple* if it lacks non-trivial proper ideals.

**Theorem 3.5.1.** *Let  $A$  be a complex commutative semi-simple completely regular unital Banach algebra such that  $\Delta(A)$  consists of infinitely many points, and let  $\sigma : A \rightarrow A$  be an algebra automorphism of  $A$ . Then  $A \rtimes_{\sigma} \mathbb{Z}$  is simple if and only if the naturally induced system  $(\Delta(A), \widetilde{\sigma})$  is minimal.*

*Proof.* Suppose first that the system is minimal, and assume that  $I$  a proper ideal of  $A \rtimes_{\sigma} \mathbb{Z}$ . Note that  $I \cap A$  is a proper  $\sigma$ - and  $\sigma^{-1}$ -invariant ideal of  $A$ . By basic theory of Banach algebras,  $I \cap A$  is contained in a maximal ideal of  $A$  (note that  $I \cap A \neq A$  as  $A$  is unital

and  $I$  was assumed to be proper), which is the kernel of an element  $\mu \in \Delta(A)$ . Now  $\widehat{I \cap A}$  is a  $\widehat{\sigma}$ - and  $\widehat{\sigma}^{-1}$ -invariant proper non-trivial ideal of  $\widehat{A}$ , all of whose elements vanish in  $\mu$ . Invariance of this ideal implies that all of its elements even annihilate the whole orbit of  $\mu$  under  $\widehat{\sigma}$ . But by minimality, every such orbit is dense and hence  $\widehat{I \cap A} = \{0\}$ . By semi-simplicity of  $A$ , this means  $I \cap A = \{0\}$ , so  $I = \{0\}$  by Corollary 3.4.5. For the converse, assume that there is an element  $\mu \in \Delta(A)$  whose orbit  $O(\mu)$  is not dense. By complete regularity of  $A$  there is a nonzero  $g \in \widehat{A}$  that vanishes on  $\overline{O(\mu)}$ . Then clearly the ideal generated by  $g$  in  $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$  consists of finite sums of elements of the form  $(f_n \delta^n) * g * (h_m \delta^m) = [f_n \cdot (g \circ \widehat{\sigma}^{-n}) \cdot (h_m \circ \widehat{\sigma}^{-n})] \delta^{n+m}$ , and hence the coefficient of every power of  $\delta$  in this ideal must vanish in  $\mu$ , whence the ideal is proper. Hence by Theorem 3.3.1,  $A \rtimes_{\sigma} \mathbb{Z}$  is not simple.  $\square$

### 3.6. Every non-zero ideal has non-zero intersection with $A'$

We shall now show that any non-zero ideal of  $A \rtimes_{\sigma} \mathbb{Z}$  has non-zero intersection with  $A'$ . This should be compared with Corollary 3.4.5, which says that a non-zero ideal may well intersect  $A$  solely in 0. There was no known analogue of this result in the context of crossed product  $C^*$ -algebras in the literature at the time this paper was submitted.

**Theorem 3.6.1.** *Let  $A$  be a complex commutative semi-simple completely regular Banach algebra, and  $\sigma : A \rightarrow A$  an automorphism. Then every non-zero ideal  $I$  in  $A \rtimes_{\sigma} \mathbb{Z}$  has non-zero intersection with the commutant  $A'$  of  $A$ , that is  $I \cap A' \neq \{0\}$ .*

*Proof.* As usual, we work in  $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$ . When  $\overline{\text{Per}^{\infty}(\Delta(A))} = \Delta(A)$ , the result follows immediately from Corollary 3.4.5. We will use induction on the number of non-zero terms in an element  $f = \sum_{n \in \mathbb{Z}} f_n \delta^n$  to show that it generates an ideal that intersects  $\widehat{A}'$  non-trivially. The starting point for the induction, namely when  $f = f_n \delta^n$  with non-zero  $f_n$ , is clear since any such element generates an ideal that even intersects  $\widehat{A}$  non-trivially, as was shown in the proof of Theorem 3.4.4. Now assume inductively that the conclusion of the theorem is true for the ideals generated by any element of  $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$  with  $r$  non-zero terms for some positive integer  $r$ , and consider an element  $f = f_{n_1} \delta^{n_1} + \dots + f_{n_{r+1}} \delta^{n_{r+1}}$ . By multiplying from the right with a suitable element we obtain an element in the ideal generated by  $f$  of the form  $g = \sum_{i=0}^{m_r} g_i \delta^i$  such that  $g_0 \neq 0$ . If some of the other  $g_i$  are zero we are done by induction hypothesis, so we may assume this is not the case. We may also assume that  $g$  is not in the commutant of  $\widehat{A}$  since otherwise we are of course also done. This means, by Theorem 3.3.3, that there is such  $j$  that  $0 < j \leq m_r$  and  $\text{supp}(g_j) \not\subseteq \text{Per}^j(\Delta(A))$ . Pick an  $x \in \text{supp}(g_j)$  such that  $x \neq \widehat{\sigma}^{-j}(x)$  and  $g_j(x) \neq 0$ . As  $\Delta(A)$  is Hausdorff we can choose an open neighbourhood  $U_x$  of  $x$  such that  $U_x \cap \widehat{\sigma}^{-j}(U_x) = \emptyset$ . Complete regularity of  $A$  implies existence of an  $h \in \widehat{A}$  such that  $h \circ \widehat{\sigma}^{-j}(x) = 1$  and  $h$  vanishes identically outside of  $\widehat{\sigma}^{-j}(U_x)$ . Now  $g * h = \sum_{i=0}^{m_r} g_i \cdot (h \circ \widehat{\sigma}^{-i}) \delta^i$ . Using complete regularity of  $A$  again we pick a function  $a \in \widehat{A}$  such that  $a(x) = 1$  and  $a$  vanishes outside  $U_x$ . We have  $a * g * h = \sum_{i=0}^{m_r} a \cdot g_i \cdot (h \circ \widehat{\sigma}^{-i}) \delta^i$ , which is in the ideal generated by  $f$ . Now  $a \cdot g_0 \cdot h$  is identically zero since  $a \cdot h = 0$ . On the other hand,  $a \cdot g_j \cdot (h \circ \widehat{\sigma}^{-j})$  is non-zero in the point  $x$ . Hence  $a * g * h$  is a non-zero element in the ideal generated by  $f$  whose number of non-zero coefficient functions is less than or equal to  $r$ . By the induction hypothesis, such an element

generates an ideal that intersects the commutant of  $\widehat{A}$  non-trivially. By Theorem 3.3.1 it follows that every non-zero ideal of  $A \rtimes_{\sigma} \mathbb{Z}$  intersects  $A'$  non-trivially.  $\square$

### 3.7. Primeness versus topological transitivity

We shall show that for certain  $A$ ,  $A \rtimes_{\sigma} \mathbb{Z}$  is prime if and only if the induced system  $(\Delta(A), \tilde{\sigma})$  is topologically transitive. The analogue of this result in the context of crossed product  $C^*$ -algebras is in [7, Theorem 5.5].

**Definition 3.7.1.** The system  $(\Delta(A), \tilde{\sigma})$  is called *topologically transitive* if for any pair of non-empty open sets  $U, V$  of  $\Delta(A)$ , there exists an integer  $n$  such that  $\tilde{\sigma}^n(U) \cap V \neq \emptyset$ .

**Definition 3.7.2.** The algebra  $A \rtimes_{\sigma} \mathbb{Z}$  is called *prime* if the intersection between any two non-zero ideals  $I, J$  is non-zero, that is  $I \cap J \neq \{0\}$ .

For convenience, we also make the following definition.

**Definition 3.7.3.** The map  $E : \widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z} \rightarrow \widehat{A}$  is defined by  $E(\sum_{n \in \mathbb{Z}} f_n \delta^n) = f_0$ .

To prove the main theorem of this section, we need the two following topological lemmas.

**Lemma 3.7.4.** *If  $(\Delta(A), \tilde{\sigma})$  is not topologically transitive, then there exist two disjoint invariant non-empty open sets  $O_1$  and  $O_2$  such that  $\overline{O_1} \cup \overline{O_2} = \Delta(A)$ .*

*Proof.* As the system is not topologically transitive, there exist non-empty open sets  $U, V \subseteq \Delta(A)$  such that for any integer  $n$  we have  $\tilde{\sigma}^n(U) \cap V = \emptyset$ . Now clearly the set  $O_1 = \bigcup_{n \in \mathbb{Z}} \tilde{\sigma}^n(U)$  is an invariant non-empty open set. Then  $\overline{O_1}$  is an invariant closed set. It follows that  $O_2 = \Delta(A) \setminus \overline{O_1}$  is an invariant open set containing  $V$ . Thus we even have that  $\overline{O_1} \cup O_2 = \Delta(A)$ , and the result follows.  $\square$

**Lemma 3.7.5.** *If  $(\Delta(A), \tilde{\sigma})$  is topologically transitive and there is an  $n_0 > 0$  such that  $\Delta(A) = \text{Per}^{n_0}(\Delta(A))$ , then  $\Delta(A)$  consists of a single orbit and is thus finite.*

*Proof.* Assume two points  $x, y \in \Delta(A)$  are not in the same orbit. As  $\Delta(A)$  is Hausdorff we may separate the points  $x, \sigma(x), \dots, \sigma^{n_0-1}(x), y$  by pairwise disjoint open sets  $V_0, V_1, \dots, V_{n_0-1}, V_y$ . Now consider the set

$$U_x := V_0 \cap \tilde{\sigma}^{-1}(V_1) \cap \tilde{\sigma}^{-2}(V_2) \cap \dots \cap \tilde{\sigma}^{-n_0+1}(V_{n_0-1}).$$

Clearly the sets  $A_x = \bigcup_{i=0}^{n_0-1} \tilde{\sigma}^i(U_x)$  and  $A_y = \bigcup_{i=0}^{n_0-1} \tilde{\sigma}^i(V_y)$  are disjoint invariant non-empty open sets, which leads us to a contradiction. Hence  $\Delta(A)$  consists of one single orbit under  $\tilde{\sigma}$ .  $\square$

We are now ready for a proof of the following result.

**Theorem 3.7.6.** *Let  $A$  be a complex commutative semi-simple completely regular unital Banach algebra such that  $\Delta(A)$  consists of infinitely many points, and let  $\sigma$  be an automorphism of  $A$ . Then  $A \rtimes_{\sigma} \mathbb{Z}$  is prime if and only if the associated system  $(\Delta(A), \tilde{\sigma})$  on the character space is topologically transitive.*

*Proof.* Suppose that the system  $(\Delta(A), \tilde{\sigma})$  is not topologically transitive. Then there exists, by Lemma 3.7.4, two disjoint invariant non-empty open sets  $O_1$  and  $O_2$  such that  $\overline{O_1} \cup \overline{O_2} = \Delta(A)$ . Let  $I_1$  and  $I_2$  be the ideals generated in  $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$  by  $k(\overline{O_1})$  (the set of all functions in  $\widehat{A}$  that vanish on  $\overline{O_1}$ ) and  $k(\overline{O_2})$  respectively. We have that

$$E(I_1 \cap I_2) \subseteq E(I_1) \cap E(I_2) = k(\overline{O_1}) \cap k(\overline{O_2}) = k(\overline{O_1} \cup \overline{O_2}) = k(\Delta(A)) = \{0\}.$$

It is not difficult to see that if  $I \subseteq \widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is an ideal and  $E(I) = \{0\}$ , then  $I = \{0\}$ . Namely, suppose  $F = \sum_{\mathbb{Z}} f_n \delta^n \in I$  and  $f_i \neq 0$  for some integer  $i$ . Since  $A$  is unital, so is  $\widehat{A}$  and thus  $\delta^{-1} \in \widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ . So  $F * \delta^{-i} \in I$  and hence  $E(F * \delta^{-i}) = f_i = 0$  which is a contradiction, so  $I = \{0\}$ . Hence  $I_1 \cap I_2 = \{0\}$  and  $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is not prime. By Theorem 3.3.1, neither is  $A \rtimes_{\sigma} \mathbb{Z}$ . Next suppose that  $(\Delta(A), \tilde{\sigma})$  is topologically transitive. Assume that  $\text{Per}^{\infty}(\Delta(A))$  is not dense. Then by Lemma 3.4.3 there is an integer  $n_0 > 0$  such that  $\text{Per}^{n_0}(\Delta(A))$  has non-empty interior. As  $\text{Per}^{n_0}(\Delta(A))$  is invariant and closed, topological transitivity implies that  $\Delta(A) = \text{Per}^{n_0}(\Delta(A))$ . This, however, is impossible since by Lemma 3.7.5 it would force  $\Delta(A)$  to consist of a single orbit and hence be finite. Thus  $\text{Per}^{\infty}(\Delta(A))$  is dense after all. Now let  $I$  and  $J$  be two non-zero proper ideals in  $A \rtimes_{\sigma} \mathbb{Z}$ . Unitality of  $A$  assures us that  $I \cap A$  and  $J \cap A$  are proper  $\sigma$ - and  $\sigma^{-1}$ -invariant ideals of  $A$  and density of  $\text{Per}^{\infty}(\Delta(A))$  assures us that they are non-zero, by Corollary 3.4.5. Consider  $A_I = \{\mu \in \Delta(A) \mid \mu(a) = 0 \text{ for all } a \in I \cap A\}$  and  $A_J = \{\nu \in \Delta(A) \mid \nu(b) = 0 \text{ for all } b \in J \cap A\}$ . Now by Banach algebra theory a proper ideal of a commutative unital Banach algebra  $A$  is contained in a maximal ideal, and a maximal ideal of  $A$  is always precisely the set of zeroes of some  $\zeta \in \Delta(A)$ . This implies that both  $A_I$  and  $A_J$  are non-empty, and semi-simplicity of  $A$  assures us that they are proper subsets of  $\Delta(A)$ . They are clearly also closed and invariant under  $\tilde{\sigma}$  and  $\tilde{\sigma}^{-1}$ . Hence  $\Delta(A) \setminus A_I$  and  $\Delta(A) \setminus A_J$  are invariant non-empty open sets. By topological transitivity we must have that these two sets intersect, hence that  $A_I \cup A_J \neq \Delta(A)$ . This means that there exists  $\eta \in \Delta(A)$  and  $a \in I \cap A, b \in J \cap A$  such that  $\eta(a) \neq 0, \eta(b) \neq 0$  and hence that  $\eta(ab) \neq 0$ . Hence  $0 \neq ab \in I \cap J$ , and we conclude that  $A \rtimes_{\sigma} \mathbb{Z}$  is prime.  $\square$

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## Chapter 4

# Dynamical systems associated with crossed products

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**Abstract.** In this paper, we consider both algebraic crossed products of commutative complex algebras  $A$  with the integers under an automorphism of  $A$ , and Banach algebra crossed products of commutative  $C^*$ -algebras  $A$  with the integers under an automorphism of  $A$ . We investigate, in particular, connections between algebraic properties of these crossed products and topological properties of naturally associated dynamical systems. For example, we draw conclusions about the ideal structure of the crossed product by investigating the dynamics of such a system. To begin with, we recall results in this direction in the context of an algebraic crossed product and give simplified proofs of generalizations of some of these results. We also investigate new questions, for example about ideal intersection properties of algebras properly between the coefficient algebra  $A$  and its commutant  $A'$ . Furthermore, we introduce a Banach algebra crossed product and study the relation between the structure of this algebra and the topological dynamics of a naturally associated system.

### 4.1. Introduction

A lot of work has been done on the connection between certain topological dynamical systems and crossed product  $C^*$ -algebras. In [13] and [14], for example, one starts with a homeomorphism  $\sigma$  of a compact Hausdorff space  $X$  and constructs the crossed product  $C^*$ -algebra  $C(X) \rtimes_{\alpha} \mathbb{Z}$ , where  $C(X)$  is the algebra of continuous complex valued functions on  $X$  and  $\alpha$  is the  $\mathbb{Z}$ -action on  $C(X)$  naturally induced by  $\sigma$ . One of many results obtained is equivalence between simplicity of the algebra and minimality of the system, provided that  $X$  consists of infinitely many points, see [2], [8], [13], [14] or, for a more general approach in the metrizable case, [15]. In [10], a purely algebraic variant of the crossed product is considered, having more general classes of algebras than merely continuous functions on

compact Hausdorff spaces as coefficient algebras. For example, it is proved there that, for such crossed products, the analogue of the equivalence between density of aperiodic points of a dynamical system and maximal commutativity of the coefficient algebra in the associated crossed product  $C^*$ -algebra is true for significantly larger classes of coefficient algebras and associated dynamical systems. In [11], further work is done in this setup, mainly for crossed products of complex commutative semi-simple completely regular Banach-algebras  $A$  (of which  $C(X)$  is an example) with the integers under an automorphism of  $A$ . In particular, various properties of the ideal structure in such crossed products are shown to be equivalent to topological properties of the naturally induced topological dynamical system on  $\Delta(A)$ , the character space of  $A$ .

In this paper, we recall some of the most important results from [10] and [11], and in a number of cases provide significantly simplified proofs of generalizations of results occurring in [11], giving a clearer view of the heart of the matter. We also include results of a new type in the algebraic setup, and furthermore start the investigation of the Banach algebra crossed product  $\ell_1^\sigma(\mathbb{Z}, A)$  of a commutative  $C^*$ -algebra  $A$  with the integers under an automorphism  $\sigma$  of  $A$ . In the case when  $A$  is unital, this algebra is precisely the one whose  $C^*$ -envelope is the crossed product  $C^*$ -algebra mentioned above.

This paper is organized as follows. In Section 4.2 we give the most general definition of the kind of crossed product that we will use throughout the first sections of this paper. We also mention the elementary result that the commutant of the coefficient algebra is automatically a maximal commutative subalgebra of the crossed product.

In Section 4.3 we prove that for any such crossed product  $A \rtimes_\psi \mathbb{Z}$ , the commutant  $A'$  of the coefficient algebra  $A$  has non-zero intersection with every non-zero ideal  $I \subseteq A \rtimes_\psi \mathbb{Z}$ . In [11, Theorem 6.1], a more complicated proof of this was given for a restricted class of coefficient algebras  $A$ .

In Section 4.4 we focus on the case when  $A$  is a function algebra on a set  $X$  with an automorphism  $\tilde{\sigma}$  of  $A$  induced by a bijection  $\sigma : X \rightarrow X$ . According to [14, Theorem 5.4], the following three properties are equivalent for a compact Hausdorff space  $X$  and a homeomorphism  $\sigma$  of  $X$ :

- The aperiodic points of  $(X, \sigma)$  are dense in  $X$ ;
- Every non-zero closed ideal  $I$  of the crossed product  $C^*$ -algebra  $C(X) \rtimes_\alpha \mathbb{Z}$  is such that  $I \cap C(X) \neq \{0\}$ ;
- $C(X)$  is a maximal abelian  $C^*$ -subalgebra of  $C(X) \rtimes_\alpha \mathbb{Z}$ .

In Theorem 4.4.5 an analogue of this result is proved for our setup. A reader familiar with the theory of crossed product  $C^*$ -algebras will easily recognize that if one chooses  $A = C(X)$  for  $X$  a compact Hausdorff space in this theorem, then the algebraic crossed product is canonically isomorphic to a norm-dense subalgebra of the crossed product  $C^*$ -algebra associated with the considered induced dynamical system.

For a different kind of coefficient algebras  $A$  than the ones allowed in Theorem 4.4.5, we prove a similar result in Theorem 4.4.6. Theorem 4.4.5 and Theorem 4.4.6 have no non-trivial situations in common (Remark 4.4.8).

In Section 4.5 we show that in many situations we can always find both a subalgebra properly between the coefficient algebra  $A$  and its commutant  $A'$  (as long as  $A \subsetneq A'$ , a

property we have a precise condition for in Theorem 4.4.5) and a non-trivial ideal trivially intersecting it, and a subalgebra properly between  $A$  and  $A'$  intersecting every non-trivial ideal non-trivially.

Section 4.6 is concerned with the algebraic crossed product of a complex commutative semi-simple Banach algebra  $A$  with the integers under an automorphism  $\sigma$  of  $A$ , naturally inducing a homeomorphism  $\tilde{\sigma}$  of the character space  $\Delta(A)$  of  $A$ . We extend results from [11].

In Section 4.7 we introduce the Banach algebra crossed product  $\ell_1^\sigma(\mathbb{Z}, A)$  for a commutative  $C^*$ -algebra  $A$  and an automorphism  $\sigma$  of  $A$ . In Theorem 4.7.4 we give an explicit description of the closed commutator ideal in this algebra in terms of the dynamical system naturally induced on  $\Delta(A)$ . We determine the characters of  $\ell_1^\sigma(\mathbb{Z}, A)$  and show that the modular ideals which are maximal and contain the closed commutator ideal are precisely the kernels of the characters.

## 4.2. Definition and a basic result

Let  $A$  be an associative commutative complex algebra and let  $\Psi : A \rightarrow A$  be an algebra automorphism. Consider the set

$$A \rtimes_\Psi \mathbb{Z} = \{f : \mathbb{Z} \rightarrow A \mid f(n) = 0 \text{ except for a finite number of } n\}.$$

We endow it with the structure of an associative complex algebra by defining scalar multiplication and addition as the usual pointwise operations. Multiplication is defined by *twisted convolution*,  $*$ , as follows;

$$(f * g)(n) = \sum_{k \in \mathbb{Z}} f(k) \cdot \Psi^k(g(n - k)),$$

where  $\Psi^k$  denotes the  $k$ -fold composition of  $\Psi$  with itself. It is trivially verified that  $A \rtimes_\Psi \mathbb{Z}$  is an associative  $\mathbb{C}$ -algebra under these operations. We call it the *crossed product of  $A$  and  $\mathbb{Z}$  under  $\Psi$* .

A useful way of working with  $A \rtimes_\Psi \mathbb{Z}$  is to write elements  $f, g \in A \rtimes_\Psi \mathbb{Z}$  in the form  $f = \sum_{n \in \mathbb{Z}} f_n \delta^n$ ,  $g = \sum_{m \in \mathbb{Z}} g_m \delta^m$ , where  $f_n = f(n)$ ,  $g_m = g(m)$ , addition and scalar multiplication are canonically defined, and multiplication is determined by  $(f_n \delta^n) * (g_m \delta^m) = f_n \cdot \Psi^n(g_m) \delta^{n+m}$ , where  $n, m \in \mathbb{Z}$  and  $f_n, g_m \in A$  are arbitrary.

Clearly one may canonically view  $A$  as an abelian subalgebra of  $A \rtimes_\Psi \mathbb{Z}$ , namely as  $\{f_0 \delta^0 \mid f_0 \in A\}$ . The following elementary result is proved in [10, Proposition 2.1].

**Proposition 4.2.1.** *The commutant  $A'$  of  $A$  is abelian, and thus it is the unique maximal abelian subalgebra containing  $A$ .*

## 4.3. Every non-zero ideal has non-zero intersection with $A'$

Throughout the whole paper, when speaking of an ideal we shall always mean a *two-sided* ideal. We shall now show that every non-zero ideal in  $A \rtimes_\Psi \mathbb{Z}$  has non-zero intersection with

$A'$ . This result, Theorem 4.3.1, should be compared with Theorem 4.4.5, which says that a non-zero ideal may well intersect  $A$  solely in 0. There was no analogue in the literature of Theorem 4.3.1 in the context of crossed product  $C^*$ -algebras at the time this paper was submitted. Note that in [11] a proof of Theorem 4.3.1 was given for the case when  $A$  was completely regular semi-simple Banach algebra, and that this proof heavily relied upon  $A$  having these properties. The present proof is elementary and valid for arbitrary commutative algebras. Note that the fact that all elements of the crossed product are *finite* sums of the form  $\sum_n f_n \delta^n$  is crucial to the argument.

**Theorem 4.3.1.** *Let  $A$  be an associative commutative complex algebra and let  $\Psi$  be an automorphism of  $A$ . Then every non-zero ideal of  $A \rtimes_{\Psi} \mathbb{Z}$  has non-zero intersection with the commutant  $A'$  of  $A$ .*

*Proof.* Let  $I$  be a non-zero ideal, and let  $f = \sum_n f_n \delta^n \in I$  be non-zero. Suppose that  $f \notin A'$ . Then there must be an  $f_{n_i}$  and  $a \in A$  such that  $f_{n_i} \cdot a \neq 0$ . Hence  $f' := (\sum_n f_n \delta^n) * \Psi^{-n_i}(a) \delta^{-n_i}$  is a non-zero element of  $I$ , having  $f_{n_i} \cdot a$  as coefficient of  $\delta^0$  and having at most as many non-zero coefficients as  $f$ . If  $f' \in A'$  we are done, so assume  $f' \notin A'$ . Then there exists  $b \in A$  such that  $F := b * f' - f' * b \neq 0$ . Clearly  $F \in I$  and it is easy to see that  $F$  has strictly less non-zero coefficients than  $f'$  (the coefficient of  $\delta^0$  in  $F$  is zero), hence strictly less than  $f$ . Now if  $F \in A'$ , we are done. If not, we repeat the above procedure. Ultimately, if we do not happen to obtain a non-zero element of  $I \cap A'$  along the way, we will be left with a non-zero monomial  $G := g_m \delta^m \in I$ . If this does not lie in  $A'$ , there is an  $a \in A$  such that  $g_m \cdot a \neq 0$ . Hence  $G * \Psi^{-m}(a) \delta^{-m} = g_m \cdot a \in I \cap A \subseteq I \cap A'$ .  $\square$

## 4.4. Automorphisms induced by bijections

Fix a non-empty set  $X$ , a bijection  $\sigma : X \rightarrow X$ , and an algebra of functions  $A \subseteq \mathbb{C}^X$  that is invariant under  $\sigma$  and  $\sigma^{-1}$ , i.e., such that if  $h \in A$ , then  $h \circ \sigma \in A$  and  $h \circ \sigma^{-1} \in A$ . Then  $(X, \sigma)$  is a discrete dynamical system (the action of  $n \in \mathbb{Z}$  on  $x \in X$  is given by  $n : x \mapsto \sigma^n(x)$ ) and  $\sigma$  induces an automorphism  $\tilde{\sigma} : A \rightarrow A$  defined by  $\tilde{\sigma}(f) = f \circ \sigma^{-1}$  by which  $\mathbb{Z}$  acts on  $A$  via iterations.

In this section we will consider the crossed product  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  for the above setup, and explicitly describe the commutant  $A'$  of  $A$ . Furthermore, we will investigate equivalences between properties of aperiodic points of the system  $(X, \sigma)$ , and properties of  $A'$ . First we make a few definitions.

**Definition 4.4.1.** For any nonzero  $n \in \mathbb{Z}$  we set

$$\begin{aligned} \text{Sep}_A^n(X) &= \{x \in X \mid \exists h \in A : h(x) \neq \tilde{\sigma}^n(h)(x)\}, \\ \text{Per}_A^n(X) &= \{x \in X \mid \forall h \in A : h(x) = \tilde{\sigma}^n(h)(x)\}, \\ \text{Sep}^n(X) &= \{x \in X \mid x \neq \sigma^n(x)\}, \\ \text{Per}^n(X) &= \{x \in X \mid x = \sigma^n(x)\}. \end{aligned}$$

Furthermore, let

$$\begin{aligned}\text{Per}_A^\infty(X) &= \bigcap_{n \in \mathbb{Z} \setminus \{0\}} \text{Sep}_A^n(X), \\ \text{Per}^\infty(X) &= \bigcap_{n \in \mathbb{Z} \setminus \{0\}} \text{Sep}^n(X).\end{aligned}$$

Finally, for  $f \in A$ , put

$$\text{supp}(f) = \{x \in X \mid f(x) \neq 0\}.$$

It is easy to check that all these sets, except for  $\text{supp}(f)$ , are  $\mathbb{Z}$ -invariant and that if  $A$  separates the points of  $X$ , then  $\text{Sep}_A^n(X) = \text{Sep}^n(X)$  and  $\text{Per}_A^n(X) = \text{Per}^n(X)$ . Note also that  $X \setminus \text{Per}_A^n(X) = \text{Sep}_A^n(X)$ , and  $X \setminus \text{Per}^n(X) = \text{Sep}^n(X)$ . Furthermore  $\text{Sep}_A^n(X) = \text{Sep}_A^{-n}(X)$  with similar equalities for  $n$  and  $-n$  ( $n \in \mathbb{Z}$ ) holding for  $\text{Per}_A^n(X)$ ,  $\text{Sep}^n(X)$  and  $\text{Per}^n(X)$  as well.

**Definition 4.4.2.** We say that a non-empty subset of  $X$  is a *domain of uniqueness* for  $A$  if every function in  $A$  that vanishes on it, vanishes on the whole of  $X$ .

For example, using results from elementary topology one easily shows that for a completely regular topological space  $X$ , a subset of  $X$  is a domain of uniqueness for  $C(X)$  if and only if it is dense in  $X$ . In the following theorem we recall some elementary results from [10].

**Theorem 4.4.3.** *The unique maximal abelian subalgebra of  $A \rtimes_{\bar{\sigma}} \mathbb{Z}$  that contains  $A$  is precisely the set of elements*

$$A' = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid f_n \upharpoonright_{\text{Sep}_A^n(X)} \equiv 0 \text{ for all } n \in \mathbb{Z} \right\}.$$

So if  $A$  separates the points of  $X$ , then

$$A' = \left\{ \sum_{n \in \mathbb{Z}} f_n \delta^n \mid \text{supp}(f_n) \subseteq \text{Per}^n(X) \text{ for all } n \in \mathbb{Z} \right\}.$$

Furthermore, the subalgebra  $A$  is maximal abelian in  $A \rtimes_{\bar{\sigma}} \mathbb{Z}$  if and only if, for every  $n \in \mathbb{Z} \setminus \{0\}$ ,  $\text{Sep}_A^n(X)$  is a domain of uniqueness for  $A$ .

We now focus solely on topological contexts. In order to prove one of the main theorems of this section, we need the following topological lemma.

**Lemma 4.4.4.** *Let  $X$  be a Baire space which is also Hausdorff, and let  $\sigma : X \rightarrow X$  be a homeomorphism. Then the aperiodic points of  $(X, \sigma)$  are dense if and only if  $\text{Per}^n(X)$  has empty interior for all positive integers  $n$ .*

*Proof.* Clearly, if there is a positive integer  $n$  such that  $\text{Per}^n(X)$  has non-empty interior, the aperiodic points are not dense. For the converse we note that we may write

$$X \setminus \text{Per}^\infty(X) = \bigcup_{n>0} \text{Per}^n(X).$$

If the set of aperiodic points is not dense, its complement has non-empty interior, and as the sets  $\text{Per}^n(\Delta(A))$  are clearly all closed since  $X$  is Hausdorff, there must exist an integer  $n_0 > 0$  such that  $\text{Per}^{n_0}(X)$  has non-empty interior since  $X$  is a Baire space.  $\square$

We are now ready to prove the following theorem.

**Theorem 4.4.5.** *Let  $X$  be a Baire space which is also Hausdorff, and let  $\sigma : X \rightarrow X$  be a homeomorphism inducing, as usual, an automorphism  $\tilde{\sigma}$  of  $C(X)$ . Suppose  $A$  is a subalgebra of  $C(X)$  that is invariant under  $\tilde{\sigma}$  and its inverse, separates the points of  $X$  and is such that for every non-empty open set  $U \subseteq X$  there is a non-zero  $f \in A$  that vanishes on the complement of  $U$ . Then the following three statements are equivalent.*

- $A$  is a maximal abelian subalgebra of  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$ ;
- $\text{Per}^\infty(X)$  is dense in  $X$ ;
- Every non-zero ideal  $I \subseteq A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is such that  $I \cap A \neq \{0\}$ .

*Proof.* Equivalence of the first two statements is precisely the result in [10, Theorem 3.7]. The first property implies the third by Proposition 4.2.1 and Theorem 4.3.1. Finally, to show that the third statement implies the second, assume that  $\text{Per}^\infty(X)$  is not dense. It follows from Lemma 4.4.4 that there exists an integer  $n > 0$  such that  $\text{Per}^n(X)$  has non-empty interior. By the assumptions on  $A$  there exists a non-zero  $f \in A$  such that  $\text{supp}(f) \subseteq \text{Per}^n(X)$ . Consider now the non-zero ideal  $I$  generated by  $f + f\delta^n$ . It is spanned by elements of the form  $a_i\delta^i * (f + f\delta^n) * a_j\delta^j$ ,  $(f + f\delta^n) * a_j\delta^j$ ,  $a_i\delta^i * (f + f\delta^n)$  and  $f + f\delta^n$ . Using that  $f$  vanishes outside  $\text{Per}^n(X)$ , so that  $f\delta^n * a_j\delta^j = a_j f\delta^{n+j}$ , we may for example rewrite

$$\begin{aligned} a_i\delta^i * (f + f\delta^n) * a_j\delta^j &= [a_i \cdot (a_j \circ \tilde{\sigma}^{-i})\delta^i] * [f\delta^j + f\delta^{n+j}] \\ &= [a_i \cdot (a_j \circ \tilde{\sigma}^{-i}) \cdot (f \circ \tilde{\sigma}^{-i})]\delta^{i+j} + [a_i \cdot (a_j \circ \tilde{\sigma}^{-i}) \cdot (f \circ \tilde{\sigma}^{-i})]\delta^{i+j+n}. \end{aligned}$$

A similar calculation for the other three kinds of elements that span  $I$  now makes it clear that any element in  $I$  may be written in the form  $\sum_i (b_i\delta^i + b_i\delta^{n+i})$ . As  $i$  runs only through a finite subset of  $\mathbb{Z}$ , this is not a non-zero monomial. In particular, it is not a non-zero element in  $A$ . Hence  $I$  intersects  $A$  trivially.  $\square$

We also have the following result for a different kind of subalgebras of  $C(X)$ .

**Theorem 4.4.6.** *Let  $X$  be a topological space,  $\sigma : X \rightarrow X$  a homeomorphism, and  $A$  a non-zero subalgebra of  $C(X)$ , invariant both under the usual induced automorphism  $\tilde{\sigma} : C(X) \rightarrow C(X)$  and under its inverse. Assume that  $A$  separates the points of  $X$  and is such that every non-empty open set  $U \subseteq X$  is a domain of uniqueness for  $A$ . Then the following three statements are equivalent.*

- $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$ ;
- $\sigma$  is not of finite order;
- Every non-zero ideal  $I \subseteq A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is such that  $I \cap A \neq \{0\}$ .

*Proof.* Equivalence of the first two statements is precisely the result in [10, Theorem 3.11]. That the first statement implies the third follows immediately from Proposition 4.2.1 and Theorem 4.3.1. Finally, to show that the third statement implies the second, assume that there exists an  $n$ , which we may clearly choose to be non-negative, such that  $\sigma^n = \text{id}_X$ . Now take any non-zero  $f \in A$  and consider the non-zero ideal  $I = (f + f\delta^n)$ . Using an argument similar to the one in the proof of Theorem 4.4.5 one concludes that  $I \cap A = \{0\}$ .  $\square$

**Corollary 4.4.7.** *Let  $M$  be a connected complex manifold and suppose the function  $\sigma : M \rightarrow M$  is biholomorphic. If  $A \subseteq H(M)$  is a subalgebra of the algebra of holomorphic functions which separates the points of  $M$  and which is invariant under the induced automorphism  $\tilde{\sigma}$  of  $H(M)$  and its inverse, then the following three statements are equivalent.*

- $A$  is maximal abelian in  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$ ;
- $\sigma$  is not of finite order;
- Every non-zero ideal  $I \subseteq A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is such that  $I \cap A \neq \{0\}$ .

*Proof.* On connected complex manifolds, open sets are domains of uniqueness for  $H(M)$ . See for example [5].  $\square$

**Remark 4.4.8.** It is worth mentioning that the required conditions in Theorem 4.4.5 and Theorem 4.4.6 can only be simultaneously satisfied in case  $X$  consists of a single point and  $A = \mathbb{C}$ . This is explained in [10, Remark 3.13]

## 4.5. Algebras properly between the coefficient algebra and its commutant

From Theorem 4.4.5 it is clear that for spaces  $X$  which are Baire and Hausdorff and subalgebras  $A \subseteq C(X)$  with sufficient separation properties,  $A$  is equal to its own commutant in the associated crossed product precisely when the aperiodic points,  $\text{Per}^\infty(X)$ , constitute a dense subset of  $X$ . This theorem also tells us that whenever  $\text{Per}^\infty(X)$  is not dense there exists a non-zero ideal  $I$  having zero intersection with  $A$ , while the general Theorem 4.3.1 tells us that every non-zero ideal has non-zero intersection with  $A'$ , regardless of the system  $(X, \sigma)$ .

**Definition 4.5.1.** We say that a subalgebra has the *intersection property* if it has non-zero intersection with every non-zero ideal.

A subalgebra  $B$  such that  $A \subsetneq B \subsetneq A'$  is said to be properly between  $A$  and  $A'$ . Two natural questions comes to mind in case  $\text{Per}^\infty(X)$  is not dense:

- (i) Do there exist subalgebras properly between  $A$  and  $A'$  having the intersection property?
- (ii) Do there exist subalgebras properly between  $A$  and  $A'$  *not* having the intersection property?

We shall show that for a significant class of systems the answer to both these questions is positive.

**Proposition 4.5.2.** *Let  $X$  be a Hausdorff space, and let  $\sigma : X \rightarrow X$  be a homeomorphism inducing, as usual, an automorphism  $\tilde{\sigma}$  of  $C(X)$ . Suppose  $A$  is a subalgebra of  $C(X)$  that is invariant under  $\tilde{\sigma}$  and its inverse, separates the points of  $X$  and is such that for every non-empty open set  $U \subseteq X$  there is a non-zero  $f \in A$  that vanishes on the complement of  $U$ . Suppose furthermore that there exists an integer  $n > 0$  such that the interior of  $\text{Per}^n(X)$  contains at least two orbits. Then there exists a subalgebra  $B$  such that  $A \subsetneq B \subsetneq A'$  which does not have the intersection property.*

*Proof.* Using the Hausdorff property of  $X$  and the fact that  $\text{Per}^n(X)$  contains two orbits we can find two non-empty disjoint invariant open subsets  $U_1$  and  $U_2$  contained in  $\text{Per}^n(X)$ . Consider

$$B = \{f_0 + \sum_{k \neq 0} f_k \delta^k : f_0 \in A, \text{supp}(f_k) \subseteq U_1 \cap \text{Per}^k(X) \text{ for } k \neq 0\}.$$

Then  $B$  is a subalgebra and  $B \subseteq A'$ . The assumptions on  $A$  and the definitions of  $U_1$  and  $U_2$  now make it clear that  $A \subsetneq B \subsetneq A'$  since there exist, for example, non-zero functions  $F_1, F_2 \in A$  such that  $\text{supp}(F_1) \subseteq U_1$  and  $\text{supp}(F_2) \subseteq U_2$ , and thus  $F_1 \delta^n \in B \setminus A$  and  $F_2 \delta^n \in A' \setminus B$ . Consider the non-zero ideal  $I$  generated by  $F_2 + F_2 \delta^n$ . Using an argument similar to the one in the proof of Theorem 4.4.5 we see that  $I \cap A = \{0\}$ . It is also easy to see that  $I \subseteq \{\sum_k f_k \delta^k : \text{supp}(f_k) \subseteq U_2\}$  since  $U_2$  is invariant. As  $U_1 \cap U_2 = \emptyset$ , we see from the description of  $B$  that  $I \cap B \subseteq A$ , so that  $I \cap B \subseteq I \cap A = \{0\}$ .  $\square$

We now exhibit algebras properly between  $A$  and  $A'$  that do have the intersection property.

**Proposition 4.5.3.** *Let  $X$  be a Hausdorff space, and let  $\sigma : X \rightarrow X$  be a homeomorphism inducing, as usual, an automorphism  $\tilde{\sigma}$  of  $C(X)$ . Suppose  $A$  is a subalgebra of  $C(X)$  that is invariant under  $\tilde{\sigma}$  and its inverse, separates the points of  $X$  and is such that for every non-empty open set  $U \subseteq X$  there is a non-zero  $f \in A$  that vanishes on the complement of  $U$ . Suppose furthermore that there exist an integer  $n > 0$  such that the interior of  $\text{Per}^n(X)$  contains a point  $x_0$  which is not isolated, and an  $f \in A$  with  $\text{supp}(f) \subseteq \text{Per}^n(X)$  and  $f(x_0) \neq 0$ . Then there exists a subalgebra  $B$  such that  $A \subsetneq B \subsetneq A'$  which has the intersection property.*

*Proof.* Define

$$B = \{\sum_{k \in \mathbb{Z}} f_k \delta^k \in A' \mid f_k(x_0) = 0 \text{ for all } k \neq 0\},$$



where  $x_0$  is as in the statement of the theorem. Clearly  $B$  is a subalgebra and  $A \subseteq B$ . Since  $x_0$  is not isolated, we can use the assumptions on  $A$  and the fact that  $X$  is Hausdorff to first find a point different from  $x_0$  in the interior of  $\text{Per}^n(X)$  and subsequently a non-zero function  $g \in A$  such that  $\text{supp}(g) \subseteq \text{Per}^n(X)$  and  $g(x_0) = 0$ . Then  $g\delta^n \in B \setminus A$ . Also, by the assumptions on  $A$  there is a non-zero  $f \in A$  with  $\text{supp}(f) \subseteq \text{Per}^n(X)$  such that  $f(x_0) \neq 0$ , whence  $f\delta^n \in A' \setminus B$ . This shows that  $B$  is a subalgebra properly between  $A$  and  $A'$ . To see that it has the intersection property, let  $I$  be an arbitrary non-zero ideal in the crossed product and note that by Theorem 4.3.1 there is a non-zero  $F = \sum_{k \in \mathbb{Z}} f_k \delta^k$  in  $I \cap A'$ . Now if for all  $k \neq 0$  we have that  $f_k(x_0) = 0$ , we are done. So suppose there is some  $k \neq 0$  such that  $f_k(x_0) \neq 0$ . Since  $f_k$  is continuous and  $x_0$  is not isolated, we may use the Hausdorff property of  $X$  to conclude that there exists a non-empty open set  $V$  contained in the interior of  $\text{Per}^n(X)$  such that  $x_0 \notin V$  and  $f_k(x) \neq 0$  for all  $x \in V$ . The assumptions on  $A$  now imply that there is an  $h \in A$  such that  $h(x_0) = 0$  and  $h(x_1) \neq 0$  for some  $x_1 \in V \subseteq \text{supp}(f_n)$ . Clearly  $0 \neq h * F \in I \cap B$ .  $\square$

**Theorem 4.5.4.** *Let  $X$  be a Baire space which is Hausdorff and connected. Let  $\sigma : X \rightarrow X$  be a homeomorphism inducing an automorphism  $\tilde{\sigma}$  of  $C(X)$  in the usual way. Suppose  $A$  is a subalgebra of  $C(X)$  that is invariant under  $\tilde{\sigma}$  and its inverse, such that for every open set  $U \subseteq X$  and  $x \in U$  there is an  $f \in A$  such that  $f(x) \neq 0$  and  $\text{supp}(f) \subseteq U$ . Then precisely one of the following situations occurs:*

- (i)  $A = A'$ , which happens precisely when  $\text{Per}^\infty(X)$  is dense;
- (ii)  $A \subsetneq A'$  and there exist both subalgebras properly between  $A$  and  $A'$  which have the intersection property, and subalgebras which do not. This happens precisely when  $\text{Per}^\infty(X)$  is not dense and  $X$  is infinite;
- (iii)  $A \subsetneq A'$  and every subalgebra properly between  $A$  and  $A'$  has the intersection property. This happens precisely when  $X$  consists of one point.

*Proof.* By Theorem 4.4.5, (i) is clear and we may assume that  $\text{Per}^\infty(X)$  is not dense. Suppose first that  $X$  is infinite and note that by Lemma 4.4.4 there exists  $n_0 > 0$  such that  $\text{Per}^{n_0}(X)$  has non-empty interior. If this interior consists of one single orbit then as  $X$  is Hausdorff every point in the interior is both closed and open, so that  $X$  consists of one point by connectedness, which is a contradiction. Hence there are at least two orbits in the interior of  $\text{Per}^{n_0}(X)$ . Furthermore, no point of  $X$  can be isolated. Thus by Proposition 4.5.2 and Proposition 4.5.3 there are subalgebras properly between  $A$  and  $A'$  which have the intersection property, and subalgebras which do not. Suppose next that  $X$  is finite, so that  $X = \{x\}$  by connectedness. Then  $\sigma$  is the identity map, and  $A = \mathbb{C}$ . In this case,  $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$  may be canonically identified with  $\mathbb{C}[t, t^{-1}]$ . Let  $B$  be a subalgebra such that  $\mathbb{C} \subsetneq B \subsetneq \mathbb{C}[t, t^{-1}]$ , and let  $I$  be a non-zero ideal of  $\mathbb{C}[t, t^{-1}]$ . We will show that  $I \cap B \neq \{0\}$  and hence may assume that  $I \neq \mathbb{C}[t, t^{-1}]$ . Since  $\mathbb{C}[t, t^{-1}]$  is the ring of fractions of  $\mathbb{C}[t]$  with respect to the multiplicatively closed subset  $\{t^n \mid n \text{ is a non-negative integer}\}$  and  $\mathbb{C}[t]$  is a principal ideal domain, it follows from [1, Proposition 3.11(i)] that  $I$  is of the form  $(t - \alpha_1) \cdots (t - \alpha_n) \mathbb{C}[t, t^{-1}]$  for some  $n > 0$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ . There exists a non-constant  $f$  in  $B$ , and then the element  $(f - f(\alpha_1)) \cdots (f - f(\alpha_n))$  is a non-zero element of

$B$ . It is clearly also in  $I$  since it vanishes at  $\alpha_1, \dots, \alpha_n$  and hence has  $(t - \alpha_1) \cdots (t - \alpha_n)$  as a factor. Hence  $I \cap B \neq \{0\}$  and the proof is completed.  $\square$

It is interesting to mention that arguments similar to the ones used in Propositions 4.5.2 and 4.5.3 work in the context of the crossed product  $C^*$ -algebra  $C(X) \rtimes_\alpha \mathbb{Z}$  where  $X$  is a compact Hausdorff space and  $\alpha$  the automorphism induced by a homeomorphism of  $X$ . We expect to report separately on this and related results in the  $C^*$ -algebra context.

## 4.6. Semi-simple Banach algebras

In what follows, we shall focus on cases where  $A$  is a commutative complex Banach algebra, and freely make use of the basic theory for such  $A$ , see e.g.[6]. As conventions tend to differ slightly in the literature, however, we mention that we call a commutative Banach algebra  $A$  *completely regular* (the term *regular* is also frequently used in the literature) if, for every subset  $F \subseteq \Delta(A)$  (where  $\Delta(A)$  denotes the character space of  $A$ ) that is closed in the Gelfand topology and for every  $\phi_0 \in \Delta(A) \setminus F$ , there exists an  $a \in A$  such that  $\widehat{a}(\phi) = 0$  for all  $\phi \in F$  and  $\widehat{a}(\phi_0) \neq 0$ . All topological considerations of  $\Delta(A)$  will be done with respect to its Gelfand topology.

Now let  $A$  be a complex commutative semi-simple completely regular Banach algebra, and let  $\sigma : A \rightarrow A$  be an algebra automorphism. As in [10],  $\sigma$  induces a map  $\tilde{\sigma} : \Delta(A) \rightarrow \Delta(A)$  defined by  $\tilde{\sigma}(\mu) = \mu \circ \sigma^{-1}$ ,  $\mu \in \Delta(A)$ , which is automatically a homeomorphism when  $\Delta(A)$  is endowed with the Gelfand topology. Hence we obtain a topological dynamical system  $(\Delta(A), \tilde{\sigma})$ . In turn,  $\tilde{\sigma}$  induces an automorphism  $\widehat{\sigma} : \widehat{A} \rightarrow \widehat{A}$  (where  $\widehat{A}$  denotes the algebra of Gelfand transforms of all elements of  $A$ ) defined by  $\widehat{\sigma}(\widehat{a}) = \widehat{a} \circ \tilde{\sigma}^{-1} = \widehat{\sigma(a)}$ . Therefore we can form the crossed product  $\widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$ .

In what follows, we shall make frequent use of the following fact. Its proof consists of a trivial direct verification.

**Theorem 4.6.1.** *Let  $A$  be a commutative semi-simple Banach algebra and  $\sigma$  an automorphism, inducing an automorphism  $\widehat{\sigma} : \widehat{A} \rightarrow \widehat{A}$  as above. Then the map  $\Phi : A \rtimes_{\sigma} \mathbb{Z} \rightarrow \widehat{A} \rtimes_{\widehat{\sigma}} \mathbb{Z}$  defined by  $\sum_{n \in \mathbb{Z}} a_n \delta^n \mapsto \sum_{n \in \mathbb{Z}} \widehat{a}_n \delta^n$  is an isomorphism of algebras mapping  $A$  onto  $\widehat{A}$ .*

We shall now conclude that, for certain  $A$ , two different algebraic properties of  $A \rtimes_{\sigma} \mathbb{Z}$  are equivalent to density of the aperiodic points of the naturally associated dynamical system on the character space  $\Delta(A)$ . The analogue of this result in the context of crossed product  $C^*$ -algebras is [14, Theorem 5.4]. We shall also combine this with a theorem from [10] to conclude a stronger result for the Banach algebra  $L_1(G)$ , where  $G$  is a locally compact abelian group with connected dual group.

**Theorem 4.6.2.** *Let  $A$  be a complex commutative semi-simple completely regular Banach algebra,  $\sigma : A \rightarrow A$  an automorphism and  $\tilde{\sigma}$  the homeomorphism of  $\Delta(A)$  in the Gelfand topology induced by  $\sigma$  as described above. Then the following three properties are equivalent:*

- *The aperiodic points  $\text{Per}^\infty(\Delta(A))$  of  $(\Delta(A), \tilde{\sigma})$  are dense in  $\Delta(A)$ ;*

- Every non-zero ideal  $I \subseteq A \rtimes_\sigma \mathbb{Z}$  is such that  $I \cap A \neq \{0\}$ ;
- $A$  is a maximal abelian subalgebra of  $A \rtimes_\sigma \mathbb{Z}$ .

*Proof.* As  $A$  is completely regular, and  $\Delta(A)$  is Baire since it is locally compact and Hausdorff, it is immediate from Theorem 4.4.5 that the following three statements are equivalent.

- The aperiodic points  $\text{Per}^\infty(\Delta(A))$  of  $(\Delta(A), \tilde{\sigma})$  are dense in  $\Delta(A)$ ;
- Every non-zero ideal  $I \subseteq \widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$  is such that  $I \cap \widehat{A} \neq \{0\}$ ;
- $\widehat{A}$  is a maximal abelian subalgebra of  $\widehat{A} \rtimes_{\tilde{\sigma}} \mathbb{Z}$ .

Now applying Theorem 4.6.1 we can pull everything back to  $A \rtimes_\sigma \mathbb{Z}$  and the result follows. □

The following result for a more specific class of Banach algebras is an immediate consequence of Theorem 4.6.2 together with [10, Theorem 4.16].

**Theorem 4.6.3.** *Let  $G$  be a locally compact abelian group with connected dual group and let  $\sigma : L_1(G) \rightarrow L_1(G)$  be an automorphism. Then the following three statements are equivalent.*

- $\sigma$  is not of finite order;
- Every non-zero ideal  $I \subseteq L_1(G) \rtimes_\sigma \mathbb{Z}$  is such that  $I \cap L_1(G) \neq \{0\}$ ;
- $L_1(G)$  is a maximal abelian subalgebra of  $L_1(G) \rtimes_\sigma \mathbb{Z}$ .

To give a more complete picture, we also include the results [11, Theorem 5.1] and [11, Theorem 7.6].

**Theorem 4.6.4.** *Let  $A$  be a complex commutative semi-simple completely regular unital Banach algebra such that  $\Delta(A)$  consists of infinitely many points, and let  $\sigma$  be an automorphism of  $A$ . Then*

- $A \rtimes_\sigma \mathbb{Z}$  is simple if and only if the associated system  $(\Delta(A), \tilde{\sigma})$  on the character space is minimal.
- $A \rtimes_\sigma \mathbb{Z}$  is prime if and only if  $(\Delta(A), \tilde{\sigma})$  is topologically transitive.

## 4.7. The Banach algebra crossed product $\ell_1^\sigma(\mathbb{Z}, A)$ for commutative $C^*$ -algebras $A$

Let  $A$  be a commutative  $C^*$ -algebra with spectrum  $\Delta(A)$  and  $\sigma : A \rightarrow A$  an automorphism. We identify the set  $\ell^1(\mathbb{Z}, A)$  with the set  $\{\sum_{n \in \mathbb{Z}} f_n \delta^n \mid f_n \in A, \sum_{n \in \mathbb{Z}} \|f_n\| < \infty\}$  and endow it with the same operations as for the finite sums in Section 4.2. Using that  $\sigma$  is isometric one easily checks that the operations are well defined, and that the natural  $\ell^1$ -norm on this algebra is an algebra norm with respect to the convolution product.

We denote this algebra by  $\ell_1^\sigma(\mathbb{Z}, A)$ , and note that it is a Banach algebra. By basic theory of  $C^*$ -algebras, we have the isometric automorphism  $A \cong \widehat{A} = C_0(\Delta(A))$ . As in Section 4.6,  $\sigma$  induces a homeomorphism,  $\tilde{\sigma} : \Delta(A) \rightarrow \Delta(A)$  and an automorphism  $\widehat{\sigma} : C_0(\Delta(A)) \rightarrow C_0(\Delta(A))$  and we have a canonical isometric isomorphism of  $\ell_1^\sigma(\mathbb{Z}, A)$  onto  $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$  as in Theorem 4.6.1.

We will work in the concrete crossed product  $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$ . We shall describe the closed commutator ideal  $\mathcal{C}$  in terms of  $(\Delta(A), \sigma)$ . In analogy with the notation used in [12], we make the following definitions.

**Definition 4.7.1.** Given a subset  $S \subseteq \Delta(A)$ , we set

$$\begin{aligned} \ker(S) &= \{f \in C_0(\Delta(A)) \mid f(x) = 0 \text{ for all } x \in S\}, \\ \text{Ker}(S) &= \left\{ \sum_{n \in \mathbb{Z}} f_n \in \ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A))) \mid f_n(x) = 0 \text{ for all } x \in S, n \in \mathbb{Z} \right\}. \end{aligned}$$

Clearly  $\text{Ker}(S)$  is always a closed subspace, and in case  $S$  is invariant, it is a closed ideal.

We will also need the following version of the Stone-Weierstrass theorem.

**Theorem 4.7.2.** *Let  $X$  be a locally compact Hausdorff space and let  $C$  be a closed subset of  $X$ . Let  $B$  be a self-adjoint subalgebra of  $C_0(X)$  vanishing on  $C$ . Suppose that for any pair of points  $x, y \in X$ , with  $x \neq y$ , such that at least one of them is not in  $C$ , there exists  $f \in B$  such that  $f(x) \neq f(y)$ . Then  $\overline{B} = \{f \in C_0(X) : f(x) = 0 \text{ for all } x \in C\}$ .*

*Proof.* This follows from the more general result [3, Theorem 11.1.8], as it is well known that the pure states of  $C_0(\Delta(A))$  are precisely the point evaluations on the locally compact Hausdorff space  $\Delta(A)$ , and that a pure state of a  $C^*$ -subalgebra always has a pure state extension to the whole  $C^*$ -algebra. By passing to the one-point compactification of  $\Delta(A)$ , one may also easily derive the result from the more elementary [4, Theorem 2.47].  $\square$

**Definition 4.7.3.** Let  $A$  be a normed algebra. An approximate unit of  $A$  is a net  $\{E_\lambda\}_{\lambda \in \Lambda}$  such that for every  $a \in A$  we have  $\lim_\lambda \|E_\lambda a - a\| = \lim_\lambda \|a E_\lambda - a\| = 0$ .

Recall that every  $C^*$ -algebra has an approximate unit such that  $\|E_\lambda\| \leq 1$  for all  $\lambda \in \Lambda$ . In general, however, an approximate identity need not be bounded. We are now ready to prove the following result, which is the analogue of the first part of [12, Proposition 4.9].

**Theorem 4.7.4.**  $\mathcal{C} = \text{Ker}(\text{Per}^1(\Delta(A)))$ .

*Proof.* It is easily seen that  $\mathcal{C} \subseteq \text{Ker}(\text{Per}^1(\Delta(A)))$ . For the converse inclusion we choose an approximate identity  $\{E_\lambda\}_{\lambda \in \Lambda}$  for  $C_0(\Delta(A))$  and note first of all that for any  $f \in C_0(\Delta(A))$  we have  $f * (E_\lambda \delta) - (E_\lambda \delta) * f = E_\lambda (f - f \circ \tilde{\sigma}^{-1}) \delta \in \mathcal{C}$ . Hence as  $\mathcal{C}$  is closed,  $(f - f \circ \tilde{\sigma}^{-1}) \delta \in \mathcal{C}$  for all  $f \in C_0(\Delta)$ . Clearly the set  $J = \{g \in C_0(\Delta(A)) \mid g \delta \in \mathcal{C}\}$  is a closed subalgebra (and even an ideal) of  $C_0(\Delta(A))$ . Denote by  $I$  the (self-adjoint) ideal of  $C_0(\Delta(A))$  generated by the set of elements of the form  $f - f \circ \tilde{\sigma}^{-1}$ . Note that  $I$  vanishes on  $\text{Per}^1(\Delta(A))$  and that it is contained in  $J$ . Using complete regularity of  $C_0(\Delta(A))$ , it is straightforward to check that for any pair of distinct points

$x, y \in \Delta(A)$ , at least one of which is not in  $\text{Per}^1(\Delta(A))$ , there exists a function  $f \in I$  such that  $f(x) \neq f(y)$ . Hence by Theorem 4.7.2  $I$  is dense in  $\ker(\text{Per}^1(\Delta(A)))$ , and thus  $\{f\delta \mid f \in \ker(\text{Per}^1(\Delta(A)))\} \subseteq \mathcal{C}$  since  $J$  is closed. So for any  $n \in \mathbb{Z}$  and  $f \in \ker(\text{Per}^1(\Delta(A)))$  we have  $(f\delta) * (E_\lambda \circ \tilde{\sigma})\delta^{n-1} = (fE_\lambda)\delta^n \in \mathcal{C}$ . This converges to  $f\delta^n$ , and hence  $\mathcal{C} \supseteq \text{Ker}(\text{Per}^1(\Delta(A)))$ .  $\square$

Denote the set of non-zero multiplicative linear functionals of  $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$  by  $\Xi$ . We shall now determine a bijection between  $\Xi$  and  $\text{Per}^1(\Delta(A)) \times \mathbb{T}$ . It is a standard result from Banach algebra theory that any  $\mu \in \Xi$  is bounded and of norm at most one. Since one may choose an approximate identity  $\{E_\lambda\}_{\lambda \in \Lambda}$  for  $C_0(\Delta(A))$  such that  $\|E_\lambda\| \leq 1$  for all  $\lambda \in \Lambda$  it is also easy to see that  $\|\mu\| = 1$ . Namely, given  $\mu \in \Xi$  we may choose an  $f \in C_0(\Delta(A))$  such that  $\mu(f) \neq 0$ . Then by continuity of  $\mu$  we have  $\mu(f) = \lim_\lambda \mu(fE_\lambda) = \mu(f) \lim_\lambda \mu(E_\lambda)$  and hence  $\lim_\lambda \mu(E_\lambda) = 1$ .

**Lemma 4.7.5.** *The limit  $\zeta := \lim_\lambda \mu(E_\lambda\delta)$  exists for all  $\mu \in \Xi$ , and is independent of the approximate unit  $\{E_\lambda\}_{\lambda \in \Lambda}$ . Furthermore,  $\zeta \in \mathbb{T}$  and  $\lim_\lambda \mu(E_\lambda\delta^n) = \zeta^n$  for all integers  $n$ .*

*Proof.* By continuity and multiplicativity of  $\mu$  we have that  $\lim_\lambda \mu(f)\mu(E_\lambda\delta) = \mu(f\delta)$  for all  $f \in C_0(X)$ . So for any  $f$  such that  $\mu(f) \neq 0$  we have that  $\lim_\lambda \mu(E_\lambda\delta) = \frac{\mu(f\delta)}{\mu(f)}$ . This shows that the limit  $\zeta$  exists and is the same for any approximate unit, and using a similar argument one easily sees that  $\lim_\lambda \mu(E_\lambda\delta^n)$  also exists and is independent of  $\{E_\lambda\}_{\lambda \in \Lambda}$ . For the rest of the proof, we fix an approximate unit  $\{E_\lambda\}_{\lambda \in \Lambda}$  such that  $\|E_\lambda\| \leq 1$  for all  $\lambda \in \Lambda$ . As we know that  $\|\mu\| = 1$ , we see that  $|\zeta| \leq 1$ . Now suppose  $|\zeta| < 1$ . It is easy to see that  $\lim_\lambda \mu(E_\lambda) = 1 = \zeta^0$ . Hence also

$$\begin{aligned} 1 &= \lim_\lambda \mu(E_\lambda)^2 = \lim_\lambda \mu(E_\lambda^2) = \lim_\lambda \mu((E_\lambda\delta) * ((E_\lambda \circ \tilde{\sigma})\delta^{-1})) \\ &= \lim_\lambda \mu((E_\lambda\delta)) \cdot \lim_\lambda \mu((E_\lambda \circ \tilde{\sigma})\delta^{-1}). \end{aligned}$$

Now as we assumed  $|\zeta| < 1$ , this forces  $|\lim_\lambda \mu([(E_\lambda \circ \tilde{\sigma})\delta^{-1}])| > 1$ , which is clearly a contradiction since  $\|\mu\| = 1$ . To prove the last statement we note that for any  $n$ ,  $\{E_\lambda \circ \tilde{\sigma}^{-n}\}_{\lambda \in \Lambda}$  is an approximate unit for  $C_0(X)$ , and that if  $\{F_\lambda\}_{\lambda \in \Lambda}$  is another approximate unit for  $C_0(X)$  indexed by the same set  $\Lambda$ , we have that  $\{E_\lambda F_\lambda\}_{\lambda \in \Lambda}$  is an approximate unit as well. Now note that  $\mu(E_\lambda\delta) \cdot \mu(E_\lambda\delta) = \mu((E_\lambda\delta) * (E_\lambda\delta)) = \mu(E_\lambda(E_\lambda \circ \tilde{\sigma}^{-1})\delta^2)$ . Using what we concluded above about independence of approximate units, this shows that  $\zeta^2 = \lim_\lambda \mu(E_\lambda\delta)^2 = \lim_\lambda \mu(E_\lambda(E_\lambda \circ \tilde{\sigma}^{-1})\delta^2) = \lim_\lambda \mu(E_\lambda\delta^2)$ . Inductively, we see that  $\lim_\lambda \mu(E_\lambda\delta^n) = \zeta^n$  for non-negative  $n$ . As  $\mu((E_\lambda\delta^{-1}) * (E_\lambda\delta)) = \mu(E_\lambda(E_\lambda \circ \tilde{\sigma}))$ , we conclude that  $\lim_\lambda \mu(E_\lambda\delta^{-1}) = \zeta^{-1}$ , and an argument similar to the one above allows us to draw the desired conclusion for all negative  $n$ .  $\square$

We may use this to see that  $\Xi = \emptyset$  if the system  $(\Delta(A), \tilde{\sigma})$  lacks fixed points. This follows from the fact that the restriction of a map  $\mu \in \Xi$  to  $C_0(\Delta(A))$  must be a point evaluation,  $\mu_x$  say, by basic Banach algebra theory. If  $x \neq \sigma(x)$  there exists an  $h \in C_0(\Delta(A))$  such that  $h(x) = 1$  and  $h \circ \sigma(x) = 0$ . By Lemma 4.7.5 we see that  $\mu(h\delta) = \lim_\lambda \mu(hE_\lambda\delta) = \lim_\lambda \mu(h)\mu(E_\lambda\delta) = h(x)\zeta = \zeta$  and likewise  $\mu(h\delta^{-1}) = \zeta^{-1}$ . But then

$$1 = \zeta^{-1}\zeta = \mu((h\delta^{-1}) * (h\delta)) = \mu(h \cdot (h \circ \sigma)) = h(x) \cdot h \circ \sigma(x) = 0,$$

which is a contradiction.

Now for any  $x \in \text{Per}^1(\Delta(A))$  and  $\xi \in \mathbb{T}$  there is a unique element  $\mu \in \Xi$  such that  $\mu(f_n \delta^n) = f_n(x) \xi^n$  for all  $n$  and by the above every element of  $\Xi$  must be of this form for a unique  $x$  and  $\xi$ . Thus we have a bijection between  $\Xi$  and  $\text{Per}^1(\Delta(A)) \times \mathbb{T}$ . Denote by  $I(x, \xi)$  the kernel of such  $\mu$ . This is clearly a modular ideal of  $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$  which is maximal and contains  $\mathcal{C}$  by multiplicativity and continuity of elements of  $\Xi$ .

**Theorem 4.7.6.** *The modular ideals of  $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$  which are maximal and contain the commutator ideal  $\mathcal{C}$  are precisely the ideals  $I(x, \xi)$ , where  $x \in \text{Per}^1(\Delta(A))$  and  $\xi \in \mathbb{T}$ .*

*Proof.* One inclusion is clear from the discussion above. For the converse, let  $M$  be such an ideal and note that it is easy to show that a maximal ideal containing  $\mathcal{C}$  is not properly contained in any proper left or right ideal. Thus as  $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))$  is a spectral algebra, [7, Theorem 2.4.13] implies that  $\ell_1^{\widehat{\sigma}}(\mathbb{Z}, C_0(\Delta(A)))/M$  is isomorphic to the complex field. This clearly implies that  $M$  is the kernel of a non-zero element of  $\Xi$ .  $\square$

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## Chapter 5

# On the commutant of $C(X)$ in $C^*$ -crossed products by $\mathbb{Z}$ and their representations

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**Abstract.** For the  $C^*$ -crossed product  $C^*(\Sigma)$  associated with an arbitrary topological dynamical system  $\Sigma = (X, \sigma)$ , we provide a detailed analysis of the commutant, in  $C^*(\Sigma)$ , of  $C(X)$  and the commutant of the image of  $C(X)$  under an arbitrary Hilbert space representation  $\tilde{\pi}$  of  $C^*(\Sigma)$ . In particular, we give a concrete description of these commutants, and also determine their spectra. We show that, regardless of the system  $\Sigma$ , the commutant of  $C(X)$  has non-zero intersection with every non-zero, not necessarily closed or self-adjoint, ideal of  $C^*(\Sigma)$ . We also show that the corresponding statement holds true for the commutant of  $\tilde{\pi}(C(X))$  under the assumption that a certain family of pure states of  $\tilde{\pi}(C^*(\Sigma))$  is total. Furthermore we establish that, if  $C(X) \subsetneq C(X)'$ , there exist both a  $C^*$ -subalgebra properly between  $C(X)$  and  $C(X)'$  which has the aforementioned intersection property, and such a  $C^*$ -subalgebra which does not have this property. We also discuss existence of a projection of norm one from  $C^*(\Sigma)$  onto the commutant of  $C(X)$ .

### 5.1. Introduction

Let  $\Sigma = (X, \sigma)$  be a topological dynamical system where  $X$  is a compact Hausdorff space and  $\sigma$  is a homeomorphism of  $X$ . We denote by  $\alpha$  the automorphism of  $C(X)$ , the algebra of all continuous complex valued functions on  $X$ , induced by  $\sigma$ , namely  $\alpha(f) = f \circ \sigma^{-1}$  for  $f \in C(X)$ . Denote by  $C^*(\Sigma)$  the associated transformation group  $C^*$ -algebra, that is, the  $C^*$ -crossed product of  $C(X)$  by  $\mathbb{Z}$ , where  $\mathbb{Z}$  acts on  $C(X)$  via iterations of  $\alpha$ . The interplay between topological dynamical systems and  $C^*$ -algebras has been intensively studied, see

for example [6, 10, 11, 12, 13]. The following result constitutes the motivating background of this paper.

**Theorem 5.1.1.** [10, Theorem 5.4] *For a topological dynamical system  $\Sigma$ , the following statements are equivalent.*

- (i)  $\Sigma$  is topologically free;
- (ii)  $I \cap C(X) \neq 0$  for every non-zero closed ideal  $I$  of  $C^*(\Sigma)$ ;
- (iii)  $C(X)$  is a maximal abelian  $C^*$ -subalgebra of  $C^*(\Sigma)$ .

Recall that a system  $\Sigma = (X, \sigma)$  is called topologically free if the set of its aperiodic points is dense in  $X$ . We say that  $C(X)$  has the intersection property for closed ideals of  $C^*(\Sigma)$  when (ii) is satisfied (cf. Definition 5.4.1).

Many significant results concerning the interplay between  $\Sigma$  and  $C^*(\Sigma)$  have been obtained under the assumption that  $\Sigma$  is topologically free. As there are important examples of topological dynamical systems that are not topologically free, rational rotation of the unit circle being a typical one, our aim is to analyze the situation around Theorem 5.1.1 for arbitrary  $\Sigma$ .

We shall be concerned in detail with the commutant of  $C(X)$ , which we denote by  $C(X)'$ , and the commutant of the image of  $C(X)$  under Hilbert space representations of  $C^*(\Sigma)$ . In a series of papers, [7, 8, 9], improved analogues of Theorem 5.1.1 have been obtained in the context of an algebraic crossed product by the integers of, in particular, commutative Banach algebras  $A$  more general than  $C(X)$ , and especially  $A'$  has been thoroughly investigated there. In that setup it is an elementary result that  $A'$  is commutative ([7, Proposition 2.1]) and thus a maximal commutative subalgebra of the corresponding crossed product, and that  $A'$  has non-zero intersection with every non-zero ideal ([9, Theorem 3.1]) even when  $A$  is an arbitrary commutative complex algebra.

Here we give an explicit description of  $C(X)' \subseteq C^*(\Sigma)$  and  $\pi(C(X))' \subseteq \tilde{\pi}(C^*(\Sigma))$  where  $\tilde{\pi} = \pi \times u$  is a Hilbert space representation of  $C^*(\Sigma)$  (Proposition 5.3.2). Moreover, we prove that these algebras constitute commutative, hence maximal commutative,  $C^*$ -subalgebras that are invariant under  $\text{Ad } \delta$  (recall that, for  $a \in C^*(\Sigma)$ ,  $\text{Ad } \delta(a) = \delta a \delta^*$ ) and  $\text{Ad } \delta^*$  respectively under  $\text{Ad } u$  and  $\text{Ad } u^*$  (here  $\delta$  denotes the canonical unitary element of  $C^*(\Sigma)$  that implements the action of  $\mathbb{Z}$  on  $C(X)$  via  $\alpha$ , and  $u = \tilde{\pi}(\delta)$ ), and determine the structure of their spectra (Theorem 5.3.6). Invariance under these automorphisms implies that their restrictions to  $C(X)'$  and  $\pi(C(X))'$ , respectively, correspond to homeomorphisms of the spectra of these algebras. Certain aspects of the associated dynamical systems are investigated (Proposition 5.3.8) and later used to prove Theorem 5.4.3:  $\pi(C(X))'$  has the intersection property for ideals, not necessarily closed or self-adjoint, under the assumption that a certain family of pure states of  $\tilde{\pi}(C^*(\Sigma))$  is total and, consequently, Corollary 5.4.4, one of the main results of this paper: regardless of the system  $\Sigma$ ,  $C(X)'$  has the intersection property for ideals of  $C^*(\Sigma)$ . It is a consequence of Proposition 5.4.2 that these algebras have the intersection property for arbitrary ideals rather than just for closed ones, which also sharpens our background result, Theorem 5.1.1. In Section 5.5 we investigate ideal intersection properties of so called intermediate subalgebras, meaning  $C^*$ -subalgebras  $B$  of  $C^*(\Sigma)$  such that  $C(X) \subseteq B \subseteq C(X)'$ . In Proposition 5.5.1 we give an abstract condition on such

$B$ , in terms of the relation of its spectrum to the spectrum of  $C(X)'$ , that it satisfies precisely when it has the intersection property for ideals. Using this, we provide in Theorem 5.5.2 a rather short alternative proof of a refined version of our aforementioned background result. In Theorem 5.5.9 we clarify completely the situation concerning  $C(X)$  and its commutant, as well as the intersection property for ideals of these algebras and intermediate subalgebras:  $C(X)'$  is the unique maximal abelian  $C^*$ -subalgebra of  $C^*(\Sigma)$  containing  $C(X)$ ,  $C(X)' = C(X)$  precisely when  $\Sigma$  is topologically free and as soon as  $C(X)'$  is strictly larger than  $C(X)$  there exist intermediate subalgebras  $B_i$  with  $C(X) \subsetneq B_i \subsetneq C(X)'$ , for  $i = 1, 2$ , such that  $B_1$  has the intersection property for ideals and  $B_2$  does not. Finally, in Section 5.6 we discuss the existence of norm one projections from  $C^*(\Sigma)$  onto  $C(X)'$ .

## 5.2. Notation and preliminaries

Throughout this paper we consider topological dynamical systems  $\Sigma = (X, \sigma)$  where  $X$  is a compact Hausdorff space and  $\sigma : X \rightarrow X$  is a homeomorphism. Here  $\mathbb{Z}$  acts on  $X$  via iterations of  $\sigma$ , namely  $x \mapsto \sigma^n(x)$  for  $n \in \mathbb{Z}$  and  $x \in X$ . We denote by  $\text{Per}^\infty(\sigma)$  and  $\text{Per}(\sigma)$  the sets of aperiodic points and periodic points, respectively. If  $\text{Per}^\infty(\sigma) = X$ ,  $\Sigma$  is called free and if  $\text{Per}^\infty(\sigma)$  is dense in  $X$ ,  $\Sigma$  is called topologically free. Moreover, for an integer  $n$  we write  $\text{Per}^n(\sigma) = \text{Per}^{-n}(\sigma) = \{x \in X : \sigma^n(x) = x\}$ , and  $\text{Per}_n(\sigma)$  for the set of all points belonging to  $\text{Per}^n(\sigma)$  but to no  $\text{Per}^k(\sigma)$  with  $|k|$  non-zero and strictly less than  $|n|$ . When  $n = 0$  we regard  $\text{Per}^0(\sigma) = X$ . We write  $\text{Per}(x) = k$  if  $x \in \text{Per}_k(\sigma)$ . Note that if  $\text{Per}(y) = k$  and  $y \in \text{Per}^n(\sigma)$  then  $k|n$ . For a subset  $S \subseteq X$  we write its interior as  $S^0$  and its closure as  $\bar{S}$ . When a periodic point  $y$  belongs to the interior of  $\text{Per}_k(\sigma)$  for some integer  $k$  we call  $y$  a periodic interior point. We denote the set of all such points by  $\text{PIP}(\sigma)$ . Note that  $\text{PIP}(\sigma)$  does not coincide with  $\text{Per}(\sigma)^0$  in general, as the following example shows. Let  $X = [0, 1] \times [-1, 1]$  be endowed with the standard subspace topology from  $\mathbb{R}^2$  and let  $\sigma$  be the homeomorphism of  $X$  defined as reflection in the  $x$ -axis. Then clearly  $\text{Per}(\sigma) = X$ . Furthermore,  $\text{Per}_1(\sigma) = [0, 1] \times \{0\}$ , hence  $\text{Per}_1(\sigma)^0 = \emptyset$ , and  $\text{Per}_2(\sigma) = X \setminus \text{Per}_1(\sigma)$ , so that  $\text{Per}_2(\sigma)^0 = \text{Per}_2(\sigma)$ . We conclude that  $\text{PIP}(\sigma) = \text{Per}_2(\sigma) \subsetneq X = \text{Per}(\sigma)^0$ . Incidentally,  $\text{Per}_2(\sigma)$  in this example also shows that the sets  $\text{Per}_n(\sigma)$  are in general not closed, as opposed to the sets  $\text{Per}^n(\sigma)$  which are easily seen to always be closed. The following lemma will be a key result in what follows.

**Lemma 5.2.1.** *The union of  $\text{Per}^\infty(\sigma)$  and  $\text{PIP}(\sigma)$  is dense in  $X$ .*

*Proof.* Suppose the union were not dense in  $X$ , and let  $Y$  be the complement of its closure. Then  $Y$  is a non-empty open subset of  $X$  and hence it is locally compact in the induced topology. Since a locally compact space is a Baire space and  $Y = \bigcup_{k=1}^\infty \text{Per}^k(\sigma) \cap Y$ , where the  $\text{Per}^k(\sigma) \cap Y$  are closed in  $Y$ , there exists a positive integer  $n$  such that  $\text{Per}^n(\sigma) \cap Y$  has non-empty interior in  $Y$  and hence in  $X$  as  $Y$  is an open subset of  $X$ . Take the minimal such integer and write it as  $n$  again. If  $n = 1$  or a prime number we arrive at a contradiction as for such  $n$  we have that  $\text{Per}^n(\sigma) = \text{Per}_n(\sigma)$  and the above then implies that  $\text{Per}_n(\sigma)$  has non-empty interior in  $X$ . Thus we assume that  $n$  is greater than one and not a prime number. Let  $k_1, k_2, \dots, k_i$  be the positive divisors of  $n$  that are strictly smaller than  $n$ . Suppose that  $\bigcup_{j=1}^i \text{Per}^{k_j}(\sigma) \cap Y$  has non-empty interior, say  $\emptyset \neq V \cap Y \subseteq \bigcup_{j=1}^i \text{Per}^{k_j}(\sigma) \cap Y$  for some

open subset  $V$  of  $X$ , so that  $V \cap Y = \bigcup_{j=1}^i \text{Per}^{k_j}(\sigma) \cap V \cap Y$ . Since  $V \cap Y$ , being open in  $X$ , is a locally compact Hausdorff space in the induced topology, hence a Baire space, there exists  $j$  such that the closure (in  $V \cap Y$ ) of  $\text{Per}^{k_j}(\sigma) \cap V \cap Y$  has non-empty interior in  $V \cap Y$ . However, since  $\text{Per}^{k_j}(\sigma) \cap V \cap Y$  is closed in  $V \cap Y$ , because  $\text{Per}^{k_j}(\sigma)$  is closed in  $X$ , and since  $V \cap Y$  is open in  $X$ , we see that  $\text{Per}^{k_j}(\sigma) \cap V \cap Y$  itself has an interior point in  $X$ . Hence  $\text{Per}^{k_j}(\sigma)$  has an interior point in  $Y$  and using the assumption on  $n$  we arrive at a contradiction since  $k_j < n$ . We conclude that  $\bigcup_{j=1}^i \text{Per}^{k_j}(\sigma) \cap Y$  has empty interior. Denote by  $U$  the interior of  $\text{Per}^n(\sigma) \cap Y$  in  $Y$ . Since  $U$  is non-empty by assumption, it follows from the above that  $U \setminus (\bigcup_{j=1}^i \text{Per}^{k_j}(\sigma) \cap Y)$  is a non-empty open subset of  $Y$  and hence of  $X$ . But  $U \setminus (\bigcup_{j=1}^i \text{Per}^{k_j}(\sigma) \cap Y) \subseteq \text{Per}^n(\sigma) \cap Y \setminus (\bigcup_{j=1}^i \text{Per}^{k_j}(\sigma) \cap Y) = \text{Per}_n(\sigma) \cap Y$ . We conclude that  $\text{Per}_n(\sigma)$  has an interior point in  $Y$ , which is a contradiction.  $\square$

We remark that when speaking of ideals we shall always mean two-sided ideals which are not necessarily closed or self-adjoint unless we state this explicitly.

With the automorphism  $\alpha : C(X) \rightarrow C(X)$  defined by  $\alpha(f) = f \circ \sigma^{-1}$  for  $f \in C(X)$ , we denote the  $C^*$ -crossed product  $C(X) \rtimes_{\alpha} \mathbb{Z}$  by  $C^*(\Sigma)$ . For simplicity, we denote the natural isomorphic copy of  $C(X)$  in  $C^*(\Sigma)$  by  $C(X)$  as well. We denote the canonical unitary element of  $C^*(\Sigma)$  that implements the action of  $\mathbb{Z}$  on  $C(X)$  via  $\alpha$  by  $\delta$ , recalling that  $\alpha(f) = \text{Ad } \delta(f)$  for  $f \in C(X)$ . By construction,  $C^*(\Sigma)$  is generated as a  $C^*$ -algebra by  $C(X)$  together with  $\delta$ . A generalized polynomial is a finite sum of the form  $\sum_n f_n \delta^n$  with  $f_n \in C(X)$ , and we shall refer to the norm-dense  $*$ -subalgebra of  $C^*(\Sigma)$  consisting of all generalized polynomials as the algebraic part of  $C^*(\Sigma)$ . We write the canonical faithful projection of norm one from  $C^*(\Sigma)$  to  $C(X)$  as  $E$  and recall that  $E$  is defined on the algebraic part of  $C^*(\Sigma)$  as  $E(\sum_n f_n \delta^n) = f_0$ . For an element  $a \in C^*(\Sigma)$  and an integer  $j$  we define  $a(j) = E(a\delta^{-j})$ , the  $j$ th generalized Fourier coefficient of  $a$ . It is a fact that  $a = 0$  if and only if  $a(j) = 0$  for all integers  $j$  and that  $a$  is thus uniquely determined by its generalized Fourier coefficients ([10, Theorem 1.3]). A Hilbert space representation of  $C^*(\Sigma)$  is written as  $\tilde{\pi} = \pi \times u$ , where  $\pi$  is the representation of  $C(X)$  on the same Hilbert space given by restriction of  $\tilde{\pi}$ , and  $u = \tilde{\pi}(\delta)$ . The operations on  $C^*(\Sigma)$  then imply that

$$\pi(\alpha(f)) = u\pi(f)u^* = \text{Ad } u(\pi(f)), \text{ for } f \in C(X).$$

We shall often make use of the dynamical system  $\Sigma_{\pi} = (X_{\pi}, \sigma_{\pi})$  derived from a representation  $\tilde{\pi}$ . As explained in [10, p. 26], we define this dynamical system as

$$X_{\pi} = h(\pi^{-1}(0)) \quad \text{and} \quad \sigma_{\pi} = \sigma|_{X_{\pi}},$$

where  $h(\pi^{-1}(0))$  means the standard hull of the kernel ideal of  $\pi$  in  $C(X)$ ;  $X_{\pi}$  is obviously a closed subset of  $X$  that is invariant under  $\sigma$  and its inverse. The system  $\Sigma_{\pi}$  is topologically conjugate to the dynamical system  $\Sigma'_{\pi} = (X'_{\pi}, \sigma'_{\pi})$  where  $X'_{\pi}$  is the spectrum of  $\pi(C(X))$  and the map  $\sigma'_{\pi}$  is the homeomorphism of  $X'_{\pi}$  induced by the automorphism  $\text{Ad } u$  on  $\pi(C(X))$ . To see this, note that the homeomorphism  $\theta : X_{\pi} \rightarrow X'_{\pi}$  induced by the isomorphism  $\pi(f) \mapsto f|_{X_{\pi}}$  between  $\pi(C(X))$  and  $C(X_{\pi})$  is such that  $\sigma'_{\pi} \circ \theta = \theta \circ \sigma_{\pi}$ . Thus we identify these two dynamical systems. Note that under this identification,  $\pi(f)$  corresponds to the restriction of the function  $f$  to the set  $X_{\pi}$ . We denote the canonical unitary element of  $C^*(\Sigma_{\pi})$  by  $\delta_{\pi}$ . We now recall three results that will be important to us throughout this paper.

**Proposition 5.2.2.** [13, Proposition 3.4.] *If  $\tilde{\pi} = \pi \times u$  is an infinite-dimensional irreducible representation of  $C^*(\Sigma)$ , then the dynamical system  $\Sigma_\pi$  is topologically free.*

**Theorem 5.2.3.** [10, Theorem 5.1.] *Let  $\tilde{\pi} = \pi \times u$  be a representation of  $C^*(\Sigma)$  on a Hilbert space  $H$ . If the induced dynamical system  $\Sigma_\pi$  is topologically free, then there exists a projection  $\epsilon_\pi$  of norm one from the  $C^*$ -algebra  $\tilde{\pi}(C^*(\Sigma))$  to  $\pi(C(X))$  such that the following diagram commutes.*

$$\begin{array}{ccc} C^*(\Sigma) & \xrightarrow{\tilde{\pi}} & \tilde{\pi}(C^*(\Sigma)) \\ E \downarrow & & \downarrow \epsilon_\pi \\ C(X) & \xrightarrow{\pi} & \pi(C(X)) \end{array}$$

**Corollary 5.2.4.** [10, Corollary 5.1.A.] *Suppose the situation is as in Theorem 5.2.3. Then the map defined by  $\pi(f) \mapsto f|_{X_\pi}$  for  $f \in C(X)$  and  $u \mapsto \delta_\pi$ , extends to an isomorphism between  $\tilde{\pi}(C^*(\Sigma))$  and  $C^*(\Sigma_\pi)$ .*

For  $x \in X$  we denote by  $\mu_x$  the functional on  $C(X)$  that acts as point evaluation in  $x$ . Since the pure state extensions to  $C^*(\Sigma)$  of the point evaluations on  $C(X)$  will play a prominent role in this paper, we shall now recall some basic facts about them, without proofs. For further details and proofs, we refer to [10, §4]. For  $x \in \text{Per}^\infty(\sigma)$  there is a unique pure state extension of  $\mu_x$ , denoted by  $\varphi_x$ , given by  $\varphi_x = \mu_x \circ E$ . The set of pure state extensions of  $\mu_y$  for  $y \in \text{Per}(\sigma)$  is parametrized by the unit circle as  $\{\varphi_{y,t} : t \in \mathbb{T}\}$ . We write the GNS-representations associated with the pure state extensions above as  $\tilde{\pi}_x$  and  $\tilde{\pi}_{y,t}$ . For  $x \in \text{Per}^\infty(\sigma)$ ,  $\tilde{\pi}_x$  is the representation of  $C^*(\Sigma)$  on  $\ell^2$ , whose standard basis we denote by  $\{e_i\}_{i \in \mathbb{Z}}$ , defined on the generators as follows. For  $f \in C(X)$  and  $i \in \mathbb{Z}$  we have  $\tilde{\pi}_x(f)e_i = f \circ \sigma^i(x) \cdot e_i$ , and  $\tilde{\pi}_x(\delta)e_i = e_{i+1}$ . For  $y \in \text{Per}(\sigma)$  with  $\text{Per}(y) = p$  and  $t \in \mathbb{T}$ ,  $\tilde{\pi}_{y,t}$  is the representation on  $\mathbb{C}^p$ , whose standard basis we denote by  $\{e_i\}_{i=0}^{p-1}$ , defined as follows. For  $f \in C(X)$  and  $i \in \{0, 1, \dots, p-1\}$  we set  $\tilde{\pi}_{y,t}(f)e_i = f \circ \sigma^i(y) \cdot e_i$ . For  $j \in \{0, 1, \dots, p-2\}$ ,  $\tilde{\pi}_{y,t}(\delta)e_j = e_{j+1}$  and  $\tilde{\pi}_{y,t}(\delta)e_{p-1} = t \cdot e_0$ . We also mention that the unitary equivalence class of  $\tilde{\pi}_x$  is determined by the orbit of  $x$ , and that of  $\tilde{\pi}_{y,t}$  by the orbit of  $y$  and the parameter  $t$ .

In what follows, we shall sometimes make use of the following important result ([12, Proposition 2]).

**Proposition 5.2.5.** *For every closed ideal  $I \subseteq C^*(\Sigma)$  there exist families  $\{x_\alpha\}$  of aperiodic points and  $\{y_\beta, t_\gamma\}$  of periodic points and parameters from the unit circle, such that  $I$  is the intersection of the associated kernels  $\ker(\tilde{\pi}_{x_\alpha})$  and  $\ker(\tilde{\pi}_{y_\beta, t_\gamma})$ .*

### 5.3. The structure of $C(X)'$ and $\pi(C(X))'$

We shall now make a detailed analysis of the commutants  $C(X)'$  of  $C(X) \subseteq C^*(\Sigma)$  and  $\pi(C(X))'$  of  $\pi(C(X)) \subseteq \tilde{\pi}(C^*(\Sigma))$ , respectively. Here  $\tilde{\pi} = \pi \times u$  is a Hilbert space representation of  $C^*(\Sigma)$  as usual. These  $C^*$ -subalgebras are defined as follows

$$C(X)' = \{a \in C^*(\Sigma) : af = fa \text{ for all } f \in C(X)\}$$

and

$$\pi(C(X))' = \{\tilde{\pi}(a) \in \tilde{\pi}(C^*(\Sigma)) : \tilde{\pi}(af) = \tilde{\pi}(fa) \text{ for all } f \in C(X)\}.$$

We will need the following topological lemma.

**Lemma 5.3.1.** *The system  $\Sigma = (X, \sigma)$  is topologically free if and only if  $\text{Per}^n(\sigma)$  has empty interior for all positive integers  $n$ .*

*Proof.* Clearly, if there is a positive integer  $n_0$  such that  $\text{Per}^{n_0}(\sigma)$  has non-empty interior, the aperiodic points are not dense. For the converse, we recall that  $X$  is a Baire space since it is compact and Hausdorff, and note that

$$\text{Per}(\sigma) = \bigcup_{n>0} \text{Per}^n(\sigma).$$

If  $\text{Per}^\infty(\sigma)$  is not dense, its complement  $\text{Per}(\sigma)$  has non-empty interior, and as the sets  $\text{Per}^n(\sigma)$  are clearly all closed, there must exist an integer  $n_0 > 0$  such that  $\text{Per}^{n_0}(\sigma)$  has non-empty interior since  $X$  is a Baire space.  $\square$

The following proposition describes the commutants  $C(X)'$  and  $\pi(C(X))'$ .

**Proposition 5.3.2.** *Let  $\tilde{\pi} = \pi \times u$  be a Hilbert space representation of  $C^*(\Sigma)$ .*

- (i)  $C(X)' = \{a \in C^*(\Sigma) : \text{supp}(a(n)) \subseteq \text{Per}^n(\sigma) \text{ for all } n\}$ . *Consequently,  $C(X)' = C(X)$  if and only if the dynamical system is topologically free.*
- (ii)  $\pi(C(X))'$  *consists of all elements  $\tilde{\pi}(a)$  such that  $\tilde{\pi}_{x_\alpha}(a) \in \pi_{x_\alpha}(C(X))$  and  $\tilde{\pi}_{y_\beta, t_\gamma}(a) \in \pi_{y_\beta, t_\gamma}(C(X))$  for all  $\alpha, \beta, \gamma$  that appear in the description of the ideal  $I = \ker(\tilde{\pi})$  as in Proposition 5.2.5.*

*Proof.* The first assertion is a direct extension of [7, Corollary 3.4] to the context of  $C^*$ -crossed products. The main steps of the proof are contained in the first part of the proof of [10, Theorem 5.4], but we reproduce them here for the reader's convenience, and because this is also our basic starting point. Let  $a \in C^*(\Sigma)$  and  $f \in C(X)$  be arbitrary. We then have, for  $n \in \mathbb{Z}$ ,

$$\begin{aligned} (fa)(n) &= E(fa\delta^{*n}) = f \cdot E(a\delta^{*n}) = f \cdot a(n), \\ (af)(n) &= E(af\delta^{*n}) = E(a\delta^{*n}a^n(f)) = E(a\delta^{*n}) \cdot a^n(f) = a(n) \cdot f \circ \sigma^{-n}. \end{aligned}$$

Hence for  $a \in C(X)'$  we have, for any  $f \in C(X)$ ,  $x \in X$  and  $n \in \mathbb{Z}$ , that

$$f(x) \cdot a(n)(x) = a(n)(x) \cdot f \circ \sigma^{-n}(x).$$

Therefore, if  $a(n)(x)$  is not zero we have that  $f(x) = f \circ \sigma^{-n}(x)$  for all  $f \in C(X)$ . It follows that  $\sigma^{-n}(x) = x$  and hence that  $x$  belongs to the set  $\text{Per}^n(\sigma)$ , whence  $\text{supp}(a(n)) \subseteq \text{Per}^n(\sigma)$ .

Conversely, if  $\text{supp}(a(n)) \subseteq \text{Per}^n(\sigma)$  for every  $n$ , it follows easily from the above that

$$(fa)(n) = (af)(n)$$

for every  $f \in C(X)$  and  $n \in \mathbb{Z}$  and hence that  $a$  belongs to  $C(X)'$ .

Moreover, by Lemma 5.3.1,  $\Sigma$  is topologically free if and only if for every nonzero integer  $n$  the set  $\text{Per}^n(\sigma)$  has empty interior. So when the system is topologically free, we see from the above description of  $C(X)$  that an element  $a$  in  $C(X)'$  necessarily belongs to  $C(X)$ . If  $\Sigma$  is not topologically free, however, some  $\text{Per}^n(\sigma)$  has non-empty interior and hence there is a non-zero function  $f \in C(X)$  such that  $\text{supp}(f) \subseteq \text{Per}^n(\sigma)$ . Then  $f\delta^n \in C(X)' \setminus C(X)$  by the above.

For the second assertion, note that for an element  $a$  in  $C^*(\Sigma)$ ,  $\tilde{\pi}(a)$  belongs to  $\pi(C(X))'$  if and only if  $af - fa$  belongs to the kernel  $I$  for every function  $f \in C(X)$ . Hence this is equivalent to saying that the image of  $a$  belongs to the commutant of the image of  $C(X)$  for all irreducible representations with respect to the indices  $\alpha, \beta, \gamma$ . This in turn is equivalent to the assertion in (ii) because when  $x$  is aperiodic we know that  $\tilde{\pi}_x$  is infinite-dimensional, whence it follows from Proposition 5.2.2 that  $\Sigma_{\pi_x}$  is topologically free, and since  $C(X_{\pi_x})$  corresponds to  $\pi_x(C(X))$  under the isomorphism in Corollary 5.2.4, part (i) of this proposition implies that  $\pi_x(C(X))' = \pi_x(C(X))$ . So  $\tilde{\pi}_{x_\alpha}(a) \in \pi_{x_\alpha}(C(X))$  for all  $\alpha$  that occur in the description of  $I$ . When  $y$  is periodic with period  $n$ , the image  $\pi_{y,t}(C(X))$  consists of the diagonal matrices in  $M_n$  and thus coincides with its commutant. We conclude that  $\tilde{\pi}_{y\beta, t_\gamma}(a) \in \pi_{y\beta, t_\gamma}(C(X))$  for all  $\beta, \gamma$  that occur in the description of  $I$ .  $\square$

Before continuing, we recall the following noncommutative version of Fejér's theorem on Cesàro sums, which we shall use in our arguments.

**Proposition 5.3.3.** [12, Proposition 1]. *The sequence  $\{\sigma_n(a)\}_{n=0}^\infty$ , where  $\sigma_n(a)$  is the  $n$ -th generalized Cesàro sum of an element  $a \in C^*(\Sigma)$ , defined by*

$$\sigma_n(a) = \sum_{-n}^n \left(1 - \frac{|i|}{n+1}\right) a(i) \delta^i,$$

*converges to  $a$  in norm.*

Actually it is known that replacing the Cesàro sums by any other summability kernel such as the de la Vallée-Poussin kernel, Jackson kernel etc, we obtain the corresponding approximation sequences converging to  $a$  in norm ([12, Proposition 1]).

In the passage following Corollary 5.2.4 we described the pure state extensions to  $C^*(\Sigma)$  of the point evaluations on  $C(X)$ . Recalling the notation introduced there, we define the following two sets:

$$\Phi = \{\varphi_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\varphi_{y,t} : y \in \text{Per}(\sigma), t \in \mathbb{T}\}.$$

$$\Phi' = \{\varphi_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\varphi_{y,t} : y \in \text{PIP}(\sigma), t \in \mathbb{T}\}.$$

We notice that a representation  $\tilde{\pi}$  of  $C^*(\Sigma)$  can be factored as  $\tilde{\pi} = \hat{\pi} \circ \tilde{\rho}$ , where  $\tilde{\rho}$  is the canonical homomorphism from  $C^*(\Sigma)$  to  $C^*(\Sigma_\pi)$  induced by the restriction map from  $C(X)$  to  $C(X_\pi)$  and  $\hat{\pi}$  is the associated homomorphism from  $C^*(\Sigma_\pi)$  onto  $\tilde{\pi}(C^*(\Sigma))$ . Writing this out, we have

$$\begin{aligned} f &\xrightarrow{\tilde{\rho}} f|_{X_\pi} \xrightarrow{\hat{\pi}} \pi(f) \text{ for } f \in C(X), \\ \delta &\xrightarrow{\tilde{\rho}} \delta_\pi \xrightarrow{\hat{\pi}} \tilde{\pi}(\delta) = u. \end{aligned}$$

Note that here the restriction of  $\hat{\pi}$  to  $C(X_\pi)$  is an isomorphism onto  $\pi(C(X))$ . In the following arguments we often use this factorization, and may then regard  $C(X_\pi)$  as an embedded subalgebra of  $\tilde{\pi}(C^*(\Sigma)) = \hat{\pi}(C^*(\Sigma_\pi))$ . Moreover, we consider the pure state extensions to  $C^*(\Sigma_\pi)$  of point evaluations on  $C(X_\pi)$  as well as the pure state extensions to  $\tilde{\pi}(C^*(\Sigma))$  of point evaluations on  $C(X_\pi)$  when the latter is viewed as an embedded subalgebra of the former. We denote the families of pure state extensions to  $C^*(\Sigma_\pi)$  corresponding to  $\Phi$  (the pure state extensions of all point evaluations on  $C(X_\pi)$ ) and to  $\Phi'$  (the pure state extensions of point evaluations on  $C(X_\pi)$  in the set of points  $\text{Per}^\infty(\sigma_\pi) \cup \text{PIP}(\sigma_\pi)$ ) by  $\Phi_\pi$  and  $\Phi'_\pi$ , respectively. The family of pure state extensions to  $\tilde{\pi}(C^*(\Sigma))$  of point evaluations on the aforementioned embedded copy of  $C(X_\pi)$  will be denoted by  $\Phi(\tilde{\pi})$ .

For the following arguments, we recall the notion of right multiplicative domain for a unital positive linear map  $\tau$  between two unital  $C^*$ -algebras  $A$  and  $B$ . We write

$$A'_\tau = \{a \in A : \tau(ax) = \tau(a)\tau(x) \ \forall x \in A\},$$

and call this set the right multiplicative domain of  $\tau$ . For a detailed account of the theory of positive linear maps between  $C^*$ -algebras, we refer to [2]. Using the fact that positive linear maps between unital  $C^*$ -algebras respect involution, the following right-sided version of [1, Theorem 3.1], which concerns left multiplicative domains, is readily concluded.

**Theorem 5.3.4.** *If  $\tau : A \rightarrow B$  is a 2-positive linear map between two unital  $C^*$ -algebras  $A$  and  $B$ , then  $A'_\tau = \{a \in A : \tau(aa^*) = \tau(a)\tau(a)^*\}$ .*

Since a state of a unital  $C^*$ -algebra is even completely positive, the above result holds in particular when  $\tau$  is a state. The Cauchy-Schwarz inequality for states on  $C^*$ -algebras implies that if a state vanishes on a positive element,  $a$ , then it also vanishes on its positive square root,  $\sqrt{a}$ . It follows from Theorem 5.3.4 that  $\sqrt{a}$ , and hence  $a$ , is in the right multiplicative domain of every state that vanishes on  $a$ . We shall make use of right multiplicative domains in the following lemma. Recall that a family of states of a  $C^*$ -algebra  $A$  is said to be total if the only positive element of  $A$  on which every state in the family vanishes, is zero.

**Lemma 5.3.5.** *The family  $\Phi(\tilde{\pi})$  is total on  $\tilde{\pi}(C^*(\Sigma))$ . Furthermore, the family  $\Phi'$  is total on  $C^*(\Sigma)$ .*

*Proof.* Suppose the family  $\Phi(\tilde{\pi})$  vanishes on  $a \geq 0$ . By the comment preceding this lemma it follows that  $\sqrt{a}$  is in the right multiplicative domain of every state in  $\Phi(\tilde{\pi})$ . We consider the closed ideal  $J$  generated by  $a$ , which by the functional calculus coincides with the closed ideal generated by  $\sqrt{a}$ . Note that this ideal is the closed linear span of elements having the form  $f u^i a g u^j = f(u^i a u^{i*})(u^j g u^j)$  for functions  $f, g \in C(X_\pi)$ . Clearly  $f$  belongs to the right multiplicative domain of every member of  $\Phi(\tilde{\pi})$  by Theorem 5.3.4. To see that  $u^i a u^{i*}$  does as well, note firstly that if  $\varphi \in \Phi(\tilde{\pi})$  is a pure state extension of the point evaluation in  $x \in X_\pi$  on  $C(X_\pi)$ , then  $\varphi \circ \text{Ad } u^i \in \Phi(\tilde{\pi})$  since it is a pure state extension of the point evaluation in  $\sigma_\pi^{-i}(x)$  on  $C(X_\pi)$  and hence  $\varphi(u^i a u^{i*}) = 0$ . As  $u^i \sqrt{a} u^{i*}$  is the positive square root of  $u^i a u^{i*}$  it follows again by the comment preceding this lemma that  $u^i \sqrt{a} u^{i*}$  is in the right multiplicative domain of every element of  $\Phi(\tilde{\pi})$  and this clearly implies that  $u^i a u^{i*}$  is as well. Hence for every  $\varphi \in \Phi(\tilde{\pi})$  we have

$$\varphi(f(u^i a u^{i*})(u^j g u^j)) = \varphi(f)\varphi(u^i a u^{i*})\varphi(u^j g u^j) = 0$$



since  $\varphi(u^i a u^{i*}) = 0$ . We conclude that every member of  $\Phi(\tilde{\pi})$  vanishes on the whole ideal  $J$ . We want to deduce that  $J$  is the zero ideal, so assume for a contradiction that it is not. Then there exists a positive element  $b \in C^*(\Sigma_\pi)$  such that  $0 \neq \hat{\pi}(b) \in J$ . Since  $\ker(\hat{\pi})$  is a closed ideal of  $C^*(\Sigma_\pi)$ , we know by Proposition 5.2.5 that it is the intersection of the kernels of a certain family of irreducible representations. So there must be at least one of the representations determining  $\ker(\hat{\pi})$  as in Proposition 5.2.5,  $\tilde{\pi}_\epsilon$  say, for which  $\tilde{\pi}_\epsilon(b) \neq 0$ . As  $\ker(\hat{\pi}) \subseteq \ker(\tilde{\pi}_\epsilon)$ , we may well-define an irreducible representation  $\tilde{\pi}$  of  $\tilde{\pi}(C^*(\Sigma)) = \hat{\pi}(C^*(\Sigma_\pi))$  by  $\tilde{\pi}(\hat{\pi}(a)) = \tilde{\pi}_\epsilon(a)$  for  $a \in C^*(\Sigma_\pi)$ . Then the functional  $\tilde{\varphi}$  defined by  $\tilde{\varphi}(\hat{\pi}(a)) = (\tilde{\pi}(\hat{\pi}(a))\zeta_\epsilon, \zeta_\epsilon) = (\tilde{\pi}_\epsilon(a)\zeta_\epsilon, \zeta_\epsilon) = \varphi_\epsilon(a)$  is a pure state acting as a point evaluation on the embedded copy of  $C(X_\pi)$  in  $\tilde{\pi}(C^*(\Sigma))$ . Hence  $\tilde{\varphi} \in \Phi(\tilde{\pi})$  and by the above  $\tilde{\varphi}(\hat{\pi}(b)) = 0$  as  $\hat{\pi}(b) \in J$ . Similarly to above, one easily concludes that  $\tilde{\varphi}(\hat{\pi}(\delta_\pi^i b \delta_\pi^{*i})) = \varphi_\epsilon(\delta_\pi^i b \delta_\pi^{*i}) = 0$  for all integers  $i$ . Writing this out, using that  $b \geq 0$ , we get  $0 = \varphi_\epsilon(\delta_\pi^i b \delta_\pi^{*i}) = (\tilde{\pi}_\epsilon(\delta_\pi^i b \delta_\pi^{*i})\zeta_\epsilon, \zeta_\epsilon) = (\tilde{\pi}_\epsilon(\sqrt{b} \delta_\pi^{*i})\zeta_\epsilon, \tilde{\pi}_\epsilon(\sqrt{b} \delta_\pi^{*i})\zeta_\epsilon) = \|\tilde{\pi}_\epsilon(\sqrt{b} \delta_\pi^{*i})\zeta_\epsilon\|^2$ . Since  $\tilde{\pi}_\epsilon$  is a representation of the kind described in the passage following Corollary 5.2.4, we know that the closed linear span of the set  $\{\tilde{\pi}_\epsilon(\delta_\pi^i)\zeta_\epsilon\}_{i \in \mathbb{Z}}$  is the whole underlying Hilbert space  $H_\epsilon$ , whence the above equality implies that  $\tilde{\pi}_\epsilon(\sqrt{b}) = 0$  and thus finally  $\tilde{\pi}_\epsilon(b) = 0$ . This is a contradiction. Hence  $J$  was the zero ideal after all. In particular,  $a = 0$ .

Next, to see that  $\Phi'$  is total on  $C^*(\Sigma)$  we suppose it vanishes on a positive element  $a \in C^*(\Sigma)$ . Let  $x \in \text{Per}^\infty(\sigma)$ . Since  $\varphi_x$  is the unique pure state extension, and hence the unique state extension, of  $\mu_x$ , we have

$$E(a)(x) = \mu_x \circ E(a) = \varphi_x(a) = 0.$$

Now let  $y \in \text{PIP}(\sigma)$ . Since the set  $\{\varphi_{y,t} : t \in \mathbb{T}\}$  exhausts all pure state extension of  $\mu_y$  to  $C^*(\Sigma)$  it follows that if all members vanish at  $a$ , its state extension  $\mu_y \circ E$  vanishes at  $a$ , too. Hence by Lemma 5.2.1  $E(a)$  vanishes on a dense subset of  $X$ , and thus  $E(a) = 0$ . As  $E$  is faithful, this implies that  $a = 0$ .  $\square$

Denote by  $\Gamma$  the spectrum of  $C(X)'$  and by  $\Gamma(\tilde{\pi})$  the spectrum of  $\pi(C(X))'$  for an arbitrary representation  $\tilde{\pi} = \pi \times u$  of  $C^*(\Sigma)$ . The following theorem clarifies the structure of  $\Gamma(\tilde{\pi})$ . As every  $C^*$ -algebra has a faithful representation it also determines  $\Gamma$ .

**Theorem 5.3.6.** *Let  $\tilde{\pi} = \pi \times u$  be a representation of  $C^*(\Sigma)$ .*

(i)  $\pi(C(X))'$  is a commutative  $C^*$ -subalgebra of  $\tilde{\pi}(C^*(\Sigma))$  (necessarily maximal abelian), invariant under  $\text{Ad } u$  and its inverse. In particular,  $C(X)'$  is a maximal abelian  $C^*$ -subalgebra of  $C^*(\Sigma)$ , invariant under  $\text{Ad } \delta$  and its inverse. Moreover, the latter is the closure of its algebraic part, i.e., of the set of generalized polynomials in  $C(X)'$ .

(ii) The spectrum  $\Gamma(\tilde{\pi})$  consists of the restrictions of the set  $\Phi(\tilde{\pi})$  of pure state extensions.

*Proof.* Clearly,  $\pi(C(X))'$  is a  $C^*$ -subalgebra of  $\tilde{\pi}(C^*(\Sigma))$ . Take two elements  $\tilde{\pi}(a)$  and  $\tilde{\pi}(b)$  in  $\pi(C(X))'$ . Then by Proposition 5.3.2(ii),  $\tilde{\pi}_{x_\alpha}(ab - ba) = \tilde{\pi}_{y_\beta, t_\gamma}(ab - ba) = 0$  for all  $\alpha, \beta, \gamma$  determining the kernel of  $\tilde{\pi}$  as in Proposition 5.2.5. Hence  $\pi(C(X))'$  is indeed commutative and clearly it is maximal abelian. Invariance of  $\pi(C(X))'$  under  $\text{Ad } u$  and its inverse follows readily. The corresponding statement about  $C(X)'$  and  $\text{Ad } \delta$  and its inverse

holds since  $C^*(\Sigma)$  has a faithful representation, but can also be obtained by an explicit calculation using Proposition 5.3.2 (i) together with Proposition 5.3.3. The last statement of assertion (i) also follows from the characterization of  $C(X)'$  in Proposition 5.3.2 (i) combined with Proposition 5.3.3. For the assertion (ii), we may assume that  $\tilde{\pi} = \hat{\pi}$ , and hence that  $C^*(\Sigma) = C^*(\Sigma_\pi)$  with  $X = X_\pi$  and  $\sigma = \sigma_\pi$ . Since every element of  $\Gamma(\tilde{\pi})$  is the restriction to  $\pi(C(X))'$  of at least one element of  $\Phi(\tilde{\pi})$ , it is sufficient to show that the restriction of each pure state in  $\Phi(\tilde{\pi})$  to  $\pi(C(X))'$  is in  $\Gamma(\tilde{\pi})$ . This is equivalent to proving that the restriction of every element of  $\Phi(\tilde{\pi})$  to  $\pi(C(X))'$  is multiplicative. For an aperiodic point  $x \in X$ , recall that there is a unique pure state extension of  $\mu_x$  from  $C(X)$  to  $C^*(\Sigma)$ , as mentioned in the passage following Corollary 5.2.4. This implies that there is a unique pure state extension of  $\mu_x$  to  $\tilde{\pi}(C^*(\Sigma))$ , since if there were two different ones,  $\psi_1$  and  $\psi_2$ , say, then  $\psi_1 \circ \tilde{\pi}$  and  $\psi_2 \circ \tilde{\pi}$  would be two different pure state extensions of  $\mu_x$  to  $C^*(\Sigma)$ , which is a contradiction. It follows that there is a unique pure state extension of  $\mu_x$  to  $\pi(C(X))'$ , namely the restriction to this subalgebra of the pure state extension of  $\mu_x$  to  $\tilde{\pi}(C^*(\Sigma))$ . The pure states of a commutative  $C^*$ -algebra are precisely its characters, so this restriction is indeed multiplicative. Now take a periodic point  $y \in X$  and consider a pure state extension  $\phi'_y$  of  $\mu_y$  to  $\tilde{\pi}(C^*(\Sigma))$ . Then the pure state  $\phi'_y \circ \tilde{\pi}$  is a pure state extension of  $\mu_y$  to  $C^*(\Sigma)$ , hence it is of the form  $\varphi_{y,t}$  for some  $t \in \mathbb{T}$ . Then  $\varphi_{y,t}$  vanishes on  $I = \ker(\tilde{\pi})$  and it follows that  $\tilde{\pi}_{y,t}$  does as well. To see this, let  $a \in I$  be a positive element. Then, as  $\varphi_{y,t}$  vanishes on  $I$ , we have that  $0 = \varphi_{y,t}(\delta^i a \delta^{*i}) = (\tilde{\pi}_{y,t}(\sqrt{a} \delta^{*i}) \zeta_{y,t}, \tilde{\pi}_{y,t}(\sqrt{a} \delta^{*i}) \zeta_{y,t})$  and hence  $\tilde{\pi}_{y,t}(\sqrt{a}) = 0$ , as  $H_{y,t}$  is the closed linear span of the set  $\{\tilde{\pi}_{y,t}(\delta^i) \zeta_{y,t}\}_{i \in \mathbb{Z}}$ . We conclude that  $\tilde{\pi}_{y,t}$  vanishes on  $I$  since a closed ideal is generated by its positive part. This implies that there are parameters  $\beta, \gamma$  appearing in the description of  $I$  as in Proposition 5.2.5 such that  $y_\beta = y$  and  $t_\gamma = t$ . Now, for an element  $\tilde{\pi}(a)$  in  $\pi(C(X))'$  we first see that

$$\phi'_y(\tilde{\pi}(a)) = \varphi_{y,t}(a) = (\tilde{\pi}_{y,t}(a) \zeta_{y,t}, \zeta_{y,t}).$$

By (ii) of Proposition 5.3.2 we can replace  $a$  by a function  $f \in C(X)$ . It follows that the rightmost side of the above equalities becomes simply  $f(y)$ . Therefore, if for two elements  $\tilde{\pi}(a)$  and  $\tilde{\pi}(b)$  in  $\pi(C(X))'$  we replace  $a$  and  $b$  with the functions  $f$  and  $g$ , we have that

$$\begin{aligned} \phi'_y(\tilde{\pi}(a)\tilde{\pi}(b)) &= \phi'_y \circ \tilde{\pi}(ab) = \phi'_y \circ \tilde{\pi}(fg) \\ &= f(y)g(y) = \phi'_y(\tilde{\pi}(a))\phi'_y(\tilde{\pi}(b)). \end{aligned}$$

Hence restrictions of pure state extensions to  $\pi(C(X))'$  always induce characters on it. This completes the proof.  $\square$

We now introduce some notation. A character in  $\Gamma(\tilde{\pi})$  is denoted by  $\gamma(x)$  if it is the restriction to  $\pi(C(X))'$  of some  $\psi_x \in \Phi(\tilde{\pi})$  such that  $\psi_x \circ \hat{\pi} = \varphi_x$  for  $x \in \text{Per}^\infty(\sigma_\pi)$ . Similarly, we denote by  $\gamma(y, t)$  a character in  $\Gamma(\tilde{\pi})$  that is the restriction to  $\pi(C(X))'$  of some  $\psi_{y,t} \in \Phi(\tilde{\pi})$  such that  $\psi_{y,t} \circ \hat{\pi} = \varphi_{y,t}$  for  $y \in \text{Per}(\sigma_\pi)$  and  $t \in \mathbb{T}$ . Clearly every element of  $\Gamma(\tilde{\pi})$  is of this form. Note that in general it may happen that  $\gamma(y, s) = \gamma(y, t)$  even though  $s \neq t$ . If  $\tilde{\pi}$  is faithful and  $\Sigma$  is topologically free, for example, we know by Theorem 5.1.1 that  $\pi(C(X))' = \pi(C(X))$  and hence  $\gamma(y, t)$  is independent of  $t$ . A slight subtlety occurs as for  $y \in \text{Per}(\sigma_\pi)$  it is not necessarily so that  $\gamma(y, t)$  appears in  $\Gamma(\tilde{\pi})$  for each  $t \in \mathbb{T}$ . In the following lemma we determine the  $\gamma(y, t)$  that appear.

**Lemma 5.3.7.**  $\Gamma(\tilde{\pi}) = \{\gamma(x) : x \in \text{Per}^\infty(\sigma_\pi)\} \cup \{\gamma(y, t) : \ker(\hat{\pi}) \subseteq \ker(\tilde{\pi}_{y,t})\}$ . For every  $y \in \text{Per}(\sigma_\pi)$  there is at least one  $t \in \mathbb{T}$  such that  $\gamma(y, t) \in \Gamma(\tilde{\pi})$ .

*Proof.* That  $\gamma(x)$  appears in  $\Gamma(\tilde{\pi})$  for every  $x \in \text{Per}^\infty(\sigma_\pi)$  follows from the fact that  $\mu_x$  has a unique pure state extension to  $C^*(\Sigma_\pi)$ , namely  $\varphi_x$ . The last statement of the lemma follows since every  $\mu_y$  has a pure state extension to  $\hat{\pi}(C^*(\Sigma_\pi))$ , hence its composition with  $\hat{\pi}$  must be of the form  $\varphi_{y,t}$  for some  $t \in \mathbb{T}$ . If  $\gamma(y, t) \in \Gamma(\tilde{\pi})$ , then it has a pure state extension  $\psi_{y,t}$  to  $\hat{\pi}(C^*(\Sigma_\pi))$  such that  $\psi_{y,t} \circ \hat{\pi} = \varphi_{y,t}$ , whence  $\varphi_{y,t}$  vanishes on  $\ker(\hat{\pi})$ . As in the proof of Theorem 5.3.6, it follows readily that  $\ker(\hat{\pi}) \subseteq \ker(\tilde{\pi}_{y,t})$ . Conversely, if  $\ker(\hat{\pi}) \subseteq \ker(\tilde{\pi}_{y,t})$  we can well-define a pure state extension  $\psi_{y,t}$  of  $\mu_y$  to  $\hat{\pi}(C^*(\Sigma_\pi))$  by  $\psi_{y,t}(\hat{\pi}(a)) = \varphi_{y,t}(a)$  for  $a \in C^*(\Sigma_\pi)$ . In this notation, the restriction of  $\psi_{y,t}$  to  $\pi(C(X))'$  is  $\gamma(y, t)$ .  $\square$

When using this notation, the homeomorphism  $\sigma(\tilde{\pi})$  of  $\Gamma(\tilde{\pi})$  induced by the automorphism  $\text{Ad } u$  of  $\pi(C(X))'$  is such that  $\sigma(\tilde{\pi})(\gamma(x)) = \gamma(\sigma_\pi(x))$  for  $x \in \text{Per}^\infty(\sigma_\pi)$  and  $\sigma(\tilde{\pi})(\gamma(y, t)) = \gamma(\sigma_\pi(y), t)$  for  $y \in \text{Per}(\sigma_\pi)$  and  $t \in \mathbb{T}$ . To see this, note that if, for example,  $\psi_{y,t} \circ \hat{\pi} = \varphi_{y,t}$ , then  $\psi_{y,t} \circ \text{Ad } u \circ \hat{\pi} = \varphi_{\sigma_\pi^{-1}(y), t}$ . Since  $\pi(C(X))'$  is invariant under  $\text{Ad } u$  and its inverse by Theorem 5.3.6(i) and  $\psi_{y,t}$  extends  $\gamma(y, t)$ , it follows that  $\psi_{y,t} \circ \text{Ad } u$  extends  $\gamma(y, t) \circ \text{Ad } u$ . We use the notation  $\tilde{\Sigma} = (\Gamma(\tilde{\pi}), \sigma(\tilde{\pi}))$  for the corresponding dynamical system. Similarly, we denote by  $\tilde{\sigma}$  the homeomorphism of  $\Gamma$  induced by the restriction of the automorphism  $\text{Ad } \delta$  to  $C(X)'$ . Note that  $\gamma(x) \in \text{Per}^\infty(\sigma(\tilde{\pi}))$  for every  $x \in \text{Per}^\infty(\sigma_\pi)$  and that  $\gamma(y, t) \in \text{Per}_k(\sigma(\tilde{\pi}))$  implies that  $y \in \text{Per}_k(\sigma_\pi)$ .

Define  $\Phi'(\tilde{\pi})$  to be the set of all pure state extensions of elements of

$$\text{Per}^\infty(\sigma(\tilde{\pi})) \cup \text{PIP}(\sigma(\tilde{\pi}))$$

to  $\tilde{\pi}(C^*(\Sigma))$ . It is important to know which elements of  $\Gamma(\tilde{\pi})$  have unique pure state extensions to elements of  $\Phi'(\tilde{\pi})$ . The following proposition sheds some light upon this question.

**Proposition 5.3.8.** *The elements of  $\text{Per}^\infty(\sigma(\tilde{\pi})) \cup \text{PIP}(\sigma(\tilde{\pi}))$  all have a unique extension to an element of  $\Phi'(\tilde{\pi})$ .*

*Proof.* As mentioned above, points in  $\text{Per}^\infty(\sigma(\tilde{\pi}))$  must be of the form  $\gamma(x)$  where  $x \in \text{Per}^\infty(\sigma_\pi)$ . By definition, there is a  $\psi_x \in \Phi'(\tilde{\pi})$  such that  $\psi_x \circ \hat{\pi} = \varphi_x$ . As  $\varphi_x$  is the unique pure state extension of  $\mu_x$  to  $C^*(\Sigma_\pi)$  we see that, if there were another extension  $\psi \in \Phi'(\tilde{\pi})$ , then  $\psi \circ \hat{\pi} = \psi_x \circ \hat{\pi}$ , and thus  $\psi = \psi_x$ . Now consider the set  $\text{Per}_k(\sigma(\tilde{\pi}))^0$  for some positive integer  $k$  and suppose that the pure states  $\psi_{y,t}$  and  $\psi_{y,s}$  induce the same point there, i.e.,  $\gamma(y, t) = \gamma(y, s) \in \text{Per}_k(\sigma(\tilde{\pi}))^0$ . Let  $F \in C(\Gamma(\tilde{\pi}))$  be a continuous function whose support is contained in that interior such that  $F(\gamma(y, t))$  is not zero. Then the element  $Fu^k$  is an element of  $\pi(C(X))'$ . To see this, let  $\pi(g) \in \pi(C(X))$  and note that the condition on the support of  $F$  implies that  $Fu^k\pi(g) = Fu^k\pi(g)u^{*k}u^k = F\pi(g) \circ \sigma(\tilde{\pi})^{-k}u^k = \pi(g)Fu^k$ . Since  $F$  belongs to the right multiplicative domain for both  $\psi_{y,t}$  and  $\psi_{y,s}$  by Theorem 5.3.4, and since  $\text{Per}(y) = \text{Per}(\gamma(y, t)) = k$ , we have

$$\begin{aligned} Fu^k(\gamma(y, t)) &= \psi_{y,t}(Fu^k) = \psi_{y,t}(F)\psi_{y,t}(u^k) = F(\gamma(y, t))\psi_{y,t} \circ \hat{\pi}(\delta_\pi^k) \\ &= F(\gamma(y, t))\varphi_{y,t}(\delta_\pi^k) = F(\gamma(y, t))t. \end{aligned}$$

On the other hand, we conclude in the same fashion that  $Fu^k(\gamma(y, s)) = F(\gamma(y, s))s$  and hence that  $t = s$ . Thus  $\psi_{y,t} = \psi_{y,s}$ . When the  $y$ -components are different, say as  $y_1$  and  $y_2$ , we can easily separate  $\psi_{y_1,t}$  and  $\psi_{y_2,s}$  by functions in  $C(X_\pi)$ . Thus all elements of  $\text{Per}_k(\sigma(\tilde{\pi}))^0$  are uniquely extended to elements of  $\Phi(\tilde{\pi})$ .  $\square$

Note that continuity of restriction maps implies that

$$\{\gamma(y, t) \in \Gamma(\tilde{\pi}) : y \in \text{PIP}(\sigma_\pi)\} \subseteq \text{PIP}(\sigma(\tilde{\pi})),$$

and clearly  $\{\gamma(x) \in \Gamma(\tilde{\pi}) : x \in \text{Per}^\infty(\sigma_\pi)\} = \text{Per}^\infty(\sigma(\tilde{\pi}))$ .

## 5.4. Ideal intersection property of $C(X)'$ and $\pi(C(X))'$

In the theory of  $C^*$ -crossed products, an important direction of research is the investigation of how the structure of closed ideals of a crossed product reflects its building block  $C^*$ -subalgebra, which in our case is  $C(X)$ . Theorem 5.1.1 sheds some light upon this question. It tells us that all non-zero closed ideals have non-zero intersection with  $C(X)$  precisely when the system is topologically free. Hence for a dynamical system that is not topologically free, for example rational rotation of the unit circle, there always exists a non-zero closed ideal whose intersection with  $C(X)$  is zero.

In this section we analyse the situation for general dynamical systems. Since Theorem 5.1.1 tells us that  $C(X)$  has non-zero intersection with every non-zero closed ideal precisely when  $C(X) = C(X)'$ , and since  $C(X) \subseteq C(X)'$  in general, it appears natural to investigate what ideal intersection properties  $C(X)'$  has for an arbitrary system. In [8, Theorem 6.1] it is proved that in an algebraic crossed product the commutant of the subalgebra corresponding to  $C(X) \subseteq C^*(\Sigma)$  always has non-zero intersection with every non-zero ideal of the crossed product. In [9, Theorem 3.1] a generalization of that result with a more elementary proof is provided.

We make the following definition.

**Definition 5.4.1.** Let  $A$  be a unital  $C^*$ -algebra and  $B$  be a commutative  $C^*$ -subalgebra of  $A$  with the same unit.

- (i)  $B$  is said to have the intersection property for closed ideals of  $A$  if for any non-zero closed ideal  $I$  of  $A$ ,  $I \cap B \neq \{0\}$
- (ii)  $B$  is said to have the intersection property for ideals of  $A$  if for any non-zero ideal  $J$  of  $A$ , not necessarily closed or self-adjoint,  $J \cap B \neq \{0\}$ .

**Proposition 5.4.2.** *The above two properties for  $B$  are equivalent.*

*Proof.* It is enough to show that (i) implies (ii). We let  $Y$  be the spectrum of  $B$  and identify  $B$  with  $C(Y)$ . Let  $J$  be a non-zero ideal of  $A$ . By assumption,  $B \cap \bar{J} \neq \{0\}$ . Since  $B \cap \bar{J}$  is then a non-zero closed ideal of  $B$ , we may, under the identification  $B \cong C(Y)$ , write  $B \cap \bar{J} = \ker(Z)$  for some closed subset  $Z \subsetneq Y$ . Take a point  $p \in Y \setminus Z$  and a positive function  $g \in C(Y)$  vanishing on  $Z$  and such that  $g(p) = 1$ . Let  $f \in C_c(\mathbb{R})$  be such that  $f$

vanishes on  $(-\infty, \frac{1}{2}]$  and  $f(1) = 1$ . Then  $f|_{(0, \infty)} \in C_c((0, \infty))$  and by the characterization of the minimal dense ideal (also known as the Pedersen ideal)  $P_{\bar{J}}$  of  $\bar{J}$  in [5, Theorem 5.6.1] (see also [3],[4]), we have that  $f(g) = f \circ g \in P_{\bar{J}}$ . Since  $P_{\bar{J}} \subseteq J$ , this implies that  $f \circ g \in J$  and since clearly also  $f \circ g \in B$  we are done.  $\square$

**Theorem 5.4.3.** *Given a representation  $\tilde{\pi}$ , if the family  $\Phi'(\tilde{\pi})$  is total on  $\tilde{\pi}(C^*(\Sigma))$ , then the algebra  $\pi(C(X))'$  has the intersection property for ideals in  $\tilde{\pi}(C^*(\Sigma))$ .*

*Proof.* By Proposition 5.4.2, it is enough to consider closed ideals. Suppose there exists a closed ideal  $I$  that has trivial intersection with  $\pi(C(X))'$ . We shall show that  $I = \{0\}$ . Let  $q$  be the quotient map from  $\tilde{\pi}(C^*(\Sigma)) = \hat{\pi}(C^*(\Sigma_{\pi}))$  to the quotient algebra  $\tilde{\pi}(C^*(\Sigma))/I$ . Then it induces an isomorphism between  $\pi(C(X))'$  and  $q(\pi(C(X))')$ . This implies that every pure state  $\varphi \in \Gamma(\tilde{\pi})$  on  $\pi(C(X))'$  is of the form  $\zeta \circ q|_{\pi(C(X))'}$ , where  $\zeta$  is a pure state of  $q(\pi(C(X))')$ . Let  $\tilde{\zeta}$  be a pure state extension of  $\zeta$  to  $q(\tilde{\pi}(C^*(\Sigma)))$ . Clearly,  $\tilde{\varphi} = \tilde{\zeta} \circ q$  is a pure state extension of  $\varphi$  to  $\tilde{\pi}(C^*(\Sigma))$ . Since pure state extensions of elements of  $\text{Per}^{\infty}(\tilde{\sigma}) \cup \text{PIP}(\tilde{\sigma}) \subseteq \Gamma(\tilde{\pi})$  to  $\tilde{\pi}(C^*(\Sigma))$  are unique by Proposition 5.3.8, it follows that every element of  $\Phi'(\tilde{\pi})$  factors through  $q$  and hence vanishes on  $I$ . As  $\Phi'(\tilde{\pi})$  was assumed to be total, we conclude that  $I = \{0\}$ .  $\square$

**Corollary 5.4.4.**  *$C(X)'$  has the intersection property for ideals of  $C^*(\Sigma)$ .*

*Proof.* Consider a faithful representation of  $C^*(\Sigma)$  and apply Lemma 5.3.5.  $\square$

The following example shows that it can happen that  $\tilde{\pi}(C(X))' \subsetneq \pi(C(X))'$  and that the former does not necessarily have the intersection property for ideals.

**Example 5.4.5.** We consider again the dynamical system mentioned in Section 5.2. Namely, let  $X = [0, 1] \times [-1, 1] \subseteq \mathbb{R}^2$  be endowed with the standard topology and let  $\sigma$  be the homeomorphism of  $X$  defined as reflection in the  $x$ -axis. Using the notation  $\Sigma_{[0,1]} = ([0, 1], \sigma|_{[0,1]})$  and noting that  $\sigma|_{[0,1]}$  is just the identity homeomorphism, we consider the map from  $C^*(\Sigma)$  to  $C^*(\Sigma_{[0,1]})$  defined on the algebraic part of  $C^*(\Sigma)$  as  $\sum_n f_n \delta^n \mapsto \sum_n f_n|_{[0,1]} \delta^n|_{[0,1]}$ , where  $\delta|_{[0,1]}$  denotes the canonical unitary element of  $C^*(\Sigma_{[0,1]})$ . It clearly extends by continuity to a surjective homomorphism  $\tilde{\pi} = \pi \times u : C^*(\Sigma) \rightarrow C^*(\Sigma_{[0,1]})$ . By Proposition 5.3.2 (i),  $C(X)'$  is the  $C^*$ -subalgebra of  $C^*(\Sigma)$  generated by  $C(X)$  and  $\delta^2$ . We also note that  $C^*(\Sigma_{[0,1]}) \cong C([0, 1] \times \mathbb{T})$ . As  $\pi(C(X)) = C([0, 1])$ , and  $C^*(\Sigma_{[0,1]})$  is commutative, we see that  $\pi(C(X))' = C^*(\Sigma_{[0,1]})$ , while  $\tilde{\pi}(C(X))'$  only contains elements  $a \in C^*(\Sigma_{[0,1]})$  with  $a(j) = 0$  for odd integers  $j$ . Hence  $\tilde{\pi}(C(X))' \subsetneq \pi(C(X))'$ . To see that  $\tilde{\pi}(C(X))'$  does not have the intersection property for ideals, consider the ideal  $I \subseteq C([0, 1] \times \mathbb{T})$  consisting of all functions that vanish on  $[0, 1] \times C$ , where  $C \subseteq \mathbb{T}$  is the closed upper halfcircle. As  $\tilde{\pi}(C(X))'$  is identified with the  $C^*$ -subalgebra of  $C([0, 1] \times \mathbb{T})$  generated by  $C([0, 1])$  and  $z^2$  it is clear that  $\tilde{\pi}(C(X))' \cap I = \{0\}$ .

## 5.5. Intermediate subalgebras

As we now know that  $C(X)'$  always has the intersection property for ideals in  $C^*(\Sigma)$ , while  $C(X)$  has it if and only if  $C(X) = C(X)'$  by Theorem 5.1.1, we shall consider

$C^*$ -subalgebras  $B$  with  $C(X) \subseteq B \subseteq C(X)'$  and in particular investigate their intersection properties with ideals. We call such a  $C^*$ -subalgebra an intermediate subalgebra. In [9, Section 5] intermediate subalgebras are studied in the context of an algebraic crossed product. The following proposition gives an abstract characterization of intermediate subalgebras which have the intersection property, and consequently of those who do not, in terms of the relation of their spectra to the spectrum of  $C(X)'$ .

**Proposition 5.5.1.** *Suppose  $B \subseteq C^*(\Sigma)$  is a  $C^*$ -subalgebra such that  $C(X) \subseteq B \subseteq C(X)'$  and denote by  $\Delta$  and  $\Gamma$  the spectrum of  $B$  and  $C(X)'$ , respectively. Then  $B$  has the intersection property for ideals if and only if for every proper closed subset  $S \subseteq \Gamma$  that is invariant under  $\tilde{\sigma}$  and its inverse, the restriction of  $S$  to  $B$  is a proper subset of  $\Delta$ .*

*Proof.* By Proposition 5.4.2, it is enough to consider closed ideals. Let  $S$  be a proper closed subset of  $\Gamma$  that is invariant under  $\tilde{\sigma}$  and its inverse. Let  $I$  be the closed ideal of  $C^*(\Sigma)$  generated by the hull  $h(S)$  of  $S$ , defined by  $h(S) = \{a \in C(X)' : s(a) = 0 \text{ for all } s \in S\}$ . We first assert that  $I \cap C(X)' = h(S)$ . Note that  $I$  is the closed linear span of elements of the form  $f \delta^i F g \delta^j = f(\delta^i F g \delta^{*i}) \delta^{i+j}$ , where  $F \in h(S)$ . As  $F g \in C(X)'$ , we can rewrite this as  $f(\delta^i F g \delta^{*i}) \delta^{i+j} = f(Fg) \circ \tilde{\sigma}^{-i} \delta^{i+j}$ . Let  $\varphi_x$  and  $\varphi_{y,t}$  be pure state extensions of points  $\gamma(x)$  and  $\gamma(y, t)$  in  $S$ . Since the element  $f(Fg) \circ \tilde{\sigma}^{-i}$  belongs to  $C(X)'$ , and hence to the right multiplicative domain for  $\varphi_x$  and  $\varphi_{y,t}$  by Theorem 5.3.4, and to  $h(S)$  by invariance of  $S$  under  $\tilde{\sigma}$  and its inverse, we have

$$\varphi_x(f \delta^i F g \delta^j) = \varphi_x(f(Fg) \circ \tilde{\sigma}^{-i}) \varphi_x(\delta^{i+j}) = 0,$$

and also

$$\varphi_{y,t}(f \delta^i F g \delta^j) = \varphi_{y,t}(f(Fg) \circ \tilde{\sigma}^{-i}) \varphi_{y,t}(\delta^{i+j}) = 0.$$

Therefore, the pure states  $\varphi_x$  and  $\varphi_{y,t}$  vanish on  $I$ , and thus  $I \cap C(X)' \subseteq h(S)$  and we conclude that  $I \cap C(X)' = h(S)$ . Denoting by  $r_B : \Gamma \rightarrow \Delta$  the restriction map, we see that  $I \cap B = h(r_B(S))$ . So if  $B$  has the intersection property, then  $h(r_B(S)) \neq \{0\}$ , which implies  $r_B(S) \subsetneq \Delta$ . On the other hand, suppose that  $B$  does not have the intersection property and hence that there exists a nonzero closed ideal  $I$  that has trivial intersection with  $B$ . Since  $C(X)'$  itself has the intersection property by Corollary 5.4.4, and since it is easy to see that  $I \cap C(X)'$  is a closed ideal of  $C(X)'$  invariant under  $\text{Ad } \delta$  and its inverse, we may write  $I \cap C(X)' = h(S)$  for some proper closed subset  $S$  of  $\Gamma$  that is invariant under  $\tilde{\sigma}$  and its inverse. Since  $B \cap I = \{0\}$  we see that  $\{0\} = B \cap I \cap C(X)' = B \cap h(S) = h(r_B(S))$ , whence  $r_B(S) = \Delta$ .  $\square$

Combining this result with Proposition 5.3.2, Theorem 5.3.6 and Corollary 5.4.4 we can provide the following alternative proof of a refined version of Theorem 5.1.1.

**Theorem 5.5.2.** *For a topological dynamical system  $\Sigma$ , the following statements are equivalent.*

- (i)  $\Sigma$  is topologically free;
- (ii)  $I \cap C(X) \neq 0$  for every non-zero ideal  $I$  of  $C^*(\Sigma)$ ;
- (iii)  $C(X)$  is a maximal abelian  $C^*$ -subalgebra of  $C^*(\Sigma)$ .

*Proof.* Since  $C(X)'$  is maximal abelian by Theorem 5.3.6 (i), equivalence of (i) and (iii) is an immediate consequence of Proposition 5.3.2 (i). The implication (iii)  $\Rightarrow$  (ii) follows from Corollary 5.4.4 since  $C(X) \subseteq C(X)'$ . To prove (ii)  $\Rightarrow$  (i), suppose that  $\text{Per}^\infty(\sigma)$  is not dense in  $X$ . Then  $C(X) \subsetneq C(X)'$  by the above, and by Theorem 5.3.2(i) it follows that there exists a positive  $k$  for which  $\text{Per}^k(\sigma)^0 \neq \emptyset$ . Let  $S = \{\gamma(x) : x \in \text{Per}^\infty(\sigma)\} \cup \{\gamma(y, 1) : y \in \text{Per}(\sigma)\}$ . It is easy to see that  $S$  is invariant under  $\tilde{\sigma}$  and its inverse, and clearly its restriction to  $C(X)$  coincides with  $X$ . We will show that  $S$  is a proper subset of  $\Gamma$ . Let  $y \in \text{Per}^k(\sigma)^0$ . Then there are positive integers  $l, r$  such that  $\text{Per}(y) = l$  and  $k = l \cdot r$ . Let  $s$  be a complex number such that  $s^r = i$  and let  $f \in C(X)$  be real-valued and such that  $\text{supp}(f) \subseteq \text{Per}^k(\sigma)^0$  and  $f(y) = 1$ . Then by Proposition 5.3.2 (i) it follows that  $f\delta^k \in C(X)'$ . To see that  $\gamma(y, s) \notin S$ , note that  $\gamma(x)(f\delta^k)$  and  $\gamma(y, 1)(f\delta^k)$  are real-valued for all  $x \in \text{Per}^\infty(\sigma)$  and  $y \in \text{Per}(\sigma)$ , while  $\gamma(y, s)(f\delta^k) = s^r \cdot f(y) = i \cdot 1$ . Hence by Proposition 5.5.1,  $C(X)$  does not have the intersection property for ideals.  $\square$

Given a dynamical system  $\Sigma$ , it seems natural to ask which kinds of *properly* intermediate  $C^*$ -subalgebras, by which we mean  $C^*$ -subalgebras  $B$  with  $C(X) \subsetneq B \subsetneq C(X)'$ , exist in  $C^*(\Sigma)$ , regarding the intersection property for ideals. In [9, Theorem 5.4], a related result is obtained in the case when an algebraic crossed product is associated with a more general dynamical system than the ones considered in this paper. In our context of  $C^*$ -crossed products, it turns out that if  $C(X) \subsetneq C(X)'$ , there always exist  $C^*$ -subalgebras  $B_1, B_2$  with  $C(X) \subsetneq B_i \subsetneq C(X)'$  for  $i = 1, 2$  such that  $B_1$  has the intersection property for ideals and  $B_2$  does not. The result will follow easily from a number of propositions in which we construct intermediate  $C^*$ -subalgebras of  $C^*(\Sigma)$  for restricted classes of  $\Sigma$ . The following two propositions are extensions of [9, Proposition 5.2] and [9, Proposition 5.3], respectively, to the context of  $C^*$ -crossed products.

**Proposition 5.5.3.** *Suppose that there exists an integer  $n > 0$  such that  $\text{Per}^n(\sigma)^0$  contains at least two orbits. Then there exists a  $C^*$ -subalgebra  $B$  of  $C^*(\Sigma)$  with  $C(X) \subsetneq B \subsetneq C(X)'$  that does not have the intersection property for ideals.*

*Proof.* Fix an integer  $n > 0$  such that  $\text{Per}^n(\sigma)^0$  contains at least two orbits. Then it is easy to see that there exist two disjoint open subsets  $U_1, U_2$  of  $\text{Per}^n(\sigma)^0$  that are invariant under  $\sigma$  and  $\sigma^{-1}$ . Define  $B = \{a \in C^*(\Sigma) : \text{supp}(a(k)) \subseteq U_1 \cap \text{Per}^k(\sigma) \text{ for } k \neq 0\}$ . Using the continuity of the projection map  $E$  and Proposition 5.3.3 it is easy to check that  $B$  is indeed a  $C^*$ -subalgebra of  $C^*(\Sigma)$ . By the explicit description of  $C(X)'$  in Proposition 5.3.2(i) we see that we have the inclusions  $C(X) \subseteq B \subseteq C(X)'$ . To see that the inclusions are strict, consider non-zero functions  $f_{n_1}, f_{n_2} \in C(X)$  with support in  $U_1$  and  $U_2$ , respectively. Then  $f_{n_1}\delta^n \in B \setminus C(X)$  and  $f_{n_2}\delta^n \in C(X)' \setminus B$  and thus  $C(X) \subsetneq B \subsetneq C(X)'$ . We now exhibit a non-zero closed ideal that has trivial intersection with  $B$ . Let  $f_{n_2}$  be as above and let  $I = (f_{n_2} - f_{n_2}\delta^n)$ , the closed ideal generated by the element  $f_{n_2} - f_{n_2}\delta^n$ . We first prove that  $I \cap C(X) = \{0\}$ . For a point  $y \in \text{Per}(\sigma)$ , consider the representation  $\tilde{\pi}_{y,1}$  of  $C^*(\Sigma)$  as described in the passage following Corollary 5.2.4. We see that for  $y \notin \text{Per}^n(\sigma)$ ,  $\tilde{\pi}_{y,1}(f_{n_2}) = 0$  whence  $\tilde{\pi}_{y,1}(f_{n_2} - f_{n_2}\delta^n) = 0$  and hence  $\tilde{\pi}_{y,1}$  vanishes on  $I$  for such  $y$ . For  $y \in \text{Per}^n(\sigma)$ , clearly  $\text{Per}(y)|n$ . Then  $\tilde{\pi}_{y,1}(\delta^n) = \tilde{\pi}_{y,1}(1) = \text{id}$  and thus  $\tilde{\pi}_{y,1}(f_{n_2} - f_{n_2}\delta^n) = 0$ , so  $\tilde{\pi}_{y,1}$  vanishes on  $I$  also in this case. For  $x \in \text{Per}^\infty(\sigma)$ , it is clear

that  $\tilde{\pi}_x(f_{n_2}) = 0$  and thus  $\tilde{\pi}_x$  vanishes on  $I$ . Suppose now that  $f \in C(X) \cap I$ . Since the family  $\{\tilde{\pi}_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\tilde{\pi}_{y,1} : y \in \text{Per}(\sigma)\}$  is easily seen to be total on  $C(X)$ , we conclude that  $f$  must be zero and thus  $I \cap C(X) = \{0\}$ . Knowing this, to conclude that  $B \cap I = \{0\}$  it suffices to show that for any integer  $j \neq 0$  and  $a \in B \cap I$  we have  $a(j) = 0$ . Clearly,  $I$  can be described as the closure of the set

$$\left\{ \left( \sum_i g_i \delta^i \right) (f_{n_2} - f_{n_2} \delta^n) \left( \sum_j h_j \delta^j \right) : \text{only finitely many } g_i, h_j \in C(X) \text{ non-zero} \right\}.$$

The facts that  $\text{supp}(f_{n_2}) \subseteq U_2$  and that  $U_2$  is invariant under  $\sigma$  and  $\sigma^{-1}$  imply that the generalized Fourier coefficients of elements of this dense subset of  $I$  all have support contained in  $U_2$ . Continuity of  $E$  then implies that for  $a \in I$  and  $i \in \mathbb{Z}$  we have  $\text{supp}(a(i)) \subseteq U_2$ . Now suppose  $a \in B \cap I$ . By the above we know that  $\text{supp}(a(i)) \subseteq U_2$  for every  $i \in \mathbb{Z}$  since  $a \in I$ , but by definition of  $B$  we also have that, for  $i \neq 0$ ,  $\text{supp}(a(i)) \cap U_2 = \emptyset$ . So the only non-zero generalized Fourier coefficient of  $a$  is  $a(0)$ , meaning that  $a \in C(X)$ . By the above, however, we know that  $I \cap C(X) = \{0\}$  so  $B \cap I = \{0\}$  as desired.  $\square$

**Proposition 5.5.4.** *Suppose that there exists an integer  $n > 0$  such that  $\text{Per}^n(\sigma)^0$  contains a point  $x_0$  that is not isolated. Then there exists a  $C^*$ -subalgebra  $B$  of  $C^*(\Sigma)$  with  $C(X) \subsetneq B \subsetneq C(X)'$  that has the intersection property for ideals.*

*Proof.* Suppose that a point  $x_0 \in X$  is as in the statement of the theorem. Define

$$B = \{a \in C(X)' : a(k)(x_0) = 0 \text{ for } k \neq 0\}.$$

Using the continuity of the projection map  $E$  and Proposition 5.3.3 it is easy to check that  $B$  is indeed a  $C^*$ -subalgebra of  $C^*(\Sigma)$  and clearly  $C(X) \subseteq B \subseteq C(X)'$ . To show that the inclusions are strict we note that the fact that  $x_0$  is not isolated implies that  $\text{Per}^n(\sigma)^0$  must be infinite, so we can choose an  $x_1 \in \text{Per}^n(\sigma)^0$  and separate  $x_0$  from  $x_1$  by two open subsets  $U_0, U_1 \subseteq \text{Per}^n(\sigma)^0$ . Take two functions  $f_{n_0}, f_{n_1} \in C(X)$  with support contained in  $U_0$  and  $U_1$ , respectively, and such that  $f_{n_0}(x_0) \neq 0$  and  $f_{n_1}(x_1) \neq 0$ . Then  $f_{n_0} \delta^n \in C(X)' \setminus B$  and  $f_{n_1} \delta^n \in B \setminus C(X)$  and thus  $C(X) \subsetneq B \subsetneq C(X)'$ . Now let  $I \subseteq C^*(\Sigma)$  be an ideal. We wish to show that  $B \cap I \neq \{0\}$ . By Corollary 5.4.4 we know that  $C(X)' \cap I \neq \{0\}$ . Suppose  $0 \neq a \in C(X)' \cap I$ . If  $a(i)(x_0) = 0$  for every integer  $i \neq 0$  then  $a \in B \cap I$  and we are done, so suppose there is an  $i \neq 0$  such that  $a(i)(x_0) \neq 0$ . Since  $x_0$  is not isolated we can find an  $x_1 \neq x_0$  such that  $a(i)(x_1) \neq 0$ . Separate  $x_0$  and  $x_1$  by open sets  $V_0, V_1$  and choose a function  $g \in C(X)$  such that  $g(x_1) = 1$  and  $\text{supp}(g) \subseteq V_1$ . The module property of  $E$  (by which we mean that if  $a \in C^*(\Sigma)$  and  $f, g \in C(X)$  we have  $E(fag) = fE(a)g$ ) implies that  $(ga)(i) = g \cdot a(i) \neq 0$ . Hence  $0 \neq ga \in B \cap I$ .  $\square$

To analyze the last possibility we need to recall two basic results from the theory of  $C^*$ -crossed products as introduced in this paper. The following result is part of [11, Proposition 3.5].

**Proposition 5.5.5.** *Suppose  $\Sigma = (X, \sigma)$  is such that  $X$  consists of a single  $\sigma$ -orbit of order  $p$ ,  $X = \{x, \sigma(x), \dots, \sigma^{p-1}(x)\}$ , where  $p$  is a positive integer. Then  $C^*(\Sigma) \cong C(\mathbb{T}, M_p)$ ,*



where  $M_p$  is the set of  $p \times p$ -matrices over the complex numbers, via the isomorphism  $\tilde{\rho}_x = \rho_x \times u$ , where

$$\rho_x(f) = \begin{pmatrix} f(x) & 0 & \dots & 0 \\ 0 & f \circ \sigma(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f \circ \sigma^{p-1}(x) \end{pmatrix}$$

for  $f \in C(X)$ , and

$$u(z) = \begin{pmatrix} 0 & 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

for  $z \in \mathbb{T}$ .

The following result is easily proved using the definition of a  $C^*$ -crossed product.

**Lemma 5.5.6.** *Suppose  $\Sigma = (X, \sigma)$  is such that  $X$  is the union of two disjoint non-empty open (hence closed) subsets  $A_1, A_2 \subseteq X$  that are invariant under  $\sigma$  and its inverse. Denoting  $\Sigma_i = (A_i, \sigma|_{A_i})$  for  $i = 1, 2$  we then have that  $C^*(\Sigma) \cong C^*(\Sigma_1) \oplus C^*(\Sigma_2)$  as  $C^*$ -algebras. Denoting the canonical unitaries of the two summands by  $\delta_i$  for  $i = 1, 2$ , an isomorphism is given by  $f \mapsto f|_{A_1} \oplus f|_{A_2}$  for  $f \in C(X)$  and  $\delta \mapsto \delta_1 \oplus \delta_2$ .*

We also make the following definition.

**Definition 5.5.7.** Let  $p$  be a positive integer. For a subset  $A \subseteq C(\mathbb{T})$  we denote by  $\text{diag}(A)$  the subset of  $C(\mathbb{T}, M_p)$  consisting of diagonal  $p \times p$ -matrices with entries from  $A$ . Given  $p$  functions  $f_0, f_1, \dots, f_{p-1} \in C(\mathbb{T})$  we denote by  $\text{diag}(f_0, f_1, \dots, f_{p-1})$  the diagonal  $p \times p$ -matrix having  $f_i$  as entry  $(i, i)$ .

**Proposition 5.5.8.** *Suppose that  $\Sigma = (X, \sigma)$  is such that  $X$  contains an orbit of order  $p$ ,  $\mathcal{O}_\sigma(x) = \{x, \sigma(x), \dots, \sigma^{p-1}(x)\}$ , consisting of isolated points. Then there exist  $C^*$ -subalgebras  $B_i$  of  $C^*(\Sigma)$  with  $C(X) \subsetneq B_i \subsetneq C(X)'$  for  $i = 1, 2$  such that  $B_1$  has the intersection property for ideals, and  $B_2$  does not.*

*Proof.* Denote the complement of  $\mathcal{O}_\sigma(x)$  in  $X$  by  $X_1$ , and let  $\sigma_1 = \sigma|_{X_1}$ . Using the notation  $\Sigma_1 = (X_1, \sigma_1)$  we then have, by Proposition 5.5.5 together with Lemma 5.5.6, that  $C^*(\Sigma) \cong C^*(\Sigma_1) \oplus C(\mathbb{T}, M_p)$ , and it is straightforward to check that  $C(X)$  is identified with  $C(X_1) \oplus \text{diag}(\mathbb{C}) \subseteq C^*(\Sigma_1) \oplus C(\mathbb{T}, M_p)$  under this isomorphism. If  $X_1$  is empty, the left summands above and in what follows are naturally zero. Since the commutant of  $\text{diag}(\mathbb{C})$  in  $C(\mathbb{T}, M_p)$  is easily seen to be  $\text{diag}(C(\mathbb{T}))$ , we conclude that  $C(X)'$  is identified with  $C(X_1)' \oplus \text{diag}(C(\mathbb{T}))$ , where  $C(X_1)'$  is the commutant of  $C(X_1)$  in  $C^*(\Sigma_1)$ , under the isomorphism. Now take two distinct points  $x_1, x_2 \in \mathbb{T}$  and consider the  $C^*$ -subalgebra  $B = \{f \in C(\mathbb{T}) : f(x_1) = f(x_2)\}$  of  $C(\mathbb{T})$ . Let

$$B_1 = C(X_1)' \oplus \text{diag}(B) \subseteq C^*(\Sigma_1) \oplus C(\mathbb{T}, M_p).$$

It is readily checked that  $B_1$  is a  $C^*$ -subalgebra of  $C^*(\Sigma_1) \oplus C(\mathbb{T}, M_p)$  such that

$$C(X_1) \oplus \text{diag}(\mathbb{C}) \subsetneq B_1 \subsetneq C(X_1)' \oplus \text{diag}(C(\mathbb{T})).$$

Now let  $I \subseteq C^*(\Sigma_1) \oplus C(\mathbb{T}, M_p)$  be a non-zero ideal. Then by Corollary 5.4.4 there exists a non-zero  $a \oplus b \in (C(X_1)' \oplus \text{diag}(C(\mathbb{T}))) \cap I$ . If  $b = 0$  then  $a \oplus b \in I \cap B_1$  and we are done, so suppose  $b \neq 0$ . Writing  $b = \text{diag}(f_0, \dots, f_{p-1})$  where the  $f_i$  are in  $C(\mathbb{T})$ , we see that at least one of the  $f_i$  is such that there exist a  $t \in \mathbb{T} \setminus \{x_1, x_2\}$  with  $f_i(t) \neq 0$ . Take an  $f \in C(\mathbb{T})$  such that  $f(t) = 1$  and  $f(x_1) = f(x_2) = 0$  and let  $M = \text{diag}(f, f, \dots, f) \in C(\mathbb{T}, M_p)$ . Then  $0 \oplus 0 \neq (0 \oplus M) \cdot (a \oplus b) = 0 \oplus M \cdot b \in I \cap B_1$ . We conclude that  $B_1$  has the intersection property. To construct an intermediate subalgebra that does not have the intersection property, take two proper closed subsets  $C_1, C_2 \subseteq \mathbb{T}$  such that  $C_1 \cup C_2 = \mathbb{T}$ . Note that connectedness of  $\mathbb{T}$  implies that  $C_1 \cap C_2 \neq \emptyset$  and that both  $C_1$  and  $C_2$  are infinite. Now let  $B_2 = C(X_1)' \oplus \text{diag}(\{f \in C(\mathbb{T}) : f \text{ is constant on } C_1\})$ . It is not hard to see that  $B_2$  is a  $C^*$ -subalgebra of  $C^*(\Sigma_1) \oplus C(\mathbb{T}, M_p)$  such that

$$C(X_1) \oplus \text{diag}(\mathbb{C}) \subsetneq B_2 \subsetneq C(X_1)' \oplus \text{diag}(C(\mathbb{T})).$$

Now let  $I = \{0\} \oplus M_p(\ker(C_2)) \subseteq C^*(\Sigma_1) \oplus C(\mathbb{T}, M_p)$ , where  $M_p(\ker(C_2))$  is the set of  $p \times p$ -matrices with entries in  $\ker(C_2)$ . This is easily seen to be a non-trivial closed ideal and clearly  $I \cap B_2 = \{0 \oplus 0\}$  as  $C_1 \cap C_2 \neq \emptyset$ .  $\square$

We can now easily derive the main result of this section.

**Theorem 5.5.9.** *For a topological dynamical system  $\Sigma = (X, \sigma)$ ,  $C(X)'$  always has the intersection property for ideals in the associated  $C^*$ -crossed product  $C^*(\Sigma)$ . Furthermore, precisely one of the following two cases occurs:*

- (i)  $\Sigma$  is topologically free. Then  $C(X) = C(X)'$ ;
- (ii)  $\Sigma$  is not topologically free. Then  $C(X) \subsetneq C(X)'$  and  $C(X)$  does not have the intersection property for ideals. In this case, there exist  $C^*$ -subalgebras  $B_i$  with  $C(X) \subsetneq B_i \subsetneq C(X)'$  for  $i = 1, 2$  such that  $B_1$  has the intersection property for ideals and  $B_2$  does not.

*Proof.* That  $C(X)'$  always has the intersection property for ideals is stated in Corollary 5.4.4. Case (i) is clear by Theorem 5.1.1. If  $\Sigma$  is not topologically free,  $C(X)$  is properly contained in  $C(X)'$  and does not have the intersection property for ideals by Theorem 5.1.1. The explicit description of  $C(X)'$  in Proposition 5.3.2(i) implies that there exists a positive integer  $n$  such that  $\text{Per}^n(\sigma)^0 \neq \emptyset$ . Suppose first that  $\text{Per}^n(\sigma)^0$  contains a non-isolated point. Then it is easy to see that the conditions in Propositions 5.5.3 and 5.5.4 are both satisfied and case (ii) follows. If  $\text{Per}^n(\sigma)^0$  contains only isolated points, the condition in Proposition 5.5.8 is satisfied and again case (ii) follows.  $\square$

## 5.6. Projections onto $C(X)'$

With the norm one projection  $E : C^*(\Sigma) \rightarrow C(X)$  in mind, one might wonder whether there exists a projection map of norm one onto  $C(X)'$ . We have the following result.

**Theorem 5.6.1.** *A necessary and sufficient condition for the existence of a projection of norm one from  $C^*(\Sigma)$  onto  $C(X)'$  is that for every positive integer  $k$  the set  $\text{Per}_k(\sigma)^0$  is closed. In this case the projection is uniquely determined and it is faithful.*

As we feel that the proof of this result is rather long in relation to its relevance, we omit it here and content ourselves with a proof in the special case when  $\Sigma = (X, \sigma)$  is such that  $\Sigma = \text{Per}_q(\sigma)$  for some positive integer  $q$ .

**Proposition 5.6.2.** *Suppose  $\Sigma = (X, \sigma)$  is such that  $\Sigma = \text{Per}_q(\sigma)$  for some positive integer  $q$ . Then there exists a unique projection  $E_0$  of norm one from  $C^*(\Sigma)$  onto  $C(X)'$ . Furthermore,  $E_0$  is faithful.*

*Proof.* Here  $\text{PIP}(\sigma) = \text{Per}_q(\sigma) = X$ . It follows from Proposition 5.3.2(i) that

$$C(X)' = \{a \in C^*(\Sigma) : a(j) = 0 \text{ if } q \text{ does not divide } j\}.$$

We define a linear map  $E_0$  from the algebraic part of  $C^*(\Sigma)$  to  $C(X)'$  by

$$E_0\left(\sum_{k=-n}^n f_k \delta^k\right) = \sum_{\{l: |lq| \leq n\}} f_{lq} \delta^{lq} \text{ for } f_k \in C(X) \text{ and } n \text{ a non-negative integer.}$$

Note that the image of  $E_0$  is dense in  $C(X)'$  by Proposition 5.3.2(i) and Proposition 5.3.3. We will show that  $E_0$  is norm-decreasing and then extend it by continuity to the whole of  $C^*(\Sigma)$ . First we show that for  $\varphi \in \Phi$  and any  $a$  in the algebraic part of  $C^*(\Sigma)$ , we have  $\varphi(a) = \varphi \circ E_0(a)$ . Write  $a = \sum_{k=-n}^n f_k \delta^k$  and take a  $\varphi_{y,t}$  with  $y \in \text{Per}_q(\sigma)$  and  $t \in \mathbb{T}$ . Denote by  $\tilde{\pi}_{y,t}$  its associated irreducible GNS-representation with cyclic unit vector  $\zeta_{y,t}$  and note that

$$\varphi_{y,t}(a) = (\tilde{\pi}_{y,t}(a)\zeta_{y,t}, \zeta_{y,t}) = \sum_{k=-n}^n (\tilde{\pi}_{y,t}(f_k \delta^k)\zeta_{y,t}, \zeta_{y,t}) = \sum_{\{l: |lq| \leq n\}} f_{lq}(y)t^l.$$

On the other hand

$$\varphi_{y,t} \circ E_0(a) = \sum_{\{l: |lq| \leq n\}} (\tilde{\pi}_{y,t}(f_{lq} \delta^{lq})\zeta_{y,t}, \zeta_{y,t}) = \sum_{\{l: |lq| \leq n\}} f_{lq}(y)t^l.$$

Now as  $E_0(a) \in C(X)'$ , it follows by Theorem 5.3.6(ii) and the above conclusion that

$$\|E_0(a)\| = \sup_{\varphi \in \Phi} |\varphi \circ E_0(a)| = \sup_{\varphi \in \Phi} |\varphi(a)| \leq \|a\|.$$

So indeed  $E_0$  is norm decreasing on algebraic elements  $a$  and thus extends by continuity to a norm one projection from  $C^*(\Sigma)$  onto  $C(X)'$ . To see that  $E_0$  is faithful, let  $b \in C^*(\Sigma)$  be a non-negative element and suppose  $E_0(b) = 0$ . By definition,  $\gamma(y, t) \circ E_0 = \varphi_{y,t} \circ E_0$  and by the above  $\varphi_{y,t} \circ E_0 = \varphi_{y,t}$ , so  $\varphi_{y,t}(b) = 0$ . Since every element of  $\Phi$  has this form and  $\Phi$  is total by Lemma 5.3.5, it follows that  $b = 0$  and hence  $E_0$  is faithful. Suppose there is another norm one projection  $E_1 : C^*(\Sigma) \rightarrow C(X)'$ . Since  $X = \text{Per}_q(\sigma)$  it follows easily

that the system  $\tilde{\Sigma} = (\Gamma, \tilde{\sigma})$  as introduced in the passage following Lemma 5.3.7 above is such that  $\Gamma = \text{Per}_q(\tilde{\sigma}) = \text{PIP}(\tilde{\sigma})$  and hence by Proposition 5.3.8 every element of  $\Gamma$  has a unique pure state extension to  $C^*(\text{Sigma})$ . Letting  $\gamma(y, t) \in \Gamma$  be arbitrary, we see that  $\gamma(y, t) \circ E_1$  is a state extension of it and since the unique pure state extension of  $\gamma(y, t)$  must also be the unique state extension of it, it follows that  $\gamma(y, t) \circ E_1 = \gamma(y, t) \circ E_0 = \varphi_{y,t}$ . Since  $\gamma(y, t)$  was arbitrary it follows that  $E_1 = E_0$  as desired.  $\square$

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## Chapter 6

# On the Banach $*$ -algebra crossed product associated with a topological dynamical system

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**Abstract.** Given an arbitrary topological dynamical system  $\Sigma = (X, \sigma)$ , where  $X$  is a compact Hausdorff space and  $\sigma$  a homeomorphism of  $X$ , we introduce and analyze the associated Banach  $*$ -algebra crossed product  $\ell^1(\Sigma)$ . The  $C^*$ -envelope of this algebra is the usual  $C^*$ -crossed product of  $C(X)$  by the integers under the automorphism of  $C(X)$  induced by  $\sigma$ . While the connections between the structure of this  $C^*$ -algebra and the properties of  $\Sigma$  are well-studied, such considerations concerning  $\ell^1(\Sigma)$  are new. We derive equivalences between topological dynamical properties of  $\Sigma$  and structural properties of  $\ell^1(\Sigma)$  that have well-known analogues in the  $C^*$ -algebra context, but also obtain a result on this so-called interplay whose counterpart in the case of  $C^*(\Sigma)$  is false.

### 6.1. Introduction

The interplay between topological dynamical systems and  $C^*$ -algebras has been intensively studied, e.g. in [13] and [14] where for an arbitrary topological dynamical system  $\Sigma = (X, \sigma)$  one associates a crossed product  $C^*$ -algebra  $C^*(\Sigma)$  with it. This is the  $C^*$ -crossed product of  $C(X)$  by the integers under the automorphism of  $C(X)$  induced by  $\sigma$ . It is shown in [8], [9] and [10] that a number of connections between topological dynamics and  $C^*$ -algebras as appearing in [13] and [14] have an analogue for a certain dense  $*$ -subalgebra  $k(\Sigma)$  of  $C^*(\Sigma)$ . Conversely, analogues of results obtained in the setup in [8], [9] and [10] have later been proven in the context of the interplay between  $\Sigma$  and  $C^*(\Sigma)$ . Namely, the

result [11, Corollary 4.4], which says that the commutant of  $C(X)$  in  $C^*(\Sigma)$  always has the intersection property for ideals, is an analogue of [9, Theorem 6.1] and [10, Theorem 3.1].

One way of obtaining  $C^*(\Sigma)$  is as the enveloping  $C^*$ -algebra of a certain Banach  $*$ -algebra,  $\ell^1(\Sigma)$ , which we define in Section 6.2. Because  $\ell^1(\Sigma)$  has sufficiently many Hilbert space representations,  $C^*(\Sigma)$  contains a  $*$ -isomorphic copy of  $\ell^1(\Sigma)$  as a dense  $*$ -subalgebra. Furthermore,  $\ell^1(\Sigma)$  contains a  $*$ -isomorphic copy of  $k(\Sigma)$  as a dense  $*$ -subalgebra. The inclusions are easily seen to be strict and we may write, up to  $*$ -isomorphisms,  $k(\Sigma) \subsetneq \ell^1(\Sigma) \subsetneq C^*(\Sigma)$ . Inspired by the fact that several theorems on the interplay between  $\Sigma$  and  $C^*(\Sigma)$  have analogues if the latter is replaced by  $k(\Sigma)$ , we investigate  $\ell^1(\Sigma)$  and in particular the connection between its structural properties and the topological dynamical properties of  $\Sigma$ . Although  $\ell^1(\Sigma)$  is the Banach  $*$ -algebra of crossed product type which is most naturally associated with  $\Sigma$ , it seems that this connection has only briefly been studied so far (cf. [10]).

In Section 6.2 we define the properties of topological dynamical systems that we shall investigate, recall the simple key result Lemma 6.2.1 and go through the construction of  $\ell^1(\Sigma)$  in detail. Furthermore, we introduce some basic definitions, and an extension theorem (Theorem 6.2.5), from the theory of ordered linear spaces which we use, together with elementary theorems on Banach  $*$ -algebras, to deduce a certain extension result for states on Banach  $*$ -algebras (Proposition 6.2.10) that will be useful to us. We also recall the explicit description of a certain collection of pure states of  $C^*(\Sigma)$ , as appearing in [13] and [14], that we shall exploit in our setup. In Section 6.3 we describe the commutant of  $C(X)$  in  $\ell^1(\Sigma)$ , which we denote by  $C(X)'$ , explicitly and conclude that  $C(X)' = C(X)$  precisely when  $\Sigma$  is topologically free (Theorem 6.3.2). We show that  $C(X)'$  is commutative, hence the unique maximal abelian Banach  $*$ -subalgebra of  $\ell^1(\Sigma)$  that contains  $C(X)$  (Proposition 6.3.3), and describe its character space (Theorem 6.3.4). We prove that  $C(X)'$  has non-zero intersection with every non-zero closed (not necessarily self-adjoint) ideal of  $\ell^1(\Sigma)$ , regardless of  $\Sigma$  (Theorem 6.3.7). In Section 6.4 we use Theorem 6.3.7 to conclude a number of analogues of results on the interplay between  $\Sigma$  and  $C^*(\Sigma)$ . Theorem 6.4.1 says that  $C(X)$  has the intersection property for closed ideals in  $\ell^1(\Sigma)$  if and only if it is a maximal abelian Banach  $*$ -subalgebra of  $\ell^1(\Sigma)$ , which is in turn equivalent to topological freeness of  $\Sigma$ . Theorem 6.4.2 states that  $\ell^1(\Sigma)$  is simple if and only if  $\Sigma$  is minimal, under the condition that  $X$  is infinite. Theorem 6.4.5 says that if  $X$  is infinite,  $\ell^1(\Sigma)$  is prime if and only if  $\Sigma$  is topologically transitive. We also give simple counter examples of Theorem 6.4.2 and Theorem 6.4.5 when  $X$  is finite. In Section 6.5 we use a deep result from abstract harmonic analysis (Theorem 6.5.1) to prove that every closed ideal of  $\ell^1(\Sigma)$  is self-adjoint if and only if  $\Sigma$  is free (Theorem 6.5.2). This is a result lacking an obvious analogon in the context of the interplay between  $\Sigma$  and  $C^*(\Sigma)$  as closed ideals of  $C^*$ -algebras are always self-adjoint.

## 6.2. Definitions and preliminaries

Throughout this paper we consider topological dynamical systems  $\Sigma = (X, \sigma)$ , where  $X$  is a compact Hausdorff space and  $\sigma : X \rightarrow X$  is a homeomorphism. Here  $\mathbb{Z}$  acts on  $X$  via iterations of  $\sigma$ , namely  $x \mapsto \sigma^n(x)$  for  $n \in \mathbb{Z}$  and  $x \in X$ . We denote by  $\text{Per}^\infty(\sigma)$  and



$\text{Per}(\sigma)$  the sets of aperiodic points and periodic points, respectively. If  $\text{Per}^\infty(\sigma) = X$ ,  $\Sigma$  is called free and if  $\text{Per}^\infty(\sigma)$  is dense in  $X$ ,  $\Sigma$  is called topologically free. Moreover, for an integer  $n$  we write  $\text{Per}^n(\sigma) = \text{Per}^{-n}(\sigma) = \{x \in X : \sigma^n(x) = x\}$ , and  $\text{Per}_n(\sigma)$  for the set of all points belonging to  $\text{Per}^n(\sigma)$  but to no  $\text{Per}^k(\sigma)$  with  $|k|$  non-zero and strictly less than  $|n|$ . When  $n = 0$  we have  $\text{Per}^0(\sigma) = X$ . We write  $\text{Per}(x) = k$  if  $x \in \text{Per}_k(\sigma)$ , with  $k > 0$ . Note that if  $\text{Per}(y) = k$ , with  $k > 0$ , and  $y \in \text{Per}^n(\sigma)$ , then  $k|n$ . For a subset  $S \subseteq X$  we denote its interior by  $S^0$  and its closure by  $\bar{S}$ . When a periodic point  $y$  belongs to the interior of  $\text{Per}_k(\sigma)$  for some positive integer  $k$  we call  $y$  a periodic interior point. We denote the set of all such points by  $\text{PIP}(\sigma)$ . Note that  $\text{PIP}(\sigma)$  does not coincide with  $\text{Per}(\sigma)^0$  in general, as the following example shows. Let  $X = [0, 1] \times [-1, 1]$  be endowed with the standard subspace topology from  $\mathbb{R}^2$  and let  $\sigma$  be the homeomorphism of  $X$  defined as reflection in the  $x$ -axis. Then clearly  $\text{Per}(\sigma) = X$ . Furthermore,  $\text{Per}_1(\sigma) = [0, 1] \times \{0\}$ , hence  $\text{Per}_1(\sigma)^0 = \emptyset$ , and  $\text{Per}_2(\sigma) = X \setminus \text{Per}_1(\sigma)$ , so that  $\text{Per}_2(\sigma)^0 = \text{Per}_2(\sigma)$ . We conclude that  $\text{PIP}(\sigma) = \text{Per}_2(\sigma) \subsetneq X = \text{Per}(\sigma)^0$ . Incidentally,  $\text{Per}_2(\sigma)$  in this example also shows that the sets  $\text{Per}_n(\sigma)$  are in general not closed, as opposed to the sets  $\text{Per}^n(\sigma)$  which are easily seen to always be closed.

The following topological lemma will be important to us throughout this paper. For a proof we refer to [11, Lemma 2.1].

**Lemma 6.2.1.** *The union of  $\text{Per}^\infty(\sigma)$  and  $\text{PIP}(\sigma)$  is dense in  $X$ .*

Given a dynamical system  $\Sigma = (X, \sigma)$  and a point  $x \in X$ , we denote by  $\mathcal{O}_\sigma(x) = \{\sigma^n(x) : n \in \mathbb{Z}\}$  the orbit of  $x$  in the system. Recall that a dynamical system  $\Sigma = (X, \sigma)$  is called minimal if every orbit of  $\Sigma$  is dense in  $X$ . It is called topologically transitive if for any pair of non-empty open sets  $U, V$  of  $X$ , there exists an integer  $n$  such that  $\sigma^n(U) \cap V \neq \emptyset$ .

We denote by  $\alpha$  the automorphism of  $C(X)$  induced by  $\sigma$  via  $\alpha(f) = f \circ \sigma^{-1}$  for  $f \in C(X)$ . Via  $n \mapsto \alpha^n$ , the integers act on  $C(X)$  by iterations. Given a topological dynamical system  $\Sigma = (X, \sigma)$ , we shall endow the set

$$\ell^1(\Sigma) = \{a : \mathbb{Z} \rightarrow C(X) : \sum_{k \in \mathbb{Z}} \|a(k)\|_\infty < \infty\},$$

where  $\|\cdot\|_\infty$  denotes the supremum norm on  $C(X)$ , with the structure of a Banach \*-algebra. As in [12], we understand a Banach \*-algebra (or involutive Banach algebra) to be a complex Banach algebra with an *isometric* involution. We define scalar multiplication and addition on  $\ell^1(\Sigma)$  as the natural pointwise operations. Multiplication is defined by convolution twisted by  $\alpha$  as follows:

$$(ab)(n) = \sum_{k \in \mathbb{Z}} a(k) \cdot \alpha^k(b(n-k)),$$

for  $a, b \in \ell^1(\Sigma)$ . We define the involution,  $*$ , as

$$a^*(n) = \overline{\alpha^n(a(-n))},$$

for  $a \in \ell^1(\Sigma)$ . The bar denotes the usual pointwise complex conjugation. Finally, we define a norm on  $\ell^1(\Sigma)$  by

$$\|a\| = \sum_{k \in \mathbb{Z}} \|a(k)\|_\infty,$$

for  $a \in \ell^1(\Sigma)$ . It is not difficult to check that when endowed with these operations,  $\ell^1(\Sigma)$  is indeed a Banach  $*$ -algebra. A useful way of working with  $\ell^1(\Sigma)$  is to write an element  $a \in \ell^1(\Sigma)$  in the form  $a = \sum_{k \in \mathbb{Z}} a_k \delta^k$ , for  $a_k = a(k)$  and  $\delta = \chi_{\{1\}}$  where, for  $n, m \in \mathbb{Z}$ ,

$$\chi_{\{n\}}(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

It is then clear that  $\delta^* = \delta^{-1}$  and that  $\delta^n = \chi_{\{n\}}$ , where  $n \in \mathbb{Z}$ . In the rest of this paper we shall use the notation  $a_k$  rather than  $a(k)$ , for  $a \in \ell^1(\Sigma)$  and  $k \in \mathbb{Z}$ . Clearly one may canonically view  $C(X)$  as a closed abelian  $*$ -subalgebra of  $\ell^1(\Sigma)$ , namely as  $\{a\delta^0 : a \in C(X)\}$ . Thus  $\ell^1(\Sigma)$  is generated as a Banach  $*$ -algebra by an isometrically isomorphic copy of  $C(X)$  and the unitary element  $\delta$ , subject to the relation  $\delta f \delta^* = \alpha(f)$  for  $f \in C(X)$ . We let  $k(\Sigma) = \{\sum_k f_k \delta^k : \text{only finitely many } f_k \text{ are non-zero}\}$  and note that  $k(\Sigma)$  is a dense  $*$ -subalgebra of  $\ell^1(\Sigma)$ . We write the canonical projection of norm one from  $\ell^1(\Sigma)$  to  $C(X)$  as  $E$ , where  $E(\sum_k a_k \delta^k) = a_0$ . Note that if  $a = \sum_k a_k \delta^k$  then  $E(a\delta^{*j}) = a_j$  for every integer  $j$ , and that  $E(fag) = fE(a)g$  for  $f, g \in C(X)$ . Furthermore it is easy to show that for a finite collection of scalars  $\lambda_i \geq 0$  and  $a_i \in \ell^1(\Sigma)$ ,  $E(\sum_i \lambda_i a_i^* a_i) \geq 0$  and that  $E(\sum_i \lambda_i a_i^* a_i) = 0$  implies  $\sum_i \lambda_i a_i^* a_i = 0$ .

In our analysis of  $\ell^1(\Sigma)$  we shall make use of the theory of ordered linear spaces in the sense of [3], whence we recall a number of basic definitions appearing there. Although all linear spaces in [3] have the reals as scalar field, we restate the definitions here for linear spaces over a field  $\mathbb{F}$ , where  $\mathbb{F}$  is  $\mathbb{R}$  or  $\mathbb{C}$ .

**Definition 6.2.2.** Let  $V$  be a linear space. A linear ordering of  $V$  is a binary relation  $\leq$ , not necessarily anti-symmetric, on  $V$  such that

- (i)  $v \leq w$  and  $w \leq u$  implies  $v \leq u$  for all  $v, w, u \in V$ ;
- (ii)  $v \leq w$  implies  $v + u \leq w + u$  for all  $v, w, u \in V$ ;
- (iii)  $v \leq w$  and  $\lambda \geq 0$  implies  $\lambda v \leq \lambda w$  for all  $v, w \in V$  and  $\lambda \in \mathbb{R}$ .

An ordered linear space is a linear space over  $\mathbb{F}$  with a linear ordering.

**Definition 6.2.3.** Let  $V$  be an ordered linear space with linear ordering  $\leq$ . A linear subspace  $W \subseteq V$  is said to be cofinal if given any  $v \in V$  with  $0 \leq v$ , there exists a  $w \in W$  such that  $v \leq w$ .

**Definition 6.2.4.** Let  $V$  be an ordered linear space with linear ordering  $\leq$ . A linear functional  $f : V \rightarrow \mathbb{F}$ , where  $\mathbb{F}$  is the scalar field of  $V$ , is said to be positive if  $0 \leq v$  implies  $0 \leq f(v)$  for all  $v \in V$ .

We are now ready to state the following extension theorem, which will be a key result for us.

**Theorem 6.2.5.** [3, Theorem 1.6.1.] *Suppose  $W$  is a cofinal linear subspace of an ordered real linear space  $V$ . Then a positive linear functional on  $W$  can be extended to a positive linear functional on  $V$ .*

To apply this theorem in our setup, we need to recall two results from the theory of Banach  $*$ -algebras. First, however, we recall the standard linear ordering on a Banach  $*$ -algebra.

**Definition 6.2.6.** Let  $A$  be a Banach  $*$ -algebra. We introduce a linear ordering on  $A$  by defining, for  $a, b \in A$ ,  $a \leq b$  if and only if  $b - a = \sum_{i=1}^n \lambda_i c_i^* c_i$  for some  $c_i \in A$  and  $\lambda_i$  non-negative real numbers. An element  $a \in A$  is called non-negative if  $0 \leq a$ .

We can now state the results we need.

**Lemma 6.2.7.** [12, Lemma I.9.8] *Let  $A$  be a unital Banach algebra. If  $a$  is an element of  $A$  such that the spectral radius of  $1 - a$  is strictly less than one, then there exists  $b \in A$  with  $b^2 = a$ . Furthermore, if  $A$  is a Banach  $*$ -algebra and if  $a$  is self-adjoint, then a self-adjoint element can be chosen as the above  $b$ .*

**Lemma 6.2.8.** [12, Lemma I.9.9] *If  $A$  is a unital Banach  $*$ -algebra, then every positive linear functional  $\omega$  of  $A$  is continuous and  $\|\omega\| = \omega(1)$ .*

Recall the following definition.

**Definition 6.2.9.** Let  $A$  be a unital Banach  $*$ -algebra. A positive linear functional  $f$  on  $A$  such that  $f(1) = 1$  is called a state (we also use this terminology for positive linear functionals on arbitrary  $*$ -subalgebras of  $A$ ). We denote by  $S(A)$  the set of states of  $A$  endowed with the weak\* topology. The space  $S(A)$  is then a convex and compact subset of the unit ball in the dual of  $A$  by Lemma 6.2.8 together with the Banach-Alaoglu theorem. We call the extreme points of  $S(A)$  pure states. If  $A$  is commutative, its pure states are precisely its characters whence we shall refer to the pure states of  $A$  as the characters of  $A$  in this situation.

The following extension result for Banach  $*$ -algebras will be a cornerstone in what follows.

**Proposition 6.2.10.** *Let  $A$  be a unital Banach  $*$ -algebra and let  $B$  be a Banach  $*$ -subalgebra containing the unit element of  $A$ . Then every state of  $B$  extends to a state of  $A$ , and every pure state of  $B$  extends to a pure state of  $A$ .*

*Proof.* Viewing the set of self-adjoint elements of  $A$  as a real linear space, it follows from Lemma 6.2.7 that the set of self-adjoint elements of  $B$  constitutes a real linear cofinal subspace of it. To see this, suppose  $a$  is a self-adjoint element of  $A$ . Then  $b = 1 - \frac{a}{\|a\| + \epsilon}$ , where  $\epsilon$  is some positive real number, is a self-adjoint element such that  $1 - b$  has norm, and hence spectral radius, strictly less than one. Hence by Lemma 6.2.7 there exists  $c \in \ell^1(\Sigma)$  such that  $b = c^*c$ , and thus  $(\|a\| + \epsilon) - a = (\|a\| + \epsilon)c^*c$ . Hence  $a \leq (\|a\| + \epsilon)$  and clearly  $(\|a\| + \epsilon) \in B$ . By Theorem 6.2.5 this implies that, when we view the set of self-adjoint elements of  $A$  as a real linear space and the set of self-adjoint elements of  $B$  as a real linear

subspace of it, positive linear functionals on the latter extend to positive linear functionals on the former. Since every element in a Banach \*-algebra can be written uniquely as  $a = a_1 + ia_2$  with  $a_1, a_2$  self-adjoint, it follows easily that a positive complex linear functional on  $B$  has a positive complex linear extension to  $A$ . Invoking Theorem 6.2.8 we thus see that states of  $B$  have state extensions to  $A$ . Now suppose that  $\varphi$  is a pure state of  $B$ . The set of state extensions of  $\varphi$  to  $A$  is a weak\*-compact convex set by the Banach-Alaoglu theorem, whence by the Krein-Milman theorem there exists an extension  $\tilde{\varphi}$  that is extreme amongst all state extensions of  $\varphi$ . To see that  $\tilde{\varphi}$  is a pure state, suppose there exist states  $\tilde{\psi}_1, \tilde{\psi}_2$  and  $\lambda$  with  $0 < \lambda < 1$  such that  $\tilde{\varphi} = \lambda\tilde{\psi}_1 + (1-\lambda)\tilde{\psi}_2$ . Denoting by  $\psi_i$ , for  $i = 1, 2$ , the restriction of  $\tilde{\psi}_i$  to  $B$  we then have  $\varphi = \lambda\psi_1 + (1-\lambda)\psi_2$ . Since  $\varphi$  is pure, we see that  $\psi_1 = \psi_2 = \varphi$ . Thus  $\tilde{\psi}_1$  and  $\tilde{\psi}_2$  also yield  $\varphi$  when restricted to  $B$ . Since  $\tilde{\varphi}$  is extreme amongst all state extensions of  $\varphi$ , we conclude that  $\tilde{\psi}_1 = \tilde{\psi}_2 = \tilde{\varphi}$ .  $\square$

Although  $\ell^1(\Sigma)$  is our main object of study, we shall sometimes refer to its enveloping  $C^*$ -algebra, which we denote by  $C^*(\Sigma)$ . For a detailed account of the general theory of the interplay between  $\Sigma$  and  $C^*(\Sigma)$  we refer to [13] and [14]. For particular aspects of it akin to the work in this paper, see [11]. A few known results on  $C^*(\Sigma)$  appearing in [13] and [14] need to be mentioned here, however. As is explained in [13, Section 3.2],  $\ell^1(\Sigma)$  has sufficiently many Hilbert space representations, whence its enveloping  $C^*$ -algebra,  $C^*(\Sigma)$ , contains a dense \*-isomorphic copy of it. It follows from [2, Theorem 2.7.5 (i)] that restriction gives a bijection between the states of  $C^*(\Sigma)$  and those of  $\ell^1(\Sigma)$ , and between the pure states of  $C^*(\Sigma)$  and those of  $\ell^1(\Sigma)$ .

For  $x \in X$  we denote by  $\mu_x$  the functional on  $C(X)$  that acts as point evaluation in  $x$ . The pure state extensions of such point evaluations to  $\ell^1(\Sigma)$  will play a prominent role in this paper. We shall exploit the fact that the pure state extensions to  $C^*(\Sigma)$  of point evaluations on  $C(X)$  have been described explicitly. We now recall some basic facts about them, without proofs. For further details and proofs, we refer to [14, §4]. For  $x \in \text{Per}^\infty(\sigma)$  there is a unique pure state extension of  $\mu_x$ , denoted by  $\varphi_x$ , given by  $\varphi_x = \mu_x \circ E$  (here  $E$  denotes the continuous extension to  $C^*(\Sigma)$  of the projection  $E : \ell^1(\Sigma) \rightarrow C(X)$  as defined above). The set of pure state extensions of  $\mu_y$  for  $y \in \text{Per}(\sigma)$  is parametrized by the unit circle as  $\{\varphi_{y,t} : t \in \mathbb{T}\}$ . We denote the GNS-representations of  $C^*(\Sigma)$  associated with the pure state extensions above by  $\tilde{\pi}_x$  and  $\tilde{\pi}_{y,t}$ , respectively. For  $x \in \text{Per}^\infty(\sigma)$ ,  $\tilde{\pi}_x$  is the representation of  $C^*(\Sigma)$  on  $\ell^2$ , whose standard basis we denote by  $\{e_i\}_{i \in \mathbb{Z}}$ , defined on the generators as follows. For  $f \in C(X)$  and  $i \in \mathbb{Z}$  we have  $\tilde{\pi}_x(f)e_i = f \circ \sigma^i(x) \cdot e_i$ , and  $\tilde{\pi}_x(\delta)e_i = e_{i+1}$ . For  $y \in \text{Per}(\sigma)$  with  $\text{Per}(y) = p > 0$  and  $t \in \mathbb{T}$ ,  $\tilde{\pi}_{y,t}$  is the representation on  $\mathbb{C}^p$ , whose standard basis we denote by  $\{e_i\}_{i=0}^{p-1}$ , defined as follows. For  $f \in C(X)$  and  $i \in \{0, 1, \dots, p-1\}$  we set  $\tilde{\pi}_{y,t}(f)e_i = f \circ \sigma^i(y) \cdot e_i$ . For  $j \in \{0, 1, \dots, p-2\}$ ,  $\tilde{\pi}_{y,t}(\delta)e_j = e_{j+1}$  and  $\tilde{\pi}_{y,t}(\delta)e_{p-1} = t \cdot e_0$ . We also mention that the unitary equivalence class of  $\tilde{\pi}_x$  is determined by the orbit of  $x$ , and that of  $\tilde{\pi}_{y,t}$  by the orbit of  $y$  and the parameter  $t$ . We shall abuse notation slightly and use the same symbol for a pure state of  $C^*(\Sigma)$  as for its restriction to  $\ell^1(\Sigma)$ , and similarly for the associated GNS-representation. Hence, denoting by  $\Phi$  the set of all pure state extensions of point evaluations on  $C(X)$  to  $\ell^1(\Sigma)$ , we have  $\Phi = \{\varphi_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\varphi_{y,t} : y \in \text{Per}(\sigma), t \in \mathbb{T}\}$ . We shall also use the subset  $\Phi'$  of  $\Phi$  defined by  $\Phi' = \{\varphi_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\varphi_{y,t} : y \in \text{PIP}(\sigma), t \in \mathbb{T}\}$ .

When speaking of closed ideals in  $\ell^1(\Sigma)$ , we shall always mean closed two-sided ideals

that are not necessarily self-adjoint. We say that  $\ell^1(\Sigma)$  is simple if it lacks proper non-zero closed ideals, and we call  $\ell^1(\Sigma)$  prime if the intersection of any pair of non-zero closed ideals is non-zero.

### 6.3. The commutant of $C(X)$

We will now analyze the commutant of  $C(X)$  in  $\ell^1(\Sigma)$ , denoted by  $C(X)'$  and defined as

$$C(X)' = \{a \in \ell^1(\Sigma) : af = fa \text{ for all } f \in C(X)\}.$$

One easily concludes that it is a Banach  $*$ -subalgebra of  $\ell^1(\Sigma)$ . We need the following topological lemma, for a proof of which we refer to [11, Lemma 3.1].

**Lemma 6.3.1.** *The system  $\Sigma = (X, \sigma)$  is topologically free if and only if  $\text{Per}^n(\sigma)$  has empty interior for all positive integers  $n$ .*

We give the following concrete description of  $C(X)'$ .

**Proposition 6.3.2.**  $C(X)' = \{\sum_k a_k \delta^k \in \ell^1(\Sigma) : \text{supp}(a_k) \subseteq \text{Per}^k(\sigma) \text{ for all } k\}$ . Consequently,  $C(X)' = C(X)$  if and only if the dynamical system is topologically free.

*Proof.* The assertion is an adaption of [8, Corollary 3.4] to our context, but we include a proof here for the reader's convenience. Suppose  $a = \sum_k a_k \delta^k \in C(X)'$ . For any  $f$  in  $C(X)$  we then have

$$\begin{aligned} f a_k &= E(f a \delta^{*k}) = E(a f \delta^{*k}) = E(a \delta^{*k} \alpha^k(f)) = \\ &= E(a \delta^{*k}) \cdot \alpha^k(f) = a_k \cdot \alpha^k(f). \end{aligned}$$

Hence for all  $x \in X$

$$f(x) a_k(x) = a_k(x) f \circ \sigma^{-k}(x).$$

Therefore, if  $a_k(x)$  is not zero we have that  $f(x) = f \circ \sigma^{-k}(x)$  for all  $f \in C(X)$ . It follows that  $\sigma^{-k}(x) = x$ , i.e.  $x$  belongs to  $\text{Per}^k(\sigma)$ . Since  $\text{Per}^k(\sigma)$  is closed,  $\text{supp}(a_k) \subseteq \text{Per}^k(\sigma)$ . Conversely, if  $\text{supp}(a_k) \subseteq \text{Per}^k(\sigma)$  for all  $k$ , then  $f(x) a_k(x) = a_k(x) f \circ \sigma^{-k}(x)$  for every  $k$  and  $x$ , so that  $(fa)_k = (af)_k$  for all  $k$ , i.e.  $a$  commutes with  $f$ . This establishes the description of  $C(X)'$ .

Moreover, by Lemma 6.3.1,  $\Sigma$  is topologically free if and only if for every nonzero integer  $k$  the set  $\text{Per}^k(\sigma)$  has empty interior. So when the system is topologically free, we see from the above description of  $C(X)'$  that an element  $a$  in  $C(X)'$  necessarily belongs to  $C(X)$ . If  $\Sigma$  is not topologically free, however,  $\text{Per}^k(\sigma)$  has non-empty interior for some non-zero  $k$  and hence there is a non-zero function  $f \in C(X)$  such that  $\text{supp}(f) \subseteq \text{Per}^k(\sigma)$ . Then  $f \delta^k \in C(X)' \setminus C(X)$  by the above.  $\square$

The following elementary result is an adaption of [8, Proposition 2.1] to our setup.

**Proposition 6.3.3.** *The commutant  $C(X)'$  of  $C(X)$  is abelian, and thus it is the unique maximal abelian Banach  $*$ -subalgebra of  $\ell^1(\Sigma)$  containing  $C(X)$ .*

*Proof.* Suppose  $a, b \in C(X)'$ . By definition of the multiplication in  $\ell^1(\Sigma)$  it follows that for an integer  $n$ ,  $(ab)_n = \sum_k a_k \cdot \alpha^k(b_{n-k})$ . As  $a \in C(X)'$ , it follows from Proposition 6.3.2 that  $(ab)_n = \sum_k a_k \cdot b_{n-k}$ . Similarly,  $(ba)_n = \sum_k b_k \cdot a_{n-k}$ . Thus  $(ab)_n = (ba)_n$  for all integers  $n$  and hence  $ab = ba$ .  $\square$

We now determine the characters of  $C(X)'$  in terms of the sets  $\Phi$  and  $\Phi'$  as introduced in Section 6.2.

**Theorem 6.3.4.** *The characters of  $C(X)'$  are precisely the restrictions of elements in  $\Phi$  to  $C(X)'$ . Furthermore, the restriction map is injective on  $\Phi'$ .*

*Proof.* By Proposition 6.2.10, characters of  $C(X)'$  have pure state extensions to  $\ell^1(\Sigma)$ . Of course, the restriction of a character of  $C(X)'$  to  $C(X)$  is a character of the latter, and since we have concluded that characters of  $C(X)'$  have pure state extensions to  $\ell^1(\Sigma)$ , it follows that every character of  $C(X)'$  is the restriction of some element of  $\Phi$ . So to prove the first assertion it suffices to show that every element of  $\Phi$  is multiplicative on  $C(X)'$ . By Proposition 6.3.2,  $C(X)'$  is the closed linear span of its monomials, by which we mean elements of the form  $f_n \delta^n$ . This implies that it suffices to check multiplicativity on the monomials of  $C(X)'$ . Let  $f_n \delta^n, f_m \delta^m \in C(X)'$  and note that by Proposition 6.3.2 this implies that  $\text{supp}(f_n) \subseteq \text{Per}^n(\sigma)$  and  $\text{supp}(f_m) \subseteq \text{Per}^m(\sigma)$ . Suppose first that  $x \in \text{Per}^\infty(\sigma)$  and consider the pure state  $\varphi_x = \mu_x \circ E$  on  $\ell^1(\Sigma)$ . By definition of  $\varphi_x$ , it follows that

$$\varphi_x(f_n \delta^n) = \begin{cases} 0, & n \neq 0; \\ f_n(x), & n = 0, \end{cases}$$

and similarly for  $f_m \delta^m$ . Furthermore  $f_n \delta^n f_m \delta^m = f_n \cdot f_m \circ \sigma^{-n} \delta^{n+m}$  and thus

$$\varphi_x(f_n \delta^n f_m \delta^m) = \begin{cases} 0, & n \neq -m; \\ f_n(x) \cdot f_m \circ \sigma^{-n}(x), & n = -m. \end{cases}$$

As  $x \in \text{Per}^\infty(\sigma)$  the support conditions on  $f_n$  and  $f_m$  imply that

$$\varphi_x(f_n \delta^n f_m \delta^m) = \begin{cases} 0, & n, m \text{ not both zero}; \\ f_n(x) f_m(x), & n = m = 0. \end{cases}$$

So clearly  $\varphi(f_n \delta^n f_m \delta^m) = \varphi(f_n \delta^n) \varphi(f_m \delta^m)$ .

Now suppose that  $y \in \text{Per}_k(\sigma)$  for some integer  $k > 0$ . Let  $t \in \mathbb{T}$  and consider  $\varphi_{y,t}$ . Then

$$\varphi_{y,t}(f_n \delta^n) = (\tilde{\pi}_{y,t}(f_n \delta^n) e_0, e_0) = \begin{cases} t^{\frac{n}{k}} f_n(y), & \text{if } k|n; \\ 0, & \text{otherwise.} \end{cases}$$

and similarly for  $f_m \delta^m$ . Furthermore

$$\varphi_{y,t}(f_n \delta^n f_m \delta^m) = \begin{cases} t^{\frac{n+m}{k}} f_n(y) f_m \circ \sigma^{-n}(y), & \text{if } k|(n+m); \\ 0, & \text{otherwise.} \end{cases}$$

As  $y \in \text{Per}_k(\sigma)$ , however, the support conditions on  $f_n$  and  $f_m$  imply that

$$\varphi_{y,t}(f_n \delta^n f_m \delta^m) = \begin{cases} t^{\frac{n+m}{k}} f_n(y) f_m(y), & \text{if } k|n \text{ and } k|m; \\ 0, & \text{otherwise.} \end{cases}$$

It follows that  $\varphi_{y,t}(f_n \delta^n f_m \delta^m) = \varphi_{y,t}(f_n \delta^n) \varphi_{y,t}(f_m \delta^m)$ .

To prove the second assertion, note firstly that by the discussion in the end of Section 6.2 we know that for  $x \in \text{Per}^\infty(\sigma)$ ,  $\varphi_x$  is the unique pure state extension of  $\mu_x$  from  $C(X)$  to  $\ell^1(\Sigma)$ . Hence no other element in  $\Phi$  than  $\varphi_x$  itself coincides with  $\varphi_x$  on  $C(X)'$ . Now suppose  $y \in \text{PIP}(\sigma)$  with  $y \in \text{Per}_k(\sigma)^0$  for some positive integer  $k$ , and let  $t \in \mathbb{T}$ . Suppose, by contradiction, that there are two pure state extensions  $\varphi_{y,s}$  and  $\varphi_{y,t}$  of  $\mu_y$ , with  $s \neq t$ , that coincide on  $C(X)'$ . Let  $f \in C(X)$  be a function such that  $f(y) = 1$  and  $\text{supp}(f) \subseteq \text{Per}^k(\sigma)$ . Then by Theorem 6.3.2 it follows that  $f \delta^k \in C(X)'$ . Furthermore

$$\varphi_{y,s}(f \delta^k) = f(y)s = s \neq t = f(y)t = \varphi_{y,t}(f \delta^k),$$

which is a contradiction.  $\square$

Before continuing our investigation of  $C(X)'$ , we state a result on ideals of  $\ell^1(\Sigma)$ .

**Proposition 6.3.5.** *Let  $I$  be a proper (not necessarily closed or self-adjoint) ideal in  $\ell^1(\Sigma)$ . Viewing  $I$  as a subset of  $C^*(\Sigma)$  under the canonical embedding, the closure of  $I$  in  $C^*(\Sigma)$  is proper as well.*

*Proof.* Since  $\ell^1(\Sigma)$  is unital and hence has no proper dense ideals, the closed ideal  $J$  generated by the set  $\{a^*a : a \in I\}$  is self-adjoint and proper in  $\ell^1(\Sigma)$ . We prove first that there is a state of  $\ell^1(\Sigma)$  that vanishes on  $J$ . To see this, consider the state, of the Banach  $*$ -subalgebra  $\mathbb{C} + J \subseteq \ell^1(\Sigma)/J$ , defined by  $\lambda + J \mapsto \lambda$  for  $\lambda \in \mathbb{C}$ . By Proposition 6.2.10 it has a state extension,  $f$  say, to  $\ell^1(\Sigma)/J$ . Denoting by  $\pi : \ell^1(\Sigma) \rightarrow \ell^1(\Sigma)/J$  the natural quotient map it is clear that  $f \circ \pi$  is a state of  $\ell^1(\Sigma)$  that vanishes on  $J$ . Suppose now that the closure of  $I$  inside  $C^*(\Sigma)$  coincides with  $C^*(\Sigma)$ . Then there is a sequence  $(a_n) \in I$  that converges to 1 in the norm of  $C^*(\Sigma)$ . The sequence  $(a_n^*a_n)$  then converges to 1 as well. The elements of this sequence, however, are in  $J$  whence the state extension of  $f \circ \pi$  to  $C^*(\Sigma)$  vanishes on 1 by continuity, a contradiction.  $\square$

We make the following definition.

**Definition 6.3.6.** A Banach  $*$ -subalgebra  $B$  of  $\ell^1(\Sigma)$  is said to have the intersection property for closed ideals if for every non-zero closed ideal  $I \subseteq \ell^1(\Sigma)$  we have  $I \cap B \neq \{0\}$ .

Finally we are ready to prove the main result of this section.

**Theorem 6.3.7.**  *$C(X)'$  has the intersection property for closed ideals in  $\ell^1(\Sigma)$ .*

*Proof.* Suppose that  $I$  is a closed non-zero ideal such that  $I \cap C(X)' = \{0\}$ . Define  $J$  to be the closed ideal generated by the set  $\{a^*a : a \in I\}$ . Then  $J$  is easily seen to be a non-zero closed self-adjoint ideal of  $\ell^1(\Sigma)$  contained in  $I$ . Hence  $\ell^1(\Sigma)/J$  is a unital Banach  $*$ -algebra. Clearly  $J \cap C(X)' = \{0\}$  and, denoting by  $\pi : \ell^1(\Sigma) \rightarrow \ell^1(\Sigma)/J$  the natural quotient map, we may algebraically identify  $C(X)'$  with its isomorphic embedding  $\pi(C(X)')$  in  $\ell^1(\Sigma)/J$  and write this embedded algebra as  $C(X)'/J$ . As in the proof of Proposition 6.2.10, one concludes by Theorem 6.2.5 together with Lemma 6.2.7 that states on  $C(X)'/J$  have state extensions to  $\ell^1(\Sigma)/J$ . We shall use this fact to show that every character of  $C(X)'$  has a pure state extension to  $\ell^1(\Sigma)$  that vanishes on  $J$ . Fix a character  $\omega$

of  $C(X)'$ . Then  $\omega \circ (\pi|_{C(X)})^{-1}$  is clearly a state on  $C(X)'/J$ . It is easy to see that the set of states of  $\ell^1(\Sigma)/J$  that extend  $\omega \circ (\pi|_{C(X)})^{-1}$  constitutes a non-empty weak\*-closed subset of the unit ball in the dual of  $\ell^1(\Sigma)/J$ , whence by the Banach-Alaoglu theorem those states form a weak\*-compact subset which is clearly also convex. By the Krein-Milman theorem this set is the closed convex hull of its extreme points, so there is a state extension  $\omega'$  of  $\omega \circ (\pi|_{C(X)})^{-1}$  to  $\ell^1(\Sigma)/J$  that is an extreme point of the set of all those state extensions of  $\omega \circ (\pi|_{C(X)})^{-1}$ . Using the same technique as in the proof of Proposition 6.2.10, one concludes that the fact that  $\omega$  is pure on  $C(X)'$  implies that  $\omega'$  is even an extreme point of the set of all states on  $\ell^1(\Sigma)/J$ , and hence a pure state of it. For completeness, we give a proof of this fact. Suppose there is a  $\lambda \in (0, 1)$  and two states  $\zeta_1, \zeta_2$  of  $\ell^1(\Sigma)/J$  such that  $\omega' = \lambda\zeta_1 + (1 - \lambda)\zeta_2$ . Restricting to  $C(X)'/J$  we get

$$\omega \circ (\pi|_{C(X)})^{-1} = \lambda\zeta_1|_{C(X)'/J} + (1 - \lambda)\zeta_2|_{C(X)'/J}$$

and hence

$$\omega = \lambda\zeta_1 \circ \pi|_{C(X)} + (1 - \lambda)\zeta_2 \circ \pi|_{C(X)}.$$

But  $\omega$  was pure so it follows that  $\zeta_1 \circ \pi|_{C(X)} = \zeta_2 \circ \pi|_{C(X)} = \omega$ . Finally we conclude that  $\zeta_1|_{C(X)'/J} = \zeta_2|_{C(X)'/J} = \omega \circ (\pi|_{C(X)})^{-1}$ , so  $\zeta_1, \zeta_2$  were extensions of  $\omega \circ (\pi|_{C(X)})^{-1}$ , whence by assumption  $\zeta_1 = \zeta_2 = \omega'$  and thus  $\omega'$  is a pure state of  $\ell^1(\Sigma)/J$ . By the GNS-construction, pure states of Banach \*-algebras correspond to irreducible Hilbert space representations. This makes it easy to see that  $\omega' \circ \pi$  is a pure state of  $\ell^1(\Sigma)$  extending  $\omega$  and vanishing on  $J$ . As restrictions to  $C(X)'$  of elements in  $\Phi'$  have unique pure state extensions to  $\ell^1(\Sigma)$  by Theorem 6.3.4, it follows that every element in  $\Phi'$  vanishes on  $J$ . This means that all pure state extensions to  $\ell^1(\Sigma)$  of point evaluations in  $\text{Per}^\infty(\sigma)$  and  $\text{PIP}(\sigma)$  vanish on  $J$ . Thus clearly  $\varphi_x = \mu_x \circ E$  vanishes on  $J$  for  $x \in \text{Per}^\infty(\sigma)$ , and  $\mu_y \circ E$ , with  $y \in \text{PIP}(\sigma)$ , vanishes on  $J$  as well by the Krein-Milman theorem since it is a state extension of  $\mu_y$  and we have already concluded that all pure state extensions of  $\mu_y$  to  $\ell^1(\Sigma)$  vanish on  $J$ . Thus the states  $\{\mu_x \circ E : x \in \text{Per}^\infty(\sigma) \cup \text{PIP}(\sigma)\}$  all vanish on  $J$ . Now suppose  $a \in J$  and fix an arbitrary  $k \in \mathbb{Z}$ . By the above  $\mu_x \circ E(a\delta^{*k}) = a_k(x) = 0$  for all  $x \in \text{Per}^\infty(\sigma) \cup \text{PIP}(\sigma)$ . By Lemma 6.2.1 we conclude that  $a_k \equiv 0$ . As  $k$  was arbitrary, it follows that  $a = 0$  and we can finally conclude that  $J = \{0\}$  and hence also that  $I = \{0\}$ . This is a contradiction, and thus  $I \cap C(X)' \neq \{0\}$  as asserted.  $\square$

## 6.4. Consequences of the intersection property of $C(X)'$

Theorem 6.3.7 allows us to prove a number of analogues of theorems on the interplay between  $\Sigma$  and  $C^*(\Sigma)$  appearing e.g. in [13] and [14]. We begin with the following analogue of [14, Theorem 5.4].

**Theorem 6.4.1.** *For a topological dynamical system  $\Sigma$ , the following statements are equivalent.*

- (i)  $\Sigma$  is topologically free;
- (ii)  $I \cap C(X) \neq 0$  for every non-zero closed ideal  $I$  of  $\ell^1(\Sigma)$ ;



(iii)  $C(X)$  is a maximal abelian Banach  $*$ -subalgebra of  $\ell^1(\Sigma)$ .

*Proof.* Equivalence of (i) and (iii) is an immediate consequence of Proposition 6.3.2 together with Proposition 6.3.3. To see that (i) implies (ii) note that, by Proposition 6.3.2, (i) implies that  $C(X) = C(X)'$  and thus (ii) follows by Theorem 6.3.7. To show that (ii) implies (i), we shall use the same technique as in the proof of [14, Theorem 5.4]. Suppose that  $\Sigma$  is not topologically free. Then by Lemma 6.3.1 there is a positive integer  $n$  such that  $\text{Per}^n(\sigma)$  has non-empty interior. Let  $f \in C(X)$  be non-zero and such that  $\text{supp}(f) \subseteq \text{Per}^n(\sigma)$  and consider the closed ideal  $I$  of  $\ell^1(\Sigma)$  generated by  $f - f\delta^n$ . Note that all representations in  $\{\tilde{\pi}_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\tilde{\pi}_{y,1} : y \in \text{Per}(\sigma)\}$  vanish on  $f - f\delta^n$  and hence on  $I$ . To see this, first note that for  $x \in \text{Per}^\infty(\sigma)$  and  $i \in \mathbb{Z}$  we have  $\tilde{\pi}_x(f - f\delta^n)e_i = f \circ \sigma^i(x)e_i - f \circ \sigma^{i+n}(x)e_{i+n} = 0$  since  $f$  is zero outside  $\text{Per}(\sigma)$ . Similarly it follows that  $\tilde{\pi}_{y,1}(f - f\delta^n) = 0$  when  $y$  does not have period dividing  $n$ . If  $y$  has period  $k$  where  $n = r \cdot k$  then for  $i \in \{0, 1, \dots, k-1\}$  we have  $\tilde{\pi}_{y,1}(f - f\delta^n)e_i = f \circ \sigma^i(y)e_i - 1^r \cdot f \circ \sigma^i(y)e_i = 0$ . This clearly implies that the family  $\Phi'' = \{\varphi_x : x \in \text{Per}^\infty(\sigma)\} \cup \{\varphi_{y,1} : y \in \text{Per}(\sigma)\}$  vanishes on  $I$ . Suppose now that  $g \in C(X) \cap I$ . For every point  $z \in X$  there is a pure state extension of  $\mu_z$  in  $\Phi''$ . Hence  $\mu_z(g) = g(z) = 0$  and we conclude that  $g \equiv 0$  and hence that  $I \cap C(X) = \{0\}$ .  $\square$

The following result is analogous to [14, Theorem 5.3], [1, Theorem VIII 3.9] and the main result in [6].

**Theorem 6.4.2.** *Suppose that  $X$  consists of infinitely many points. Then  $\ell^1(\Sigma)$  is simple if and only if  $\Sigma = (X, \sigma)$  is minimal.*

*Proof.* Suppose that  $\Sigma$  is not minimal. Then there is a point  $x \in X$  such that  $\overline{\mathcal{O}_\sigma(x)} \neq X$ . Note that  $\overline{\mathcal{O}_\sigma(x)}$  is invariant under  $\sigma$  and its inverse. Define

$$I = \{a \in \ell^1(\Sigma) : a_k \in \ker(\overline{\mathcal{O}_\sigma(x)}) \text{ for all integers } k\}$$

where  $\ker(\overline{\mathcal{O}_\sigma(x)}) = \{f \in C(X) : f \text{ vanishes on } \overline{\mathcal{O}_\sigma(x)}\}$ . It is easy to see that  $I$  is a proper non-zero closed ideal of  $\ell^1(\Sigma)$ , which is thus not simple. Conversely, suppose that  $\ell^1(\Sigma)$  is not simple and let  $I$  be a proper non-zero closed ideal of it. If  $\Sigma$  is minimal the fact that  $X$  is infinite clearly implies that  $\Sigma$  is topologically free, since  $X = \text{Per}^\infty(\sigma)$ . By Theorem 6.4.1 it follows that  $I \cap C(X) \neq \{0\}$ . It is not difficult to see that  $I \cap C(X)$  is a closed ideal of  $C(X)$  that is invariant under  $\alpha$  and its inverse. It is clearly proper since  $I \cap C(X) = C(X)$  would imply that  $I = \ell^1(\Sigma)$ . As the closed ideals of  $C(X)$  are precisely the kernels of closed subsets of  $X$  we may write  $I \cap C(X) = \ker(C)$ , where  $C$  is some proper non-empty closed subset of  $X$ . It also follows that  $C$  is invariant under  $\sigma$  and its inverse, since  $I \cap C(X)$  is invariant under  $\alpha$  and its inverse. This contradicts the minimality of  $\Sigma$ .  $\square$

The assertion does not hold if we drop the condition that  $X$  be infinite. Consider for example the case when  $\Sigma = (\{x\}, \text{id})$ . Then  $\ell^1(\Sigma)$  is easily seen to be isometrically  $*$ -isomorphic to  $\ell^1$ , which is not simple since its character space is non-empty. The system  $\Sigma$ , however, is trivially minimal.

We conclude this section by proving the analogue of [14, Theorem 5.5]. To do this, we need the following two easy topological lemmas.

**Lemma 6.4.3.** *If  $\Sigma = (X, \sigma)$  is not topologically transitive, then there exist two disjoint non-empty open sets  $O_1$  and  $O_2$ , both invariant under  $\sigma$  and its inverse, such that  $\overline{O_1} \cup \overline{O_2} = X$ .*

*Proof.* As the system is not topologically transitive, there exist non-empty open sets  $U, V \subseteq X$  such that for any integer  $n$  we have  $\sigma^n(U) \cap V = \emptyset$ . Now clearly the set  $O_1 = \bigcup_{n \in \mathbb{Z}} \sigma^n(U)$  is a non-empty open set invariant under  $\sigma$  and  $\sigma^{-1}$ . Then  $\overline{O_1}$  is a closed set invariant under  $\sigma$  and  $\sigma^{-1}$ . It follows that  $O_2 = X \setminus \overline{O_1}$  is an open set, invariant under  $\sigma$  and  $\sigma^{-1}$ , containing  $V$ . Thus we even have that  $\overline{O_1} \cup O_2 = X$ , and the result follows.  $\square$

**Lemma 6.4.4.** *If  $\Sigma = (X, \sigma)$  is topologically transitive and there is an  $n > 0$  such that  $X = \text{Per}^n(\sigma)$ , then  $X$  consists of a single orbit and is thus finite.*

*Proof.* Assume two points  $x, y \in X$  are not in the same orbit. As  $X$  is Hausdorff we may separate the points  $x, \sigma(x), \dots, \sigma^{n-1}(x), y$  by pairwise disjoint open sets  $V_0, V_1, \dots, V_{n-1}, V_y$ . Now consider the set

$$U_x := V_0 \cap \sigma^{-1}(V_1) \cap \sigma^{-2}(V_2) \cap \dots \cap \sigma^{-n+1}(V_{n-1}).$$

Clearly the sets  $A_x = \bigcup_{i=0}^{n-1} \sigma^i(U_x)$  and  $A_y = \bigcup_{i=0}^{n-1} \sigma^i(V_y)$  are disjoint non-empty open sets, both invariant under  $\sigma$  and  $\sigma^{-1}$ , which leads us to a contradiction. Hence  $X$  consists of one single orbit under  $\sigma$ .  $\square$

**Theorem 6.4.5.** *Suppose that  $X$  consists of infinitely many points. Then  $\ell^1(\Sigma)$  is prime if and only if  $\Sigma = (X, \sigma)$  is topologically transitive.*

*Proof.* Suppose first that the system  $\Sigma$  is not topologically transitive. Then there exists, by Lemma 6.4.3, two disjoint non-empty open sets  $O_1$  and  $O_2$ , both invariant under  $\sigma$  and  $\sigma^{-1}$ , such that  $\overline{O_1} \cup \overline{O_2} = X$ . Let  $I_1$  and  $I_2$  be the closed ideals generated in  $\ell^1(\Sigma)$  by  $\ker(\overline{O_1})$  and  $\ker(\overline{O_2})$  respectively. It is then not difficult to see that, for  $i = 1, 2$ , we have

$$I_i = \left\{ \sum_n f_n \delta^n \in \ell^1(\Sigma) : f_n \in \ker(\overline{O_i}) \text{ for all } n \right\}$$

and hence that  $E(I_i) = \ker(\overline{O_i})$ . Hence

$$E(I_1 \cap I_2) \subseteq E(I_1) \cap E(I_2) = \ker(\overline{O_1}) \cap \ker(\overline{O_2}) = \ker(\overline{O_1} \cup \overline{O_2}) = \ker(X) = \{0\}.$$

Now note that if  $I$  is an ideal and  $E(I) = \{0\}$ , then  $I = \{0\}$ . Namely, suppose that  $a \in I$ . Then for an arbitrary integer  $n$  we have that  $a_n = E(a \delta^{*n}) = 0$  and hence  $a = 0$ . Applying this to  $I_1 \cap I_2$ , we see that  $I_1 \cap I_2 = \{0\}$ , hence  $\ell^1(\Sigma)$  is not prime. Next suppose that  $\Sigma$  is topologically transitive. We claim that  $\Sigma$  is topologically free. If not, then by Lemma 6.3.1 there is an integer  $n > 0$  such that  $\text{Per}^n(\sigma)$  has non-empty interior. As  $\text{Per}^n(\sigma)$  is invariant under  $\sigma$  and  $\sigma^{-1}$  and closed, topological transitivity implies that  $X = \text{Per}^n(\sigma)$ . This, however, is impossible since by Lemma 6.4.4 it would force  $X$  to consist of a single orbit and hence be finite. Thus  $\Sigma$  is topologically free after all. Now let  $I$  and  $J$  be two non-zero proper closed ideals in  $\ell^1(\Sigma)$  and assume by contradiction that  $I \cap J = \{0\}$ . Then

$I \cap C(X)$  and  $J \cap C(X)$  are proper closed ideals of  $C(X)$  with zero intersection that are invariant under  $\alpha$  and its inverse, and topological freeness of  $\Sigma$  assures us that they are non-zero, by Theorem 6.4.1. This implies that there are proper non-empty closed subsets  $C_1, C_2$  of  $X$  that are invariant under  $\sigma$  and its inverse and such that  $I \cap C(X) = \ker(C_1)$  and  $J \cap C(X) = \ker(C_2)$ . Now  $\{0\} = I \cap J \cap C(X) = \ker(C_1) \cap \ker(C_2) = \ker(C_1 \cup C_2)$  whence  $C_1 \cup C_2 = X$ . Since  $C_2$  is proper and closed,  $(C_1)^0 \supset X \setminus C_2 \neq \emptyset$ . Since  $C_1$  is proper and closed,  $X \setminus C_1$  is open and non-empty. Invariance of  $C_1$  under  $\sigma$  and its inverse implies that  $\sigma^n((C_1)^0) \cap (X \setminus C_1) = \emptyset$  for all integers  $n$ . This contradicts topological transitivity of  $\Sigma$  and we conclude that  $I \cap J \neq \{0\}$ . It follows that  $\ell^1(\Sigma)$  is prime.  $\square$

This theorem is also false if the condition that  $X$  be infinite is dropped. Again, consider the case when we have  $\Sigma = (\{x\}, \text{id})$ . As mentioned after the proof of Theorem 6.4.2,  $\ell^1(\Sigma)$  is isometrically  $*$ -isomorphic to  $\ell^1$ . It is well known that the character space of  $\ell^1$  can be identified with  $\mathbb{T}$  with its standard topology. Consider two proper non-empty closed subsets  $C_1, C_2$  of  $\mathbb{T}$  such that  $C_1 \cup C_2 = \mathbb{T}$ . Then  $\ker(C_i) = \{a \in \ell^1 : a \text{ is annihilated by every character in } C_i\}$ , for  $i = 1, 2$ , are two proper closed ideals of  $\ell^1$  that are non-zero by regularity of  $\ell^1$ . Semi-simplicity of  $\ell^1$  implies that  $\ker(C_1) \cap \ker(C_2) = \ker(\mathbb{T}) = \{0\}$  and we conclude that  $\ell^1$  is not prime. Trivially, however,  $\Sigma$  is topologically transitive.

## 6.5. Closed ideals of $\ell^1(\Sigma)$ which are not self-adjoint

We will determine when all closed ideals of  $\ell^1(\Sigma)$  are self-adjoint. Our approach is based on the following special case of the rather deep general result [7, Theorem 7.7.1].

**Theorem 6.5.1.**  *$\ell^1$  contains a closed ideal which is not self-adjoint.*

Now we can establish the following.

**Theorem 6.5.2.** *Every closed ideal of  $\ell^1(\Sigma)$  is self-adjoint if and only if  $\Sigma$  is free.*

*Proof.* Suppose  $\Sigma$  is free, hence in particular topologically free, and let  $I \subseteq \ell^1(\Sigma)$  be a non-zero closed ideal. Denote by  $\pi : \ell^1(\Sigma) \rightarrow \ell^1(\Sigma)/I$  the natural quotient map. Then  $I \cap C(X)$  is easily seen to be a closed ideal of  $C(X)$  that is invariant under  $\alpha$  and its inverse and that is non-zero by Theorem 6.4.1. Hence we may write  $I \cap C(X) = \ker(X_\pi)$  for some closed subset  $X_\pi$  of  $X$  that is invariant under  $\sigma$  and its inverse. Denote by  $\sigma_\pi$  the restriction of  $\sigma$  to  $X_\pi$  and write  $\Sigma_\pi = (X_\pi, \sigma_\pi)$ . We shall show that  $\pi$  can be factored in a certain way. Denote by  $\phi : \ell^1(\Sigma) \rightarrow \ell^1(\Sigma_\pi)$  the  $*$ -homomorphism defined by  $\sum_k f_k \delta^k \mapsto \sum_k f_k|_{X_\pi} \delta^k$ . By Tietze's extension theorem every function in  $C(X_\pi)$  can be extended to a function in  $C(X)$ , and an easy application of Urysohn's lemma shows that one can choose an extension whose norm is arbitrarily close to the norm of the function one extends. Using this, it is not difficult to show that the map  $\Psi : \ell^1(\Sigma_\pi) \rightarrow \ell^1(\Sigma)/I$  defined by  $\sum_k f_k \delta^k \mapsto \sum_k \tilde{f}_k \delta^k + I$ , where the  $\tilde{f}_k$  are such that  $\sum_k \tilde{f}_k \delta^k \in \ell^1(\Sigma)$  and  $\tilde{f}_k|_{X_\pi} = f_k$ , is a well-defined contractive homomorphism. We note that  $\pi = \Psi \circ \phi$ . Since  $\ker(\Psi)$  is a closed ideal of  $\ell^1(\Sigma_\pi)$ ,  $\ker(\Psi) \cap C(X_\pi) = \{0\}$  and  $\Sigma_\pi$  is free, hence topologically free, it

follows from Theorem 6.4.1 that  $\Psi$  is injective. Thus  $I = \ker(\pi) = \ker(\phi)$  and the latter is self-adjoint since  $\phi$  is a  $*$ -homomorphism.

Conversely, suppose that some  $x \in X$  has period  $p > 0$ . We will use the  $p$ -dimensional GNS-representations  $\tilde{\pi}_{x,z}$ , where  $z \in \mathbb{T}$ , of  $\ell^1(\Sigma)$ , which are associated with the periodic point  $p$  as described in Section 6.2, to construct a continuous surjective  $*$ -homomorphism  $\Psi : \ell^1(\Sigma) \rightarrow M_p(\ell^1)$ . Here  $M_p(\ell^1)$  has its natural structure as a  $*$ -algebra and is a Banach space under the norm  $\|A\| = \max_{1 \leq i, j \leq p} \|A_{i,j}\|$ , where  $A \in M_p(\ell^1)$ . Then, if  $I$  is a closed non-self-adjoint ideal of  $\ell^1$  as in Theorem 6.5.1,  $M_p(I)$  is a closed non-self-adjoint ideal of  $M_p(\ell^1)$  and hence, since  $\Psi$  is surjective,  $\Psi^{-1}(M_p(I))$  is a closed non-self-adjoint ideal of  $\ell^1(\Sigma)$ . To construct  $\Psi$  we first note that  $\ell^1$  is isomorphic, as a  $*$ -algebra, to the algebra  $\text{AC}(\mathbb{T})$  of continuous functions on  $\mathbb{T}$  with an absolutely convergent Fourier series. The isomorphism is the Fourier transform, given by  $\mathcal{F}((\dots, a_{-1}, a_0, a_1, \dots))(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ . It yields a natural  $*$ -isomorphism  $j : M_p(\ell^1) \rightarrow M_p(\text{AC}(\mathbb{T}))$ . We will construct a surjective  $*$ -homomorphism  $\theta : \ell^1(\Sigma) \rightarrow M_p(\text{AC}(\mathbb{T}))$  and then  $j^{-1} \circ \theta$  will be the desired continuous  $*$ -homomorphism  $\Psi$ . To define  $\theta$ , we recall that the GNS-representations  $\tilde{\pi}_{x,z} : \ell^1(\Sigma) \rightarrow M_p(\mathbb{C})$  are such that

$$\tilde{\pi}_{x,z}(f) = \begin{pmatrix} f(x) & 0 & \dots & 0 \\ 0 & f \circ \sigma(x) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f \circ \sigma^{p-1}(x) \end{pmatrix}$$

for  $f \in C(X)$ , and

$$\tilde{\pi}_{x,z}(\delta) = \begin{pmatrix} 0 & 0 & \dots & 0 & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

One sees that

$$\tilde{\pi}_{x,z}(\delta^2) = \begin{pmatrix} 0 & 0 & \dots & 0 & z & 0 \\ 0 & 0 & \dots & 0 & 0 & z \\ 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 \end{pmatrix}, \quad \tilde{\pi}_{x,z}(\delta^3) = \begin{pmatrix} 0 & 0 & \dots & z & 0 & 0 \\ 0 & 0 & \dots & 0 & z & 0 \\ 0 & 0 & \dots & 0 & 0 & z \\ 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 & 0 \end{pmatrix},$$

etc., and that

$$\tilde{\pi}_{x,z}(\delta^p) = \begin{pmatrix} z & 0 & \dots & 0 & 0 \\ 0 & z & \dots & 0 & 0 \\ 0 & 0 & z & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & z \end{pmatrix}.$$

Calculating, it is not difficult to see that

$$\tilde{\pi}_{x,z}(\sum_k f_k \delta^k) = \begin{pmatrix} \sum_k f_{kp}(x)z^k & \cdots & \sum_k f_{(k-1)p+1}(x)z^k \\ \sum_k f_{kp+1}(\sigma(x))z^k & \cdots & \sum_k f_{(k-1)p+2}(\sigma(x))z^k \\ \sum_k f_{kp+2}(\sigma^2(x))z^k & \cdots & \sum_k f_{(k-1)p+3}(\sigma^2(x))z^k \\ \vdots & \vdots & \vdots \\ \sum_k f_{(k+1)p-1}(\sigma^{p-1}(x))z^k & \cdots & \sum_k f_{kp}(\sigma^{p-1}(x))z^k \end{pmatrix}.$$

This makes it obvious that the representations  $\tilde{\pi}_{x,z}$  combine naturally to a  $*$ -homomorphism  $\theta : \ell^1(\Sigma) \rightarrow M_p(\text{AC}(\mathbb{T}))$ . Moreover,  $\theta$  is surjective. Indeed, since each complex number  $f_k(\sigma^i(x))$  occurs only once (somewhere in row  $i + 1$ ) as a Fourier coefficient in the matrix of  $\theta(\sum_k f_k \delta^k)$ , one sees that prescribing the image of  $\theta(\sum_k f_k \delta^k)$  amounts to prescribing the numbers  $f_k(\sigma^i(x))$  in an unambiguous way. The Urysohn lemma therefore implies that  $\theta$  is surjective. Finally, defining  $s(i, j) \in \mathbb{Z}$  such that for all  $z \in \mathbb{T}$  the  $(i, j)$  entry of  $\theta(\sum_k f_k \delta^k)(z)$  is  $\sum_{n=-\infty}^{\infty} f_{np+s(i,j)}(\sigma^{i-1}(x))z^n$ , we see that the  $(i, j)$  entry of  $j^{-1} \circ \theta(\sum_k f_k \delta^k)$  is the sequence  $\{f_{np+s(i,j)}(\sigma^{i-1}(x))\}_{n=-\infty}^{\infty}$ . This makes it obvious that  $j^{-1} \circ \theta : \ell^1(\Sigma) \rightarrow M_p(\ell^1)$  is not only a surjective  $*$ -homomorphism but also continuous as desired.  $\square$

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# Samenvatting

Dit proefschrift gaat over verbanden tussen dynamische systemen en daarmee geassocieerde algebra's van gekruist produkt type, met als centrale thema het vaststellen van equivalenties tussen structureigenschappen van de gekruiste produkten enerzijds en eigenschappen van de dynamica anderzijds. Het voornaamste startpunt hierbij is een dynamisch systeem  $\Sigma = (X, \sigma)$ , waarbij  $X$  een compacte Hausdorff ruimte is en  $\sigma$  een homeomorfisme van  $X$ . De gehele getallen werken dan op  $X$  door iteratie van  $\sigma$  en met deze actie wordt al sinds lange tijd een  $C^*$ -algebra van gekruist produkt type,  $C^*(\Sigma)$ , geassocieerd. We concentreren ons echter niet alleen op  $C^*(\Sigma)$ , maar ook op een Banach  $*$ -algebra,  $\ell^1(\Sigma)$ , en een niet-volledige  $*$ -algebra,  $k(\Sigma)$ , die beide van gekruist produkt type zijn en via  $*$ -isomorfismen als dichte deelalgebra's in  $C^*(\Sigma)$  ingebed kunnen worden. De  $C^*$ -algebra  $C^*(\Sigma)$  is dan de zogenaamde omhullende  $C^*$ -algebra van  $\ell^1(\Sigma)$ . Voor ieder van deze drie algebra's onderzoeken we het verband tussen de structuur en de dynamica van  $\Sigma$ . Bij  $C^*(\Sigma)$  past dit onderzoek in een lange traditie; het beschouwen van  $\ell^1(\Sigma)$  en  $k(\Sigma)$  is nieuw.

Alhoewel  $C^*$ -algebra's technisch aantrekkelijke eigenschappen hebben die niet gelden voor willekeurige Banach  $*$ -algebra's zoals  $\ell^1(\Sigma)$ , heeft  $\ell^1(\Sigma)$  daarentegen ten opzichte van  $C^*(\Sigma)$  het voordeel dat zijn elementen expliciet gegeven zijn als een convergente reeks. Dit geldt ook voor de algebra  $k(\Sigma)$ , waar de sommaties zelfs eindig zijn. Beide algebra's zijn daardoor beter toegankelijk voor expliciete berekeningen dan  $C^*(\Sigma)$ . De algebra's  $C^*(\Sigma)$ ,  $\ell^1(\Sigma)$  en  $k(\Sigma)$  bevatten alle een kopie van  $C(X)$  waarvan we de commutant,  $C(X)'$ , in alledrie de algebra's gedetailleerd onderzoeken. In het bijzonder kijken we naar de doorsnijdingseigenschappen van deze commutant met idealen in deze algebra's. We bewijzen onder andere dat, onafhankelijk van het systeem  $\Sigma$ ,  $C(X)'$  altijd een doorsnede ongelijk aan nul heeft met ieder ideaal van de algebra in kwestie dat ongelijk aan nul is; in  $\ell^1(\Sigma)$  verkrijgen we dit resultaat alleen voor gesloten idealen. We gebruiken deze nieuwe stellingen om een aantal bekende verbanden tussen  $\Sigma$  en  $C^*(\Sigma)$  op een alternatieve en bevredigendere manier te begrijpen en eveneens om analoge van deze verbanden af te leiden voor  $\ell^1(\Sigma)$  en  $k(\Sigma)$ . De analogie is overigens niet volledig: zo blijkt dat  $\ell^1(\Sigma)$  gesloten idealen kan hebben die niet zelfgeadjeungeerd zijn. Dit verschijnsel, dat voor  $C^*(\Sigma)$  als  $C^*$ -algebra uiteraard niet aan de orde is, treedt in  $\ell^1(\Sigma)$  op dan en slechts dan als  $\Sigma$  periodieke punten heeft. Een aantal basisresultaten die betrekking hebben op  $C(X)'$  en die ten grondslag liggen aan de genoemde verbanden tussen structuur en dynamica, zijn sterk afhankelijk van het feit dat we deze commutant samen met zijn ruimte van maximale idealen zowel in  $\ell^1(\Sigma)$  als in  $C^*(\Sigma)$  expliciet kunnen beschrijven.

Verder onderzoeken we algebra's van gekruist produkt type die geassocieerd zijn met

paren  $(B, \Psi)$ , waarbij  $B$  een Banach algebra is en  $\Psi$  een automorfisme van  $B$ . Deze familie van algebra's generaliseert de algebra's  $k(\Sigma)$ . Wanneer de Banach algebra  $B$  commutatief, semisimpel en regulier is, kunnen vragen over het gekruiste produkt vertaald worden naar algebra's van het type  $k(\Sigma)$ , waarbij het dynamische systeem de ruimte van maximale idealen van  $B$  als topologische ruimte heeft. Een aantal structureigenschappen van het gekruiste produkt blijkt dan equivalent te zijn met dynamische eigenschappen in de ruimte van maximale idealen.

# Curriculum Vitae

Christian Svensson werd op 9 september 1980 te Varberg in Zweden geboren. Na zijn eindexamen aan Elof Lindälvs Gymnasium in Kungsbacka verhuisde hij in augustus 1999 naar Noord-Zweden om aan Luleå University of Technology de MSc-opleiding tot natuurkundig ingenieur te volgen. In juni 2001 verhuisde hij naar Zuid-Zweden om zijn opleiding voort te zetten aan Lund University. In het academisch jaar 2003-2004 studeerde hij als Erasmusstudent wiskunde en wiskundige logica aan University of Leeds in Groot-Brittannië. Daarna keerde hij terug naar Lund om bij Doc. dr. S.D. Silvestrov aan zijn masterscriptie “Commuting differential operators and commuting elements in non-commutative algebras” te werken, waarop hij in februari 2005 afstudeerde. Vervolgens begon hij als Assistent in Opleiding aan de Universiteit Leiden bij Prof. dr. S.M. Verduyn Lunel en Dr. M.F.E. de Jeu, die zijn dagelijkse begeleider was, in samenwerking met Doc. dr. S.D. Silvestrov in Lund. Het sinds die tijd verrichte onderzoek heeft geleid tot dit proefschrift. De eerste vier maanden van zijn promotietijd bracht hij door aan de Universiteit van Antwerpen, waar hij een reeks minicursussen over niet-commutatieve meetkunde volgde. Verder volgde hij zomerscholen in Ottawa en Kopenhagen en bezocht hij congressen in Canada, Nederland en Zweden, waarop hij ook voordrachten gaf over zijn onderzoek.