ANALYSIS OF FACE-HOMOGENEOUS RANDOM WALKS ON LOW DIMENSIONAL LATTICES.

Proefschrift

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Chapter 1

Introduction

1.1 Introduction

The main object of this thesis is face-homogeneous random walks on low dimensional lattices. Face-homogeneity is a property of a random walk. It can be considered as a natural step for generalizing homogeneity. Our analysis is focused on two models of face-homogeneous random walks in dimension two. For these models we study the Large deviations bounds, fluid (Euler) limit and almost closed sets. All these subjects are independent fields of probability theory. Nevertheless we will show how they are connected to each other. Especially we will use results on almost closed sets for studying the so-called Euler limit and the Large deviations bounds.

In section 1.2 we will explain what we understand by a random walk and we will give the definition of face-homogeneity. In sections 1.3 and 1.4 we will introduce the reader to the fluid (Euler) limits and theory of Large Deviations. Some simple examples will be provided as well. In these sections we will also collect known results for homogeneous random walks and we will give an overview of the new results obtained in this thesis.

Chapter 2 is a theoretical background for this thesis. It contains some results from the theory of Markov chains and martingales that we will need for our analysis. Introduction to the theory of almost closed sets is given in section 2.3. Chapters 3, 4, 5 and 6 are the main contribution of the thesis and they can be read independently.

Chapter 3 is a slight modification of the paper A. Hordijk, N. Popov, F. M. Spieksma (2002). *Discrete scattering and almost closed sets for simple face-homogeneous random walks.* Technical Report 2002-26, Leiden University. In this chapter we will show the relation between discrete scattering of the fluid limit and atomic almost closed sets.

Chapter 4 is mainly based on the paper N.Popov, F.M.Spieksma (2002). *Non-existence of a stochastic fluid limit for a cycling random walk.* Technical Report, MI 2002-25, Leiden University. Here we study the fluid limit of a face-homogeneous random walk on $\mathbf{Z}^2$. In this chapter we will also discuss the structure of almost closed sets for this random walk.

In Chapter 5 we analyse the Large Deviations bounds in the quarter plane. This chapter has appeared as A.Hordijk, N. Popov (2003) *Large deviations bounds for face-homogeneous random walks in the quarter plane.* Probability in the Engineering and Informational Sciences

17(3): 369-395.

Finally Chapter 6 focuses on applications of the results in Chapter 5. It has appeared with the exception of minor modifications as A. Hordijk, N. Popov (2003). *Large deviations analysis of a coupled processors system.* Probability in the Engineering and Informational Sciences 17(3): 397-409.

We expect the reader to be familiar with basic probability theory, i.e. with the concepts of a random variable, distribution, expectation, independence, probability space, convergence in probability and almost sure convergence, conditional probability, independent identically distributed (i.i.d.) random variables, Law of Large Numbers (LLN) etc.

1.2 Face-homogeneous random walks

In the next sections we will give a more technical introduction to this thesis.

When speaking of a random walk, we will always understand it to be defined on a countable set $\boldsymbol{I}$, called the *state space.* The elements of $\boldsymbol{I}$ are called *states.* In our analysis, $\boldsymbol{I}$ will be a subset of the lattices

$$\mathbf{Z}^p = \big\{ i = (i_1, \cdots, i_p) : \ i_1, \cdots, i_p \text{ are integers} \big\},$$

$$\mathbf{Z}^p_+ = \big\{ i \in \mathbf{Z}^p : \ i_1 \geq 0, \cdots, i_p \geq 0 \big\}.$$

One can view a random walk as a particle jumping (walking) on $\boldsymbol{I}$ from one state to the other. The jumps occur at discrete time moments $n = 1, 2 \cdots$. The initial position of a particle (a random walk) is supposed to be fixed and given. We denote it by ξ_0. Thus starting from state ξ_0, the particle (the random walk) jumps randomly to another state. This first jump (the size of the first jump) will be denoted by y_1. So, at time $n = 1$, the particle is in state $\xi_0 + y_1$. Clearly, y_1 is a random variable with values in $\mathbf{Z}^p$. At the next moment $n = 2$, the particle (the random walk) makes a jump y_2 and it enters state $\xi_0 + y_1 + y_2$. By continuation we get a sequence of random variables $y_1, y_2, \cdots$, called *the jumps (*or *the jump variables).* The sum

$$\xi_n \triangleq \xi_0 + y_1 + \cdots + y_n$$

is called *the position of the particle (the random walk) at time* n. Clearly, ξ_n is a random variable as well. The main property of the random walk that we consider, is time-homogeneity, i.e. given that the particle is in state i at time n, the probability of a jump to state j does not depend on n :

$$\mathsf{P}\{\xi_{n+1} = j \,|\, \xi_n = i\} = p_{i,j} \text{ for any } n = 0, 1, 2, \cdots .$$

The numbers $p_{i,j}$ are called *transition probabilities* (from i to j). They define a stochastic matrix $(p_{i,j}), i, j \in \boldsymbol{I}$, i.e.

$$\sum_{j \in \boldsymbol{I}} p_{i,j} = 1 \text{ for any } i \in \boldsymbol{I} \text{ and } p_{i,j} \geq 0.$$

Thus we say that a random walk is defined if a state space $\boldsymbol{I}$, a stochastic matrix $(p_{i,j}), i, j \in \boldsymbol{I}$, and an initial state $\xi_0 \in \boldsymbol{I}$ are given. We will denote a random walk by $\{\xi_n\}$ (or just ξ_n without brackets).

In this thesis we will consider only random walks with *bounded jumps*. This means that there is a constant $D > 1$, such that $\|\xi_{n+1} - \xi_n\| < D$ for any $n \geq 0$, as here $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^p$:

$$\|i\| = \sqrt{i_1^2 + \cdots + i_p^2},\ i = (i_1, \cdots, i_p).$$

We will also restrict ourself to the case of *irreducible* random walks. Irreducibility of a random walk means that from any state we can reach any other state in a finite number of jumps with positive probability (see definition of irreducibility in section 2.1).

From homogeneity to face-homogeneity

The simplest and frequently used class of random walks is homogeneous random walks. Recall that a random walk $\{\xi_n\}$ is called *homogeneous* if the jumps $y_1, y_2, \cdots$ are i.i.d. random variables. The homogeneity property means that the transition probabilities $p_{i,j}$ depend only on the difference $j - i$, i.e.

$$p_{i,j} = p_{j-i} \text{ for all } i, j \in \boldsymbol{I},$$

which means that the jumps y_n have the same distribution

$$\mathsf{P}\{y_n = k\} = p_k \text{ for all } k \text{ and } n.$$

The theory of homogeneous random walks is well developed. For a homogeneous random walk the Large deviations bounds, the fluid limit and almost closed sets have quite simple structures. We will describe these structures in the next sections.

However it is not always possible to define an irreducible homogeneous random walk on a desirable state space. Check for example $\boldsymbol{I} = \{0, 1, 2, \cdots\}$. A natural step for generalizing homogeneity is to divide the state space into regions (faces) of homogeneity.

Definition 1.1

A random walk $\{\xi_n\}$ is called face-homogeneous*, if the state space $\boldsymbol{I}$ can be partitioned into a finite collection of disjoint subsets $\boldsymbol{\Lambda} \subseteq \boldsymbol{I}$, where $\{\xi_n\}$ behaves as a homogeneous random walk. More precisely, the jumps from states in subset $\boldsymbol{\Lambda}$ are i.i.d. or in other words, the transition probabilities $p_{i,j}$ depend only on the difference $j - i$ and on the subset $\boldsymbol{\Lambda}$, to which i belongs :*

$$p_{i,j} = p_{j-i}^{\boldsymbol{\Lambda}} \text{ for any } i \in \boldsymbol{\Lambda}, j \in \boldsymbol{I}. \tag{1.2.1}$$

The subsets $\boldsymbol{\Lambda}$ are called faces *(or* homogeneity faces*).*

For a homogeneous random walk the state space itself is a homogeneity face.

Example 1.2.1 Let $\boldsymbol{I}$ be the set of all non-negative integers $\{0, 1, 2, \cdots\}$. Suppose that for any $i > 0$ transition probabilities $p_{i,j}$ depends only on the difference $j - i$, i.e there are two different types of transition probabilities $p_{i,j} = p_{j-i}^{+}, i > 0$, and $p_{0,j} = p_j^0$. Then we have two faces of homogeneity

$$\boldsymbol{\Lambda}^0 = \{0\} \text{ and } \boldsymbol{\Lambda}^+ = \{1, 2, \cdots\}.$$

In other words, we have two types of jump variables. The first type corresponds to a jump of size y^0 from the origin with distribution $\mathsf{P}\{y^0 = k\} = p_k^0$. The second type corresponds to jumps of size y^i from state $i > 0$, with the distribution $\mathsf{P}\{y^i = k\} = p_k^+$. Here the variables $y^1, y^2, \cdots$ are i.i.d.

Now we define *the mean drift of the random walk* as the vector function $m = (m_1, \cdots, m_p) : \boldsymbol{I} \to \mathbf{R}^p$ with

$$m(i) \triangleq \sum_{j \in \boldsymbol{I}} (j - i) p_{i,j}. \tag{1.2.2}$$

Since we consider only random walks with bounded jumps, the mean drift exists and it is finite. This function will play a major role in our analysis. The value $m(i)$ is called *the mean drift at state* i. If $\{\xi_n\}$ is face-homogeneous, then for a given face $\boldsymbol{\Lambda}$ the mean drift $m(i)$ is the same for any $i \in \boldsymbol{\Lambda}$. This drift is called *the mean drift on face* $\boldsymbol{\Lambda}$ and we denote it by $m^{\boldsymbol{\Lambda}}$. So we have that

$$m^{\boldsymbol{\Lambda}} = \sum_{k \in \mathbf{Z}^p} k p_k^{\boldsymbol{\Lambda}}.$$

In the theory of random walks it is very important to know if a given random walk is ergodic or transient (see definitions in section 2.1.2). The definitions of ergodicity and transience are written in terms of the series generated by transition probabilities. It appears that for many models of face-homogeneous random walks it is possible to give necessary and sufficient conditions of ergodicity and transience via the drifts $m^{\boldsymbol{\Lambda}}$. In the models of our analysis these conditions are very simple.

1.3 Euler limit

The topic of this section is the large-time behavior of a random walk ξ_n. It appears that the behavior of a random walk for sufficiently large n can be described often by a deterministic function known in the literature as fluid approximation or Euler limit. To define the Euler limit (or fluid approximation) we have to scale the random walk simultaneously in space and in time. To illustrate the nature of this scaling we consider a simple example.

Example 1.3.1 *(homogeneous random walk)*

Suppose that $y_1, y_2, \cdots$, is a sequence of i.i.d. random variables with values in $\mathbf{Z}$ and finite expectation $\mathsf{E}y_1$. The well-known strong Law of Large Numbers (LLN) tells that

$$\frac{y_1 + \cdots + y_n}{n} \to \mathsf{E}y_1 \text{ almost surely as } n \to +\infty.$$

Let $\xi_n = y_0 + y_1 + \cdots + y_n$. Then $\{\xi_n\}$ is a homogeneous random walk. Let us fix any $x \in \mathbf{R}$ and $\tau > 0$. Now we scale the initial state ξ_0 and the time n by setting

$$\xi_0 = [xN] \text{ and } n = [\tau N],$$

where $[xN]$, $[\tau N]$ denote the integer parts of xN and τN respectively. Clearly,

$$\frac{\xi_0}{N} = \frac{[xN]}{N} \to x \ \text{ and } \ \frac{n}{N} = \frac{[\tau N]}{N} \to \tau$$

as $N \to +\infty$. Then by the strong LLN the limit

$$\lim_{N\to+\infty} \frac{\xi_{[\tau N]}}{N} = \lim_{N\to+\infty} \frac{[xN] + y_1 + \cdots + y_{[\tau N]}}{N}$$

almost surely exists and it equals $x + \tau \mathsf{E} y_1$. This limit is called the *fluid (or Euler) limit.*

We can easily define the Euler limit for a random walk of any dimension.

Definition 1.2 *Consider a random walk* $\{\xi_n\}$ *on a countable set* $\boldsymbol{I} \subseteq \mathbf{Z}^p$*. Let* $\xi_0 = [xN], x \in \mathbf{R}^p$*. Then for any* $\tau > 0$ *the limit*

$$u(x,\tau) \triangleq \lim_{N\to+\infty} \frac{\xi_{[\tau N]}}{N}$$

is called the fluid limit (or Euler limit)*, provided it exists. Any realization* $u(x,\cdot)$ *is called* a fluid (Euler) path starting at point x.

In this definition we do not specify the type of convergence for the Euler limit. Any type of convergence is wellcome. But mainly we will talk about almost surely convergence or convergence in probability.

Next we would like to mention some questions concerning the Euler limit. The typical problems are "does Euler limit exist for each x and τ?", "is it deterministic or random?" Example 1.3.1 shows that the Euler limit of a homogeneous random walk almost surely exists for any x, τ, and it is a deterministic linear function of x, τ . For a face-homogeneous random walk the Euler limit can be random as well. If it is deterministic, then it is a piece-wise linear function of τ, as we will see below. The next logical question in the case of a random limit is "what is the limiting distribution and how does it depend on x and τ?"

In the following example we will illustrate all these problems for a simple model of face-homogeneous random walk. In this model the random walk has two different types of the jumps (transition probabilities).

Example 1.3.2 Let the state space be the set of integers **Z** with three homogeneity faces :

$$\mathbf{\Lambda}^- = \{\cdots, -2, -1\},\ \mathbf{\Lambda}^0 = \{0\} \text{ and } \mathbf{\Lambda}^+ = \{1, 2, \cdots\},$$

i.e. the transition probabilities are given by

$$p_{i,j} = \begin{cases} p^-_{j-i} & \text{if } i < 0, \\ p^0_{j-i} & \text{if } i = 0, \\ p^+_{j-i} & \text{if } i > 0. \end{cases}$$

To our homogeneity faces correspond three different mean drifts :

$$m^- = \sum_k kp^-_k,\ m^0 = \sum_k kp^0_k, \text{ and } m^+ = \sum_k kp^+_k.$$

The most interesting case of this model is the case $m^- < 0 < m^+$, i.e. the drifts have the opposite directions. It appears that in this case the Euler limit is random for $x = 0$ and it is deterministic for any $x \neq 0$. This case is described in the following lemma.

Lemma 1.3.1 *Let $m^- < 0 < m^+$. Then for any $\tau > 0, x \in \mathbf{R}$ and $\xi_0 = [xN]$ the Euler limit*

$$u(x,\tau) \triangleq \lim_{N\to\infty} \frac{\xi_{[\tau N]}}{N}$$

exists almost surely, where

$$u(x,\tau) = \begin{cases} x + \tau m^- & \text{if } x < 0, \\ x + \tau m^+ & \text{if } x > 0. \end{cases}$$

If $x = 0$ then $u(0,\tau)$ is random and it takes the values τm^- and τm^+ with positive probabilities such that

$$\mathsf{P}\{u(0,\tau) = \tau m^-\} + \mathsf{P}\{u(0,\tau) = \tau m^+\} = 1.$$

The proof of this lemma is given in section 3.3. □

One can ask what makes the value $x = 0$ so special? The essential difference between $x = 0$ and the other values of x is that the initial state $\xi_0 = [xN]$ is fixed (independent on N) in the case $x = 0$. As it will be proved in section 3.3, for any fixed initial state $\xi_0 = i$ the sequence $\frac{1}{N}\xi_{[\tau N]}$ converges almost surely to a random variable. This variable takes the same values τm^- and τm^+ for any fixed ξ_0. The essential difference is that the probabilities

$$\mathsf{P}\{\frac{\xi_{[\tau N]}}{N} \to \tau m^-\} \text{ and } \mathsf{P}\{\frac{\xi_{[\tau N]}}{N} \to \tau m^+\}$$

depend on ξ_0. The behavior of a random walk with a fixed initial state is connected with the theory of almost closed sets. The reader will find more details of this case in Chapter 3.

So to analyse the Euler limit one has to separate the case of the Euler limit with a fixed initial state ξ_0 and the case of the Euler limit with the scaling $\xi_0 = [xN], x \neq 0$.

One more interesting example of the Euler limit is given in section 3.4 In that example a random walk changes faces infinitely often. Nevertheless, the Euler limit exists. Moreover, it is deterministic and a linear function of τ.

New results.

In Chapter 4 we consider a random walk on the two dimensional lattice $\mathbf{Z}^2$. The axes $\{(x_1, 0) : x_1 \in \mathbf{R}\}$ and $\{(0, x_2) : x_2 \in \mathbf{R}\}$ divide this plane into four quarter planes. We suppose that these quarter planes are homogeneity faces and the mean drifts $m^{\mathbf{\Lambda}}$ are of the form shown in Figure 1.1. For this model we distinguish two cases : the ergodic random walk and the transient random walk. We prove that the random walk is ergodic if $C < 1$ and it is transient if $C > 1$, where C is a positive constant defined explicitely by the values of the drifts $m^{\mathbf{\Lambda}}$ (see formula 4.2.1).

It is known for this model and it is also proved in Chapter 4 that for any $\tau > 0$ and any $x \in \mathbf{R}^2$ inside quarter plane the fluid limit $u(x,\tau)$ exists in probability and it is deterministic. Moreover, the function $u(x,\tau)$ is linear on the faces and it is a piece-wise linear function of t. The image of $u(x,\cdot)$ is depicted in Figure 1.2 and it is called the Euler path.

For a fixed initial state the Euler limit of transient ξ_n does not exist in general (if ξ_n is ergodic the Euler limit is equal to 0). Nevertheless we have proved for $x = 0$ that in

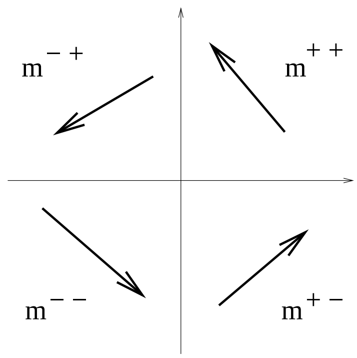

Figure 1.1: the mean drifts

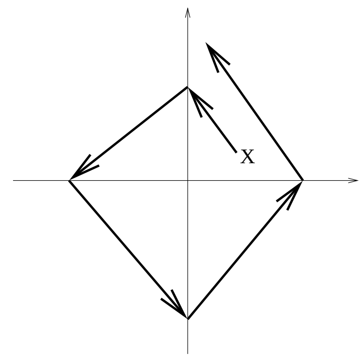

Figure 1.2: the Euler path starting at x

the limit the sequence $\frac{1}{N}\xi_{[\tau N]}$ is placed on a closed curve $\mathcal{I}(\tau)$, which is isomorphic to a circle. In Chapter 4 we give an explicit equation of this curve and it is depicted in Figure 1.3. Moreover, we prove that if the Euler limit exists for $x = 0$ then it is not deterministic, but defines a probability measure μ_τ on the curve $\mathcal{I}(\tau)$. We calculate this measure explicitly. It appears that it is absolutely continuous (w.r.t. the Lebesgue measure on $\mathcal{I}(\tau)$).

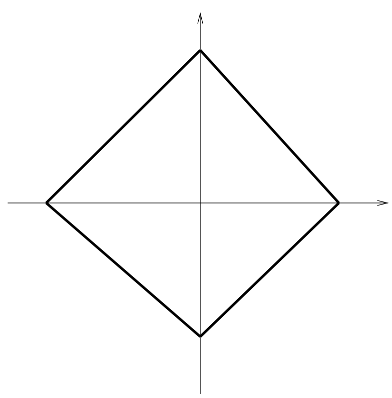

Figure 1.3: Isochrone

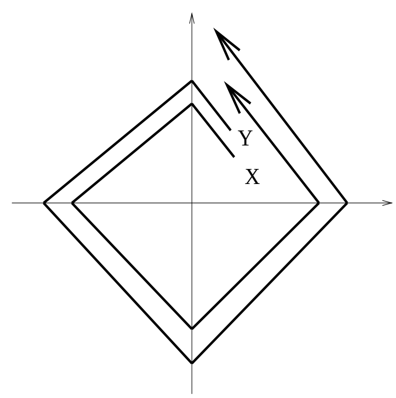

Figure 1.4: two Euler paths starting at x, y

For this model we also study so-called sojourn subsets of the state space. We will prove that the region between any two Euler paths forms a sojourn set. Our conjecture for this model is that this region is an almost closed set as well. If this is true then it will imply that the state space $\mathbf{Z}^2$ is non-atomic.

In Chapter 3 we will derive some results for characterizing the structure of so-called almost closed sets for face-homogeneous random walks. We will present a conjecture on the relation between discrete scattering of the fluid limit and the absence of non-atomic almost closed sets. We will illustrate the conjecture with random walks having simple *and* non-simple decomposition into almost closed sets.

1.4 Large Deviations theory

Large Deviations theory concerns estimates of probabilities of rare events. These rare events depend on the time parameter N, which tends to infinity. Their probabilities decrease to 0 exponentially quickly as a function of N. Briefly speaking, the LD problem is to estimate this exponent via lower and upper bounds (LD bounds) for these probabilities. Moreover, the LD bounds as functions of N have to be sharp.

Before we introduce the LD notations, we would like to go back to the Euler limit. In Example 1.3.1 the position $\xi_{[\tau N]}$ scaled by the factor N converges almost surely to the deterministic limit $x + v\tau$, where $v = \mathsf{E}\, y_1$. It implies that for any $\delta > 0$ and $\tau > 0$

$$\mathsf{P}\{|\frac{\xi_{[\tau N]}}{N} - x - v\tau| < \delta\} \to 1 \text{ as } N \to \infty.$$

One can ask how fast this probability goes to 1. For a homogeneous random walk $\{\xi_n\}$ with bounded jumps $y_1, y_2, \cdots$, we can apply Kolmogorov's inequality to obtain a lower bound:

$$\mathsf{P}\{\max_{1\le t\le[\tau N]} |\frac{\xi_t}{N} - x - v\frac{t}{N}| < \delta\} \ge 1 - \frac{\mathrm{Var}\{y_1\}}{\delta^2 N}, \tag{1.4.1}$$

where $\mathrm{Var}\{y_1\}$ denotes the variance of y_1. LD theory gives more precise (accurate) estimates for the probability in the left-hand side of (1.4.1) then Kolmogorov's inequality.

On other hand, if we take another v (which is not defined by the Euler limit) then the probability in the left-hand side of (1.4.1) converges to 0, i.e. the event becomes rare. In this case LD theory also provides us with sharp estimates. For $v \ne \mathsf{E}y_1$ it is proved in [6] and [28] that

$$\mathsf{P}\{\max_{1\le t\le[\tau N]} |\frac{\xi_t}{N} - x - v\frac{t}{N}| < \delta\} = \exp\{-\mathcal{L}_\tau(v)N + o(N)\},$$

where $o(N)/N \to 0$ as $N \to +\infty$ and the value $\mathcal{L}_\tau(v) > 0$ depends on the distribution of y_1. The function $\mathcal{L}_\tau(\cdot)$ is called *the local rate function.* Here the LD problem is to determine $\mathcal{L}_\tau(v)$ explicitly as a function of τ and v. For the homogeneous random walk this problem has been solved. The answer is the following (see [6],[28]):

$$\mathcal{L}_\tau(v) = \tau \sup_{\alpha\in\mathrm{R}}\{\alpha v - H(\alpha)\},$$

with $H(\alpha)$ the *cumulant generating function* (cgf), i.e.

$$H(\alpha) \triangleq \log \mathsf{E}\exp\{\alpha y_1\} = \log\big(\sum_{k\in\mathbf{Z}} p_k \exp\{\alpha k\}\big).$$

Note that the event in the left-hand side of (1.4.1) observes the position ξ_l scaled by N in δ-neighborhood of a linear path

$$\varphi(s) = x + vs,\ s \in [0, \tau].$$

Instead of a linear path, we can take an arbitrary continuous path in consideration. Through all our LD analysis we will consider the events of the following type :

$$A_N = \{\xi_0 = [xN], \max_{1\le t\le[\tau N]} \|\frac{\xi_t}{N} - \varphi(\frac{t}{N})\| < \delta\},\ \delta > 0, \tag{1.4.2}$$

where x, τ, δ are given and fixed. Event A_N means that at times $t = 0, 1, \cdots, [\tau N]$, the random walk ξ_t scaled by N takes values in δ-neighborhood of a continuous path

$$\varphi(s) : [0, \tau] \to \mathbf{R}.$$

Then the rate function $\mathcal{L}_\tau(\cdot)$ depends on φ. For homogeneous randoms on **Z**, it is proved in [6],[15] that

$$\mathcal{L}_\tau(\varphi) = \begin{cases} \int_0^\tau \sup_\alpha \{\alpha\varphi'(t) - H(\alpha)\} dt & \text{if } \varphi \text{ is absolutely continuous} \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4.3}$$

where $\varphi'(t)$ denotes the derivative of $\varphi(t)$. The function $\mathcal{L}_\tau(\cdot)$, which is defined for an arbitrary continuous path, is called *the global rate function*.

Note that there is an explicit expression for $\mathcal{L}_\tau(\cdot)$ similar to (1.4.3) for a homogeneous random walk on $\mathbf{Z}^n$.

The determination of the local rate function becomes complicated if a random walk is not homogeneous. However, for many models assuming certain conditions the LD problem has been solved (see [6],[13],[14],[15],[28]).

We continue with a simple example of a face-homogeneous random walk on the non-negative integers.

Example 1.4.1 Let us consider a random walk on $\mathbf{Z}_+$ with two homogeneity faces :

$$\mathbf{\Lambda}^0 = \{0\} \text{ and } \mathbf{\Lambda}^+ = \{1, 2, \cdots\}.$$

Thus we have two types of transition probabilities

$$p_{i,j} = \begin{cases} p_j^0 & \text{if } i = 0, \\ p_{j-i}^+ & \text{if } i > 0. \end{cases} \tag{1.4.4}$$

In this model the main difficulty is to define the rate function $\mathcal{L}_\tau(\cdot)$ at $\varphi \equiv 0$. We will call this path *the zero path*. In [15] and [28] it has been proved that

$$\mathcal{L}_\tau(\varphi \equiv 0) = -\tau \inf_\alpha \{H^0(\alpha) \vee H^+(\alpha)\},$$

where $H^{\mathbf{\Lambda}}(\alpha), \mathbf{\Lambda} = 0, +$, is *the cumulant generating function on the face* $\mathbf{\Lambda}$, defined as follows:

$$H^0(\alpha) \triangleq \log(\sum_{k \in \mathbf{Z}} p_k^0 \exp\{\alpha k\}) \text{ and } H^+(\alpha) \triangleq \log(\sum_{k \in \mathbf{Z}} p_k^+ \exp\{\alpha k\}). \tag{1.4.5}$$

We also would like to mention that if the random walk is recurrent (see definition 2.4) then $\mathcal{L}_\tau(0) = 0$. Hence, the only case of interest is the rate function $\mathcal{L}_\tau(\cdot)$ at $\varphi \equiv 0$ for the transient random walk.

Similarly to one dimensional case, [15] gives an expression for the rate function for a face-homogeneous random walk on $\mathbf{Z}_+ \times \mathbf{Z}^m$ with two homogeneity faces.

New results. LD for the quarter plane.

In [15] the LD problem has been also analysed for a face homogeneous random walk on the integer quarter plane lattice $\mathbf{Z}_+^2$ with four homogeneity faces $\mathbf{\Lambda}^0, \mathbf{\Lambda}^1, \mathbf{\Lambda}^2, \mathbf{\Lambda}^3$ (see figure 1.5). Here $\mathbf{\Lambda}^0$ is the origin (the point $(0,0)$), $\mathbf{\Lambda}^1$ is the horizontal boundary without the origin, $\mathbf{\Lambda}^2$ is the vertical boundary without the origin, $\mathbf{\Lambda}^3$ is the open quarter plane. A complete description of the model is given in section 5.2.

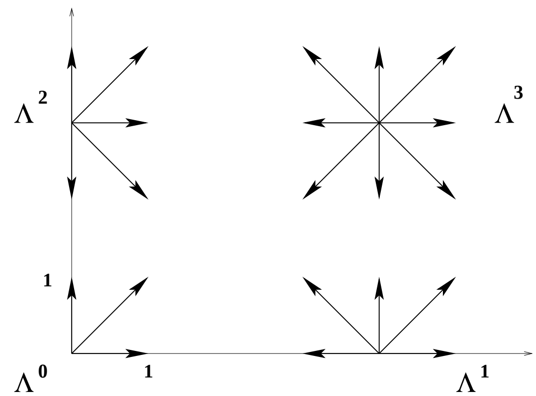

Figure 1.5: transition probabilities on the faces

In [15] the rate function for this model has been obtained for all absolutely continuous paths φ except the path $\varphi \equiv 0$ in the case when the random walk is transient. In Chapter 5 we solve this case under some conditions. The expression for $\mathcal{L}_\tau(0)$ is given in terms of four cumulant generating functions $H^0, H^1, H^2, H^3 : \mathbf{R}^2 \to \mathbf{R}$, that correspond to the homogeneity faces. To give the formula of $\mathcal{L}_\tau(0)$ we define two points $\alpha^1, \alpha^2 \in \mathbf{R}^2$ as follows

$$\alpha^1 = \arg\min_{\alpha \in \mathrm{R}^2} \{H^1(\alpha) \vee H^3(\alpha)\},$$
$$\alpha^2 = \arg\min_{\alpha \in \mathrm{R}^2} \{H^2(\alpha) \vee H^3(\alpha)\}.$$

In Chapter 5 we will prove that α^1 and α^2 are finite. Let

$$\hat{\alpha} = \arg\max\{H^3(\alpha^1), H^3(\alpha^2)\}.$$

In Chapter 5 we prove that under condition

$$H^{\mathbf{\Lambda}}(\hat{\alpha}) \leq \{H^3(\alpha^1) \vee H^3(\alpha^2)\} \text{ for any } \mathbf{\Lambda} = 0, 1, 2, \tag{1.4.6}$$

the local rate function (at $\varphi \equiv 0$) has the following expression

$$\mathcal{L}_\tau(0) = -\tau\{H^3(\alpha^1) \vee H^3(\alpha^2)\}.$$

We would like to mention that the same problem has been studied in [14], where the expression for $\mathcal{L}_\tau(0)$ is given in terms of the convergence parameter of associated local transform matrices. Using our approach it is easy to compute numerically the value $\mathcal{L}_\tau(0)$ for a given model.

In Chapter 6 we apply the results of Chapter 5 to a coupled processors system. This system is a mathematical model for two processors that handle two types of jobs. The arrival processes are supposed to be Markovian and the service times are assumed to be exponentially distributed. If there are jobs of both types then both processors proceed independently but if the buffer of one job-type is empty then the processor of the other job-type changes its service speed. This kind of coupled processors model allows for various specifications as: “the full capacity given to the nonempty buffer” and “switched off processors if one of the buffers is empty”. It turns out that for these special coupled processors models we can give closed-form expressions for $\mathcal{L}_\tau(0)$ in terms of the arrival and service rates. These expressions turn out to be surprisingly simple. As a by-product we show that the extra condition (1.4.6), is indeed not necessary in these specific models. Also we characterize the condition (1.4.6) for the general coupled processors model.

Chapter 2

Basic definitions and tools

In this Chapter we will give necessary notations concerning Markov chains, Large deviations and almost closed sets. We also collect some necessary results from the literature [3],[5],[6],[7],[28].

2.1 Markov chains

2.1.1 Definitions

In this subsection we will give the definitions of a Markov chain, irreducibility and aperiodicity.

Let $\boldsymbol{I}$ be a countable set and $(p_{i,j})$ be a stochastic matrix on $\boldsymbol{I} \times \boldsymbol{I}$.

Definition 2.1 *The pair* $\{I, (p_{i,j})\}$ *is called* a discrete-time Markov chain (MC).

Given $I, (p_{i,j})$ and a distribution μ_0 on $\boldsymbol{I}$, we define now the probability space $(\Omega, \Sigma, \mathsf{P}_{\mu_0})$. Let

Ω denote the path space, that is the set of all paths

$$\Omega = \{\omega = (\omega_0, \omega_1, \cdots) : \omega_n \in I, n \in \mathbf{Z}_+\};$$

Σ denote the standard σ-algebra generated by cylindric sets

$$\{i_0, \cdots, i_n\} \triangleq \{\omega \in \Omega : \omega_0 = i_0, \cdots, \omega_n = i_n\}, n \in \mathbf{Z}_+;$$

P_{μ_0} denote the probability measure on (Ω, Σ) such that for any cylindric set $\{i_0, \cdots, i_n\}$ we have

$$\mathsf{P}_{\mu_0}\{i_0, \cdots, i_n\} \triangleq \mu_0(i_0) \times p_{i_0,i_1} \times p_{i_1,i_2} \times \cdots \times p_{i_{n-1},i_n}.$$

Definition 2.2

A sequence of random variables $\{\xi_n, n \in \mathbf{Z}_+\}$ *defined on the probability space* $(\Omega, \Sigma, \mathsf{P}_{\mu_0})$ *by*

$$\xi_n(\omega) = \omega_n \text{ for any } \omega \in \Omega \text{ and } n \in \mathbf{Z}_+.$$

is called a discrete-time Markov chain with initial distribution μ_0.

When $\mu_0(i) = 1$ for a given state $i \in I$, we will simply write P_i and E_i for the the corresponding probability and expectation operators.

Remark 2.1.1 *We would like to say that in this thesis a random walk is a Markov chain. Note that we defined a random walk on a subset of* $\mathbf{Z}^p$*. A discrete-time Markov chain is more general object, which is defined here on an arbitrary countable state space.*

The random variable ξ_n is called *the state of the the Markov chain at time* $n = 0, 1, \cdots$.

Definition 2.3 *A MC* $\{I, (p_{i,j})\}$ *is called* irreducible *if for any* $i, j \in I$ *there exists a finite sequence of states* $i_1, \cdots, i_k$ *such that*

$$p_{i,i_1} \times p_{i_1,i_2} \times \cdots \times p_{i_k,j} > 0.$$

By $p_{i,j}^{(n)}$ denote the n-step transition probability, i.e.

$$p_{i,j}^{(n)} = \mathsf{P}\{\xi_n = j | \xi_0 = i\} = \sum_{(i_1,\cdots,i_{n-1})} p_{i,i_1} \times p_{i_1,i_2} \times \cdots \times p_{i_{n-1},j}.$$

Next choose any $i \in I$. Let $n_1(i) < n_2(i) < \cdots$ be the positive integers for which $p_{i,i}^{n_k(i)} > 0$, and let $d(i)$ denote its greatest common divisor.

Theorem 2.1 *For an irreducible Markov chain the number* $d(i), i \in I$, *does not depend on* i *and therefore it is called* the period of the chain. *The chain is called* aperiodic *if* $d = 1$.

Throughout our analysis in all models we will assume the following condition to hold.

Assumption 2.1.1 *The Markov chain is irreducible and aperiodic.*

2.1.2 Ergodicity, null recurrence and transience.

In this subsection we will recall classification of a Markov chain $\{I, (p_{i,j})\}$ in terms of ergodicity, null recurrence and transience. We will also give sufficient conditions for a MC to be ergodic, transient or null recurrent.

Let

$$f_n(i,j) \triangleq \mathsf{P}\{\omega : \xi_k(\omega) \neq j, 0 < k < n; \xi_n(\omega) = j \,|\, \xi_0 = i\}$$

denote the probability that the MC starting at i reaches j for the first time at time $n \geq 1$. Then

$$f_{i,j} \triangleq \sum_{n=1}^{+\infty} f_n(i,j)$$

is the probability that the MC starting at i visits j at all. Clearly, $f_{i,j} \leq 1$ for any $i, j \in I$. Additionally, define

$$g_{i,j} \triangleq \sum_{n=1}^{+\infty} n f_n(i,j).$$

It is the mean time to reach j for the first time, when starting at i.

Definition 2.4 *An irreducible aperiodic MC is called*

1. recurrent *if* $f_{i,j} = 1$*, at least for one pair* (i, j)*;*

2. transient *if* $f_{i,j} < 1$ *for some pair* (i, j)*;*

3. ergodic *if* $g_{i,j} < +\infty$*, at least for one pair* (i, j)*;*

4. null recurrent *if* $f_{i,j} = 1$ *and* $g_{i,j} = +\infty$ *for at least for one pair* (i, j)*;*

5. non-ergodic *if* $g_{i,j} = +\infty$ *for least for one pair* (i, j).

Intuitively it is reasonable to expect that only in the ergodic case the long run average proportion of time spent in each point is positive. In the case of transience, the Markov chain *escapes* from a compact subset of the states and in the case of null recurrent the Markov chain will return to each compact set infinitely often, but this occurs so rarely that the long mean fraction of time spent in such set is negligible. The following stronger result is true.

An initial distribution p_0 on $\boldsymbol{I}$ is called *stationary distribution* if the position of the Markov chain at any time t has distribution p_0, and then the Markov chain itself is called *stationary.* Consider the equation

$$\pi_j = \sum_{i \in I} \pi_i p_{i,j}, \;\; j \in I, \tag{2.1.1}$$

where $\pi = (\pi_i \geq 0, i \in I)$ is unknown vector.

Theorem 2.2 *For any* $i, j \in I$ *the limits*

$$\pi_j = \lim_{n \to \infty} p_{i,j}^{(n)}$$

exist and are independent of i*. Moreover, they are non-negative solutions of (2.1.1) with* $\sum_j \pi_i \leq 1$.

The MC is ergodic iff there exists a positive solution of (2.1.1), up to a multiplicative constant. The normalized solution $\sum_j \pi_j = 1$ *is a stationary distribution for the Markov chain with positive mass in each point and it is the unique. In particular,*

$$\pi_i = \frac{1}{g_{i,i}}, \; i \in I.$$

□

The following lemma shows the relation between the mean drift (1.2.2) and a solution of (2.1.1).

Lemma 2.1.1 *Let π_i be a solution of (2.1.1). Then $\sum_{i\in I}\pi_i m(i)=0$.*

Proof. Taking (1.2.2) and (2.1.1) in account we get

$$\sum_i \pi_i m(i) = \sum_i \pi_i \sum_j (j-i)p_{i,j} = \sum_i \pi_i m(i) = \sum_i \pi_i \sum_j (j-i)p_{i,j} =$$
$$= \sum_j j \sum_i \pi_i p_{i,j} - \sum_i \pi_i \sum_j i p_{i,j} = \sum_j j\pi_j - \sum_i i\pi_i = 0.$$

□

In the following, we recall well-known Lyapunov-Foster criterion for ergodicity and sufficient condition for transience (see Theorems 2.2.4 and 2.2.7 of [7]).

Theorem 2.3 *The Markov chain $\{\xi_n\}$ is ergodic if and only if there exist positive functions $f:\boldsymbol{I}\to\mathbf{R}_+$, $k:\boldsymbol{I}\to\mathbf{Z}_+$, a constant $\epsilon>0$ and a finite set $A\subset\boldsymbol{I}$, such that*

$$\mathsf{E}\{f(\xi_{n+k(\xi_n)})\,|\,\xi_n=i\}\begin{cases}\le f(i)-\epsilon k(i), & i\notin A,\\ <\infty, & i\in A.\end{cases}$$

□

Theorem 2.4 *The Markov chain $\{\xi_n\}$ is transient if there exist a positive function $f(i), i\in\boldsymbol{I}$, a* bounded *function $k:\boldsymbol{I}\to\mathbf{Z}_+$, constants $\epsilon, C', d>0$, such that*

$$\mathsf{E}\{f(\xi_{n+k(\xi_n)})\,|\,\xi_n=i\}\ge f(i)+\epsilon,\quad i\in\{j\,|\,f(j)>C'\}\ne\emptyset,$$

and $|f(i)-f(j)|>d$ implies $\mathsf{P}\{\xi_{n+1}=j\,|\,\xi_n=i\}=0$.

□

The functions $f(\cdot)$ mentioned in these theorems are usually called *Lyapunov functions*. Next theorem is a simple case of theorem 2.1.8 in [7].

Theorem 2.5 *The Markov chain $\{\xi_n\}$ is null recurrent if there exist a non-negative function $f(i), i\in\boldsymbol{I}$, and a finite subset $A\subset\boldsymbol{I}$ such that the following conditions hold:*

1. *for some $\alpha\in(1;2]$*

$$\sup_{i\in\boldsymbol{I}}\mathsf{E}\{|f(\xi_{n+1})-f(\xi_n)|^\alpha\,|\,\xi_n=i\}=C<\infty;$$

2. $sup_{i\notin A}f(i)>sup_{i\in A}f(i)$ *and* $\lim_{i\to\infty}f(i)=\infty$*;*

3.

$$\mathsf{E}\{f(\xi_{n+1})-f(\xi_n)\,|\,\xi_n=i\}=0 \text{ for all } i\notin A.$$

□

2.2 Martingale bounds

This section reviews some known results on convergence and bounds for martingales.

Recall that a sequence of random variables ξ_n *converges in probability* to the random variable ξ if for any $\epsilon > 0$, the probability $\mathsf{P}\{\|\xi_n - \xi\| < \epsilon\}$ approaches one as n approaches infinity. Almost surely convergence asserts a bit more. A sequence of random variables ξ_n *converges almost surely* to the random variable ξ if $\mathsf{P}\{\omega : \xi_n(\omega) \to \xi(\omega)\} = 1$, or in other words, for any $\epsilon > 0$, $\|\xi_n - \xi\| > \epsilon$ only a finite number of times with probability 1. We shall give a sufficient condition for a sequence of random variables to converge almost surely.

Lemma 2.2.1 *If $\sum_{n=1}^{\infty} \mathsf{P}\{\|\xi_n - \xi\| > \frac{1}{r}\}$ converges for any positive integer r, then the sequence ξ_n converges almost surely to the random variable ξ.*

Many convergence results of sums of i.i.d. random variables extend to convergence for martingales. We recall the definition of a martingale.

Definition 2.5 *Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and let $\mathcal{F}_n, n = 0, 1, \cdots$ be an increasing set of sub-σ-algebras of $\mathcal{F}$. By $\mathcal{F}_\infty$ we denote the σ-field generated by $\mathcal{F}_n, n = 0, 1, \cdots$. Assume $\{M_n\}$ to be a sequence of random variables on Ω satisfying*

i) *M_n is measurable with respect to $\mathcal{F}_n$;*

ii) *summability: $\mathsf{E}\|M_n\| < \infty$.*

The sequence $\{M_n, \mathcal{F}_n\}$ is called a martingale if in addition, $\mathsf{E}\{M_{n+1} \,|\, \mathcal{F}_n\} = M_n$ for any $n \geq 0$.

Theorem 2.6 *(Kolmogorov's inequality for martingales)*

If $\{M_l\}$ is a martingale then for each $p \geq 1$ and $a > 0$

$$\mathsf{P}\{\max_{l \leq n} \|M_l\| > a\} \leq \frac{\mathsf{E}\|M_n\|^p}{a^p}.$$

Lemma 2.2.2 *(Azuma-Hoeffding inequality, [29] E14.2 p.237)*

Let $\{M_l\}$ be a martingale, the increments $M_l - M_{l-1}$ (with $M_0 = 0$) of which are bounded in absolute value by c_l. Then for any $x > 0$ we have

$$\mathsf{P}\{\sup_{1 \leq l \leq n} M_n \geq x\} \leq \exp\{-\frac{x^2}{2\sum_{l=1}^{n} c_l^2}\}. \tag{2.2.1}$$

2.3 Almost closed sets and invariant σ-algebra

We recall definitions and results from Chung's exposition [5], §I.17.

Let $\mathcal{A} \subset \Sigma$ be any Borel sub-σ-algebra of sets. A set $C \in \mathcal{A}$ is called *atomic* with respect to $\mathcal{A}$ if $\mathsf{P}_{\mu_0}(C) > 0$ and C does not contain two disjoint subsets in $\mathcal{A}$ with positive probability. It is called *completely non-atomic*, if $\mathsf{P}_{\mu_0}(C) > 0$ and it does not contain any atomic subsets (in $\mathcal{A}$). In the latter case for each given $c, 0 < c < \mathsf{P}_{\mu_0}\{C\}$, there exists a subset M in $\mathcal{A}$ of C such that $\mathsf{P}_{\mu_0}\{M\} = c$.

The following lemma is well-known (cf. [2], [5]).

Lemma 2.3.1 *The path space Ω can be represented by means of a denumerable collection of disjoint sets belonging to $\mathcal{A}$:*

$$\Omega = \cup_{n=0} C_n,$$

where some of the C_n may be absent. If present, then C_0 is completely non-atomic, and C_n, $n \geq 1$, are atomic. This decomposition is unique modulo sets of zero measure. Hence $\sum_n \mathsf{P}_{\mu_0}\{C_n\} = 1$.

We will call $\mathcal{A}$ simple if Ω is atomic w.r.t. $\mathcal{A}$, i.e. $C_1 = \Omega$. Note that $\mathsf{P}_{\mu_0}(C)$ being positive or zero does *not* depend on the initial measure μ_0.

Bearing in mind our interest in the long run behavior of the Markov chain, we will consider the sub-σ-algebra of invariant sets. To this end, introduce the time shift $\mathcal{T}$ on Ω:

$$\mathcal{T}(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots).$$

A set $C \in \Sigma$ is called *invariant* whenever $\mathcal{T}^{-1}C = C$. The class of invariant sets is a sub-σ-algebra of Σ denoted by $\mathcal{G}$.

One can find the decomposition of Ω w.r.t $\mathcal{G}$ through a decomposition of the state space $\boldsymbol{I}$ into so-called almost closed sets. Let $A \subset \boldsymbol{I}$ and define two corresponding invariant sets by

$$\begin{aligned}
\underline{\mathcal{L}}(A) &\triangleq \liminf_{n\to\infty}\{\xi_n \in A\} = \cup_{m\geq 0} \cap_{t\geq m} \{\omega \mid \xi_t(\omega) \in A\} \\
\overline{\mathcal{L}}(A) &\triangleq \limsup_{n\to\infty}\{\xi_n \in A\} = \cap_{m\geq 0} \cup_{t\geq m} \{\omega \mid \xi_t(\omega) \in A\}.
\end{aligned}$$

Thus, we say that $\omega \in \underline{\mathcal{L}}(A)$ iff $\xi_t(\omega) \in A$ eventually and $\omega \in \overline{\mathcal{L}}(A)$ iff $\xi_t(\omega) \in A$ infinitely often. We will write sometimes $\{\xi_t \in A \text{ i.o.}\}$ instead of $\overline{\mathcal{L}}(A)$. Note also that

$$\underline{\mathcal{L}}(A) \subset \overline{\mathcal{L}}(A) \;\text{ and }\; \underline{\mathcal{L}}(A^c) = \left(\overline{\mathcal{L}}(A)\right)^c,$$

where we use the symbol ^{c} to denote complementation both in I and $\mathcal{G}$.

We call a set A *transient* if $\mathsf{P}_{\mu_0}\{\overline{\mathcal{L}}(A)\} = 0$. The set A is called *almost closed* if it is not transient and

$$\mathsf{P}_{\mu_0}\{\underline{\mathcal{L}}(A)\} = \mathsf{P}_{\mu_0}\{\overline{\mathcal{L}}(A)\}\;(>0). \tag{2.3.1}$$

By $\mathcal{A}$ denote the class of almost closed sets and transient sets. It is an algebra of sets and transient sets are an ideal of this algebra. The following theorem exhibits (see [5], §I.17) the relation between $\mathcal{G}$ and $\mathcal{A}$.

Theorem 2.7 *To each invariant set $C \in \mathcal{G}$ there corresponds a transient or almost closed set $A \in \mathcal{A}$, unique up to transient sets, such that*

$$C = \overline{\mathcal{L}}(A) = \underline{\mathcal{L}}(A),$$

according as C is null or not. The correspondence is an isomorphism of algebras. In particular, one can choose the set A by setting

$$A = \big\{ i \in \boldsymbol{I} \mid \mathsf{P}_i\{C\} > a \big\},$$

for $0 < a < 1$ arbitrary.

Note that for any $A \in \mathcal{A}$ and any initial state i one has

$$\mathsf{P}_i\{\underline{\mathcal{L}}(A)\} = \mathsf{P}_i\{\overline{\mathcal{L}}(A)\} = \lim_{n\to\infty} \mathsf{P}_i\{\xi_n \in A\}. \tag{2.3.2}$$

The above correspondence with invariant sets, motivates calling an almost closed set $A \in \mathcal{A}$ *atomic* w.r.t. $\mathcal{A}$, if it does not contain two disjoint almost closed sets, and *completely non-atomic*, if it does not contain any atomic subset. As a consequence of Lemma 2.3.1 and Theorem 2.7 the state space $\boldsymbol{I}$ can be partitioned into a denumerable set of disjoint almost closed sets

$$\boldsymbol{I} = \cup_{n=0}^{\infty} A_n, \tag{2.3.3}$$

where A_0 (if present) is completely non-atomic and A_i, $i \geq 1$, is atomic (if present). The decomposition is unique modulo transient sets. Moreover, one has

$$1 = \sum_{n=0}^{\infty} \mathsf{P}_i\{\underline{\mathcal{L}}(A_n)\} = \sum_{n=0}^{\infty} \mathsf{P}_i\{\overline{\mathcal{L}}(A_n)\} = \sum_{n=0}^{\infty} \lim_{t\to\infty} \mathsf{P}_i\{\xi_t \in A_n\}. \tag{2.3.4}$$

The Markov chain (the random walk) is called *simple* whenever the state space consisting of a single atomic almost closed set. The following lemma follows from theorem 3 in [3].

Lemma 2.3.2 *The state space of a homogeneous random walk is simple.*

Example 2.3.1 *(Non-atomic state space, see [3] or [5], §I.17)*
Consider the random walk

$$\xi_t = \frac{1}{2} + \sum_{i=1}^{t} \frac{x_i}{2^{i+1}},$$

where the random variables $x_1, x_2, \cdots$, are independent and assume the values -1 and 1 with probability $1/2$. The state space is the set of numbers of the form numbers $m2^{-n}$ with $m = 1, 3, \cdots 2^n - 1, n \geq 1$. It is easy to see that the state space does not contain any atomic almost closed set.

Finally we would like to emphasize the relationship between the invariant σ-algebra $\mathcal{G}$ and the class of the bounded harmonic functions.

Theorem 2.8 ([5] §I.17 Theorem 5) *To every bounded harmonic function $h : \boldsymbol{I} \to \mathbf{R}$, i.e. $h(i) = \sum_{j\in\boldsymbol{I}} p_{i,j} h(j)$, there corresponds a $\mathcal{G}$-measurable random variable ζ such that*

$$h(\zeta_n) = \mathsf{E}\big\{\zeta \mid \zeta_n\big\},\ a.s.$$

and conversely, where $\zeta = \{\zeta_0, \zeta_1, \ldots\}$.

Corollary 2.3.1 *The invariant algebra (or the state space) is simple if and only if all the bounded harmonic functions are constant.*

For aperiodic and irreducible chains, this immediately shows that the process is simple whenever the chain is recurrent (see §10.13 in [29]).

2.4 Large deviations principle for a random walk on $\mathbf{Z}_+^2$

In this section we will give the definition of LD principle for a random walk on $\mathbf{Z}_+^2$ in the form as it was given in [15]. Moreover, we collect a useful theorem from [15].

Let $\{\xi_n\}$ be a random walk on $\mathbf{Z}_+^2$ with bounded jumps and four homogeneity faces:

$$\mathbf{\Lambda}_0 = \{(0,0) \in \mathbf{Z}^2\}, \qquad \mathbf{\Lambda}_3 = \{i \in \mathbf{Z}^2 \mid i_1 > 0, i_2 > 0\},$$
$$\mathbf{\Lambda}_1 = \{i \in \mathbf{Z}^2 \mid i_1 > 0, i_2 = 0\}, \qquad \mathbf{\Lambda}_2 = \{i \in \mathbf{Z}^2 \mid i_1 = 0, i_2 > 0\}.$$

For any $\tau > 0$ consider the set $C\big([0;\tau], \mathbf{R}_+^2\big)$ of all continuous functions $\varphi : [0;\tau] \to \mathbf{R}_+^2$ with the metric

$$d(\varphi_1, \varphi_2) \triangleq \sup_{0 \le t \le \tau} \|\varphi_1(t) - \varphi_2(t)\|.$$

Definition 2.6 *A function $\mathcal{L}_\tau, \tau > 0$, mapping the space $C\big([0;\tau], \mathbf{R}_+^2\big)$ into $\mathbf{R} \cup \{+\infty\}$ is called rate function if*

(i) $\mathcal{L}_\tau(\varphi) \ge 0$;

(ii) *$\mathcal{L}_\tau(\cdot)$ is lower semicontinuous, i.e. the set $\{\varphi : \mathcal{L}_\tau(\varphi) \le a\}$ is compact for any real a.*

Consider for any $\tau, s \ge 0$ and $x \in \mathbf{R}_+^2$ the set

$$\Phi_{x,\tau}(s) = \{\varphi \in C\big([0;\tau], \mathbf{R}_+^2\big) : \varphi(0) = x \text{ and } \mathcal{L}_\tau(\varphi) \le s\}.$$

Definition 2.7 *We say that the random walk $\{\xi_n\}$ satisfies the LD principle with rate function $\mathcal{L}_\tau$ if for any $\tau \ge 0$ and $x \in \mathbf{R}_+^2$ the following three conditions hold :*

(a) *(compactness) for any $s \ge 0$ the set $\Phi_{x,\tau}(s)$ is compact ;*

(b) *(upper bound)*

for any $\delta > 0, s_0 > 0, \delta' > 0$ there exists N_0 such that for any $N > N_0, 0 < s < s_0$ the following estimate holds

$$\mathsf{P}\big\{\xi_0 = [xN], \sup_{t=0,\cdots,[\tau N]} \|\frac{\xi_t}{N} - \varphi(\frac{t}{N})\| \ge \delta \text{ for any } \varphi \in \Phi_{x,\tau}(s)\big\} \le \exp\{\delta' N - Ns\};$$

(c) *(lower bound)*

for any $\delta > 0, s_0 > 0, \delta' > 0$ there exists N_0 such that for any $N > N_0, \varphi \in \Phi_{x,\tau}(s_0)$ the following estimate holds

$$\mathsf{P}\big\{\xi_0 = [xN], \sup_{t=0,\cdots,[\tau N]} \|\frac{\xi_t}{N} - \varphi(\frac{t}{N})\| < \delta\big\} \ge \exp\{-\delta' N - N\mathcal{L}_\tau(\varphi)\}.$$

Now we need some notations to formulate theorem 2.9. By $v^{\mathbf{\Lambda}}$ denote the projection of $v \in \mathbf{R}^2$ on face $\mathbf{\Lambda}$. Let a function

$$L(\cdot,\cdot) : \mathbf{R}_+^2 \times \mathbf{R}^2 \to \mathbf{R}_+ \cup \{+\infty\}$$

be given such that

$$L(x,v) = L(y,w) \text{ iff } x,y \in \mathbf{\Lambda} \text{ and } v^{\mathbf{\Lambda}} = w^{\mathbf{\Lambda}} \text{ for some face } \mathbf{\Lambda}.$$

For an absolutely continuous path $\varphi : [0,\tau] \to \mathbf{R}_+^2$, we denote its velocity vector by $\varphi'(t)$. For any $\tau > 0$, we define the function

$$\mathcal{L}_\tau(\varphi) = \begin{cases} \int_0^\tau L\big(\varphi(t), \varphi'(t)\big)dt & \text{if the path } \varphi \text{ is absolutely continuous,} \\ +\infty & \text{otherwise .} \end{cases} \tag{2.4.1}$$

By $\Phi_{x,\tau}$ we denote the set of all continuous paths $\varphi : [0,\tau] \to \mathbf{R}_+^2, \varphi(0) = x$, such that $\mathcal{L}_\tau(\varphi) < \infty$. The following theorem has been proved in [15] (see theorem 3.1.1).

Theorem 2.9 *(LD theorem)*

Let for the random walk $\{\xi_n\}$ the functions $\mathcal{L}_\tau, \tau > 0$, (as defined in (2.4.1)) satisfy the following conditions:

(i) *$\mathcal{L}_\tau$ are lower semicontinuous;*

(ii) *for any $\delta' > 0$ one can find $\delta > 0, \sigma > 0$ such that for any $x, z \in \mathbf{R}_+^2$ with $|x - z| < \sigma$ and for any* **linear** *path $\varphi \in \Phi_{x,\tau}$ the following estimate holds*

$$\mathsf{P}\big\{\xi_0 = [zN], \sup_{t=0,\cdots,[\tau N]} \|\frac{\xi_t}{N} - \varphi(\frac{t}{N})\| < \delta\big\} \le \exp\{+\delta' N - N\mathcal{L}_\tau(\varphi)\}$$

for all sufficiently large N;

(iii) *for any $\delta > 0, \delta' > 0$ one can find $\sigma > 0$ such that for any $x, z \in \mathbf{R}_+^2$ with $|x - z| < \sigma$ and for any* **linear** *path $\varphi \in \Phi_{x,\tau}$ the following estimate holds*

$$\mathsf{P}\big\{\xi_0 = [zN], \sup_{t=0,\cdots,[\tau N]} \|\frac{\xi_t}{N} - \varphi(\frac{t}{N})\| < \delta\big\} \ge \exp\{-\delta' N - N\mathcal{L}_\tau(\varphi)\}$$

for all sufficiently large N;

(iv) *for any $x \in \mathbf{R}_+^2$, $\varphi \in \Phi_{w,\tau}$ and for any $\epsilon > 0$ there exists a piecewise linear path $\tilde{\varphi} \in \Phi_{x,\tau}$ such that*

$$\sup_{0 \le t \le \tau} \|\varphi(t) - \tilde{\varphi}(t)\| < \epsilon \text{ and } |\mathcal{L}_\tau(\varphi) - \mathcal{L}_\tau(\tilde{\varphi})| < \epsilon. \tag{2.4.2}$$

Then the random walk $\{\xi_n\}$ satisfies LD principle with the rate function $\mathcal{L}_\tau$.

Remark 2.4.1 *In all the models that we will consider the condition (i) and (iv) of theorem 2.9 can be easily derived from the properties of $L(x,v)$ by standard methods (see for example [16]).*

Chapter 3

Discrete scattering

This Chapter is a slight modification of the paper A. Hordijk, N. Popov, F. M. Spieksma (2002). *Discrete scattering and almost closed sets for simple face-homogeneous random walks.* Technical Report 2002-26, Leiden University.

3.1 Introduction

In this Chapter we would like to highlight the relation between almost closed sets and scattering properties of face-homogeneous random walks over its sets of fluid (Euler) paths.

Let be given a discrete time, irreducible Markov chain $\{\xi_n\}_{n=0,1,\ldots}$, on a countable state space $\boldsymbol{I} \subset \mathbf{Z}^p$ with stationary transition probabilities

$$p_{i,j} = \mathsf{P}\{\xi_{n+1} = j \,|\, \xi_n = i\}.$$

For unravelling the transient behavior of this Markov chain, it seems of interest to study the *almost closed sets* (see section 2.3). Blackwell (cf. [3]) has used the almost closed set structure for determining the structure of the invariant σ-algebra. A consequence is a characterization of the Poisson boundary of the Markov chain (cf. [17]), which (essentially) is the probability space restricted to the sub-σ-algebra of invariant sets with induced measure.

Here we would like to highlight the relation between almost closed sets and scattering properties of face-homogeneous random walks over its set of fluid (Euler) paths. We will focus on the following aspect of transient face-homogeneous random walks. Consider the time-space scaled process

$$\frac{\xi_{[tN]}}{N} \text{ as } N \to \infty,$$

where the initial position ξ_0 is given and fixed. Assume that it has a limit (in distribution), $u(t)$ say. This limit may be stochastic. Recall that any realization of $u(\cdot)$ is called a fluid limit or Euler path. Since $\boldsymbol{I} \subset \mathbf{Z}^p$, we have that $u(t) \in \mathbf{R}^p, t \geq 0$. As a natural extension of a set $A \subset \mathbf{Z}^p$ to $\mathbf{R}^p$ we will take for instance the convex hull A^h of A in $\mathbf{R}^p$.

Let $\{A_i\}_{i=0}^{\infty}$ be the almost closed sets from the decomposition (2.3.3). Our conjecture has the following form (modulo technical conditions).

Conjecture 3.1.1 *Assume that there exist finitely many almost closed sets and no non-atomic ones. Then to each almost closed set A corresponds precisely one path $u(\cdot)$, such that $u(t) \in A^h$, for $t \geq T$, for some finite time T. Moreover, for any fixed $i \in \boldsymbol{I}$*

$$\mathsf{P}\Big\{ \lim_{N\to\infty} \frac{\xi_{[tN]}}{N} = u(t) \,|\, \xi_0 = i \Big\} = \mathsf{P}\Big\{ \liminf_{N\to\infty} \{\xi_N \in A\} \,|\, \xi_0 = i \Big\}, \tag{3.1.1}$$

that is, the probability of selecting a given path is given by the absorption probability of the corresponding almost closed set. These probabilities are called the scattering probabilities.

We will call this 'discrete scattering'. In case of an uncountably infinite number of Euler paths starting at a given point, we have 'continuous scattering'. Chapter 4 and paper [24] discuss an example of continuous scattering.

The chapter's aim is to bring into the limelight a connection between the invariant set structure (i.e. the bounded harmonic functions) and Euler paths. This relation is implicit in for instance the computation of the Poisson boundary by Kurkova [18] for face-homogeneous random walks in mainly dimension 2. It is therefore not surprising, that the proof techniques used in [18] seem often to be similar to the ones used here.

In order to present the main idea of discrete scattering, we discuss some simple examples with both simple and non-simple decomposition, using proof techniques that seem to be applicable to more general cases. These proof techniques are based on the existence of well-behaved Lyapunov functions.

The basic definitions and results from [3] and [5] concerning almost closed sets are given in section 2.3. In section 3.2 we will derive a number of tools that seem basic to us for studying the almost closed set structure and Euler limit behavior for face-homogeneous random walks in general. These use certain techniques from Feller [8] and Lyapunov function techniques from [7].

As an application, section 3.3 will study face-homogeneous random walks on the integer line.

In section 3.4 we consider a face-homogeneous random walk on $\mathbf{Z}_+ \times \mathbf{Z}$. We will use a martingale limit theorem to show that the fluid limit exists almost surely and it is equal to so-called 'second vector field'.

Finally section 3.5 aims to show validity of the conjecture for two versions of the models studied in Chapter 6. The first is the "coupled processors system with switched off processors whenever a queue is empty". The second is the same model with additional input whenever a queue is empty. To be clear, for all models in Chapter 6 the conjecture can be shown to hold. However, the derivations are analogous to the one presented here and we prefer to focus on the two versions mentioned.

3.2 Tools

Sojourn sets and well-behaved Lyapunov functions

For characterizing almost closed sets, and consequently invariant sets, the concept 'sojourn set' (cf. [8], [5]) seems the more manageable one.

The set S of states is called a *sojourn set* iff $\mathsf{P}_i\{\underline{\mathcal{L}}(S)\} > 0$ for some $i \in \boldsymbol{I}$. In this case, for every a, $0 < a < 1$, define the set

$$S^{(a)} = \{i \in \boldsymbol{I} \,|\, \mathsf{P}_i\{\underline{\mathcal{L}}(S)\} > a\}.$$

The following theorem (cf. [5], §I.17) holds.

Theorem 3.1 *If S is a sojourn set, then for every a, $0 < a < 1$, the set $S \cap S^{(a)}$ is almost closed, $S^{(a)} - S \cap S^{(a)}$ is transient and*

$$\mathsf{P}_i\{\underline{\mathcal{L}}(S)\} = \lim_{n\to\infty} \mathsf{P}_i\{\xi_n \in S \cap S^{(a)}\}. \tag{3.2.1}$$

In essence, this reduces the problem of the almost closed set structure by the more easier one of determining the sojourn set structure. By means of the following lemma (cf. [7], [25]) certain sets attracting probability mass can be identified as sojourn sets. For convenience we will give the proof here, since it does not occur in explicit form in these references. The proof itself is a standard construction of transforming additive Lyapunov functions into multiplicative ones, as will be used repeatedly.

Lemma 3.2.1 *Suppose there exist a function $f : \boldsymbol{I} \to \mathbf{R}$, a set $B \subset \boldsymbol{I}$, a step function $k : \boldsymbol{I} \to \mathbf{Z}_+$ and constants $d, \epsilon > 0$, $C \geq 0$, such that*

i) *the f-jumps are bounded by d, i.e. $|f(\xi_{n+1}) - f(\xi_n)| \leq d$, a.s. for any $n \geq 0$;*

ii) *the step function is uniformly bounded, i.e. $\sup_i k(i) < \infty$;*

iii) *$B \subset \{i \,|\, f(i) \leq C\} \neq \boldsymbol{I}$;*

iv) *the f-drift outside B is strictly positive, i.e.*

$$\mathsf{E}\{f\big(\xi_{n+k(\xi_n)}\big) - f(\xi_n) \,|\, \xi_n = i\} \geq \epsilon, \quad i \in \boldsymbol{I} \setminus B. \tag{3.2.2}$$

Then the set $\{i \,|\, f(i) > C + c\}$ is almost closed for any $c \geq 0$.

Proof. Denote $B' = \{i \,|\, f(i) \leq C\}$, then $B' \supset B$. Denote the entrance time of B' by T, i.e.

$$T = \begin{cases} n & \text{if } n = \inf\{l \geq 0 \,|\, \xi_{l-1} \notin B', \xi_l \in B'\} \\ +\infty & \text{if } \xi_n \notin B \text{ for all } n. \end{cases}$$

First assume that $k(i) \equiv 1$ and note that $\exp\{y\} < 1 + y + 3y^2/2$, whenever $|y| < 1$. Using the drift condition (3.2.2), we have for any sufficiently small constant $h > 0$ and $i \notin B'$ that

$$\begin{aligned} &\mathsf{E}\big\{\exp\{-h\big(f(\xi_{n+1}) - f(\xi_n)\big)\} \,|\, \xi_n = i\big\} \leq \\ &\quad\leq \mathsf{E}\Big\{1 - h\big(f(\xi_{n+1}) - f(\xi_n)\big) + \frac{3h^2\big(f(\xi_{n+1}) - f(\xi_n)\big)^2}{2} \,|\, \xi_n = i\Big\} \\ &\quad\leq 1 - h\epsilon + \frac{3(hd)^2}{2} \leq \exp\{-\gamma\}, \end{aligned}$$

for some positive constant $\gamma > 0$. Hence, denoting $g(i) = \exp\{-hf(i)\}$, and iterating, we have for $\xi_0 = i \notin B'$

$$\mathsf{E}_i\{g(\xi_n)\mathbf{1}_{\{T>n-1\}}\} \le g(i)\exp\{-\gamma n\}, \quad t > 0. \tag{3.2.3}$$

Since $\{T > n-1\} \supset \{T = n\}$ and $g \ge 0$, it follows that

$$\mathsf{E}_i\{g(\xi_n)\mathbf{1}_{\{T=n\}}\} \le g(i)\exp\{-\gamma n\}, \quad n > 0.$$

For $j \in B'$ we have $g(j) \ge \exp\{-hC\}$, so that

$$\mathsf{P}_i\{T = n\}\exp\{-hC\} \le g(i)\exp\{-\gamma n\}, \quad n > 0.$$

Multiplying both sides by $\exp\{hC\}$ and taking the summation over $n \ge 1$, we find that

$$\mathsf{P}_i\{T < \infty\} \le \frac{\exp\{hC - hf(i)\}}{1 - \exp\{-\gamma\}}.$$

Consequently, we get that for any $i \notin B'$

$$\mathsf{P}_i\{T = \infty\} = 1 - \mathsf{P}_i\{T < \infty\} \ge 1 - \frac{\exp\{hC - hf(i)\}}{1 - \exp\{-\gamma\}}$$

By the positive drift condition, necessarily the set $\{i \notin B'\}$ is infinite and

$$\sup_{i \notin B'} f(i) = \infty.$$

This implies $\{i \in \boldsymbol{I} \mid f(i) > c\} \neq \emptyset$, for any constant c. Let $C' > C$, such that

$$\alpha = 1 - \frac{\exp\{h(C - C')\}}{1 - \exp\{-\gamma\}} > 0.$$

For i with $f(i) > C'$, we have for $A = \{i \in \boldsymbol{I} \mid f(i) > C\}$ that

$$\mathsf{P}_i\{\underline{\mathcal{L}}(A)\} \ge \mathsf{P}_i\{f(\xi_n) > C \text{ for all } n > 0\} = \mathsf{P}_i\{T = \infty\} \ge \alpha. \tag{3.2.4}$$

By irreducibility, it follows that A is a sojourn set.

Next we would like to show almost closedness. This will follow from Theorem 3.1, if we can show that $A = A \cap A^{(a)}$ for some $a > 0$. By condition (i) and (iv), for any $i \in A$, the probability of reaching the set $\{j \in \boldsymbol{I} \mid f(j) \ge f(i) + \epsilon\}$ after the next jump is at least ϵ/d. Hence, for any $i \in A$,

$$\mathsf{P}_i\{f(\xi_k) > C'\} \ge \left(\frac{\epsilon}{d}\right)^k, \text{ with } k = \left\lceil \frac{C' - C}{\epsilon} \right\rceil.$$

This yields for any $i \in A$ that

$$\mathsf{P}_i\{\underline{\mathcal{L}}(A)\} \ge \sum_{\{y \mid f(y) > C'\}} \mathsf{P}_i\{\xi_k = y\}\mathsf{P}_y\{\underline{\mathcal{L}}(A)\} \ge \left(\frac{\epsilon}{d}\right)^k \alpha, \tag{3.2.5}$$

and, consequently the desired assertion holds for $a = (\epsilon/d)^k \cdot \alpha$.

Clearly, by condition (iii) the exception set B is contained in $\boldsymbol{I} \setminus A$, and so one may take any constant $C + c$, $c \geq 0$, instead of C.

Let us now assume $k(\cdot)$ to be a general step function. We will give the proof again for the constant C. Consider the embedded Markov chain $\tilde{\xi}_n$ at successive instants $0, k(\xi_0), k\big(\xi_{k(\xi_0)}\big), \ldots$. The above derivation implies (3.2.4) for the embedded chain, for the constants $C + d \cdot \sup_i k(i)$, and C', such that

$$\alpha = 1 - \exp\{h(C + d \cdot \sup_i k(i) - C')\}/\big(1 - \exp\{-\gamma\}\big) > 0.$$

Boundedness conditions (i) and (ii) imply that

$$\mathsf{P}_i\{\inf_{n>0} f(\xi_n) > C\} \geq \mathsf{P}_i\{\inf_{n>0} f(\tilde{\xi}_n) > C + d \cdot \sup_i k(i)\},$$

thus implying (3.2.4) for the chain ξ_n and constant C, for the constants C' and α that have been chosen for the embedded chain $\tilde{\xi}_n$. The remainder follows as before. □

A sufficient condition for transience of an infinite set can be derived in a similar fashion.

Lemma 3.2.2 *Suppose there exist a function $f : \boldsymbol{I} \to \mathbf{R}$, a set $A \subset \boldsymbol{I}$ and a finite step function $k : \boldsymbol{I} \to \mathbf{Z}_+$, such that*

i) *the step function is uniformly bounded, i.e.* $\sup_i k(i) < \infty$*;*

ii) *the set A is almost closed;*

iii) *the f-drift outside A is strictly negative, i.e.*

$$\mathsf{E}\{f(\xi_{n+k(\xi_n)}) - f(\xi_n) \,|\, \xi_n = i\} \leq -\epsilon, \quad i \in \boldsymbol{I} \setminus A; \tag{3.2.6}$$

iv) $f(i) \geq 0$ *on* $\big(\boldsymbol{I} \setminus A\big) \cup \{j \in A \,|\; p_{i,j} > 0, \text{for some } i \in \boldsymbol{I} \setminus A\}$.

Then the set $\boldsymbol{I} \setminus A$ is transient.

Proof. Note that if the set A would be finite, then (3.2.6) is simply a generalized version of Lyapunov-Foster criterion for positive recurrence of the Markov chain. Checking that proof, it follows that the time T to hit A (from a state $i \notin A$) is finite a.s. and has finite expectation, whether A be finite or not, provided that (iv) holds. (In fact, one only needs positivity of f on the set $\{j \in A \,|\; p_{i,j} > 0, \text{for some } i \in \boldsymbol{I} \setminus A\}$. It seems however, that non-negativity of f on the whole of $\boldsymbol{I} \setminus A$ would be implied this combined with the drift condition (3.2.6).)

Let A^c denote complementation of A, i.e. $A^c = \{i \in \boldsymbol{I} : i \notin A\}$. From the fact that the time T to hit A (from a state $i \notin A$) is finite with probability 1, it follows that $\mathsf{P}_i\{T = \infty\} = 0$ for any $i \notin A$. It implies that $\mathsf{P}\{\underline{\mathcal{L}}(A^c)\} = 0$. On other hand

$$\underline{\mathcal{L}}(A^c) = \big(\overline{\mathcal{L}}(A)\big)^c.$$

Hence, $\mathsf{P}\{\overline{\mathcal{L}}(A)\} = 1$. Since the set A is almost closed, we have that $\mathsf{P}\{\underline{\mathcal{L}}(A)\} = 1$. So the set $\boldsymbol{I} \setminus A$ is transient. □

In applications as face-homogeneous random walks, one often can construct a suitable Lyapunov function on the whole state space, satisfying (3.2.2) outside at most a compact set.

Lemma 3.2.3 *Suppose that conditions (i),(ii) and (iv) of Lemma 3.2.1 are satisfied, where $B = \emptyset$. Then for any constant $C \in \mathbf{R}$ the set $\{j \in \boldsymbol{I} \mid f(j) \leq C\}$ is transient.*

Proof. By definition of transient set we need to show that $\mathsf{P}\{\cap_{m\geq 0} \cup_{n>m} f(\xi_n) \leq C\} = 0.$ It is easy to see that for $\xi_0 = i$

$$\mathsf{P}_i\{\bigcap_{m\geq 0}\bigcup_{n>m} f(\xi_n) \leq C\} = \lim_{m\to\infty} \mathsf{P}_i\{\bigcup_{n>m} f(\xi_n) < C\} \leq \lim_{m\to\infty} \bigcup_{n>m} \mathsf{P}_i\{f(\xi_n) < C\}.$$

Let $g(i) = \exp\{-hf(i)\}$ with $h > 0$. By Chebeshev's inequality we have that for any $C \in \mathbf{R}$ and any $h > 0$

$$\mathsf{P}_i\{f(\xi_n) < C\} = \mathsf{P}_i\big(g(\xi_n) > \exp\{-hC\}\big) \leq \exp\{hC\}\mathsf{E}_i\, g(\xi_n). \tag{3.2.7}$$

Since condition (iv) of Lemma 3.2.1 is satisfied on the whole state space, we get the following estimate $\mathsf{E}_i\, g(\xi_n) < g(i)\exp\{-\gamma n\}$, which can be derived similar to (3.2.3). Hence,

$$\lim_{m\to\infty} \sum_{n>m} \mathsf{P}_i\{f(\xi_n) < C\} \leq \exp\{hC - hf(i)\} \lim_{m\to\infty} \sum_{n>m} \exp\{-\gamma n\} = 0.$$

Therefore the set $\{j \in \boldsymbol{I} \mid f(j) \leq C\}$ is transient. □

Now we will derive atomicity of almost closed sets for a special subclass of face-homogeneous random walks.The main idea is that 'far away into' an almost closed set, the transition probabilities at the boundary of and outside the almost closed set do not matter anymore. Hence, one might change them so as to end up with a simple atomic chain. As a result, the original almost closed set must have been atomic as well.

To set this up, we would like to slightly dwell on how to compute the probabilities of sojourn sets, as has been extensively studied in Feller [8]. The proofs of the statements mentioned below, can be found in [8].

For quoting the necessary details, we prefer to introduce notation allowing the analytic approach used by Feller. First write P for the transition matrix of the Markov chain to be considered. In what follows, it is allowed to be a *substochastic* matrix. The restricted probability matrix P_A to set A is defined by

$$P_{A,\,i,j} = \begin{cases} p_{i,j} & i,j \in A \\ 0, & \text{otherwise} \end{cases}$$

The n-step (restricted) transition probabilities and matrix are denoted by superscript (n).

For any $A \subset \boldsymbol{I}$ the following limits exist as vectors on $\boldsymbol{I}$

$$\begin{aligned} \boldsymbol{\sigma}_A &= \lim_{n\to\infty} P_A^{(n)} \mathbf{1}_{\{A\}} \\ s_A &= \lim_{n\to\infty} P^{(n)} \boldsymbol{\sigma}_A. \end{aligned}$$

Then, up to a constant factor, σ_A is the maximum bounded harmonic function on A with respect to the restricted transition matrix P_A. The probabilistic interpretation is that

$$\boldsymbol{\sigma}_A(i) = \mathsf{P}_i\{\xi_n \in A, n \geq 0\}.$$

As a consequence, $s_A(i)$ denotes the probability that $\xi_n(i) \in A$ eventually, i.e. $s_A(i) = \mathsf{P}_i\{\underline{\mathcal{L}}(A)\}$ (harmonicity of $\boldsymbol{\sigma}_A$ w.r.t. P_A should be used for showing this directly). Thus A is a sojourn set if and only if $s_A \neq 0$ and then we will refer to it as the *sojourn solution* corresponding to A. Additionally, $s_A \geq \sigma_A$ is harmonic on $\boldsymbol{I}$ with respect to P. One has that

$$\sup_{i\in \boldsymbol{I}} s_A(i) = \sup_{i\in \boldsymbol{I}} \boldsymbol{\sigma}_A(i) = 1,$$

for sojourn set A. It follows that a sojourn set must be infinite.

The sojourn set $B \subset \boldsymbol{I}$ is said to be *representative* whenever there is $0 < \eta < 1$, such that $s_B(i) > \eta$ for all $i \in B$. A representative set B enjoys the property that the probability of eventually ending up in B equals the limiting probability of B, which is shown to exist, i.e.

$$s_B(i) = \lim_{n\to\infty} \mathsf{P}_i\{\xi_n \in B\}!$$

This does not hold for an arbitrary sojourn set, since that would imply almost closedness of the sojourn set.

For any sojourn set A and any $0 < \eta < 1$ put

$$\begin{aligned} A^\eta &= A^{(\eta)} \cap A = \{i \in A \mid s_A(i) > \eta\} \\ A_\eta &= \{i \mid \boldsymbol{\sigma}_A(i) > \eta\}. \end{aligned}$$

Then $s_A = s_{A^\eta} = s_{A_\eta}$ and so we have that A^η is representative by definition, as well as almost closed (by the fact that the limiting probability of A^η exists as well as by Theorem 3.1). In fact, for any set B, $A \supset B \supset A_\eta$ we have $s_B = s_A$! We give an example of representative sets with a nice structure.

Remark 3.2.1 *Under either the conditions of Lemma 3.2.1 or those of Lemma 3.2.3, the set* $A = \{f(i) > C + c\}$ *is representative, i.e.* $A = A^\eta$ *for some* $0 < \eta < 1$*. This follows from the proof of Lemma 3.2.1, equation (3.2.5).*

The next statement is evident, but has not been explicitly proved in neither [3] nor [8]. We will need it in the sequel.

Lemma 3.2.4 *Let two almost closed sets* A, $B \subset \boldsymbol{I}$ *be given. Then the symmetric difference of* A *and* B *is a transient set if and only if* $s_A = s_B$.

Proof. A set S is transient if and only if $\mathsf{P}_i\{\overline{\mathcal{L}}(S)\} = 0$ for some state i, and hence for all. It follows that any subset of a transient set is transient.

Suppose that A and B differ by at most a transient set. This implies that $A \cap B \neq \emptyset$, otherwise A and B could not have been almost closed. Hence, $A \cap B$ is almost closed and $s_A \geq s_{A\cap B} > 0$. On the other hand, for any $i \in \boldsymbol{I}$,

$$\begin{aligned} s_A(i) &= \mathsf{P}_i\{\overline{\mathcal{L}}(A)\} \\ &\leq \mathsf{P}_i\{\overline{\mathcal{L}}(A \setminus B)\} + \mathsf{P}_i\{\overline{\mathcal{L}}(A \cap B)\} = s_{A\cap B}(i), \end{aligned}$$

and so $s_A = s_B$.

Suppose that $s_A = s_B$. Then $A^{(\eta)} = B^{(\eta)}$. By Theorem 3.1, the sets $A^{(\eta)} \setminus A^\eta$ and $A^{(\eta)} \setminus B^\eta$ are transient. It follows that $A^\eta \cap B^\eta \neq \emptyset$, otherwise A^η and B^η would have been transient themselves. Hence, $A^\eta \cap B^\eta$ is almost closed. Moreover, $\big(A^\eta \setminus A^\eta \cap B^\eta\big) \subset A^{(\eta)} \setminus B^\eta$ is transient. Thus, A^η and $A^\eta \cap B^\eta$ are both almost closed sets differing a transient set. By the foregoing, $s_{A^\eta} = s_{A^\eta \cap B^\eta}$. Because of $s_A = s_{A^\eta}$, we have $s_A = s_{A^\eta \cap B^\eta}$.

Clearly, for any $i \in \boldsymbol{I}$,

$$\begin{aligned}\overline{\mathcal{L}}\big(A \setminus (A^\eta \cap B^\eta)\big) \subset \Big(\overline{\mathcal{L}}\big(A \setminus (A^\eta \cap B^\eta)\big) \cap \overline{\mathcal{L}}(A^\eta \cap B^\eta)\Big) \\ \cap \quad \cup \Big(\overline{\mathcal{L}}\big(A \setminus (A^\eta \cap B^\eta)\big) \cap \overline{\mathcal{L}}(\boldsymbol{I} \setminus A)\Big) \cup \underline{\mathcal{L}}\big(A \setminus (A^\eta \cap B^\eta)\big).\end{aligned}$$

Given $\xi_0 = i$, the probability of the first event equals 0 because of almost closedness of $A^\eta \cap B^\eta$, the probability of the second event equals 0 because of almost closedness of A and the probability of the latter equals 0, since $s_A = s_{A^\eta \cap B^\eta}$. Thus, $A \setminus (A^\eta \cap B^\eta)$ is a transient set. Similarly, $B \setminus (A^\eta \cap B^\eta)$ is a transient set. Consequently, $\big(A \cup B\big) \setminus (A^\eta \cap B^\eta)$ is transient. The symmetric difference of A and B is contained in this set and is therefore transient. □

Our method of checking atomicity relies on the lemma following here.

Lemma 3.2.5 *Let $S \subset \boldsymbol{I}$. Let $\{\tilde{\xi}_n\}$ be an irreducible and aperiodic Markov chain on a denumerable space $\widetilde{\boldsymbol{I}} \supset \boldsymbol{I}$, such that*

$$p_{i,j} = \tilde{p}_{i,j}, \quad i \in S, \text{ all } j \in \boldsymbol{I},$$

where $\tilde{p}_{i,j} = \mathsf{P}\{\tilde{\xi}_{n+1} = j \,|\, \tilde{\xi}_n = i\}$, $i, j \in \widetilde{\boldsymbol{I}}$, denote the transition probabilities of $\tilde{\xi}_n$. Then for any $A \subset S$, we have that A is a sojourn set for ξ_n if and only if it is a sojourn set for $\tilde{\xi}_n$. In particular, if $A \subset S$ is almost closed for ξ_n, then it contains an almost closed set $A' \subset A$ for $\tilde{\xi}_n$, and vice versa.

Proof. Denote all quantifiers for the chain $\tilde{\xi}_n$ by $\tilde{\cdot}$. For $A \subset S$, denote $\tilde{A}^\eta = \{i \in A \,|\, \tilde{s}_A(i) > \eta\}$ and similarly $\tilde{A}_\eta = \{i \,|\, \tilde{\boldsymbol{\sigma}}_A(i) > \eta\}$.

Suppose $A \subset S$ is sojourn for ξ_n. By assumption $\boldsymbol{\sigma}_A = \tilde{\boldsymbol{\sigma}}_A$. Since $\tilde{s}_A \geq \tilde{\boldsymbol{\sigma}}_A \neq 0$, A is sojourn for the chain $\tilde{\xi}_n$. The analogous argument applies when assuming that A is sojourn for $\tilde{\xi}_n$. This proves the first statement.

Suppose A is almost closed for ξ_n. Then it is sojourn for $\tilde{\xi}_n$ by virtue of the first statement. The representative set $\tilde{A}^\eta$, $A \supset \tilde{A}^\eta \supset \tilde{A}_\eta = A_\eta$, is now almost closed for the chain $\tilde{\xi}_n$. □

Corollary 3.2.1 *If a face of homogeneity $\boldsymbol{\Lambda}$ is an almost closed set then it is atomic.*

Proof. Let a face of homogeneity $\boldsymbol{\Lambda} \subset \mathbf{Z}^p$ be an almost closed set for a Markov chain $\{\xi_n\}$. Consider a Markov chain $\{\tilde{\xi}_n\}$ on $\mathbf{Z}^p$ defined by transition probabilities of $\boldsymbol{\Lambda}$. Clearly, $\{\tilde{\xi}_n\}$ is homogeneous. The state space of a homogeneous MC is always atomic (see [3]).

Suppose that $\boldsymbol{\Lambda}$ contains two disjoint almost closed sets B_1 and B_2 for $\{\xi_n\}$. It means that $\boldsymbol{\Lambda}$ can not be atomic almost closed set for $\{\xi_n\}$. Clearly, B_1 are B_2 are sojourn set for $\{\xi_n\}$. By the previous lemma these sets are also sojourn for the homogeneous Markov chain $\{\tilde{\xi}_n\}$. From theorem 3.1 it follows that any sojourn set contains an almost closed set. Therefore B_k contains almost closed sets $A_k, k = 1, 2$, for $\{\tilde{\xi}_n\}$. Since B_1, B_2 are disjoint, the sets A_1, A_2 are disjoint as well. It contradicts that $\{\tilde{\xi}_n\}$ has an atomic state space. Hence, $\mathbf{\Lambda}$ does not contain two disjoint almost closed sets and so it is atomic for the original Markov chain $\{\xi_n\}$. □

For proving atomicity by means of the previous lemma, we need to specify the class of face-homogeneous random walks to be considered. This reduces to specifying space and homogeneity faces.

Let $\boldsymbol{I} = \prod_{l=1}^{p} \boldsymbol{I}_l$, with $\boldsymbol{I}_l \in \{\mathbf{Z}, \mathbf{Z}^+\}$ and let a face $\mathbf{\Lambda}$ be determined by the vector $\lambda = \{\lambda_1, \ldots, \lambda_p\}$ with values $\lambda_l \in \{0, +\}$ when $\boldsymbol{I}_l = \mathbf{Z}^+$ and $\lambda_l \in \{-, 0, +\}$, whenever $\boldsymbol{I}_l = \mathbf{Z}$. Indeed, $i \in \mathbf{\Lambda}$ whenever $\mathrm{sgn}(i_l) = \lambda_l$, $l = 1, \ldots, p$. Recall that transition probabilities $p_{i,j}$ on a $\mathbf{\Lambda}$ depend only on the difference $j - i$, i.e.

$$p_{i,j} = p^{\mathbf{\Lambda}}_{j-i} \text{ for any } i \in \mathbf{\Lambda}.$$

Define the projection operator $\mathsf{proj}^{\mathbf{\Lambda}}$ on $\boldsymbol{I}$ by putting its l-th component equal to

$$(\mathsf{proj}^{\mathbf{\Lambda}}(i))_l = \begin{cases} i_l, & \lambda_l \neq 0 \\ 0 & \lambda_l = 0. \end{cases}$$

For any face $\mathbf{\Lambda}$ we define the *induced* chain $\xi^{\mathbf{\Lambda}}_n$ as follows. If $\lambda_l = 0$, define $\boldsymbol{I}^{\mathbf{\Lambda}}_l = \boldsymbol{I}_l$. If $\lambda_l \neq 0$, then $\boldsymbol{I}_l = \{0\}$. Fix $i_0 \in \mathbf{\Lambda}$ and define $\boldsymbol{I}^{\mathbf{\Lambda}} = i_0 + \prod \boldsymbol{I}^{\mathbf{\Lambda}}_l$. This space is orthogonal to i_0. The transition probabilities are now obtained from the transition probabilities of ξ_n by orthogonal projection onto $\boldsymbol{I}^{\mathbf{\Lambda}}$: for $i_0 + i^{\mathbf{\Lambda}} \in \boldsymbol{I}^{\mathbf{\Lambda}}$ and $i_0 + j^{\mathbf{\Lambda}} \in \boldsymbol{I}^{\mathbf{\Lambda}}$

$$\mathsf{P}\{\xi^{\mathbf{\Lambda}}_{n+1} = i_0 + j^{\mathbf{\Lambda}} \,|\, \xi^{\mathbf{\Lambda}}_n = i_0 + i^{\mathbf{\Lambda}}\} = \sum_{\substack{j \in \boldsymbol{I}: \\ j - (i_0 + j^{\mathbf{\Lambda}}) \in \mathbf{\Lambda}}} p_{i_0 + i^{\mathbf{\Lambda}}, j}.$$

It is also convenient to have an ordering of faces: $\mathbf{\Lambda}' \succ \mathbf{\Lambda}$, whenever $\lambda'_l = \lambda_l$, if $\lambda_l \neq 0$.

The next lemma shows atomicity of almost closed subsets of $\cup_{\mathbf{\Lambda}' \succeq \mathbf{\Lambda}} \mathbf{\Lambda}'$, for a given face $\mathbf{\Lambda}$, provided that the induced chain $\xi^{\mathbf{\Lambda}}_n$ is ergodic. Ergodicity is mainly used to show that a transformed face-homogeneous random walk that is homogeneous with respect to the non-zero coordinates of λ, has only one atomic closed class.

Lemma 3.2.6 *Suppose that ξ_n is a face-homogeneous random walk on $\boldsymbol{I} = \prod_{l=1}^{p} \boldsymbol{I}_l$, with the above specified state space and homogeneity faces. Suppose that $\xi^{\mathbf{\Lambda}}_n$ is ergodic, for some face $\mathbf{\Lambda}$. Assume the existence of a sojourn set $S \subset \boldsymbol{I}$, with $\mathsf{proj}^{\mathbf{\Lambda}}(S) \subset \mathbf{\Lambda}$. Up to transient sets, the representative set $S^\eta \subset S$ is the unique almost closed set contained in S, and so it is atomic.*

Proof. Observe that the set $S^\eta = \{i \in S \,|\, s_S(i) > \eta\}$ is almost closed. We have to show that, up to transient sets, S does not contain any other almost closed set. The proof is based on applying the previous lemma for a suitable transformed chain.

We may assume that $\lambda_l \neq 0$ for $l = 1, \ldots, s$ and $\lambda_l = 0$, $l = s+1, \ldots, p$ (the case $s = p$ is allowed), for λ defining $\mathbf{\Lambda}$.

Define a transformed chain $\tilde{\xi}_n$ on a state space $\widetilde{\boldsymbol{I}} \supset \boldsymbol{I}$ as follows. Let $\widetilde{\boldsymbol{I}} = \mathbf{Z}^s \times \prod_{l=s+1}^p \boldsymbol{I}_i$. Again all quantifiers of this transformed Markov chain will denoted by $\tilde{\cdot}$.

For any $\tilde{i} \in \widetilde{\boldsymbol{I}}$ there exists a unique face $\mathbf{\Lambda}' \succeq \mathbf{\Lambda}$, such that $\operatorname{sgn}(\tilde{i}_l) = \operatorname{sgn}(\lambda'_l)$, $l > s$, where λ' defines $\mathbf{\Lambda}'$. Put

$$\tilde{p}_{\tilde{i},\tilde{i}+j} = p_j^{\mathbf{\Lambda}'}, \text{ for all } j.$$

By construction, the transition probabilities of ξ_n and $\tilde{\xi}_n$ coincide on the subset S. Then from Lemma 3.2.5 it follows that S is also a sojourn set for the transformed chain.

We will study the almost closed set structure for the latter. Since sojourn solutions are bounded harmonic functions, for our purpose it is sufficient to show the existence of only one bounded harmonic function (up to a multiplicative factor) and then use Lemma 3.2.4.

Define the embedded Markov chain on $\widetilde{\boldsymbol{I}}^{\mathsf{e}} = \mathbf{Z}^s \times \{0\} \times \cdots \times \{0\}$:

$$\tilde{p}^e_{i,j} = \mathsf{P}\{\tilde{\xi}_T = j \,|\, \tilde{\xi}_0 = i\}, \quad i, j \in \widetilde{\boldsymbol{I}}^{\mathsf{e}},$$

with

$$T = \inf\{n \geq 1 \,|\, \tilde{\xi}_k \notin \widetilde{\boldsymbol{I}}^{\mathsf{e}}, 1 \leq k < n, \tilde{\xi}_n \in \widetilde{\boldsymbol{I}}^{\mathsf{e}}\}$$

the first entrance time of the $\widetilde{\boldsymbol{I}}^{\mathsf{e}}$. Ergodicity of the induced chain $\xi_n^{\mathbf{\Lambda}}$ implies that $T < \infty$ with probability 1. Clearly, the embedded chain is a homogeneous random walk on $\widetilde{\boldsymbol{I}}^{\mathsf{e}}$. Since $\tilde{\xi}_n$ is aperiodic and irreducible, the embedded chain must be aperiodic and irreducible as well. Hence, by lemma 2.3.2 and corollary 2.3.1, there is only one bounded harmonic function, f^e say, for the embedded chain, and it is constant, say $f^e \equiv 1$.

Going back to the chain $\tilde{\xi}_n$, take a bounded harmonic function f on $\widetilde{\boldsymbol{I}}$. Then

$$f(i) = \sum_{j \in \widetilde{\boldsymbol{I}}^{\mathsf{e}}} \tilde{p}^e_{i,j} f(j) \text{ for any } i \in \widetilde{\boldsymbol{I}}^{\mathsf{e}}. \tag{3.2.8}$$

Indeed, using that T is finite with probability 1, we get by iteration of $\{T = n\}, n \geq 1$, that

$$\begin{aligned} f(i) &= \sum_{j \in \widetilde{\boldsymbol{I}}} \tilde{p}_{i,j} f(j) = \sum_{j \in \widetilde{\boldsymbol{I}}^{\mathsf{e}}} \tilde{p}_{i,j} f(j) + \sum_{j \notin \widetilde{\boldsymbol{I}}^{\mathsf{e}}} \tilde{p}_{i,j} \sum_{k \in \widetilde{\boldsymbol{I}}} \tilde{p}_{j,k} f(k) \\ &= \sum_{j \in \widetilde{\boldsymbol{I}}^{\mathsf{e}}} f(j) \tilde{p}_{i,j} + \sum_{k \in \widetilde{\boldsymbol{I}}^{\mathsf{e}}} f(k) \sum_{j \notin \widetilde{\boldsymbol{I}}^{\mathsf{e}}} \tilde{p}_{i,j} \tilde{p}_{j,k} + \sum_{k \notin \widetilde{\boldsymbol{I}}^{\mathsf{e}}} \sum_{j \notin \widetilde{\boldsymbol{I}}^{\mathsf{e}}} \tilde{p}_{i,j} \tilde{p}_{j,k} f(k) \\ &= \sum_{j \in \widetilde{\boldsymbol{I}}^{\mathsf{e}}} f(j) \sum_{n \geq 1} \mathsf{P}\{\tilde{\xi}_n = j \,|\, \tilde{\xi}_0 = i, T = n\} = \sum_j \tilde{p}^e_{i,j} f(j). \end{aligned}$$

So (3.2.8) means that f is harmonic for the embedded chain. Hence, it must be equal to 1 on $\widetilde{\boldsymbol{I}}^{\mathsf{e}}$ (up to a factor). Equation (3.2.8) implies that $f \equiv 1$ on $\tilde{\boldsymbol{I}}$.

As a consequence, $\tilde{\boldsymbol{I}}$ is an atomic almost closed set. By virtue of Lemma 3.2.5, the set S can only contain one almost closed class (up to transient sets). The set S^η is one such set for each $\eta \in (0,1)$. It is therefore atomic. □

So far, we have considered the problem of suitable methods for determining the almost closed set structure. Now we would like to turn to the problem of determining Euler limit paths and convergence to these of the time-scaled process.

Martingales

The other main tool to be used is a.s. convergence to 0 of a time scaled sequence of random vectors, the components of which form a martingale sequence. We will use two approaches.

The first approach is to find a suitable Lyapunov function $f : \boldsymbol{I} \to \mathbf{R}$ such that for some constant $v \in \mathbf{R}$ and any almost closed set A

$$v = \mathsf{E}\{f(\xi_{n+1}) - f(\xi_n) \,|\, \xi_n = i\} \text{ for all } i \in A.$$

It appears that for many models the function f can be chosen linear on almost closed sets and piecewise linear on state space. For the sequence $f(\xi_n)$ we construct a martingale M_n. From the convergence of M_n/n we will find the fluid limit. We will apply this method for face-homogeneous random walks on **Z** and $\mathbf{Z}_+ \times \mathbf{Z}$.

The second approach will be applied as follows. Let $\xi_0 = i$ be given, then

$$U(i;n) = i + \sum_{l=1}^{n} \mathsf{E}_i\{\xi_l - \xi_{l-1} \,|\, \xi_{l-1}\}, \quad n = 1, \ldots$$

is a random discrete time path in $\mathbf{R}^p$ when ξ_n is a vector of dimension p. The sequence $\xi_n - U(i;n)$ is a sequence with martingale components. A.s. convergence of the time scaled process to 0, together with a.s. convergence of $U(i;[tN])/N$, $N \to \infty$, implies that $\xi_{[tN]}/N$ a.s. converges to the same limit, where argument i denotes the initial position. Scaling by time is not always the right scaling, so we quote the following theorem from [9], theorem 2.18 in general form.

Theorem 3.2 *Let $(\Omega', \mathcal{F}', \mathsf{P}')$ be a probability space and let $\{\mathcal{F}'_n\}_{n=1,\ldots}$ be an increasing sequence of sub-σ-fields of $\mathcal{F}'$. Let $\{M_n, \mathcal{F}'_n, n \geq 1\}$ be a martingale and $\{T_n\}_{n=1,\ldots}$ a non-decreasing sequence of positive random variables such that T_n is $\mathcal{F}'_{n-1}$ measurable for each n. Then $M_n/T_n \to 0$, $n \to \infty$, a.s. on the set where*

$$\big\{\lim_{n\to\infty} T_n = \infty, \sum_{n=1}^{\infty} \mathsf{E}\{(M_n - M_{n-1})^2 \,|\, \mathcal{F}'_{n-1}\}/T_n^2 < \infty\big\}.$$

An immediate consequence for face homogeneous random walks is summarized below.

Corollary 3.2.2

Let ξ_n be a face-homogeneous random walk with bounded jumps and $\xi_0 = i$. Then

$$\lim_{N\to\infty} \frac{\xi_{[tN]} - U(i;[tN])}{N} = 0, \text{ a.s.}$$

In particular, we have that $U(i;[tN]) = i + \sum_{n=0}^{[tN]-1} m(\xi_n)$.

Note, that the corollary is a statement on vector processes, whereas Theorem 3.2 is a statement on one-dimensional processes. Since the space where the vector processes live, has finite dimension, generalizing Theorem 3.2 is a straightforward business.

The final step is now to study the limits (provided they exist) of $U(i;[tN])/N$ along paths of the process ξ_n. We assume that the induced chain $\xi_n^{\mathbf{\Lambda}}$ is ergodic. Say it has the stationary measure $\pi^{\mathbf{\Lambda}}$. Let $i_0 \in \mathbf{\Lambda}$ Define

$$v^{\mathbf{\Lambda}} = \sum_{\mathbf{\Lambda}' \supset \mathbf{\Lambda}} \sum_{i_0 + j^{\mathbf{\Lambda}} \in \mathbf{\Lambda}'} \pi^{\mathbf{\Lambda}}(i_0 + j^{\mathbf{\Lambda}}) m^{\mathbf{\Lambda}'}.$$

This is called the *second vector field on* $\mathbf{\Lambda}$ ([7]). By plugging in the definition of the drifts $m^{\mathbf{\Lambda}'}$, it follows that $v_l^{\mathbf{\Lambda}} = 0$, when $\lambda_l = 0$. For convenience, denote $\Omega_i = \{\omega \in \Omega \mid \omega_0 = i\}$.

Lemma 3.2.7

Assume the conditions of Lemma 3.2.6, that is suppose that $\{\xi_n\}$ *is a face-homogeneous random walk on* $\boldsymbol{I} = \prod_{l=1}^{p} \boldsymbol{I}_l$*, with the state space and homogeneity faces specified in the previous paragraph. Suppose that* $\xi_n^{\mathbf{\Lambda}}$ *is ergodic, for some face* $\mathbf{\Lambda}$*. Assume the existence of a sojourn set* $S \subset \boldsymbol{I}$*, with* $\mathsf{proj}^{\mathbf{\Lambda}}(S) \subset \mathbf{\Lambda}$*. Then*

$$\lim_{N\to\infty} \frac{U(i;[tN])(\omega)}{N} = v^{\mathbf{\Lambda}} \cdot t, \text{ for almost all } \omega \in \underline{\mathcal{L}}(A) \cap \Omega_i, \tag{3.2.9}$$

for $A \subset S$ *almost closed.*

Proof. We will use the transformed chain defined in the proof of Lemma 3.2.6. Again we use $\tilde{\cdot}$ to denote all quantifiers for $\tilde{\xi}_n$.

Let reward $m(i)$ be paid, whenever the process is in state $i \in \tilde{\boldsymbol{I}}$. Then the random vectors having time and total reward earned between the n-th and $(n+1)$-th visits $\widetilde{\boldsymbol{I}}^{\mathsf{e}}$ as components, are i.i.d. vectors, for $n = 1, 2, \ldots$. The total expected reward between two successive visits of $\widetilde{\boldsymbol{I}}^{\mathsf{e}}$ equals $v^{\mathbf{\Lambda}}/\pi_{i_0}^{\mathbf{\Lambda}}$, where $i_0 \in \mathbf{\Lambda}$ is any fixed reference point, orthogonally to which the induced state space (for the original chain) has been built.

By a delayed version of the so-called Renewal Reward theorem (cf. [23]), one has for the chain $\tilde{\xi}_n$ that

$$\begin{aligned}\frac{\tilde{U}(i;[tN])}{N} &= \frac{i + \sum_{n=1}^{[tN]} \mathsf{E}_i\{\tilde{\xi}_n - \tilde{\xi}_{n-1} \mid \tilde{\xi}_{n-1}\}}{N} \\ &= \frac{i + \sum_{n=1}^{[tN]} \mathsf{E}_i\{m(\tilde{\xi}_{n-1})\}}{N} \to v^{\mathbf{\Lambda}} \cdot t, \quad N \to \infty, \text{ a.s.}\end{aligned}$$

Note that, the Renewal Reward theorem has been formulated in [23] for one-dimensional processes. Extension to finite-dimensional processes is straightforward. For any subset $\widetilde{\Omega'}_i$ of the path space $\widetilde{\Omega}_i$ of the transformed chain it follows that the symmetric difference of the sets

$$\Big\{\tilde{\omega} \,\Big|\, \lim_{N\to\infty} \frac{\tilde{U}(i;[tN])(\tilde{\omega})}{N} = v^{\mathbf{\Lambda}} \cdot t\Big\} \cap \widetilde{\Omega'}_i \quad \text{and} \quad \widetilde{\Omega'}_i$$

is a null-set for the chain $\tilde{\xi}_t$. Let $i \in S$. The paths of ξ_n and $\tilde{\xi}_n$ restricted to $\widetilde{\Omega'}_i \subset S^\infty \cap \Omega_i$ have equal probabilities. Hence the symmetric difference of the sets

$$\Omega_i(S) = \Big\{\omega \,\Big|\, \lim_{N\to\infty} \frac{U(i;[tN])(\omega)}{N} = v^{\mathbf{\Lambda}} \cdot t\Big\} \cap S^\infty \cap \Omega_i \quad \text{and} \quad S^\infty \cap \Omega_i$$

is a null-set for the chain ξ_t.

Let $A \subset S$ be almost closed. The sets A and S^η differ a transient set by Lemma 3.2.6, and so by Lemma 3.2.4, we have $s_A = s_{S^\eta}$. Hence, the symmetric difference of sets $\underline{\mathcal{L}}(S) \cap \Omega_i$ and $\underline{\mathcal{L}}(A) \cap \Omega_i$ is a null-set, since their probabilities are equal, and the second set is contained in the first. So, it suffices to prove the assertion of the lemma for the set $\underline{\mathcal{L}}(S) \cap \Omega_i$. For $\omega \in \underline{\mathcal{L}}(S)$ there exists a finite time n_ω such that $\omega_n \in S$, for $n \geq n_\omega$ and $\omega_{n_\omega - 1} \notin S$. Write $\omega_S = (\omega_{n_\omega}, \omega_{n_\omega+1}, \ldots\}$. We will show that the set

$$\Omega_{S,i} = \Big\{\omega \in \underline{\mathcal{L}}(S) \cap \Omega_i \,|\, \frac{U(\omega_{n_\omega}, [tN])(\omega_S)}{N} \not\to v^{\mathbf{\Lambda}} \cdot t\Big\}$$

is a null-set. Indeed,

$$\begin{aligned} \mathsf{P}\{\Omega_{S,i}\} &\leq \sum_{n=1}^{\infty}\sum_{y\in S} \mathsf{P}\Big\{\omega \in \Omega_i \,:\, n_\omega = n, \omega_n = y, \frac{U(y,[tN])(\omega_n,\ldots)}{N} \not\to v^{\mathbf{\Lambda}} \cdot t\Big\} \\ &\leq \sum_{n=1}^{\infty}\sum_{y\in S} \mathsf{P}\{\xi_0 = i\}\mathsf{P}_i\{\xi_n = y\}\mathsf{P}_y\{S^\infty \cap \Omega_y \setminus \Omega_y(S)\}. \end{aligned}$$

The last probability equals 0 by the foregoing and so $\Omega_{S,i}$ is a null-set. For almost all $\omega \in \underline{\mathcal{L}}(S) \cap \Omega_i$ we now have

$$\begin{aligned} \frac{U(i;[tN])(\omega)}{N} &= \frac{i + \sum_{k=0}^{n_\omega - 1} m(\omega_k)}{N} - \frac{\omega_{n_\omega}}{N} \\ &+ \quad \frac{U(\omega_{n_\omega};[tN]-n_\omega)(\omega_S) - U(\omega_{n_\omega};[tN])(\omega_S)}{N} + \frac{U(\omega_{n_\omega};[tN])(\omega_S)}{N} \\ &\to \quad v^{\mathbf{\Lambda}} \cdot t, \quad N \to \infty. \end{aligned}$$

The conclusion follows. □

We finally piece together the results from this section.

Corollary 3.2.3

Under the conditions of Lemma 3.2.6, we have for any initial state $\xi_0 = i$ that

$$\lim_{N\to\infty} \frac{\xi_{[tN]}(\omega)}{N} = v^{\mathbf{\Lambda}} \cdot t, \quad \text{for almost all } \omega \in \underline{\mathcal{L}}(A) \cap \Omega_i,$$

for $A \subset S$ almost closed. In other words,

$$\mathsf{P}_i\{\lim_{N\to\infty} \frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}} \cdot t \,|\, \underline{\mathcal{L}}(A) \cap \Omega_i\} = 1.$$

3.3 Random walk on the integers

Consider an irreducible, face-homogeneous random walk $\{\xi_n\}$ on the integers **Z** with three homogeneity faces:

$$\mathbf{\Lambda}^+ = \{1,2,\ldots\}, \quad \mathbf{\Lambda}^0 = \{0\}, \quad \mathbf{\Lambda}^- = \{-1,-2,\ldots\}.$$

This means that the transition probabilities take three different forms:

$$p_{i,j} = \begin{cases} p^+_{j-i}, & i > 0 \\ p^0_j, & i = 0 \\ p^-_{j-i}, & i < 0. \end{cases}$$

The corresponding means jumps will be denoted by m^+, m^0 and m^- respectively. We assume that none equals 0. We also assume that the jumps are bounded.

One can see that for $i > 0$ sufficiently large and t comparatively small, the random walk inside $\mathbf{\Lambda}^+$ behaves like a homogeneous one. The same observations apply to the face $\mathbf{\Lambda}^-$. Therefore we associate with it the following dynamical system on $\mathbf{R}$. Let $u : \mathbf{R} \to \mathbf{R}$ be a continuous mapping with

$$\frac{d}{dt}u(t) = \begin{cases} m^+, & u(t) > 0 \\ m^-, & u(t) < 0 \end{cases}, \quad t \neq 0,$$

and initial condition $u(0) = 0$. The more interesting case is, when $m^- < 0 < m^+$. Then from point 0, two Euler paths start, one into either direction. This suggests the occurrence of two atomic almost closed sets: $\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^-$. This is indeed the case.

Theorem 3.3 *Assume that $m^- < 0 < m^+$. Then the random walk is transient. The almost closed set decomposition is given by $\mathbf{Z} = \mathbf{\Lambda}^+ \cup \mathbf{\Lambda}^-$, with $\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^-$ both atomic.*

For any given initial state $\xi_0 = i$ the fluid limit $\xi_{[tN]}/N$ exists almost surely with

$$\mathsf{P}_i\{\omega : \frac{\xi_{[tN]}(\omega)}{N} \to tm^-\} = \mathsf{P}_i\{\underline{\mathcal{L}}(\mathbf{\Lambda}^-)\} \text{ and } \mathsf{P}_i\{\omega : \frac{\xi_{[tN]}(\omega)}{N} \to tm^+\} = \mathsf{P}_i\{\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\}$$

i.e. conjecture 3.1.1 holds.

Proof. Suppose for simplicity that $t = 1$ and $i = 0$.

The random walk is transient by theorem 2.4 with $f(i) = |i|, k(i) = 1, C' = 0$ and $\epsilon = min\{-m^-, m^+\}$.

Almost closedness of $\mathbf{\Lambda}^+$ follows by application of lemma 3.2.1, where $f(i) = i, k(i) = 1, C = 0, \epsilon = m^+$. The same holds for $\mathbf{\Lambda}^-$ with $f(i) = -i, k(i) = 1, C = 0, \epsilon = |m^-|$. By lemma 3.2.1 these sets are atomic. Since ξ_n is transient, $\mathbf{\Lambda}^0$ is transient since it is a finite set. So $\mathbf{\Lambda}^-$ and $\mathbf{\Lambda}^+$ are the only (modulo transient sets) disjoint atomic almost closed sets.

Now let $f(i) = i/m^+$ if $i \geq 0$, and $f(i) = i/m^-$ if $i < 0$. It is easy to see that

$$\mathsf{E}\{f(\xi_{n+1}) - f(\xi_n) \,|\, \xi_n \neq 0\} = 1.$$

If also $1 = \tilde{m} \triangleq \mathsf{E}\{f(\xi_{n+1}) \,|\, \xi_n = 0\}$ then the sequence $f(\xi_n) - n$ is a zero-mean martingale. Hence, by martingale limit theorem 3.2 the sequence $f(\xi_n)/n$ converges almost surely to 1.

Suppose that $\tilde{m} \neq 1$. By δ_n denote the number of times that the random walk hits zero before time n, i.e.

$$\delta_n = \sum_{k=0}^{n-1} \mathbf{1}_{\{\xi_k = 0\}}.$$

Transience of $\{\xi_n\}$ implies that the random walk visits every state infinitely often with probability 0. In particular, we have that

$$\mathsf{P}\{\omega : \xi_n(\omega) = 0 \text{ i.o.}\} = \mathsf{P}\{\omega : \lim_{n\to\infty} \delta_n(\omega) = \infty\} = 0,$$

i.e. δ_n is almost surely finite. It is east to check that the sequence

$$M_n = f(\xi_n) + (1 - \tilde{m})\delta_n - n$$

is a martingale with bounded jumps. Since δ_n/n converges almost surely to 0, we conclude that

$$1 = \lim_{n\to\infty} \frac{f(\xi_n)}{n} = \lim_{n\to\infty} f\big(\frac{\xi_n}{n}\big) \quad \text{almost surely.}$$

The function $f(\cdot)$ takes 1 only at two points m^- and m^+. Note that since $\mathbf{\Lambda}^0$ is transient, the paths, which hit $\mathbf{\Lambda}^0$ infinitely often, do not play any role in almost surely convergence. Hence,

$$\mathsf{P}\{\omega : f\big(\frac{\xi_n(\omega)}{n}\big) \to 1\} = \mathsf{P}\{\omega : \frac{\xi_n(\omega)}{n} \to m^-\} + \mathsf{P}\{\omega : \frac{\xi_n(\omega)}{n} \to m^+\}.$$

So in limit the sequence ξ_n/n takes almost surely the values m^- and m^+. Clearly,

$$\{\omega : \xi_n(\omega)/n \to m^-\} \subset \underline{\mathcal{L}}(\mathbf{\Lambda}^-) \text{ and } \{\omega : \xi_n(\omega)/n \to m^+\} \subset \underline{\mathcal{L}}(\mathbf{\Lambda}^+).$$

Then we have that

$$1 = \mathsf{P}\{\omega : \xi_n(\omega)/n \to m^-\} + \mathsf{P}\{\omega : \xi_n(\omega)/n \to m^+\} \le \mathsf{P}\{\underline{\mathcal{L}}(\mathbf{\Lambda}^-)\} + \mathsf{P}\{\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\} = 1.$$

From this inequality we conclude that

$$\mathsf{P}\{\omega : \xi_n(\omega)/n \to m^-\} = \mathsf{P}\{\underline{\mathcal{L}}(\mathbf{\Lambda}^-)\} \text{ and } \mathsf{P}\{\omega : \xi_n(\omega)/n \to m^+\} = \mathsf{P}\{\underline{\mathcal{L}}(\mathbf{\Lambda}^+)\}.$$

This completes the proof of Conjecture 3.1.1 for our model. □

The following case is quite simple and it can be proved similar to theorem 3.3.

Theorem 3.4 *Assume that m^+, $m^- > 0$. Then the set $\mathbf{\Lambda}^+$ is almost closed, and the set $\mathbf{\Lambda}^-$ is transient. The process is simple and atomic. Conjecture 3.1.1 holds.*

Proof. We wish to apply Lemmas 3.2.3 and 3.2.6, as well as Corollary 3.2.3. The proof is then reduced to constructing a suitable Lyapunov function. Let $f(i) = i/m^+$, for $i > 0$ and $f(i) = i/m^-$ for $i \le 0$.

Let $B = \{0\}$, $C = 0$, and $k \equiv 1$. Then the set A from Lemma 3.2.3 is precisely Λ^+. The conditions of this lemma hold, if (3.2.2) holds for some $\epsilon > 0$.

For $i > 0$ we have

$$\mathsf{E}\{f(\xi_{n+1}) - f(\xi_n) \,|\, \xi_t = i\} = \frac{1}{m^+}\mathsf{E}\{\xi_{n+1} - \xi_n \,|\, \xi_n = i\} = \frac{m^+}{m^+} = 1.$$

For $i < 0$, this expectation equals 1 as well. We can take $\epsilon = 1$, and so the conditions of Lemma 3.2.3 hold. Application of this lemma, yields almost closedness of Λ^+ and transience of $\Lambda^- \cup \{0\}$, hence of Λ^-.

Similar to derivation in theorem 3.3 one can show that $\xi_{[tN]}/N$ converges almost surely to tm^+, i.e. conjecture 3.1.1 holds. □

3.4 Random walk on $\mathbf{Z}_+ \times \mathbf{Z}$

Here we will give a nice non-trivial example of the Euler limit. This limit is obtained through the *second vector field*. We will use the results of this section for our LD analysis in $\mathbf{Z}_+^2$. The theory of the second vector field has been introduced and studied in [7].

We start with the model description. Let ξ_n be a random walk on the state space

$$\mathbf{Z}_+ \times \mathbf{Z} = \{i = (i_1, i_2) : i_1 \in \mathbf{Z}_+, i_2 \in \mathbf{Z}\}$$

with two homogeneity faces:

$$\mathbf{\Lambda}^0 = \{i \in \mathbf{Z}_+ \times \mathbf{Z} : i_1 = 0\} \text{ and } \mathbf{\Lambda}^+ = \{i \in \mathbf{Z}_+ \times \mathbf{Z} : i_1 > 0\}.$$

This means that the transition probabilities are given by

$$p_{i,j} = \begin{cases} p^+_{j-i} & \text{if } i_1 \geq 1, \\ p^0_{j-i} & \text{if } i_1 = 0. \end{cases} \tag{3.4.1}$$

Clearly, $\mathbf{Z}_+ \times \mathbf{Z}$ is the half plane, where $\mathbf{\Lambda}^0$ is the vertical axis and $\mathbf{\Lambda}^+$ is the open half plane.

To our transition probabilities (3.4.1) there correspond two mean drift vectors $m^0 = (m^0_1, m^0_2)$ and $m^+ = (m^+_1, m^+_2)$. They are given by

$$m^0 = \sum_{k \in \mathbf{Z}^2} k p^0_k \text{ and } m^+ = \sum_{k \in \mathbf{Z}^2} k p^+_k.$$

The most interesting case of this model is described in the following theorem.

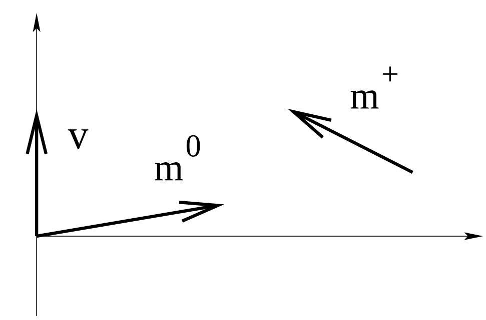

Figure 3.1: the second vector field v and the mean drifts m^0, m^+

Theorem 3.5 *Let $m^+_1 < 0 < m^0_1$ and $\xi_0 = (0, 0)$. Then for any $t > 0$*

$$\frac{\xi_{[tN]}}{N} \to tv \textit{ almost surely},$$

where the vector $v = (v_1, v_2) \in \mathbf{R}^2$ is defined as follows

$$v \triangleq \pi_0 m^0 + (1 - \pi_0) m^+ \textit{ with } \pi_0 = \frac{-m^+_1}{m^0_1 - m^+_1}.$$

The vector v is called the second vector field. *Note that $v_1 = 0$.*

Proof. First we provide the reader with the intuition for the proof of theorem 3.5.

When the random walk starts on the axis $\mathbf{\Lambda}^0$, it can not go far away from $\mathbf{\Lambda}^0$ deep into the face $\mathbf{\Lambda}^+$, because the first component of m^+ is negative, i.e. the random walk is always shifted back to the face $\mathbf{\Lambda}^0$ by the drift m^+. Hence, the random walk fluctuates along the axis $\mathbf{\Lambda}^0$ jumping from one face to the other. One can ask for the mean drift along the axis $\mathbf{\Lambda}^0$. It is defined exactly by the vector v, which is the linear combination of the mean drifts m^0 and m^+. The coefficients π_0 and $1 - \pi_0$ denote how often on average the random walk visits the faces $\mathbf{\Lambda}^0$ and $\mathbf{\Lambda}^+$ respectively.

For simplicity we suppose that $t = 1$. To show that the fluid limit almost surely exists we will construct a zero-mean martingale M_n. By the martingale limit theorem, M_n/n converges almost surely to 0. From this convergence we will find out the fluid limit.

Let us consider the first component of $\xi_n = (\xi_{n,1}; \xi_{n,2})$. Clearly, $\xi_{n,1}$ is defined on $\mathbf{Z}_+$ with transition probabilities

$$\hat{p}_{i_1,j_1} = \sum_{j_2 \in \mathbf{Z}} p_{(i_1,i_2),(j_1,j_2)},$$

where by condition (3.4.1) the sum does not depend on $i_2 \in \mathbf{Z}$ for any given $i_1,\ j_1 \in \mathbf{Z}_+$. It is easy to see that $\xi_{n,1}$ is a face-homogeneous random walk with the faces $\{0\}$ and $\{1, 2, 3, \cdots\}$ and the mean drifts m_1^0 and m_1^+.

Since $m_1^+ < 0$, the random walk $\xi_{n,1}$ is ergodic by lemma 2.3 (take $f(x) = i, k(i) = 1, \epsilon = m^+$ and $A = \{0\}$). Ergodicity implies that there is a stationary distribution, $\{\pi_{i_1} \geq 0, i_1 \in \mathbf{Z}_+\}$ say. The stationary distribution satisfies the following equations (see lemma 2.1.1):

$$\sum_{i_1} \pi_{i_1} m_1(i_1) = 0 \ \text{ and } \sum_{i_1} \pi_{i_1} = 1.$$

Hence,

$$0 = \sum_{i_1 \geq 0} \pi_{i_1} m_1(i_1) = \pi_0 m_1^0 + \sum_{i_1 > 0} \pi_{i_1} m_1^+ = \pi_0 m_1^0 + (1 - \pi_0) m_1^+.$$

From this equation we obtain that $\pi_0 = \frac{-m_1^+}{m_1^0 - m_1^+}$. Recall that $m_1^0 > 0$ and $m_1^+ < 0$, so that $\pi_0 > 0$ always.

In particular,

$$\pi_0 = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \mathbf{1}_{\{\xi_{l,1} = 0\}},$$

i.e. $\xi_{n,1}/n$ converges almost surely to 0.

Let us consider the second component $\xi_{n,2}$. We have that

$$v_2 = \pi_0 m_2^0 + (1 - \pi_0) m_2^+ = \frac{m_2^+ m_1^0 - m_1^+ m_2^0}{m_1^0 - m_1^+}.$$

Now we construct a function $f : \mathbf{Z}_+ \times \mathbf{Z} \to \mathbf{R}$ with the property

$$\mathsf{E}\{f(\xi_{n+1}) - f(\xi_n) \,|\, \xi_n = i\} = const \text{ for any state } i.$$

It appears that for the linear function

$$f(i) = \frac{m_2^+ - m_2^0}{m_1^0 - m_1^+} i_1 + i_2$$

it holds that

$$\mathsf{E}\{f(\xi_{n+1}) - f(\xi_n) \,|\, \xi_n = i\} = \frac{m_2^+ - m_2^0}{m_1^0 - m_1^+} \times m_1(i) + m_2(i) = v_2 \text{ for any state } i.$$

Then the sequence $f(\xi_n) - nm^+$ is a zero-mean martingale with $f(\xi_0) = 0$. By martingale limit theorem 3.2 the sequence $\frac{1}{n} f(\xi_n)$ converges almost surely to v_2. On other hand,

$$\frac{f(\xi_n)}{n} = \frac{m_2^+ - m_2^0}{m_1^0 - m_1^+} \times \frac{\xi_{n,1}}{n} + \frac{\xi_{n,2}}{n}.$$

We proved that $\frac{1}{n}\xi_{n,1}$ converges almost surely to 0. Hence, we conclude that $\frac{1}{n}\xi_{n,2}$ converges almost surely to v_2. □

3.5 Coupled Processors system

In this section we will illustrate the validity of Conjecture 3.1.1 for two special face-homogeneous random walks on the quarter plane. A characterization of the almost closed set structure for face-homogeneous random walks on $\mathbf{Z}_+^2$ can be found in [18].

As in the previous section, the first version has only one atomic closed class and the second has two.

3.5.1 Switched off processors whenever a queue is empty

Consider a system of two processors indexed by 1 and 2. Let $\lambda_1 > 0$ and $\lambda_2 > 0$ denote their input rates. Whenever both queues are non-empty, processor i works at speed $\mu_i > 0$, $i = 1, 2$. The moment queue 1 empties, processor 2 is switched off, and vice versa. We consider the time-discretized version obtained by uniformization, so that we may assume

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 \leq 1.$$

This model is a face-homogeneous random walk on $\mathbf{Z}_+^2$ with four homogeneity faces

$$\begin{aligned} \mathbf{\Lambda}_0 &= \{(0,0)\}, & \mathbf{\Lambda}_3 &= \{i \in \mathbf{Z}^2 \,|\, i_1 > 0, i_2 > 0\}, \\ \mathbf{\Lambda}_1 &= \{i \in \mathbf{Z}^2 \,|\, i_1 > 0, i_2 = 0\}, & \mathbf{\Lambda}_2 &= \{i \in \mathbf{Z}^2 \,|\, i_1 = 0, i_2 > 0\}. \end{aligned}$$

The jump probabilities from points in $\mathbf{\Lambda}_l$, $l = 0, 1, 2$ are given by

$$p_i^{\mathbf{\Lambda}_l} = \begin{cases} \lambda_1, & i_1 = 1, i_2 = 0, \\ \lambda_2, & i_1 = 0, i_2 = 1, \\ 1 - \lambda_1 - \lambda_2, & i_1 = i_2 = 0. \end{cases}$$

On face $\mathbf{\Lambda}_3$ they are given by

$$p_i^{\mathbf{\Lambda}_3} = \begin{cases} \lambda_1, & i_1 = 1, i_2 = 0, \\ \lambda_2, & i_1 = 0, i_2 = 1, \\ \mu_1, & i_1 = -1, i_2 = 0, \\ \mu_2, & i_1 = 0, i_2 = -1, \\ 1 - \lambda_1 - \lambda_2 - \mu_1 - \mu_2, & i_1 = i_2 = 0. \end{cases}$$

In the two-dimensional model, the drift vector $m^{\mathbf{\Lambda}}$ has two components $m_1^{\mathbf{\Lambda}}$ and $m_2^{\mathbf{\Lambda}}$. The same applies to the field $v^{\mathbf{\Lambda}}$ for an ergodic induced chain $\xi_n^{\mathbf{\Lambda}}$. So far, we have not made any assumptions on the parameters. Let us assume first that $\lambda_i < \mu_i$, $i = 1, 2$. Then the induced chains $\xi_n^{\mathbf{\Lambda}_1}$ and $\xi_n^{\mathbf{\Lambda}_2}$ are easily checked to be ergodic. They have a one dimensional state space. For instance, for $\mathbf{\Lambda}_1$ one can take $(1, 0) + (0, \mathbf{Z}_+)$ and so we can identify it with $\mathbf{Z}^+$. It has jump probabilities

$$\mathsf{P}\{\xi_{n+1}^{\mathbf{\Lambda}_1} = j \,|\, \xi_n^{\mathbf{\Lambda}_1} = i\} = \begin{cases} \lambda_2, & j = i + 1 \\ \mu_2 \mathbf{1}_{\{i>0\}}, & j = i - 1 \\ 1 - \lambda_2 - \mu_2 \mathbf{1}_{\{i>0\}}, & j = i, \end{cases}$$

and so ergodicity follows, since $\lambda_2 < \mu_2$. Moreover, $\pi_0^{\mathbf{\Lambda}_1} = 1 - \lambda_2/\mu_2$. Hence,

$$\begin{aligned} v^{\mathbf{\Lambda}_1} &= \big(1 - \frac{\lambda_2}{\mu_2}\big) m^{\mathbf{\Lambda}_1} + \frac{\lambda_2}{\mu_2} m^{\mathbf{\Lambda}_3} \\ &= (1 - \frac{\lambda_2}{\mu_2}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + \frac{\lambda_2}{\mu_2} \begin{pmatrix} \lambda_1 - \mu_1 \\ \lambda_2 - \mu_2 \end{pmatrix} \\ &= \frac{1}{\mu_2} \begin{pmatrix} \lambda_1\mu_2 - \mu_1\lambda_2 \\ 0 \end{pmatrix}. \end{aligned}$$

Similarly $\xi_n^{\mathbf{\Lambda}_2}$ is an ergodic induced chain with

$$v^{\mathbf{\Lambda}_2} = \frac{1}{\mu_1} \begin{pmatrix} 0 \\ \lambda_2\mu_1 - \mu_2\lambda_1 \end{pmatrix} = \frac{\mu_2}{\mu_1} \begin{pmatrix} 0 \\ -v_1^{\mathbf{\Lambda}_1} \end{pmatrix}.$$

Suppose that $v_1^{\mathbf{\Lambda}_1} > 0$, in other words, $\lambda_1/\mu_1 > \lambda_2/\mu_2$. Then $v_2^{\mathbf{\Lambda}_2} < 0$. Clearly, $\xi^{\mathbf{\Lambda}_3}$ is ergodic since it is a chain living on a one-point set. Hence, $v^{\mathbf{\Lambda}_3} = m^{\mathbf{\Lambda}_3}$. The assertion in Corollary 3.2.3 suggests defining a continuous dynamical system $u(\cdot)$ on $\mathbf{R}^2$ satisfying

$$\frac{d^+}{dt} u(t) = v^{\mathbf{\Lambda}_l}, \quad u(t) \in \mathbf{\Lambda}_l, l \neq 0, \tag{3.5.1}$$

for initial condition $u(0) = 0$, where d^+/dt denotes the right derivative. It is uniquely defined under the above conditions. Define $S_1 = \{i \in \mathbf{Z}_+^2 \,|\, i_2 < (\mu_2/\mu_1) i_1\}$. Note that it contains $\mathbf{\Lambda}_1$!

Theorem 3.6 *Assume $\lambda_i < \mu_i$, $i = 1, 2$, and $\lambda_1\mu_2 > \lambda_2\mu_1$. Then the set S_1 is almost closed and atomic, and $\mathbf{Z}_+^2 \setminus S_1$ is transient. The process is simple. Conjecture 3.1.1 holds.*

Proof. Almost closedness of S_1 and transience of $\mathbf{Z}_+^2 \setminus S_1$ will follow from Lemma 3.2.3, by checking the conditions of this lemma.

Let $B = \{(0,0)\}$, $C = 0$, $k \equiv 1$ and $f(i) = \mu_2 i_1 - \mu_1 i_2$. Then

$$S_1 = \{i \in \mathbf{Z}_+^2 \mid f(i) > C\}.$$

We only need to check that (3.2.2) holds for all $i \neq (0,0)$, and some $\epsilon > 0$. For any $i \neq (0,0)$

$$\mathsf{E}\{f(\xi_{n+1}) - f(\xi_n) \mid \xi_n = i\} = \lambda_1\mu_2 - \lambda_2\mu_1 > 0,$$

so that (3.2.2) holds for $i \neq (0,0)$ and $\epsilon = \lambda_1\mu_2 - \lambda_2\mu_1$.

Next, by Remark 3.2.1, S_1 is representative. Also $\mathsf{proj}^{\mathbf{\Lambda}_1}(S_1) \subset \mathbf{\Lambda}_1$. The conditions of Lemma 3.2.6 are satisfied for set $S = S_1$ and face $\mathbf{\Lambda}_1$. It follows that S_1 is atomic, and so the process is simple.

The validity of Conjecture 3.1.1 finally follows from Corollary 3.2.3, by the following decomposition:

$$\begin{aligned}
&\mathsf{P}\{\lim_{N\to\infty} \frac{\xi_{[tN]}}{N} = tv^{\mathbf{\Lambda}_1}\} = \\
&\quad = \mathsf{P}\{\lim_{N\to\infty} \frac{\xi_{[tN]}}{N} = tv^{\mathbf{\Lambda}_1} \mid \underline{\mathcal{L}}(S_1) \cap \Omega_i\}\mathsf{P}\{\underline{\mathcal{L}}(S_1) \cap \Omega_i\} + \\
&\quad\quad + \mathsf{P}\{\lim_{N\to\infty} \frac{\xi_{[tN]}}{N} = tv^{\mathbf{\Lambda}_1} \mid \Omega_i \setminus \underline{\mathcal{L}}(S_1)\}\mathsf{P}\{\Omega_i \setminus \underline{\mathcal{L}}(S_1)\} \\
&\quad = \mathsf{P}\{\underline{\mathcal{L}}(S_1) \cap \Omega_i\} = 1,
\end{aligned}$$

since $\Omega_i \setminus \underline{\mathcal{L}}(S_1)$ is a null-set. □

3.5.2 Switched off processors with additional input

The previous model has a nice 'simple' structure. In a system with arrival control, it seems not unnatural to allowing more customers or jobs to enter a queue, whenever it is empty. Indeed, this might reduce idle server time. Upon allowing this, 'non-simple' structures may appear.

Keeping the rates for 2 non-empty queues equal to the previous model, we allow arrival rates λ_1^l, λ_2^l on faces $\mathbf{\Lambda}_l$, $l = 0,1,2$. In this case, the jump probabilities from points in $\mathbf{\Lambda}_l$, $l = 0,1,2$ are given by

$$p_i^{\mathbf{\Lambda}_l} = \begin{cases} \lambda_1^l, & i_1 = 1, i_2 = 0, \\ \lambda_2^l, & i_1 = 0, i_2 = 1, \\ 1 - \lambda_1^l - \lambda_2^l, & i_1 = i_2 = 0. \end{cases}$$

Again assuming $\lambda_1 < \mu_1$, $\lambda_2 < \mu_2$ the induced chains $\xi_t^{\mathbf{\Lambda}_l}$, $l = 1,2$, are ergodic. Identifying their state space with $\mathbf{Z}_+$, the induced chain $\xi_n^{\mathbf{\Lambda}_1}$, has jump probabilities

$$\mathsf{P}\{\xi_{n+1}^{\mathbf{\Lambda}_1} = j \,|\, \xi_n^{\mathbf{\Lambda}_1} = i\} = \begin{cases} \lambda_2^1 \mathbf{1}_{\{i=0\}} + \lambda_2 \mathbf{1}_{\{i>0\}}, & j = i+1 \\ \mu_2 \mathbf{1}_{\{i>0\}}, & j = i-1 \\ 1 - \lambda_2^1 \mathbf{1}_{\{i=0\}} + \lambda_2 \mathbf{1}_{\{i>0\}} - \mu_2 \mathbf{1}_{\{i>0\}}, & j = i. \end{cases}$$

Now we have, $\pi_0^{\mathbf{\Lambda}_1} = (\mu_2 - \lambda_2)/(\mu_2 - \lambda_2 + \lambda_2^1)$. This yields

$$\begin{aligned} v^{\mathbf{\Lambda}_1} &= \frac{\mu_2 - \lambda_2}{\mu_2 - \lambda_2 + \lambda_2^1} m^{\mathbf{\Lambda}_1} + \frac{\lambda_2^1}{\mu_2 - \lambda_2 + \lambda_2^1} m^{\mathbf{\Lambda}_3} \\ &= \frac{\mu_2 - \lambda_2}{\mu_2 - \lambda_2 + \lambda_2^1} \begin{pmatrix} \lambda_1^1 \\ \lambda_2^1 \end{pmatrix} + \frac{\lambda_2^1}{\mu_2 - \lambda_2 + \lambda_2^1} \begin{pmatrix} \lambda_1 - \mu_1 \\ \lambda_2 - \mu_2 \end{pmatrix} \\ &= \frac{1}{\mu_2 - \lambda_2 + \lambda_2^1} \begin{pmatrix} \lambda_1^1(\mu_2 - \lambda_2) - \lambda_2^1(\mu_1 - \lambda_1) \\ 0 \end{pmatrix}. \end{aligned}$$

For $\mathbf{\Lambda}_2$ we get similarly,

$$v^{\mathbf{\Lambda}_2} = \frac{1}{\mu_1 - \lambda_1 + \lambda_1^2} \begin{pmatrix} 0 \\ -\lambda_1^2(\mu_2 - \lambda_2) + \lambda_2^2(\mu_1 - \lambda_1) \end{pmatrix}.$$

In order that both $v_1^{\mathbf{\Lambda}_1}$ and $v_2^{\mathbf{\Lambda}_2}$ are positive, it is sufficient to require

$$\frac{\lambda_1^2}{\lambda_2^2} < \frac{\mu_1 - \lambda_1}{\mu_2 - \lambda_2} < \frac{\lambda_1^1}{\lambda_2^1}. \tag{3.5.2}$$

Using (3.5.1) we can define a continuous dynamical system $u(t)$ for initial condition $u(0) = (0,0)$. However, it is not uniquely defined for $t \geq 0$. We have two possible realizations, both occurring with positive probability. Choose a_1 and a_2 satisfying

$$\frac{\lambda_1^2}{\lambda_2^2} < a_2 < \frac{\mu_1 - \lambda_1}{\mu_2 - \lambda_2} < \frac{1}{a_1} < \frac{\lambda_1^1}{\lambda_2^1}.$$

Let $S_1 = \{i \in \mathbf{Z}_+^2 \,|\, a_1 i_1 > i_2\}$ and $S_2 = \{i \in \mathbf{Z}_+^2 \,|\, i_1 < a_2 i_2\}$. Then $S_1 \cap S_2 \neq \emptyset$.

Theorem 3.7 *Assume $\lambda_i < \mu_i$, $i = 1, 2$, and (3.5.2). Then each set S_i is atomic and almost closed. The state space has the almost closed set decomposition $\mathbf{Z}_+^2 = S_1 \cup S_2$. Conjecture 3.1.1 holds.*

Proof. The first statement on almost closedness of S_1 and S_2 follows from a by now straightforward application of Lemma 3.2.1 with the functions $f_1(i) = a_1 i_1 - i_2$ and $f_2(i) = -i_1 + a_2 i_2$ respectively, step-function $k \equiv 1$, and constant $C = 0$.

Atomicity follows from Remark 3.2.1 by applying Lemma 3.2.6 for sets S_1 and S_2 and faces $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ respectively. Note again that $\mathsf{proj}^{\mathbf{\Lambda}_l}(S_l) = \mathbf{\Lambda}_l$.

Since $S_1 \cup S_2 = \mathbf{Z}_+^2$, they are the unique almost closed sets, modulo transient ones. This proves that the almost closed set decomposition consists of two atomic sets.

The final proof of (3.1.1) is again by a conditioning argument along the same lines as in the proof of Theorem 3.6. For completeness we give it here for the path $tv^{\mathbf{\Lambda}_1}$:

$$
\begin{aligned}
&\mathsf{P}_i\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}_1}t\} = \\
= \quad & \mathsf{P}_i\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}_1}t \,|\, \underline{\mathcal{L}}(S_1)\cap\Omega_i\}\mathsf{P}_i\{\underline{\mathcal{L}}(S_1)\cap\Omega_i\} + \\
+ \quad & \mathsf{P}_i\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}_1}t \,|\, \underline{\mathcal{L}}(S_2)\cap\Omega_i\}\mathsf{P}_i\{\underline{\mathcal{L}}(S_2)\cap\Omega_i\} + \\
+ \quad & \mathsf{P}_i\{\lim_{N\to\infty}\frac{\xi_{[tN]}}{N} = v^{\mathbf{\Lambda}_1}t \,|\, \Omega_i\setminus\big(\underline{\mathcal{L}}(S_1)\cup\underline{\mathcal{L}}(S_2)\big)\}\mathsf{P}_i\{\Omega_i\setminus\underline{\mathcal{L}}(S_1)\cup\underline{\mathcal{L}}(S_2)\} \\
= \quad & \mathsf{P}_i\{\underline{\mathcal{L}}(S_1)\cap\Omega_i\} = \mathsf{P}_i\{\underline{\mathcal{L}}(S_1)\},
\end{aligned}
$$

since $\mathsf{P}_i\{\lim_{N\to\infty}\xi_{[tN]}/N = v^{\mathbf{\Lambda}_1}t \,|\, \underline{\mathcal{L}}(S_2)\cap\Omega_i\} = 0$ and $\mathsf{P}_i\{\Omega_i\setminus\underline{\mathcal{L}}(S_1)\cup\underline{\mathcal{L}}(S_2)\} = 0$. This completes the proof of Conjecture 3.1.1. □

Conclusion

This chapter has shown Conjecture 3.1.1 for interesting face-homogeneous random walks on **Z** and $\mathbf{Z}_+^2$. In particular, we have provided tools for characterizing the almost closed set structure and fluid limits, in case of an almost closed set decomposition of the state space into *atomic* sets. The construction of suitable Lyapunov functions is the yet un-tackled Achilles' heel of completing the characterization of such face-homogeneous random walks. Partial results exist, using in fact the continuous dynamical systems defined by the second vector field for sub-chains (cf. for instance the basic reference [7]).

Chapter 4

Non-existence of a stochastic fluid limit

This chapter has appeared as N.Popov, F.M.Spieksma (2002). *Non-existence of a stochastic fluid limit for a cycling random walk.* Technical Report, MI 2002-25, Leiden University.

4.1 Introduction

In the framework of a dynamical systems or fluid approximation approach to queueing networks we consider an example of continuous scattering. In other words, an example with an infinite number of Euler (or fluid) trajectories starting at the same point, $\mathsf{0}$ say (by $\mathsf{0}$ we denote further the state (0,0)). Our example is the simplest possible, but nevertheless non-trivial. This example is similar to the case of the random walk in $\mathbf{Z}_+^3$, (see [7]) with escape to infinity by cyclically rotating over all three faces of the octant. The well-known and simplest example in the queueing setting exhibiting this phenomenon, is a priority queueing network (cf. [20]). For terminological reasons we consider the two dimensional analogs of these cases.

Our model is a random walk $\{\xi_n\}_{n=0,1,\dots}$ in $\mathbf{Z}^2$, with homogeneous jumps inside each of the four quarter planes (see Figure 1.1, where the drifts are shown). By virtue of the law of large numbers, we know that for any sufficiently small time intervals of length t, the following limit (in distribution) of the space-time scaled process exists

$$\frac{\xi_{[tN]}}{N} \xrightarrow{\mathfrak{D}} u(x;t), \quad N \to \infty, \tag{4.1.1}$$

whenever the random walk starts at $\xi_0 = [xN]$, provided that $x \in \mathbf{R}^2$ is a given point in the interior of any of the quadrants. Here $u(x;t)$, $t > 0$, is defined by the drifts and the initial condition $u(x,0) = x$. We will call this function the deterministic dynamical system, since it has the property

$$u\big(u(x;t);s\big) = u(x;t+s) \text{ for any } t,s \in \mathbf{R}_+ \text{ and } x \neq \mathsf{0}.$$

The ergodicity and transience conditions for this random walk in terms of the dynamical system are quite obvious: if the trajectories of the dynamical system go to infinity then the random walk is transient and if they converge to 0, then the random walk is ergodic. Such trajectories are called *Euler or fluid paths*.

In case the Euler paths converge to 0, it is clear that ξ_N/N converges to 0 in distribution and even a.s., for any *fixed* initial point. Indeed, using techniques from [26] and [25] one can show that the walk is exponentially ergodic. Thus, once having reached a bounded set, the walk can only move outside it with exponentially small probability.

This chapter studies the case of *diverging Euler paths*, where the situation is quite different. The random walk ξ_n starting at a given point x, turns out to spread out over all Euler trajectories. This is the reason why the time scaled process cannot converge in distribution. Indeed, if the process at macro-time is close to a certain Euler path, then scaling it linearly in time yields a cyclically rotating point along a closed curve that is homeomorphic to a circle. In this light, it is not surprising that for the limit distribution to exist, it has to be invariant with respect to the dynamical system associated with Euler paths.

Note that, as a consequence, in general there cannot be any convergence in distribution of the time scaled process $\{\xi_{[tN]}/N \mid 0 \le t \le T\}$ over macro-time intervals.

The main tool used here is an extension of Kolmogorov's inequality for i.i.d. random variables. This is a technical derivation, complicated by dispersion occurring whenever an axis is passed. In the way, we show that sets of cones defined by Euler paths, are so-called *sojourn sets*.

This chapter is the starting point for further analysis of the space decomposition into closed sets (cf. Chung [5]) and scattering phenomena. The underlying idea for face-homogeneous random walks is the following. Decomposition into atomic sets should be equivalent to the occurrence of a discrete set of Euler paths, over which the process scatters in the long run. The scattering probabilities are then equal to the absorption probabilities of the atomic sets of the decomposition. More details can be found in Chapter 3. Ours will turn out to be an example of the state space being a single non-atomic set and there is continuous scattering over the (continuous) set of all Euler trajectories. In [24] it will be proved that each trajectory turns out to be chosen randomly by a probability measure μ_q on the set of all trajectories starting at a given point q. Identifying each point of a circle around the origin with a trajectory, one can then say that the Poisson boundary is isomorphic to a circle.

4.2 The model

We consider an aperiodic irreducible Markov chain $\{\xi_n\}_{n=0,1,\dots}$ on the state space $\mathbf{Z}^2$ in discrete time. If the random walk starts at point $x \in \mathbf{Z}^2$, then this position at time n will be denoted by $\xi_n(x)$. We will assume the random walk to be *face-homogeneous*, i.e. the transition probabilities from two states, the components of which have the same sign (+,-, or 0) are equal. For any $v \in \mathbf{R}$, let $\mathrm{sgn}(v) = +, -, 0$ whenever $v > 0$, $v < 0$ or $v = 0$ respectively. Then we can denote the nine faces by Q^{ab}, $a, b \in \{+, -, 0\}$, where

$$Q^{ab} = \{x \in \mathbf{R}^2 \mid \mathrm{sgn}(x_1) = a, \mathrm{sgn}(x_2) = b\}.$$

Further, denote the mean drift in a point $i \in Q^{ab} \cap \mathbf{Z}^2$ by

$$m(i) \equiv m^{ab} = (m_1^{ab}, m_2^{ab}) = \sum_j (j-i)\mathsf{P}\{\xi_{n+1} = j \,|\, \xi_n = i\}.$$

Since the transitions on the half axes have a minor influence on the large time behavior of the process, we will assume that the drifts from points on axes coincide with the counterclockwise quadrant following the half-axis in question.

Summarizing, we assume that the one step transition probabilities

$$p_{i,j} = \mathsf{P}\{\xi_{n+1} = j \,|\, \xi_n = i\}$$

satisfy

(i) homogeneity condition $p_{i,j} = p^{ab}_{j-i}$, for $i \in Q^{ab} \cap \mathbf{Z}^2$;

(ii) boundedness of jumps $p^{ab}_k = 0$ unless $-1 \le k_1, k_2 \le 1$.

(iii) drift condition $m^{+0} = m^{++}$, $m^{0+} = m^{-+}$, $m^{-0} = m^{--}$, $m^{0-} = m^{+-}$.

We also assume that

$$m_1^{++} < 0,\ m_2^{++} > 0,\ m_1^{-+} < 0,\ m_2^{-+} < 0,\ m_1^{--} > 0,\ m_2^{--} < 0,\ m_1^{+-} > 0,\ m_2^{+-} > 0,$$

i.e. these drifts are of the form shown in Figure 1.1. The transience and ergodicity regions in the parameter space are a.s. determined by the value of the constant

$$C = \frac{m_2^{++} m_1^{-+} m_2^{--} m_1^{+-}}{m_1^{++} m_2^{-+} m_1^{--} m_2^{+-}}. \tag{4.2.1}$$

Lemma 4.2.1 *The random walk is transient if $C > 1$. It is ergodic if $C < 1$.*

We will show this result in section 4.4.2. Our basic assumption can now be reformulated as follows.

Assumption 4.2.1 *It holds that $C > 1$, and so the random walk is transient.*

Next we define the continuous time dynamical system $u(x;t)$ associated with our Markov chain $\{\xi_n\}$. It is a continuous mapping $u : \mathbf{R}^2 \setminus \{0\} \times \mathbf{R} \to \mathbf{R}^2$, defined by the mean drift vector field and initial condition $u(x;0) = x$. More precisely, $u(x;t)$ is a *continuous* piecewise linear function of t with

$$\begin{cases} u(x;0) = x & x \neq 0 \\ \dfrac{d}{dt} u(x;t) = m(u(x;t)), & t \in \mathbf{R},\ \text{if } u(x;t) \in Q^{ab},\ a,b = +,-. \end{cases} \tag{4.2.2}$$

For $x \in Q^{ab}, a, b = +, -$, and t sufficiently small, we have that $u(x;t) = x + tm^{ab}$. We call this mapping dynamical system, since it satisfies the following property:

$$u\big(u(x;s);t\big) = u(x;s+t), \quad x \neq 0, s, t > 0. \tag{4.2.3}$$

One more important property of $u(x;t)$ is *homogeneity*:

$$u(\alpha x; \alpha t) = \alpha u(x;t), \quad x \neq 0, \alpha > 0, t \in \mathbf{R}. \tag{4.2.4}$$

Note that the dynamical system cycling off to infinity coincides with $C > 1$. Thus transience in this case is evident. Formally: $\|u(x;t)\| \to \infty$, as $t \to +\infty$, $x \neq 0$ (see Figure 1.2). Similarly, $\|u(x;t)\| \to 0$, as $t \to -\infty$.

At this point we will introduce some other notations that will be frequently used.

Euler or *fluid paths* are defined to be the trajectories of u. The Euler path Γ_x starting at $x \neq 0$ equals

$$\Gamma_x = \{u(x;t),\, t \in \mathbf{R}\}.$$

Note that $\mathbf{R}^2 = \cup_{x \in \mathcal{I}} \Gamma_x$ and two Euler paths intersect only at the point 0.

By *the cycle time* we understand the mapping $\tau : \mathbf{R}^2 \setminus \{0\} \to \mathbf{R}_+$ given by

$$\tau(x) = \min\big\{t > 0 \,|\, u(x;t) \in \{\alpha x \,|\, \alpha > 0\}\big\},$$

so that $\tau(x)$ is the time that the dynamical system starting at x needs to pass precisely one cycle (see figure 4.1). It appears (see lemma 4.4.1) that

$$u\big(x; \tau(x)\big) = Cx,$$

By homogeneity of u, the cycle time is homogeneous as well:

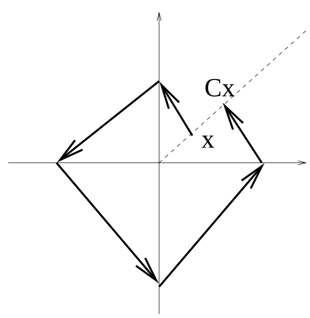

Figure 4.1: a cycle from point x to point Cx

$$\tau(\alpha x) = \alpha \tau(x), \quad \alpha > 0. \tag{4.2.5}$$

In lemma 4.4.1 we will prove that $\tau(x)$ is a piecewise linear function of x.

Finally, we introduce the ‘Euler distance to 0’.

Definition 4.1 *For $x \neq 0$ the Euler distance $r(x)$ to the point 0 is defined by*

$$r(x) = \inf\{t > 0 \,|\, u(x;-t) = 0\}.$$

The set $\mathcal{I}(t) = \{x \in \mathbf{R}^2 \,:\, t = r(x)\}$, $t > 0$, will be called the isochrone *at (Euler) distance t.*

The following lemma establishes a simple relation between cycle time and Euler distance.

Lemma 4.2.2 *We have that*

$$r(x) = \frac{\tau(x)}{C-1},$$

with C defined in (4.2.1). Hence, points at the isochrone $\mathcal{I}(s)$ at distance s have cycle time $s(C-1)$.

Proof. Let us calculate the time $r(x)$ to reach zero from the point x. First note that $u(x;t) = y$ implies $u(y;-t) = x$, for any $x \in \mathbf{R}^2 \setminus \{0\}$ and $t > 0$.

Then from definition of $r(\cdot)$ and $\tau(\cdot)$ it follows directly that

$$r(x) = r(xC^{-1}) + \tau(xC^{-1}), \tag{4.2.6}$$

since the cycle time is measured forwards in time. As $r(y) \to 0$ as $y \to 0$, we get

$$r(x) = \sum_{k=1}^{\infty} \tau(xC^{-k}) = \tau(x) \sum_{k=1}^{\infty} C^{-k} = \tau(x)C^{-1}\frac{1}{1-C^{-1}} = \frac{\tau(x)}{C-1}.$$

□

The following general result holds. It is the starting point for our analysis.

Proposition 4.2.1 *The Euler limit with $\xi_0 = [xN]$*

$$\lim_{N\to\infty} \frac{\xi_{[tN]}}{N} \stackrel{\mathfrak{D}}{=} u(x;t)$$

exists in distribution for any $x = (x_1, x_2) \neq 0$ and $t \geq 0$.

For small values of t, this follows from the law of large numbers. For larger t this statement is evident, although we do not know of any existing proof for this particular model. This is a general feature of convergence results of this type: in spite of its being an obvious fact, so far there are only proofs tuned to special classes of walks.

We will prove this result in the next section, as it will be the basis of our further analysis of the random walk starting at a *fixed* point,

4.3 Euler limit estimates for starting points $x \neq 0$

For convenience we denote by

$$\xi_t(x)$$

the position of the random walk at time t with initial state $\xi_0 = x$.

For any initial point $x \neq 0$ we will first estimate the distance $\|\xi_t(x) - u(x;t)\|$ and then we will prove Proposition 4.2.1. If we would have a completely homogeneous random walk, than we could immediately apply (modulo generalization to vector sequences with martingale components) the Azuma-Hoeffding inequality for martingales (see Lemma 2.2.2). In our case of only *face*-homogeneity, the sequence

$$M_{t+1} = \xi_{t+1}(x) - U(x;t),$$

with

$$U(x;t) = \sum_{n=0}^{t} m\big(\xi_n(x)\big)$$

is a sequence, the components of which form a martingale sequence. However, $U(x;t)$ itself is random. So, in order to get bounds on the distance $\|\xi_t(x) - u(x;t)\|$ by using the Azuma-Hoeffding-inequality (see lemma 2.2.2), we have to bound $\|U(x;t) - u(x;t)\|$. Unfortunately, our investigations did not result in essentially stronger results than the bounds from Lemma 4.3.1 below. Since this was our earlier approach, we have decided to leave it intact, and use the martingale approach later on, where we will need it.

The main problem in this derivation is the following. Starting at a point far away from the origin, it takes a long time before any axis is crossed. Inside quadrants, the walk behaves as a sum of i.i.d. random variables, and the law of large numbers applies.

However, the chosen drifts guarantee that sooner or later some axis is hit. For some time the walk is controlled by two different distributions, until (exponentially quick!) it goes into the next quadrant. Again after some time the law of large number regime applies, based on another jump distribution than before. The time between passing from one law of large number regime to another allows for some dispersion that should be controlled.

Boundedness of the number of face-transitions in Lemma 4.3.1 is mainly used in order to control the total amount of dispersion. An additional problem will arise later, since Euler paths are not parallel, but their distance blows up a factor C each cycle.

For any $x \in \mathbf{R}^2 \setminus \{0\}$ we define

$$n(x) = \inf_{t<\tau(x)} \|u(x;t)\|.$$

In the following lemma we estimate the probability for the random walk to be inside a tube of radius v of the path $u(x,s), 0 \leq s \leq t$, where $t > 0$ is any fixed number. The radius v should be sufficiently small :

(i) $v < n(x)$, meaning that the v-tube does not contain 0 ;

(ii) $2v < n(x) \cdot |C-1| = \inf_{s<\tau(x)} \|Cu(x,s) - u(x,s)\|$,
meaning that "neighboring parts" of the v-tube do not intersect.

This yields the following upper bound for v

$$v < n(x) \cdot \min\{\frac{|C-1|}{2}, 1\}. \tag{4.3.1}$$

For convenience we reformulate this upper bound in terms of the cycle time $\tau(x)$. Just note that there exists a positive constant g such that

$$n(x) > g \cdot \tau(x) \;\text{ for all } x \in \mathbf{R}^2 \setminus \{0\}$$

and therefore there exists a constant $\theta = \theta(C) > 0$ such that

$$v < \theta \cdot \tau(x) \tag{4.3.2}$$

implies (4.3.1).

Lemma 4.3.1 (Extension of Kolmogorov's inequality) *There exists a constant $c > 0$ such that for any constant v satisfying (4.3.2), the following holds*

$$\mathsf{P}\{\max_{0\le k\le t} \|\xi_k(x) - u(x;k)\| \le v\} \ge 1 - c \cdot \frac{t}{v^2}, \tag{4.3.3}$$

for any time t and any sufficiently large initial state $x \in \mathbf{Z}^2$. The constant c depends only on the number of face-transitions of the trajectory $u(x;s), 0 \le s \le t$.

Clearly, inequality (4.3.3) is trivially true whenever $v \le \sqrt{c \cdot t}$.

Proof. It is sufficient to prove the assertion for $x \in Q^{++}$. Let

$$t_0 = \frac{x_1}{|m_1^{++}|}$$

be the first time that the dynamical system $u(x;\cdot)$ hits Q^{0+}. Similarly, denote by

$$t_1 = \frac{x_2 + t_0 m_2^{++}}{|m_2^{-+}|} + t_0$$

the first time that the dynamical system hits Q^{-0}. In the time interval $[0, t_1)$ the dynamical system starting at x has precisely one face-transition.

Fix a constant $w > 1$. First we consider the case of $x_1, x_2 > w$ and x_1 is sufficiently large. The proof will establish the Euler limit estimate for this given x, $t < t_1$ and $w > 1$ such that the disc $\{y \in \mathbf{R}^2 \mid \|u(x;t) - y\| \le w \cdot \beta\}$ is contained in the interior of Q^{-+}, for some constant $\beta > 1$. The estimate for general t then follows by a finite glueing procedure applied to subpaths ending and starting inside a quadrant.

We consider three cases $t \le t_0 - w|m_1^{++}|^{-1}$, the dispersion situation

$$t_0 - w|m_1^{++}|^{-1} < t \le t_0 + 4\gamma^{-1}w, \text{ where } \gamma = \min\{|m_1^{++}|, |m_1^{-+}|\},$$

and $t_0 + 4\gamma^{-1}w < t < t_1$.

1. Case of $t < t_0 - w|m_1^{++}|^{-1}$.

Before the first face-transition occurs, the random walk inside a quadrant Q^{ab}, $a, b = +, -$, behaves as the sum of i.i.d. random vectors, i.e.

$$\xi_l(x) = x + \sum_{n=1}^{l} \eta_n^{ab},$$

where η_n^{ab} are i.i.d. random vectors with distribution

$$\mathsf{P}\{\eta_n^{ab} = q\} = p_q^{ab}$$

and with expectation $\mathsf{E}\eta_n = m^{ab}$ and variance vectors D^{ab}, $a, b = +, -$. Let

$$\sqrt{D} = \max\{\|D^{++}\|, \|D^{-+}\|, \|D^{--}\|, \|D^{+-}\|\}.$$

Denote $S_k^{ab} = (S_{k,1}^{ab}, S_{k,2}^{ab}) = \sum_{n=1}^{k} \eta_n^{ab}$. By Kolmogorov's inequality for random vectors we have for any $w > 0$ and any integer $t < t_0 - w|m_1^{++}|^{-1}$

$$\mathsf{P}\{\|S_k^{ab} - km^{ab}\| \leq w, \text{ for all } k \leq t\} \geq 1 - D \cdot \frac{t}{w^2}, \tag{4.3.4}$$

for the constant D and for any $a, b = +, -$.

This tube of radius w is contained in Q^{++}: indeed $u(x; t_0 - w|m_1^{++}|^{-1}) = x + m^{++}(t_0 - w|m_1^{++}|^{-1})$ has first component equal to $x_1 + m_1^{++} t_0 + w = w$ and the second component is increasing in the time variable. Thus the probability of all trajectories inside this tube for the unrestricted S_l-process and our random walk are equal. We find that

$$\mathsf{P}\{\max_{k \leq t} \|\xi_k(x) - u(x;k)\| \leq w\} \geq 1 - D \cdot \frac{t}{w^2}. \tag{4.3.5}$$

So for $t < t_0 - w|m_1^{++}|^{-1}$ the statement of the lemma holds for $c = D$ and $v = w$.

2. Case of $t_0 + 4\gamma^{-1}w \geq t \geq t_0 - w|m_1^{++}|^{-1}$.

Since the jumps have size at most one both horizontally and vertically, the ℓ^2 norm of the jump size is bounded by $\sqrt{2}$. Hence, in time

$$w(|m_1^{++}|^{-1} + 4\gamma^{-1}) \leq w \cdot 5\gamma^{-1},$$

the covered distance has norm at most $5\sqrt{2}\gamma^{-1}w$. Consequently, (4.3.5) implies that

$$\mathsf{P}\{\max_{k \leq t} \|\xi_k(x) - u(x;k)\| \leq w + 5\sqrt{2}w\gamma^{-1}\} \geq 1 - D \cdot \frac{t_0 - w|m_1^{++}|^{-1}}{w^2} > 1 - D \cdot \frac{t}{w^2}.$$

Putting $v = (1 + 5\sqrt{2}\gamma^{-1})w$, this implies

$$\mathsf{P}\{\max_{k \leq t} \|\xi_k(r) - u(r;k)\| \leq v\} \geq 1 - D \cdot \frac{t}{w^2} \geq 1 - c \cdot \frac{t}{v^2}, \tag{4.3.6}$$

with $c = D \cdot (1 + 5\sqrt{2}\gamma^{-1})^2$.

3. Case of $t_1 > t > t_0 + 4\gamma^{-1}w$.

Denote by

$$\tau_0 = \min\{k > 0 \,|\, \xi_k(x) \in Q^{0+}\}$$

the first hitting time of the axis Q^{0+} for the random walk we find by Kolmogorov's inequality that

$$\mathsf{P}\{C(x,w)\} \geq 1 - D \cdot \frac{t_0 + w|m_1^{++}|^{-1}}{w^2} \tag{4.3.7}$$

where

$$C(x,w) = \left\{ \begin{array}{l} \max_{k \leq \tau_0} \|\xi_k(x) - u(x,k)\| \leq w \\ t_0 - w|m_1^{++}|^{-1} \leq \tau_0 \leq t_0 + w|m_1^{++}|^{-1} \end{array} \right\}.$$

The event $C(x,w)$ implies that

$$\xi_{\tau_0}(x) \in C^*(x,w) = \{y \in \mathbf{Z}^2 \,|\, y_1 = 0,\ |y_2 - u_2(x;t_0)| \le w(1 + |m_1^{++}|^{-1})\}.$$

This is because of the event $C(x,w)$ implying that τ_0 occurs earliest at time $t_0 - w|m_1^{++}|^{-1}$. At that time, the difference between the second coordinates of $\xi_{\tau_0}(x)$ and $u(x;\tau)$ is at most w and the maximum occurred deviation in the time interval till t_0 is at most $w|m_1^{++}|^{-1}$.

Consider the random walk starting at $q \in C^*(x,w)$. We will show that the random walk starting at point q leaves the axis Q^{0+} at exponential speed. To this end, consider the event

$$\{\xi_{[\alpha w],1}(q) > -w\},$$

with α to be determined later on. Consider the process $S_k = \xi_{k,1}(q), k = 0,1,2...$ We have

$$\mathsf{E}(S_{k+1} - S_k | S_k) \le -\gamma$$

with γ as above. By virtue of Theorem 2.1.7 in [7], for any positive $\delta_1 < \gamma$ there exist constants $h > 0$, $\delta_2 > 0$ such that for any k

$$\mathsf{P}\{\xi_{k,1}(q)) > -\delta_1 k\} \le \exp\{hq_1 - \delta_2 k\} = \exp\{-\delta_2 k\}, \tag{4.3.8}$$

the latter equality holding because of $q_1 = 0$. Choose $\delta_1 = \gamma/2$ and suitable corresponding constants h, δ_2. Set $\alpha = 3/(2\delta_1) = 3\gamma^{-1}$ and $k = [\alpha w]$. This choice ensures that $t > t_0 + (4/\gamma)w > \tau_0 + [\alpha w]$.

For all $w > 1$ we have

$$\delta_1[\alpha w] \ge \frac{\gamma}{2}\Big(\frac{3}{\gamma}w - 1\Big) = \frac{3}{2}w - \frac{\gamma}{2} \ge w,$$

since $\gamma \le 1$ by definition. This implies

$$\mathsf{P}\{\xi_{[\alpha w],1}(q) > -w\} \le \mathsf{P}\{\xi_{[\alpha w],1}(q) > -\delta_1[\alpha w]\} \le \exp\{-\delta_2[\alpha w]\}, \tag{4.3.9}$$

for $w > 1$. Clearly by decreasing δ_2, (4.3.9) can be made to be satisfied for *all* $w > 0$. By the boundedness of jumps we have

$$\mathsf{P}\{\xi_{[\alpha w]}(q) \in A(q,w)\} \ge 1 - \exp\{-\delta_2[\alpha w]\}, \tag{4.3.10}$$

with

$$A(q,w) = \left\{ y \in \mathbf{Z}^2 \;\middle|\; \begin{array}{l} -[\alpha w] \le y_1 \le -w \\ |y_2 - q_2| \le [\alpha w] \end{array} \right\} \subset Q^{-+}.$$

Next note that the event $\{\xi_{[\alpha w]}(q) \in A(q,w)\}$, $q \in C^*(x,w)$, implies

$$\{\max_{1 \le k \le [\alpha w]} \|\xi_k(q) - u(q;k)\| \le w\sqrt{(2\alpha)^2 + (\frac{3}{2}\alpha)^2} = \frac{5}{2}\alpha w\}.$$

As a consequence, the event $C(x,w) \cap \{\xi_{\tau_0 + [\alpha w]}(x) \in A(\xi_{\tau_0}(x), w)\}$ implies

$$\Big\{ \max_{\tau_0 \le k \le \tau_0 + [\alpha w]} \|\xi_k(x) - u(x;k)\| \le w\Big(1 + \frac{5}{2}\alpha\Big)\Big\}, \tag{4.3.11}$$

so that the dispersion deviation is well-controlled.

Next we bound the maximum deviation between dynamical system and walk after time $\tau_0 + [\alpha w]$. Choose any point $p \in A(q, w)$ and let

$$\tau = t - \tau_0 - [\alpha w].$$

Given the event $C(x, w)$, we have $\tau \geq 0$ and

$$|\tau - (t - t_0 - [\alpha w])| = |\tau_0 - t_0| \leq w|m_1^{++}|^{-1}. \tag{4.3.12}$$

So for any realization s of τ we have (given $C(x, w)$)

$$\mathsf{P}\{\max_{k\leq s} \|\xi_k(p) - (p + km^{-+})\| \leq w\} \geq 1 - D \cdot \frac{t - t_0 - [\alpha w] + w|m_1^{++}|^{-1}}{w^2}. \tag{4.3.13}$$

We will now apply a gluing procedure. To this end, note that combining $p \in A(q, w)$ and $q \in C^*(x, w)$ yields the estimate

$$\begin{aligned}\|p + sm^{-+} - u(x; t)\| &\leq \|u(x; t_0) + sm^{-+} - u(x; t)\| + \|p - u(x; t_0)\| \\ &\leq |t_0 + s - t|\|m^{-+}\| + \sqrt{\alpha^2 w^2 + (w(1 + |m_1^{++}|^{-1}) + \alpha w)^2} \\ &\leq w\Big\{(\alpha + |m_1^{++}|^{-1})\|m^{-+}\| + \sqrt{\alpha^2 + (1 + |m_1^{++}|^{-1} + \alpha)^2}\Big\}, \end{aligned} \tag{4.3.14}$$

for all w, given the event $C(x, w)$.

Recall $\alpha = 3\gamma^{-1} > 3$, where $\gamma = \min\{|m_1^{++}|\,,\, |m_1^{-+}|\}$. Therefore, $|m_1^{++}|^{-1} < \alpha$ and so (4.3.14) implies

$$\|p + sm^{-+} - u(x; t)\| \leq w\{2\alpha\|m^{-+}\| + \sqrt{\alpha^2 + (1 + 2\alpha)^2}\}$$

Choose now $w = \beta^{-1} v$ with

$$\beta = \max\{2\alpha\|m^{-+}\| + \sqrt{\alpha^2 + (1 + 2\alpha)^2}\,,\, 1 + \frac{5}{2}\alpha\}. \tag{4.3.15}$$

Putting (4.3.7), (4.3.10), (4.3.11), (4.3.13) and (4.3.14) together, we find using the Markovian property

$$\begin{aligned}\mathsf{P}\{\max_{k\leq t} \|\xi_k(x) - u(x; k)\| \leq v\} \geq& \\ \geq& \Big(1 - D \cdot \frac{t_0 + w|m_1^{++}|^{-1}}{w^2}\Big)\Big(1 - \exp\{-\delta_2 \alpha w\}\Big) \times \\ &\times \Big(1 - D \cdot \frac{t - t_0 - [\alpha w] + w|m_1^{++}|^{-1}}{w^2}\Big) \\ \geq& 1 - D \cdot \frac{t}{w^2} + D\frac{[\alpha w]}{w^2} - D\frac{2w|m_1^{++}|^{-1}}{w^2} - \exp\{-\delta_2 \alpha w\}. \end{aligned} \tag{4.3.16}$$

The constant D comes from the variance of the jump distributions, but clearly all previous inequalities continue to hold, if we make D larger. Thus we can assume that D is so large, that

$$D\frac{[\alpha w]}{w^2} - D\frac{2w|m_1^{++}|^{-1}}{w^2} - \exp\{-\delta_2 \alpha w\} > 0$$

for any $w > 1$. Replacing w by $\beta^{-1}v$, we get

$$\mathsf{P}\{\max_{k \le t} \|\xi_k(x) - u(x;k)\| \le v\} \ge 1 - D \cdot \beta^2 \cdot \frac{t}{v^2}.$$

So we put $c = c(x,t) = D \cdot \beta^2$.

Thus we have proved the assertion for the case that x_1, $x_2 > \beta^{-1}v$, $x \in Q^{++}$. The case of one of x_1, x_2 smaller than $\beta^{-1}v$ is treated similarly to the above analysis, case 3, when the random walk enters a neighborhood of the order v of an axis. The case of x_1 and x_2 both smaller than $\beta^{-1}v$ cannot occur when the initial state x is sufficiently large.

This proves the assertion of the Lemma for $t < t_1$. □

Proof of Proposition 4.2.1

Since the point $u(x;t)$ is the outcome of a degenerate random vector, convergence in probability and convergence in distribution are equivalent. We check convergence in probability. Fix any $x \in \mathbf{R}^2 \setminus \{0\}$, any $t > 0$. We have to check for any (sufficiently small) $\sigma > 0$ that

$$\mathsf{P}\{\|\frac{\xi_{[tN]}([xN])}{N} - u(x;t)\| > \sigma\} = \mathsf{P}\{\|\xi_{[tN]}([xN]) - u(xN;tN)\| > \sigma N\} \to 0,$$

as $N \to \infty$.

In order to apply Lemma 4.3.1, we have to consider the point $u([xN];[tN])$ instead of $u(xN;tN)$. Since $u(x,t)$ is continuous on $x \neq 0$, we have that for any $x \neq 0$ and any t

$$u(x,t) = \lim_{N\to\infty} u(\frac{[xN]}{N};\frac{[tN]}{N}) = \lim_{N\to\infty} \frac{1}{N} u([xN];[tN]).$$

It means that for any given $x \neq 0, t$ and $\sigma > 0$

$$\|u(xN;tN) - u([xN];[tN])\| < \frac{\sigma}{2}N.$$

Hence,

$$\{\|\frac{\xi_{[tN]}([xN])}{N} - u(x;t)\| > \sigma\} \subset \{\|\xi_{[tN]}([xN]) - u([xN];[tN])\| > \frac{\sigma}{2}N\}$$

Clearly, for any given positive $\epsilon < 1/2, \sigma$ we have that $N^{\epsilon+1/2} < \sigma N/2$ for all sufficiently big N. By Lemma 4.3.1 there exists a constant $c > 0$, such that

$$\begin{aligned} &\mathsf{P}\big\{\|\xi_{[tN]}([xN]) - u([xN];[tN])\| > \frac{\sigma}{2}N\big\} \le \\ &\qquad \le \quad \mathsf{P}\{\,\|\xi_{[tN]}([xN]) - u([xN];[tN])\| > N^{\epsilon+1/2}\} \le c \cdot \frac{[tN]}{N^{2\epsilon+1}} \le \frac{ct}{N^{2\epsilon}} \end{aligned}$$

for all sufficiently big N. The result follows from the fact that $\lim_{N\to\infty} ct/N^{2\epsilon} = 0$. □

4.4 Criteria for ergodicity and transience

This section will show the validity of Lemma 4.2.1. For proving ergodicity and transience, the construction of a suitable Lyapunov function suffices. In the case of non-zero drifts, one can often use the Euler distance to 0as a Lyapunov function.

To this end, we need studying properties of the Euler distance $r(x)$ as a function of the initial point x, or equivalently, of the cycle time $\tau(x)$ (cf. Lemma 4.2.2).

4.4.1 Cycle time and isochrone

The cycle time is easily calculated explicitly. To this end, define the constants

$$\begin{aligned}
\tau_1 &= \frac{1}{|m_1^{++}|} + \frac{m_2^{++}}{m_1^{++}m_2^{-+}} + \frac{m_2^{++}|m_1^{-+}|}{m_1^{++}m_2^{-+}m_1^{--}} + \frac{m_2^{++}m_1^{-+}m_2^{--}}{m_1^{++}m_2^{-+}m_1^{--}m_2^{+-}},\\
\tau_2 &= \frac{1}{|m_2^{-+}|} + \frac{m_1^{-+}}{m_2^{-+}m_1^{--}} + \frac{m_1^{-+}|m_2^{--}|}{m_2^{-+}m_1^{--}m_2^{+-}} + \frac{m_1^{-+}m_2^{--}m_1^{+-}}{m_2^{-+}m_1^{--}m_2^{+-}m_1^{++}},\\
\tau_3 &= \frac{1}{m_1^{--}} + \frac{|m_2^{--}|}{m_1^{--}m_2^{+-}} + \frac{m_2^{--}m_1^{+-}}{m_1^{--}m_2^{+-}m_1^{++}} + \frac{m_2^{--}m_1^{+-}m_2^{++}}{m_1^{--}m_2^{+-}m_1^{++}|m_2^{-+}|},\\
\tau_4 &= \frac{1}{m_2^{+-}} + \frac{m_1^{+-}}{m_2^{+-}|m_1^{++}|} + \frac{m_1^{+-}m_2^{++}}{m_2^{+-}m_1^{++}m_2^{-+}} + \frac{m_1^{+-}m_2^{++}|m_1^{-+}|}{m_2^{+-}m_1^{++}m_2^{-+}m_1^{--}}.
\end{aligned}$$

One can see that $\tau_1, \tau_2, \tau_3, \tau_4$ are positive.

Further we define the vectors $\tau^{ab} = (\tau_1^{ab}, \tau_2^{ab}) \in \mathbf{R}^2, ab \neq 00$, such that

$$\tau^{++} = \tau^{+0} = (\tau_1, \tau_2), \tau^{-+} = \tau^{0+} = (-\tau_3, \tau_2),$$
$$\tau^{--} = \tau^{-0} = (-\tau_3, -\tau_4), \tau^{+-} = \tau^{0-} = (\tau_1, -\tau_4).$$

By direct calculations, the inner product of τ^{ab} and m^{ab} is equal to $C-1$, i.e.

$$\tau_1^{ab} m_1^{ab} + \tau_2^{ab} m_2^{ab} = C - 1.$$

For calculating the cycle time, we need to associate with any $x \neq 0$ the sequence of successive (deterministic) times $t_i(x)$ that the Euler path $u(x;\cdot)$ changes face. For instance, when $x \in Q^{++}$,

$$\left.\begin{aligned}
t_0(x) &= \frac{1}{|m_1^{++}|}x_1\\
t_1(x) - t_0(x) &= \frac{1}{|m_2^{-+}|}(x_2 + m_2^{++}t_0(x))\\
t_2(x) - t_1(x) &= \frac{|m_1^{-+}|}{m_1^{--}}\big(t_1(x) - t_0(x)\big)\\
t_3(x) - t_2(x) &= \frac{|m_2^{--}|}{m_2^{+-}}\big(t_2(x) - t_1(x)\big)\\
t_4(x) - t_3(x) &= \frac{m_1^{+-}}{|m_1^{++}|}\big(t_3(x) - t_2(x)\big)\\
t_{i+1}(x) - t_i(x) &= C\big(t_{i-3}(x) - t_{i-4}(x)\big), \quad i \geq 4.
\end{aligned}\right\} \tag{4.4.1}$$

Lemma 4.4.1 *The cycle time $\tau(x)$ is given by the continuous function*

$$\tau(x) = \tau_1^{ab} x_1 + \tau_2^{ab} x_2, \quad x \in Q^{ab}, \quad ab \neq 00. \tag{4.4.2}$$

Moreover, $u(x; \tau(x)) = Cx$ with C as defined in (4.2.1).

As a consequence, for any $x \in Q^{ab}, ab = +, -, 0,$

$$\tau(x + m(x)) - \tau(x) = \tau(m(x)) = \tau_1^{ab} m_1(x) + \tau_2^{ab} m_2(x) = C - 1. \tag{4.4.3}$$

Proof of Lemma 4.4.1. Let us prove (4.4.2). Suppose we start at some point $x \in Q^{++}$. Using (4.4.1), note that the dynamical system starting at x hits Q^{0+} for the first time at time $t_0(x) = x_1/|m_1^{++}|$ at the point $(0, x_2 + t_0(x) m_2^{++})$, in short

$$u(x; t_0(x)) = \big(0, x_2 + t_0(x) m_2^{++}\big) = \big(0, x_2 + \frac{m_2^{++}}{|m_1^{++}|} x_1\big).$$

Similarly, at time

$$t_1(x) = t_0(x) + \frac{1}{|m_2^{-+}|}\big(x_2 + \frac{m_2^{++}}{|m_1^{++}|} x_1\big)$$

the dynamical system $u(\cdot)$ hits Q^{-0} for the first time at the point

$$u(x; t_1(x)) = \big(\frac{m_1^{-+}}{|m_2^{-+}|}\big(x_2 + \frac{m_2^{++}}{|m_1^{++}|} x_1\big), 0\big).$$

At time

$$t_2(x) = t_1(x) + \frac{m_1^{-+}}{m_2^{-+} m_1^{--}}\big(x_2 + \frac{m_2^{++}}{|m_1^{++}|} x_1\big)$$

Q^{0-} is hit for the first time at the point

$$u(x; t_2(x)) = \big(0, \frac{m_1^{-+} m_2^{--}}{m_2^{-+} m_1^{--}}\big(x_2 + \frac{m_2^{++}}{|m_1^{++}|} x_1\big)\big).$$

Finally, at time

$$t_3(x) = t_2(x) + \frac{m_1^{-+} m_2^{--}}{|m_2^{-+}| m_1^{--} m_2^{+-}}\big(x_2 + \frac{m_2^{++}}{|m_1^{++}|} x_1\big)$$

Q^{+0} is hit for the first time, at point

$$u(x; t_3(x)) = \big(\frac{m_1^{-+} m_2^{--} m_1^{+-}}{|m_2^{-+}| m_1^{--} m_2^{+-}}\big(x_2 + \frac{m_2^{++}}{|m_1^{++}|} x_1\big), 0\big).$$

We will calculate time and place where the line αx, $\alpha > 0$, is crossed for the first time by the dynamical system $u(\cdot)$ starting at the point $u(x, t_3(x))$. In order that $u\big(u(x, t_3(x)), \tau\big) = \alpha x$ for some $\alpha > 0$ and τ minimal,

$$\tau = \alpha \frac{1}{m_2^{++}} x_2.$$

Hence
$$\frac{m_1^{-+}m_2^{--}m_1^{+-}}{|m_2^{-+}|m_1^{--}m_2^{+-}}\big(x_2+\frac{m_2^{++}}{|m_1^{++}|}x_1\big)+\alpha\frac{m_1^{++}}{m_2^{++}}x_2=\alpha x_1,$$
and so
$$\alpha=\frac{m_1^{-+}m_2^{--}m_1^{+-}m_2^{++}}{m_2^{-+}m_1^{--}m_2^{+-}m_1^{++}},$$
thus giving the constant C. Note that we can leave out absolute signs. Furthermore,
$$\tau=\frac{m_1^{-+}m_2^{--}m_1^{+-}}{m_2^{-+}m_1^{--}m_2^{+-}m_1^{++}}x_2.$$
By combination of the above relations we then find $\tau(x)=t_3(x)+\tau=\tau_1^{++}x_1+\tau_2^{++}x_2$. The proof for the other cases goes similarly. Continuity only needs to be checked at face-transitions. But this is evident. □

For convenience, we extend the cycle time τ as a continuous function on $\mathbf{R}^2$ by setting
$$\tau(0)=0.$$
The form of the isochrone is now an easy consequence of the above combined with Lemma 4.2.2. We have
$$\begin{aligned}\mathcal{I}(t) &= \{x\in\mathbf{R}^2\,|\,t=r(x)\}\\ &= \{x|\,\tau_1^{ab}x_1+\tau_2^{ab}x_2=t(C-1),x\in Q^{ab},\ a,b,=+,-,0\}.\end{aligned}\qquad(4.4.4)$$
This immediately yields the following result.

Lemma 4.4.2 *For any $t>0$, the isochrone $\mathcal{I}(t)$ is closed and homeomorphic to a circle in $\mathbf{R}^2$.*

Proof. We will prove that isochrone is closed.

By virtue of Lemma 4.4.1, the set $t=\tau(x)/(C-1)$ is a straight line on each of the closures of faces $\bar{Q}^{++}$, $\bar{Q}^{-+}$, $\bar{Q}^{--}$, $\bar{Q}^{+-}$.

It is an easy computation to check that line segments from neighboring quadrants intersect at one and the same point of an axis. □

4.4.2 Lyapunov functions

There is an interesting construction in [7] for Lyapunov functions for ergodicity and transience of face-homogeneous random walks on the lattice in $\mathbf{Z}_+^d$. Out of a Lipschitz-continuous and non-negative function that increases by at least ϵ in the direction of the generalized drift ("second vector field" in the terminology of this book), they show how to construct a Lyapunov function for Foster's criterion for ergodicity. If this function decreases, then it yields a Lyapunov function for transience. As mentioned before, a typical function with this property is the Euler distance.

The proofs go through in our case, but here we prefer to cite the ergodicity and transience criteria used and we will explicitly construct convenient Lyapunov functions.

Proof of Lemma 4.2.1.

Let us first prove the transience in case of $C > 1$.

We will use lemma 2.4 as a sufficient condition of transience. Set $f(x) = \tau(x)$ and $k(x) = 1$. We only need to show that

$$\mathsf{E}\{\tau(\xi_{t+1}) - \tau(\xi_t) \,|\, \xi_t = x\} \geq C - 1, \text{ for any } x \neq \mathsf{0}.$$

Note that $\tau(x)$ is linear inside each of the quadrants. Then for $x \in Q^{ab}$, $a, b \neq 0$, inside one of the quadrants, we have

$$\mathsf{E}\{\tau(\xi_{t+1}) - \tau(\xi_t) \,|\, \xi_t = x\} = \mathsf{E}\{\tau(\xi_{t+1} - \xi_t) \,|\, \xi_t = x\} = \tau_1^{ab} m_1^{ab} + \tau_2^{ab} m_2^{ab} = C - 1,$$

by Lemma 4.4.1 and (4.4.3) .

For $x \in Q^{ab}$, a or $b = 0$, but $\{a, b\} \neq \{0, 0\}$, on one of the axes, the situation is slightly more complicated.

Let $x \in Q^{+0}$. The mean drift m_2^{++} is defined by the jumps into Q^{++} and Q^{+-}. Denote

$$p = \sum_{j \in Q^{++}} \mathsf{P}\{\xi_{t+1} = j \,|\, \xi_t = x \in Q^{+0}\} \text{ and } q = \sum_{j \in Q^{+-}} \mathsf{P}\{\xi_{t+1} = j \,|\, \xi_t = x \in Q^{+0}\}.$$

Since the jumps are bounded by 1, we have that $m_2^{+0} = p - q$. Recall that

$$\tau(x) = \begin{cases} \tau_1 x_1 + \tau_2 x_2 & \text{if } x \in Q^{++} \cup Q^{+0}, \\ \tau_1 x_1 - \tau_4 x_2 & \text{if } x \in Q^{+-} \cup Q^{+0}. \end{cases}$$

Hence, taking in account that $m^{++} = m^{+0}$ by assumption, we get that

$$\begin{aligned} &\mathsf{E}\{\tau(\xi_{t+1}) - \tau(\xi_t) \,|\, \xi_t = x \in Q^{+0}\} = \\ &\quad = \quad p\tau_2 + q\tau_4 + \tau_1 m_1^{++} = \tau_2 m^{+0} + \tau_1 m_1^{++} + q(\tau_2 + \tau_4) \\ &\quad = \quad = C - 1 + q(\tau_2 + \tau_4) \geq C - 1. \end{aligned}$$

For $x \in Q^{0+}, Q^{0-}, Q^{-0}$ we get a similar estimate. So the condition of lemma 2.3 is satisfied with $\epsilon = C - 1 > 0$. Hence, the random walk is transient if $C - 1 > 0$.

The proof of ergodicity in case of $C < 1$, is analogous. Only the step function $k(x)$ has to be chosen greater then 1. □

4.5 Preliminary results for a fixed initial point

4.5.1 Convergence of the scaled Euler distance

The previous section uses the Euler distance $\tau(x)$ as a Lyapunov function. Using a technique of transforming (additive) Lyapunov functions for ergodicity and transience into (multiplicative) Lyapunov functions for exponential ergodicity and transience (cf. [26], [25]), one can get exponential bounds on the probability of deviating from a given level set of $r(\cdot)$. This can be used to show the following lemma.

Lemma 4.5.1 *For any fixed initial point $\xi_0 \in \mathbf{Z}^2$ the limit*

$$\lim_{N\to\infty} r(\frac{\xi_{[tN]}}{N})$$

almost surely exists and equals t.

Instead of the line of proof sketched in the above, there are reasons for choosing a martingale approach and then to apply the Azuma-Hœffding inequality (see lemma 2.2.2). To this end, we need to make a simplifying assumption. We would like to point out that the results go through in the more general case, but to a cost of increased tediousness of proofs.

In this subsection we therefore assume that

$$\mathsf{E}\{\tau(\xi_{n+1}) - \tau(\xi_n) \,|\, \xi_n = x\} = C - 1, \text{ for any } x. \tag{4.5.1}$$

Note that this holds by (4.4.3) for any $x \in Q^{ab}$, $a, b \neq 0$. One way to construct models where (4.5.1) is satisfied, is to impose that transitions from one quadrant to the next quadrant in clockwise direction cannot occur (under the initial drift condition (iii) of section 4.2).

Proof of Lemma 4.5.1 under condition (4.5.1)
We will prove the statement for $t = 1$. The general statement then easily follows. Indeed, $r\big(\xi_N(x)/N\big) \to 1$, a.s., implies the same for any subsequence and so $r\big(\xi_{[tN]}(x)/[tN]\big) \to 1$, a.s. Note that homogeneity of the function τ implies homogeneity of the function r. Hence, by multiplying by t, $r\big(t\xi_{[tN]}(x)/[tN]\big) \to t$, a.s. Clearly, the sequence $r\big(\xi_{[tN]}/N\big)$, $N = 1, 2, \ldots$, has the same a.s. limit.

For the proof of the case $t = 1$, we will prove that the sequence

$$M_n = r(\xi_n) - r(\xi_0) - n \tag{4.5.2}$$

is a zero-mean martingale. We have

$$\mathsf{E}\{M_{n+1} - M_n \,|\, M_n\} = \frac{1}{C-1}\mathsf{E}\{\tau(\xi_{n+1}) - \tau(\xi_n) \,|\, \xi_n\} - 1.$$

By virtue of (4.5.1), we get

$$\mathsf{E}\{M_{n+1} - M_n \,|\, M_n\} = 0.$$

Recall that the jumps of ξ_t are bounded in absolute value. Then the increments $M_l - M_{l-1}$ are bounded in absolute value as well.

From martingale limit theory it follows immediately that $n^{-1}M_n \to 0$ almost surely (see theorem 3.2) and so $r(\xi_n/n) \to 1$ almost surely. □

A consequence of this result is that any limiting distribution of the sequence $\xi_{[tN]}/N$ is concentrated on the isochrone $\mathcal{I}(t)$. The following lemma provides an estimate on the speed of convergence of $r(\xi_N/N)$.

Lemma 4.5.2 *Let $\epsilon > 0$ and $\delta \in (1/2, 1)$. Under condition (4.5.1), there exists $\gamma > 0$ such that*

$$\mathsf{P}\{|r(\xi_N) - r(\xi_0) - N| \le \epsilon N^{\delta} \textit{ for all } N \ge M\} \ge \gamma, \tag{4.5.3}$$

for some M.

Proof. The zero-mean martingale $\{M_l\}_l$ from (4.5.2) satisfies the conditions of Lemma 2.2.2, for constants $c_l = c$, for some constant c. Take $x = \epsilon n^\delta$ in (2.2.1) with $\epsilon > 0$ and $\delta \in (1/2, 1)$. Then (2.2.1) implies

$$\mathsf{P}\{|M_n| \geq \epsilon n^\delta\} < 2\exp\{-\frac{\epsilon^2}{2c^2} n^{2\delta-1}\}. \tag{4.5.4}$$

Taking (4.5.4) into account we obtain for any m

$$\mathsf{P}\{|M_n| \leq \epsilon n^\delta \text{ for all } n \geq m\} \geq 1 - \sum_{n\geq m} \mathsf{P}\{|M_n| \geq \epsilon n^\delta\} \geq 1 - 2\sum_{n \geq m} \exp\{-\frac{\epsilon^2 n^{2\delta-1}}{2c^2}\}.$$

Since $2\delta - 1 > 0$, the series converges. And so, there exists $\gamma > 0$ and m such that

$$2\sum_{n\geq m} \exp\{-\frac{\epsilon^2}{2c^2} n^{2\delta-1}\} < 1 - \gamma.$$

4.5.2 Absorption between any two Euler paths

Next we will prove that the region between any two (non-identical) Euler paths is a sojourn set. This will be the crucial step in showing that the scaled process cannot have any limiting distribution in general.

We need some notation to define the region between two Euler paths. Let $x, y \in \mathcal{I}(s)$, $s > 0$. By $[x \rightsquigarrow y]$ we denote the set of points $r \in \mathcal{I}(s)$, that we pass when moving *anticlockwise* from x to y along $\mathcal{I}(s)$, including x and y. Using a round bracket instead of a straight one, excludes the corresponding end point.

Similarly, the set $[\Gamma_x \rightsquigarrow \Gamma_y]$ denote the set of paths between Γ_x and Γ_y in anticlockwise direction, i.e.

$$[\Gamma_x \rightsquigarrow \Gamma_y] = \{u(z;t) \,|\, z \in [x \rightsquigarrow y], t \in \mathbf{R}\}.$$

Again we may replace (one of) the straight brackets by round ones, thus excluding the corresponding 'end' path.

Our goal is hence to show that for any $x, y \in \mathcal{I}$ and any initial state $p \in \mathbf{Z}^2$ there exists N such that

$$\mathsf{P}\{\xi_n \in (\Gamma_x \rightsquigarrow \Gamma_y) \text{ for all } n \geq N \,|\, \xi_0 = p\} > 0, \tag{4.5.5}$$

in other words, the set $(\Gamma_x \rightsquigarrow \Gamma_y)$ is a *sojourn set* in Feller's terminology (cf. [8], [5]). For proving this, we will use expanding tubes $\mathcal{T}^\gamma(p)$ containing a given Euler path $\{u(p;t), t \geq 0\}$ and contained between two Euler paths, $\mathcal{T}^\gamma(p) \subseteq (\Gamma_x \rightsquigarrow \Gamma_y)$. The parameter γ indicates the 'width' of the tube. For these tubes we will show

$$\mathsf{P}\{\xi_n \in \mathcal{T}^\gamma(p), n \geq 0 \,|\, \xi_0 = p\} > 0, \tag{4.5.6}$$

which clearly implies (4.5.5).

A connected subset $A \subseteq \mathcal{I}(s)$ will be called an *interval on the isochrone* $\mathcal{I}(s)$. Take any interval $A \subset \mathcal{I}(s)$, $s > 0$. All points of A have the same cycle time $(C-1)s$. Therefore, the dynamical system u maps the interval A to the interval $CA \subset \mathcal{I}(Cs)$. In the following, we will have to deal with neighborhoods of an interval $A \subset \mathcal{I}(s)$.

The action of u on the disc

$$\mathcal{O}(A,\alpha) = \{x \in \mathbf{R}^2 \mid \inf_{y\in A} \|x-y\| < \alpha\}, \quad \alpha > 0,$$

transforms it in a rather complicated way. Most points of $\mathcal{O}(A,\alpha)$ have different cycle times and the shape of ℓ^2-balls is not consistent with the piecewise linearity of the dynamical system. To deal with this problem, we first construct an invariant under u that can be thought of as a kind of "angle" between two Euler paths. Fix any reference point $x_0 \neq 0$ and corresponding reference Euler path Γ_{x_0}. Let $x_s = \Gamma_{x_0} \cap \mathcal{I}(s)$. Define

$$\psi_{x_0} : \mathbf{R}^2 \setminus \{0\} \to [0, C-1]$$

by

$$\psi_{x_0}(x) = \inf\Big\{t \geq 0 \,|\, x = \frac{u(x_s; st)}{1+t}\Big\}, \quad x \in \mathcal{I}(s).$$

Let us prove that ψ_{x_0} is indeed invariant under the action of u.

Lemma 4.5.3 *For $x \in \mathcal{I}(s)$ we have*

$$\psi_{x_0}\big(u(x;t)\big) = \psi_{x_0}(x), \quad t > 0.$$

Proof. The function $t \to u(x_s; st)/(1+t)$ is continuous and periodic. Hence the infinum is a minimum, say it is assumed for t', i.e.

$$\psi_{x_0}(x) = \inf\Big\{t \geq 0 \,|\, x = \frac{u(x_s; st)}{1+t}\Big\}.$$

By the construction of this function, this value t' is the unique value on $[0, C-1)$ for which

$$x = \frac{u(x_s; st')}{1+t'}.$$

The assertion follows from

$$\begin{aligned} u(x; v) &= u\Big(\frac{u(x_s; st')}{1+t'}; v\Big) = \frac{u\big(u(x_s; st'); v(1+t')\big)}{1+t'} = \\ &= \frac{u\big(u(x_s; v); (s+v)t'\big)}{1+t'} = \frac{u\big(x_{s+v}; (s+v)t'\big)}{1+t'}. \end{aligned}$$

Now we will construct a special (open) *time tube* of an interval $A \subset \mathcal{I}(s)$, which we will denote by $\mathcal{T}(A,\rho)$, $\rho < s$. Its pre-image is in fact a rectangle in the (ψ, r)-plane. For $A = \mathcal{I}(s)$ denote $\mathcal{T}(\mathcal{I}(s),\rho) = \{x \in \mathbf{R}^2 \mid |r(x) - s| < \rho\}$. For $A = [x \rightsquigarrow y]$,

$$\begin{aligned} \mathcal{T}(A,\rho) &= \{p \in \mathbf{R}^2 \,|\, 0 \leq \psi_x(p) \leq \psi_x(y), |r(p) - s| < \rho\} \\ &= \{p = u(q; w) \,|\, q \in A, |w| < \rho\}. \end{aligned}$$

If $\mathcal{T}(A,\rho) \subset Q^{ab}$, then it is a parallelogram containing the set A (cf. Figure 1.4).

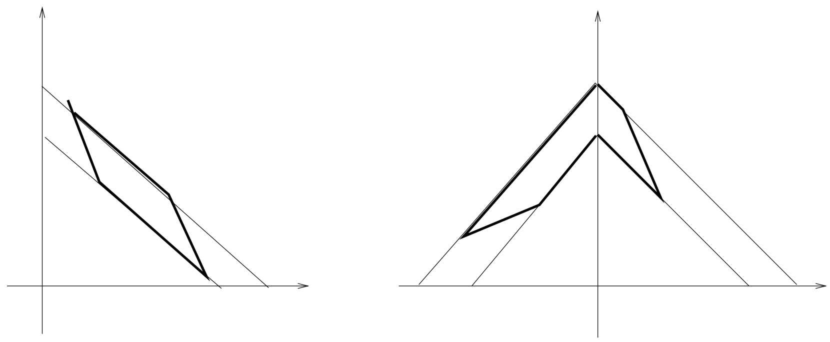

Figure 4.2: time tubes

By invariance of ψ_x, time tubes remain time tubes under the action of u, i.e.

$$u\big(\mathcal{T}(A,\rho);t\big) = \mathcal{T}\big(u(A;t),\rho\big), \quad t>0, \quad \text{any interval } A \subset \mathcal{I}(s). \tag{4.5.7}$$

In particular, $u\big(\mathcal{T}(A,\rho);\tau(A)\big) = \mathcal{T}(CA,\rho)$. Note that the pre-images (in the (ψ,r)-plane) of $\mathcal{T}(A,\rho)$ and $\mathcal{T}(CA,\rho)$ are identical!

Now let us see how these time tubes relate to open balls containing an interval $A \subset \mathcal{I}(s)$. Denote the restriction of $\mathcal{O}(A,\alpha)$ to $\mathcal{I}(s)$ by

$$\mathcal{O}_s(A,\alpha) = \mathcal{O}(A,\alpha) \cap \mathcal{I}(s).$$

This is clearly an interval on $\mathcal{I}(s)$.

Properties of time tubes

For any two intervals $A, B \subset \mathcal{I}(s)$ and sufficiently small $\rho > 0$ we have that

P1 there exist positive constants $\nu < 1 < \sigma$, not depending on s, such that for any $\alpha > 0$

$$\mathcal{T}\big(\mathcal{O}_s(A,\nu\alpha),\nu\alpha\big) \subset \mathcal{O}(A,\alpha) \subset \mathcal{T}\big(\mathcal{O}_s(A,\sigma\alpha),\sigma\alpha\big). \tag{4.5.8}$$

P2 $\mathcal{T}(\mathcal{O}_{\gamma s}(\gamma A,\gamma\alpha),\gamma\rho) = \gamma\mathcal{T}(\mathcal{O}_s(A,\alpha),\rho)$, for any $\gamma > 0$.

P3 $\mathcal{T}(A,\rho)\cap\mathcal{T}(B,\rho) = \mathcal{T}(A\cap B,\rho)$ and $\mathcal{T}(A,\rho)\cup\mathcal{T}(B,\rho) = \mathcal{T}(A\cup B,\rho)$.

As a consequence one can define also time tubes of unions of intervals on the same isochrone by the union of the corresponding time tubes.

Let $1/2 < \delta < 1$ be given, as well as the constants ν and σ from property **P1**. For constructing a time tube between two given Euler paths, we also need a deviation factor $\gamma > 0$. For any initial point $p \in \mathbf{R}^2$, $p \neq 0$, define

$$\mathcal{T}_k^\gamma(p) = \left\{ \mathcal{O}_{C^k r(p)+s}\big(u(x;s),\sigma\gamma\cdot\tau^\delta(C^k p)\big) \;\middle|\; \begin{array}{l} x \in \mathcal{O}_{C^k r(p)}(C^k p,\alpha_k^\gamma), \\ -\rho_{k+1}^\gamma < s < C^k\tau(p) + \rho_{k+1}^\gamma \end{array} \right\}$$

$k = 0, 1, \ldots$, with

$$\begin{aligned}\alpha_0^\gamma &= 0\\ \alpha_k^\gamma &= C\alpha_{k-1}^\gamma + \sigma\gamma \cdot \tau^\delta(C^{k-1}p)\end{aligned}$$

and

$$\begin{aligned}\rho_0^\gamma &= 0,\\ \rho_k^\gamma &= \rho_{k-1}^\gamma + \sigma\gamma \cdot \tau^\delta(C^{k-1}p),\end{aligned}$$

for $k = 1, \ldots$. Note that

$$\begin{aligned}\alpha_k^\gamma &= \sigma\gamma \cdot \tau^\delta(C^{k-1}p) \cdot (1 + C^{1-\delta} + \cdots + C^{(k-1)(1-\delta)}) = \sigma\gamma \cdot \tau^\delta(p)\frac{C^k - C^{k\delta}}{C - C^\delta}\\ \rho_k^\gamma &= \sigma\gamma \cdot \tau^\delta(p) + \cdots C^{(k-1)\delta}\sigma\gamma \cdot \tau^\delta(p) = \sigma\gamma \cdot \tau^\delta(p)\frac{C^{k\delta} - 1}{C^\delta - 1}.\end{aligned}$$

Additionally, we set $\mathcal{T}^\gamma(p) = \cup_{k=0}^\infty \mathcal{T}_k^\gamma(p)$.

A rough explanation of these quantities is the following. Compare the random walk and the dynamical system starting at a sufficiently large point p. The parameter α_k^γ is the cumulative dispersion during the first k cycles parallel to isochrones, and ρ_k^γ the associated cumulative (time)dispersion along the dynamical system having almost all probability mass. The first one blows up by a factor C, each time a cycle has been passed, in addition to 'noise' incurred while passing the last cycle, which scales by a factor in the power δ. The second only consists of the added 'noise' term, because of (4.5.7). The set $\mathcal{T}_k^\gamma(p)$ contains the set of 'most likely' realizations of the random walk *during* the $(k+1)$th cycle of the dynamical system.

We will first prove the next theorem.

Theorem 4.1 *Let $1/2 < \delta < 1$, $\gamma > 0$. Then there exist positive constants c', depending on δ, and d', depending on δ and γ, such that for any sufficiently big initial point $p \in \mathbf{Z}^2$ and any t*

$$\mathsf{P}\Big\{\xi_n \in \mathcal{T}^\gamma(p),\ \textit{for all } n < t \textit{ and } |r(p)+t-r(\xi_t)| < d'\sigma\gamma\cdot t^\delta \,\Big|\, \xi_0 = p\Big\} \geq 1-\frac{c'}{\gamma^2}\cdot\tau^{1-2\delta}(p), \tag{4.5.9}$$

with σ from Property **P1**.

Proof. Denote by $t_k = t_k(p)$ the time the dynamical system u requires for passing precisely k cycles when starting at p. Then $t_0 = 0$ and

$$t_k = t_{k-1} + C^{k-1}\tau(p) = \frac{C^k - 1}{C - 1}\tau(p) = (C^k - 1)r(p). \tag{4.5.10}$$

We will use an induction argument. To this end we first need some further notation. For notational convenience we will suppress the dependence on γ in our notation.

Let $A_0 = \{p\}$, and

$$A_k = \mathcal{O}_{C^k r(p)}(CA_{k-1}, \sigma\gamma\tau^\delta(C^{k-1}p)), \quad k \geq 1. \tag{4.5.11}$$

The underlying idea is that starting in a point from A_{k-1} after the $(k-1)$th cycle completion, the random walk can deviate from the dynamical system by a 'distance' of at most $\sigma\gamma \cdot \tau^{\delta}(C^{k-1}p)$ (with high probability). Moreover, the set $A_k \subset \mathcal{I}(C^k p)$ is the intersection of the set $\mathcal{T}_{k-1}(p)$, $k \geq 1$, with the isochrone of the point $C^k p$:

$$A_k = \mathcal{O}_{C^k r(p)}(C^k p, \alpha_k). \tag{4.5.12}$$

Furthermore, by (4.5.7)

$$u\big(\mathcal{T}(A_k, \rho_k); \tau(C^k p)\big) = \mathcal{T}\left(u\big(A_k; \tau(C^k p)\big), \rho_k\right) = \mathcal{T}(CA_k, \rho_k) \subset \mathcal{T}(A_{k+1}, \rho_k).$$

It suffices to show the following statement. There exist a constant c'' and $k \geq 1$ with $t_{k-1} < t \leq t_k$, such that

$$\begin{aligned} \mathsf{P}\left\{ \begin{array}{l} \xi_n(p) \in \cup_{l=0}^{k-1} \mathcal{T}_l(p), n \leq t \\ \xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^{k-1}r(p)+s}\big(u(A_{k-1}; s), \sigma\gamma s^{\delta}\big), \rho_{k-1} + \sigma\gamma s^{\delta}\big) \end{array} \right\} \\ \geq \quad 1 - \frac{c''}{\gamma^2} \tau^{1-2\delta}(p) \cdot \frac{C^{k(1-2\delta)} - 1}{C^{1-2\delta} - 1}, \end{aligned} \tag{4.5.13}$$

where $s = t - t_{k-1}$.

Let us first argue that (4.5.13) implies (4.5.9). Indeed,

$$\xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^{k-1}r(p)+s}\big(u(A_{k-1}; s), \sigma\gamma s^{\delta}\big), \rho_{k-1} + \sigma\gamma s^{\delta}\big), \quad s = t - \rho_{k-1},$$

implies

$$|r(\xi_t) - r(p) - t| < \rho_{k-1} + \sigma\gamma s^{\delta},$$

since $r(C^{k-1}p) + s = t_{k-1} + r(p) + s = t + r(p)$. Further, provided that $\tau(p) > 1$, we have $\tau^{\delta}(p) < \tau(p)$. So, for $k \geq 2$ we have by (4.5.9) and (4.5.10) that

$$\begin{aligned} \frac{\rho_{k-1}}{\sigma\gamma t_{k-1}^{\delta}} &= \frac{C^{(k-1)\delta} - 1}{C^{\delta} - 1} \cdot \frac{(C-1)^{\delta}}{(C^{k-1} - 1)^{\delta}} \\ &= \frac{(C-1)^{\delta}}{C^{\delta} - 1} \cdot \frac{1 - C^{-(k-1)\delta}}{(1 - C^{-(k-1)})^{\delta}} \to \frac{(C-1)^{\delta}}{C^{\delta} - 1}, \end{aligned}$$

as $k \to \infty$. Hence there is a constant, $d'' > 1$ say, such that $\rho_{k-1}/\sigma\gamma \leq d'' t_{k-1}^{\delta}$. Clearly, for $k = 1$, $\rho_{k-1} = 0 = t_{k-1} \leq d'' t_{k-1}$. As a consequence,

$$\rho_{k-1} + \sigma\gamma s^{\delta} < \sigma\gamma\big(\frac{\rho_{k-1}}{\sigma\gamma} + s^{\delta}\big) < \sigma\gamma\big(d'' t_{k-1}^{\delta} + s^{\delta}\big) < 2d''\sigma\gamma(t_{k-1} + s)^{\delta} \leq 2d''\sigma\gamma t^{\delta}.$$

Note that $1 - 2\delta < 0$, and so $C^{k(1-2\delta)} \downarrow 0$, as $k \to \infty$. Thus (4.5.13) implies for any t and $k \geq 1$ with $t_{k-1} < t \leq t_k$, that

$$\mathsf{P}\left\{ \begin{array}{l} \xi_n(p) \in \cup_{l=0}^{k-1} \mathcal{T}_l(p),\ n \leq t \\ |r\big(\xi_t(p)\big) - t - r(p)| < 2d''\sigma\gamma \cdot t^{\delta} \end{array} \right\} \geq 1 - \frac{c''}{\gamma^2(1 - C^{1-2\delta})} \cdot \tau^{1-2\delta}(p).$$

Putting $c' = c''/(1 - C^{1-2\delta})$ and $d' = 2d''$, proves that the assertion from the lemma follows from (4.5.13). We will now show the validity of (4.5.13) by induction to the number of cycles.

Let first $t \leq t_1$. Then u has at most 5 face transitions, when starting at p. By Lemma 4.3.1 (for convenience we use a version with $<$-sign instead of the given one with $\leq$-sign, but the proof is analogous) there exists a constant c, such that for any sufficiently large p and for any constant v satisfying (4.3.2)

$$\mathsf{P}\big\{\max_{n\leq t}\|\xi_n(p) - u(p;n)\| < v\big\} \geq 1 - c\cdot\frac{t}{v^2}.$$

Put $v = \gamma t^\delta$. Obviously (4.3.2) is satisfied for initial state p sufficiently big. Thus

$$\big\{\max_{0\leq n\leq t}\|\xi_n(p) - u(p;n)\| < \gamma t^\delta\big\} = \cap_{n=0}^t\big\{\xi_n(p) \in O\big(u(p;n),\gamma t^\delta\big)\big\}.$$

By **P1**,

$$O\big(u(p;n),\gamma t^\delta\big) \subset \mathcal{T}\big(\mathcal{O}_{r(p)+n}\big(u(p;n),\sigma\gamma t^\delta\big),\sigma\gamma t^\delta\big).$$

In turn, this implies that

$$\begin{aligned}\big\{\max_{0\leq n\leq t}\|\xi_n(p) - u(p;n)\| < \gamma t^\delta\big\} \quad &\subset \quad \bigcap_{n=0}^t\big\{\xi_n(p) \in \mathcal{T}\big(\mathcal{O}_{r(p)+n}\big(u(p;n),\sigma\gamma t^\delta\big),\sigma\gamma t^\delta\big)\big\}\\ &\subset \quad \mathcal{T}_0(p)\cap\big\{\xi_t(p)\in O\big(u(p;t),\gamma t^\delta\big)\big\}.\end{aligned}$$

Hence,

$$\mathsf{P}\left\{\begin{array}{l}\xi_n(p)\in\mathcal{T}_0(p),\ n\leq t\\ \xi_t(p)\in\mathcal{T}\big(\mathcal{O}_{r(p)+t}\big(u(p;t),\sigma\gamma t^\delta\big),\sigma\gamma t^\delta\big)\end{array}\right\} \geq 1 - \frac{c}{\gamma^2}t^{1-2\delta} \geq 1 - \frac{c}{\gamma^2}\tau^{1-2\delta}(p),$$

so that (4.5.13) holds for $t \leq t_1$, when choosing $c'' = c$. Do note, that the term

$$1 - (c''/\gamma^2)\tau^{1-2\delta}(p) > 0$$

for sufficiently large p.

Assume now that the statement holds for time periods $t \leq t_K$ for some integer $K > 1$. We will show that (4.5.13) holds for $t_K < t \leq t_{K+1}$ as well.

Note, that the event in the left-hand side of (4.5.13) implies the event

$$\begin{aligned}\xi_{t_K}(p) \in \mathcal{T}\big(\mathcal{O}_{C^K r(p)}\big(u(A_{K-1};\tau(C^{K-1}p),\sigma\gamma\tau^\delta(C^{K-1}p)\big),\rho_{K-1}+\sigma\gamma\tau^\delta(C^{K-1}p)\big)\\ = \mathcal{T}(A_K,\rho_K).\end{aligned}$$

Suppose $\xi_{t_K} = q \in \mathcal{T}(A_K,\rho_K)$. For bounding the probability of a large deviation of the random walk between t_K and t from the dynamical system starting at q, we would like to apply Lemma 4.3.1 with the *same constant* c as for the case of $t \leq t_1$. In particular, we would like to bound

$$\mathsf{P}\{\max_{0\leq n\leq s}\|\xi_n(q) - u(q;n)\| < \gamma s^\delta\},$$

where $s = t - t_k$.

The constant c depends on the number of face-transitions. Clearly, the number of face-transitions between time t_K and t is at most 5 like before. As an additional requirement we need to check that γs^δ satisfies (4.3.2).

Observe that $s \le \tau(C^K p)$ and $r(q) > r(C^K p) - \rho_K$. Moreover, by (4.5.9) we have

$$\begin{aligned} \rho_K &= \frac{(C-1)^\delta}{C^\delta - 1} \sigma\gamma r^\delta(p) \cdot (C^{K\delta} - 1) \\ &< \frac{(C-1)^\delta}{C^\delta - 1} \sigma\gamma r^\delta(C^K p). \end{aligned}$$

Hence,

$$\frac{s}{\tau(q)} \le \frac{r(C^K p)}{r(C^K p) - \rho_K} \le \frac{1}{1 - \sigma\gamma r^{\delta-1}(C^K p) \cdot (C-1)^\delta/(C^\delta - 1)}.$$

For any $\epsilon > 0$, this is smaller than $1+\epsilon$, provided that p is big enough. As a consequence, for $K \ge 1$, we have $\gamma s^\delta < \theta \cdot \tau(q)$, any $q \in \mathcal{T}(A_K, \rho_K)$, and $s = t - t_K \le t_{K+1} - t_K$, provided p is sufficiently big, for θ the constant from condition (4.3.2). Thus we can apply Lemma 4.3.1 with the same constant c as in the above. This yields that

$$\mathsf{P}\{\max_{0\le n\le s} \|\xi_n(q) - u(q;n)\| < \gamma s^\delta\} \ge 1 - \frac{c}{\gamma^2} \cdot s^{1-2\delta} \ge 1 - \frac{c''}{\gamma^2} \cdot \tau^{1-2\delta}(C^K p). \quad (4.5.14)$$

By (4.5.7)

$$u(q;n) \in u\big(\mathcal{T}(A_K, \rho_K); n\big) = \mathcal{T}(u(A_K;n), \rho_K), \quad 0 \le n \le s.$$

By **P1**

$$\mathcal{O}(u(q;n), \gamma s^\delta) \subset \mathcal{T}\big(\mathcal{O}_{C^K r(p)+n}\big(u(A_K;n), \sigma\gamma s^\delta\big), \rho_K + \sigma\gamma s^\delta\big) \subset \mathcal{T}_K(p).$$

As a consequence, given that $q \in \mathcal{T}(A_K, \rho_K)$,

$$\begin{aligned} &\left\{ \begin{array}{l} \xi_n(q) \in \mathcal{T}_K(p), 0 \le n \le s \\ \xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^K r(p)+s}\big(u(A_K;s), \sigma\gamma s^\delta\big), \rho_K + \sigma\gamma s^\delta\big) \end{array} \right\} \supset \\ &\qquad \supset \bigcap_{n=0}^{s} \{\xi_n(q) \in \mathcal{O}(u(q;n), \gamma s^\delta)\} = \{\max_{0\le n\le s} \|\xi_n(q) - u(q;n)\| < \gamma s^\delta\} \end{aligned}$$

For $t_K < t \le t_{K+1}$ we finally have

$$\begin{aligned} &\mathsf{P}\left\{ \begin{array}{l} \xi_n(p) \in \cup_{l=0}^{K} \mathcal{T}_l(p), n \le t \\ \xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^K r(p)+s}\big(u(A_K;s), \sigma\gamma s^\delta\big), \rho_K + \sigma\gamma s^\delta\big) \end{array} \right\} \ge \\ &\ge \sum_{q \in \mathcal{T}(A_K, \rho_K)} \mathsf{P}\{\xi_{t_K}(p) = q,\, \xi_n(p) \in \bigcup_{l=0}^{K-1} \mathcal{T}_l(p), 0 \le n \le t_K\} \times \\ &\qquad \times \mathsf{P}\{\max_{0\le n\le s} \|\xi_n(q) - u(q;s)\| \le \gamma s^\delta\} \\ &\ge \left(1 - \frac{c''}{\gamma^2} \tau^{1-2\delta}(p) \cdot \frac{C^{K(1-2\delta)} - 1}{C^{1-2\delta} - 1}\right) \times \left(1 - \frac{c''}{\gamma^2} \tau^{1-2\delta}(p) \cdot C^{K(1-2\delta)}\right) \\ &\ge 1 - \frac{c''}{\gamma^2} \tau^{1-2\delta}(p) \cdot \frac{C^{(K+1)(1-2\delta)} - 1}{C^{1-2\delta} - 1}, \end{aligned}$$

where (4.5.14) and the induction assumption were used for the second inequality. This shows (4.5.13), as we wanted. □

For showing (4.5.5) as well as for future reference, we will examine the sets $\mathcal{T}^\gamma(p)$ more closely. We use the notation from the proof of Theorem 4.1. Suppose that $p \in \mathcal{I}(s)$, $s \geq 1$, i.e. $r(p) = s$. Fix $\gamma > 0$ and use σ from **P1**. Define by T_n^s the projection of the time tube $\mathcal{T}_n^\gamma(p)$ *along Euler paths* onto the isochrone $\mathcal{I}(s)$ at distance s:

$$T_n^s = \{x \in \mathcal{I}(s) \,|\, \Gamma_x \cap \mathcal{T}_n^\gamma(p) \neq \emptyset\}.$$

Since $T_n^s \subset \mathcal{I}(s)$, and $\mathcal{I}(s)$ compact, $\limsup_{n\to\infty} T_n^s$ exists as an open subset of $\mathcal{I}(s)$. We will determine the limsup.

The projections T_n^s are determined by the set (cf. (4.5.12))

$$\mathcal{O}_{C^n r(p)-\rho_{n+1}^\gamma}\big(u(A_n; -\rho_{n+1}^\gamma), \sigma\gamma \cdot \tau^\delta(C^n p)\big).$$

Projecting $u(A_n; -\rho_{n+1}^\gamma)$ back to $\mathcal{I}(s)$ is the same as projecting A_n back along the Euler path to $\mathcal{I}(s)$. This map is given by (cf. (4.5.12))

$$\begin{aligned} u\big(u(A_n; -\rho_{n+1}^\gamma); 1 - C^n r(p) + \rho_{n+1}^\gamma\big) &= u(A_n; 1 - C^n r(p)) = \mathcal{O}_s(p, \alpha_n^\gamma / C^n) \\ &= \mathcal{O}_s\Big(p, \sigma\gamma \cdot \tau^\delta(p) \frac{1 - C^{-n(1-\delta)}}{C - C^\delta}\Big) \\ &\uparrow \mathcal{O}_s\big(p, \sigma\gamma \frac{\tau^\delta(p)}{C - C^\delta}\big), \; n \to \infty. \qquad (4.5.15) \end{aligned}$$

Further, whenever $\rho_{n+1}^\gamma < \tau(C^{n-1}p)$, one has

$$u\Big(\mathcal{O}_{C^n r(p)-\rho_{n+1}^\gamma}\big(u(A_n; -\rho_{n+1}^\gamma), \sigma\gamma \cdot \tau^\delta(C^n p)\big); \rho_{n+1}^\gamma\Big) \subset \mathcal{O}_{C^n r(p)}\big(A_n, \sigma\gamma\tau^\delta(C^n p) \cdot C\big).$$

Hence,

$$\begin{aligned} & u\Big(\mathcal{O}_{C^n r(p)-\rho_{n+1}^\gamma}\big(u(A_n; -\rho_{n+1}^\gamma), \sigma\gamma \cdot \tau^\delta(C^n p)\big); 1 - C^n r(p) + \rho_{n+1}^\gamma\Big) \\ &\subset \mathcal{O}_s\Big(u(A_n; 1 - C^n r(p)), \sigma\gamma \cdot \frac{\tau^\delta(C^n p)}{C^{n-1}}\Big) = \mathcal{O}_s\Big(p; \frac{\alpha_n^\gamma}{C^n} + \sigma\gamma \cdot \frac{\tau^\delta(C^n p)}{C^{n-1}}\Big) \\ &\subset \mathcal{O}_s\Big(p, \sigma\gamma \cdot \tau^\delta(p) \frac{1 + C^{-n(1-\delta)}(C^2 - C^{1+\delta} - 1)}{C - C^\delta}\Big) \\ &\to \mathcal{O}_s\big(p, \sigma\gamma \cdot \tau^\delta(p) \frac{1}{C - C^\delta}\big), \; n \to \infty. \end{aligned}$$

This holds for all sufficiently large p. Similarly,

$$\begin{aligned} & u\Big(\mathcal{O}_{C^n r(p)-\rho_{n+1}^\gamma}\big(u(A_n; -\rho_{n+1}^\gamma), \sigma\gamma \cdot \tau^\delta(C^n p)\big); 1 - C^n r(p) + \rho_{n+1}^\gamma\Big) \\ &\supset \mathcal{O}_s\Big(p, \sigma\gamma \cdot \tau^\delta(p) \frac{1 + C^{-n(1-\delta)}(C - C^\delta - 1)}{C - C^\delta}\Big) \\ &\to \mathcal{O}_s\big(p, \sigma\gamma \cdot \tau^\delta(p) \frac{1}{C - C^\delta}\big), \; n \to \infty. \end{aligned}$$

It follows that

$$\limsup_{n\to\infty} T_n^s = \mathcal{O}_s\Big(p, \sigma\gamma \cdot \tau^\delta(p)\frac{1}{C - C^\delta}\Big). \tag{4.5.16}$$

In particular, for all sufficiently large p

$$\mathcal{O}_s\big(p, \sigma\gamma \cdot \tau^\delta(p)\frac{1 + \min\{0, C^{-n(1-\delta)}(C - C^\delta - 1)\}}{C - C^\delta}\big) \subset \inf_{m\ge n} T_n^s \subset \sup_{m\ge n} T_n^s \subset$$

$$\subset \quad \mathcal{O}_s\Big(p, \sigma\gamma \cdot \tau^\delta(p)\frac{1 + \max\{0, C^{-n(1-\delta)}(C^2 - C^{1+\delta} - 1)\}}{C - C^\delta}\Big). \tag{4.5.17}$$

As a consequence, for all sufficiently large p

$$\begin{aligned}
&\big\{\Gamma_x \,|\, x \in \mathcal{O}_s\big(p, \sigma\gamma \cdot \tau^\delta(p)\frac{1 + \min\{0, C^{-n(1-\delta)}(C - C^\delta - 1)\}}{C - C^\delta}\big)\big\} \cap \{q \,|\, r(q) > C^n s\} \\
&\subset \bigcup_{m\ge n} T_m^\gamma(p) \qquad (4.5.18)\\
&\subset \big\{\Gamma_x \,|\, x \in \mathcal{O}_s\big(p, \sigma\gamma \cdot \tau^\delta(p)\frac{1 + \max\{0, C^{-n(1-\delta)}(C^2 - C^{1+\delta} - 1)\}}{C - C^\delta}\big)\big\}.
\end{aligned}$$

The construction together with Theorem 4.1 show the validity of the following Lemma.

Lemma 4.5.4 *For any $x, y \in \mathcal{I}$, $x \neq y$, and any point $p \in \mathbf{Z}^2$, there exists a time $N > 0$, such that (4.5.5) is valid, that is, $(\Gamma_x \rightsquigarrow \Gamma_y)$ is a sojourn set.*

Proof. Let $x, y \in \mathcal{I}$ be given, $x \neq y$, as well as $p \in \mathbf{Z}$. Choose any $p' \in (x \rightsquigarrow y)$, $p' \neq x, y$. Let $1/2 < \delta < 1$ be given. Note that $\tau(p') = C - 1$. Choose $\gamma > 0$ such that

$$\mathcal{O}_1\big(p', \sigma\gamma \cdot (C-1)^\delta \max\{1/(C - C^\delta), C\}\big) \subset (x \rightsquigarrow y).$$

Then by (4.5.18) $\mathcal{T}^\gamma(p') \subset (\Gamma_x \rightsquigarrow \Gamma_y)$.

By Theorem 4.1 there exists a positive constant c', such that for any sufficiently large initial point q and any time t,

$$\mathsf{P}\{\xi_n \in \mathcal{T}^\gamma(q) \,|\, \xi_0 = q\} \ge 1 - \frac{c'}{\gamma^2} \cdot \tau^{1-2\delta}(q).$$

Choose $q = C^{k'} p'$ for some large enough k'. Then it is easily checked that

$$\mathcal{T}^\gamma(q) \subset \mathcal{T}^\gamma(p') \subset (\Gamma_x \rightsquigarrow \Gamma_y).$$

Hence,

$$\mathsf{P}\{\xi_n \in (\Gamma_x \rightsquigarrow \Gamma_y) \,|\, \xi_0 = q\} \ge 1 - c' \cdot \tau^{1-2\delta}(q).$$

Because of irreducibility, there is a path of positive probability from the selected point p to q, say it has length m and the probability equals π. Then

$$\mathsf{P}\{\xi_n \in (\Gamma_x \rightsquigarrow \Gamma_y), n > m \,|\, \xi_0 = p\} \ge \pi \cdot \big(1 - c'\tau^{1-2\delta}(q)\big).$$

□

4.6 Non-existence of the limit for the time scaled process

4.6.1 Large initial points

Lemma 4.5.4 shows, that the random walk starting at a large but *fixed* point will end up with positive probability "close" to any Euler path. This suggests that scaling back along the Euler path will yield convergence in distribution, as will be the subject of the paper [24]. This scaling is given by

$$\xi_t \to u\big(\xi_t; t+1-r(\xi_t)\big).$$

The limit distribution provides the probability mass of sets of Euler paths that the process may end up in. In general, the scattering is called discrete or continuous whenever the limit distribution (provided it exists!) under this scaling is discrete or continuous.

The time scaled process can only converge, when with time scaled Euler paths $\{u(x;t)/t\}_{t\geq 0}$ one can associate precisely one point. In our case, time scaling yields a cycling set of points of the isochrone, as time goes by, thus accounting for non-convergence of the time scaled process.

We will prove the non-convergence property for all sufficiently large initial points p.

Theorem 4.2 *For all initial points $p \in$ **Z**, except possibly a compact set, the time scaled process $\xi_{[tN]}(p)/N$ does* not *converge in distribution for any macro time t.*

Proof. Fix a reference point $x_0 \in \mathcal{I}$. For $m \in$ **N** to be determined later, split the isochrone $\mathcal{I}(s)$ into m 'equal' parts as follows: $\mathcal{I}(s) = \cup_{l=1}^m \mathcal{I}_l(s)$, with

$$\mathcal{I}_l(s) = \Big\{x \in \mathcal{I}(s) \,\Big|\, \frac{l-1}{m} \leq \frac{\psi_{x_0}(x)}{C-1} < \frac{l}{m}\Big\},$$

where ψ_{x_0} is the 'angle' defined in section 4.5.2. Note that $\mathcal{I}_l(s) = u\big(\mathcal{I}_l(1); s-1\big)$ by virtue of Lemma 4.5.3. Moreover, for $t = t_k + ((r-l)/m)C^k(C-1)$, $l \leq r < l+m$

$$\frac{1}{t+1} u\big(\mathcal{I}_l(1); t\big) = \mathcal{I}_{r\,(\mathrm{mod}\ m)}(1),$$

with $t_k = C^k - 1$ is the time for an Euler path starting at a point of $\mathcal{I}$ to pass precisely k cycles.

Fix a point $q \in \mathcal{I}_1^\circ(1)$, where superscript $^\circ$ denotes the interior of a set. Again we use the notation from the proof of Theorem 4.1 and σ from **P1**. Let $\epsilon \ll 1$. Choose $\gamma > 0$ and $1/2 < \delta < 1$. By the proof of Theorem 4.1, there exists a constant c'' such that (4.5.13) holds for any sufficiently big initial point p. By (4.5.17), for all r big enough, one can take the generic point $p = \lfloor C^r q \rfloor$, such that

i) (4.5.13) holds, with

$$\mathsf{P}\big\{\xi_t(p) \in \mathcal{T}\big(\mathcal{O}_{C^{r+k}+s}\big(u(A_k;s), \sigma\gamma s^\delta\big), \rho_k^\gamma + \sigma\gamma s^\delta\big\} > 1-\epsilon,\ t_{r+k} \leq t < t_{r+k+1}, \tag{4.6.1}$$

where $s = t - t_{r+k}$;

ii) $\mathcal{O} = \mathcal{O}_1\big(q, \sigma\gamma \cdot C^{r\delta - r}(C-1)^\delta \max\{1/(C - C^\delta), C\}\big) \subset \mathcal{I}_1^\circ(1)$, so that

$$\begin{aligned} \mathcal{T}^\gamma(p) &\subset \big\{\Gamma_x \,|\, x \in \mathcal{O}_1(q, \sigma\gamma \cdot C^{r\delta - r}(C-1)^\delta \max\{1/(C - C^\delta), C\})\big\} \\ &\subset \big\{\Gamma_x \,|\, x \in \mathcal{I}_1^\circ(1)\big\}. \end{aligned}$$

Define the infinite sequence $m_k \in \{0, 1, \ldots, m-1\}$, $k = 1, \ldots$. Then one can find an increasing sequence of times t'_k with $t_k < t'_k < t_{k+1}$, such that $u(\mathcal{O}; t'_k)/(1 + t'_k) \subset \mathcal{I}_{m_k}^\circ(1)$. For k large enough, this implies for $\beta_k = \rho_k^\gamma + \sigma\gamma \cdot (t'_k - t_k)^\delta$ that

$$\bigcup_{t'_k - \beta_k < t' < t'_k + \beta_k} \frac{1}{1 + t'} u(\mathcal{O}; t') \subset \mathcal{I}_{m_k}^\circ(1),$$

since $(t'_k \pm \beta_k)/t'_k \to 1$, as $k \to \infty$. Form the cones $C_l = \{\lambda \mathcal{I}_l(1), \lambda > 0\}$. In other words, k large enough

$$\bigcup_{t'_k - \beta_k < t' < t'_k + \beta_k} u(\mathcal{O}; t') \subset C_{m_k}^\circ.$$

As a result we have for k large enough

$$\mathsf{P}\{\xi_{t'_k}(p) \in C_{m_k}\} \geq 1 - \epsilon.$$

In words, the process keeps on cycling through different cones, as time goes by. But this implies that the sequence $\xi_{t'_k}(p)/t'_k$ cannot converge in distribution.

By using finitely many different choices of the reference point x_0, non convergence can be shown for all p outside a compact set. □

4.6.2 Invariant measure

The question is left, whether there can be convergence at all, and under what conditions. It turns out, that if the scaled process converges in distribution, then the limiting distribution should be invariant with respect to the dynamical system. We prove the latter.

Let an initial point p be given. Let $\mathcal{B}^2$ denote the σ-algebra of Borel sets of $\mathbf{R}^2$. So, we assume that the sequence $\xi_N(p)/N$ converges in distribution to a random vector ξ on $(\mathbf{R}^2, \mathcal{B}^2)$ with distribution $\mu(A) = \mathsf{P}\{\xi \in A\}$ for any Borel-measurable subset of $\mathbf{R}^2$. Note further, that this is equivalent to the sequence $\xi_{[tN]}(p)/N$ converging in distribution to the vector $t\xi$.

Remind that the Euler distance $r(\xi_N(p)/N)$ of the time scaled process $\xi_N(p)/N$ a.s. converges to 1. Moreover, $\xi_N(p)/N \in A$ iff $\xi_N(p) \in NA$. This suggests to first study the measure μ of cone-type sets, defined by two non-intersecting curves starting at the origin. By virtue of Lemma 4.5.1, it is tempting to state that the measure μ is concentrated on the isochrone $\mathcal{I}$. One can identify it with a measure on $\mathcal{I}$ only if certain smoothness properties hold. This will follow from the analysis below.

With each interval $[x \rightsquigarrow y] \subset \mathcal{I}(s)$ one can associate a cone

$$A_{[x \rightsquigarrow y]} = \{\lambda[x \rightsquigarrow y] \,|\, \lambda > 0\}.$$

Analogously we define cones associated with open, or half open connected subsets of $\mathcal{I}$. Note that $q \in A_{[x \rightsquigarrow y]}$ if and only if $u\big(q; (t/s) \cdot r(q)\big) \in A_{u([x \rightsquigarrow y];t)}$. Further, write $\delta A = \bar{A} \setminus A^\circ$ for the boundary of set A.

We will work under condition 4.5.1, but this is only because of the necessary bounds already being available.

Lemma 4.6.1 *Assume that ξ_N satisfies condition 4.5.1. One has $\mu\big\{\delta A_{[x \rightsquigarrow y]}\big\} = 0$, for any cone $A_{[x \rightsquigarrow y]}$.*

Proof. Let $p = 0$ for simplicity. The case of arbitrary p complicates the choice of suitable constants in the estimates, but otherwise is not essentially different. Also, drop the dependence on p in the notation.

It is sufficient to prove that $\mu(\lambda \cdot x \,|\, \lambda > 0\} = 0$, for any $x \in \mathcal{I}$. Write $A_x = \{\lambda x \,|\, \lambda > 0\}$. Assume that $\mu\{A_x\} = \epsilon > 0$ for some $x \in \mathcal{I}$. We will prove that in that case $\mu\{A_y\} \geq \epsilon$ for any $y \in \mathcal{I}$. This contradicts finiteness of the measure μ.

Choose any $t < C - 1$. Write $B = A_{u(x;t)} = A_{u(x;t)/(1+t)}$. We will show that $\mu\{B\} \geq \mu\{A_x\}$. These two sets are related by the map $y \rightarrow u(y; t \cdot r(y))$. So, in order that $\xi_{[(1+t)N]}$ be sufficiently close to B, ξ_N needs to be close enough to A.

For our purpose it is sufficient to construct open sets A_γ, B_γ, with $A_\gamma \downarrow A_x$, and $\overline{B}_\gamma \downarrow B$, as $\gamma \rightarrow 0$, such that

$$\liminf_{N\to\infty} \mathsf{P}\{\xi_{[(1+t)N]} \in B_\gamma\} \geq \liminf_{N\to\infty} \mathsf{P}\{\xi_N \in A_\gamma\}. \tag{4.6.2}$$

Indeed, by assumption this implies

$$\begin{aligned}
\mu\{A_\gamma\} &\leq \liminf_{N\to\infty} \mathsf{P}\{\xi_N \in A_\gamma\} \\
&\leq \liminf_{N\to\infty} \mathsf{P}\{\xi_{[(1+t)N]} \in \overline{B}_\gamma\} \\
&\leq \limsup_{N\to\infty} \mathsf{P}\{\xi_{[(1+t)N]} \in \overline{B}_\gamma\} \\
&\leq \mu\{\overline{B}_\gamma\}.
\end{aligned}$$

As a consequence

$$\mu\{B\} = \lim_{\gamma\to 0} \mu\{\overline{B}_\gamma\} \geq \lim_{\gamma\to 0} \mu\{A_\gamma\} = \mu\{A_x\}.$$

Now, two factors for causing dispersion have to be taken into account in constructing suitable sets. The first is that the 'most likely paths' at point $q \in A_\gamma$, with $r(q)$ large, are 'close' to the Euler paths $u(q; \cdot)$ by Theorem 4.1, but not equal to them. These 'most likely paths' should end up in B_γ at time $[t \cdot r(q)]$.

Use the notation from the proof of Theorem 4.1 as well as σ, ν from Property **P1**. Choose $\gamma > 0$, $1/2 < \delta < 1$. Set $A_\gamma = \cup_{q \in A_x} \mathcal{O}_{r(q)}(q, \gamma)$. This set is open. For any point $q \in \mathbf{R}^2$, write $d(q, \gamma) = \sigma\gamma t^\delta \cdot r^\delta(q)$. Then by (4.5.13) the 'most likely paths' starting at point $q \in A_\gamma$, end up in the set $\mathcal{T}\big(\mathcal{O}_{r(q)(1+t)}\big(u(q; t \cdot r(q)), d(q,\gamma)\big), d(q,\gamma)\big)$ at time $t \cdot r(q)$.

The second fact to be taken into account, is that at time N, we only have that $r(\xi_N)$ is approximately equal to N. Starting at point $q \in A_\gamma$ with $r(q) \approx N$, one should therefore have that the 'most likely paths' end up in B_γ at time $[tN]$ instead of at time $[t\cdot r(q)]$. We will estimate the difference in position at the different times.

Let $1/2<\eta<\delta$, and $\epsilon>0$. Denote $C_N=\{x\in\mathbf{R}^2\,|\,|r(x)-N|<\epsilon N^\eta\}$. By virtue of (4.5.4) using the martingale defined in (4.5.2), there exists a constant d, such that

$$\mathsf{P}\{\xi_N\in C_N\}\ge 1-2\exp\{-d\epsilon^2N^{2\eta-1}\}. \tag{4.6.3}$$

For $q\in C_N$, $|[r(q)]-N|<\epsilon N^\eta$. The difference in positions at time $[tN]$ and $t\cdot r(q)$ is then bounded by

$$\|\xi_{[tN]}(q)-\xi_{[t\cdot r(q)]}(q)\|\le\sqrt{2}\epsilon\cdot tN^\eta. \tag{4.6.4}$$

Then for $q\in A_\gamma\cap C_N$, the 'most likely paths' starting at q end up in

$$\mathcal{O}\big(u(q;t\cdot r(q)),\frac{1}{\nu}d(q,\gamma)+\sqrt{2}\epsilon t\cdot N^\eta\big)$$

at time $[tN]$. For $q\in C_N$ we have that $N^\eta\le 2r^\eta(q)$. Combine this to define

$$B_\gamma=\cup_{q\in A_\gamma}\mathcal{O}\big(u(q;t\cdot r(q)),\frac{1}{\nu}d(q,\gamma)+2\sqrt{2}\epsilon t\cdot r^\eta(q)\big).$$

B_γ is an open set, as can be deduced by explicitly writing the equation for its boundary. Observe that (4.6.3) implies

$$\mathsf{P}\{\xi_N\in A_\gamma\cap C_N\}\ge\mathsf{P}\{\xi_N\in A_\gamma\}-2\exp\{-d\epsilon^2N^{2\eta-1}\}.$$

Combination with (4.5.13) for $k=1$ yields the existence of a constant c such that for all large values of N

$$\begin{aligned}
\mathsf{P}\{\xi_{[(1+t)N]}\in B_\gamma\}&\ge\sum_{q\in A_\gamma\cap C_N}\mathsf{P}\{\xi_N=q\}\mathsf{P}\{\xi_{[tN]}(q)\in B_\gamma\}\\
&\ge\sum_{q\in A_\gamma\cap C_N}\mathsf{P}\{\xi_N=q\}\Big(1-\frac{c}{\gamma^2}(C-1)^{1-2\delta}r^{1-2\delta}(q)\Big)\\
&\ge\sum_{q\in A_\gamma\cap C_N}\mathsf{P}\{\xi_N=q\}\Big(1-\frac{c}{\gamma^2}(C-1)^{1-2\delta}(N-\epsilon N^\eta)^{1-2\delta}\Big)\\
&\ge\big(\mathsf{P}\{\xi_N\in A_\gamma\}-2\exp\{-d\epsilon^2N^{2\eta-1}\}\big)\big(1-\frac{c}{\gamma^2}(C-1)^{1-2\delta}(N-\epsilon N^\eta)^{1-2\delta}\big).
\end{aligned}$$

(4.6.2) immediately follows by taking $\liminf_{N\to\infty}$ on both sides. □

Desired invariance immediately follows.

Theorem 4.3 *Assume that ξ_N satisfies condition 4.5.1. The measure μ is invariant with respect to each cone $A_{[x\rightsquigarrow y]}$, $x,y\in\mathcal{I}$, i.e.*

$$\mu(A_{[x\rightsquigarrow y]})=\mu\big(A_{[u(x;t)\rightsquigarrow u(y;t)]}\big),\quad t\ge 0. \tag{4.6.5}$$

The same applies for open and half open cones, by virtue of Lemma 4.6.1.

Proof of Theorem 4.3. It is sufficient to assume that $t \le C-1$. Assume $p=0$, and select $x, y \in \mathcal{I}$. Write $A = A_{[x \rightsquigarrow y]}$ and $B = A_{[u(x;t) \rightsquigarrow u(y;t)]}$ and drop the dependence on p in the notation. Again conditioning on the state at time N, we can use precisely the same procedure as in the proof of the previous lemma, to obtain that $\mu(B) \ge \mu(A)$.

Now, write $x' = u(x;t)/(1+t)$ and $y' = u(y;t)/(1+t)$. Then note that we can map $B = A_{[x' \rightsquigarrow y']}$ along the dynamical system to A by

$$A = A_{[u(x';(C-1-t)/(1+t)) \rightsquigarrow u(y';(C-1-t)/(1+t))]}.$$

This implies that $\mu(A) \ge \mu(B)$. □

It is now straightforward to compute the measure μ: it should be homogeneous with respect to the set of cones $A_{[x \rightsquigarrow y]}$. This can be seen, by splitting up the isochrone into k intervals that require the same time to be 'crossed'. By Theorem 4.3 their measures are equal and should be equal to $1/k$. The following corollary gives the general formula.

Corollary 4.6.1 *For* $x, y \in \mathcal{I}$,

$$\mu\{A_{[x \rightsquigarrow y]}\} = \frac{\psi_x(y)}{C-1}.$$

It is now an easy consequence, that the measure μ is smooth with respect to *any* cone. Take for instance a cone A' of the form

$$A' = \{\mathcal{O}_{r(q)}(q, \gamma r^{\delta}(q) \mid q \in A_{[x \rightsquigarrow y]}\},$$

for some positive γ and $0 < \delta < 1$. Choose monotone sequences $\{x_n\}_n$, $\{y_n\}_n \subset \mathcal{I}$, with $A_{[x_n \rightsquigarrow y_n]} \downarrow A_{[x \rightsquigarrow y]}$, $n \to \infty$. Then up to compact sets we have for any n

$$A_{[x \rightsquigarrow y]} \subset A_{\gamma} \subset A_{[x_n \rightsquigarrow y_n]}.$$

Hence,

$$\mu\{A_{[x \rightsquigarrow y]}\} \le \mu\{A_{\gamma}\} \le \mu\{A_{[x_n \rightsquigarrow y_n]}\}.$$

and by taking the limit $n \to \infty$ we obtain that $\mu\{A_{[x \rightsquigarrow y]}\} = \mu\{A_{\gamma}\}$. □

A final remark should be made: if for some point p the time scaled process $\xi_N(p)/N$ converges in distribution to μ, then this μ is also the scattering measure on the set of Euler paths, for initial point p.

Chapter 5

Large deviations bounds in the quarter plane

This chapter has appeared as A.Hordijk, N. Popov (2003). *Large deviations bounds for face-homogeneous random walks in the quarter plane.* Probability in the Engineering and Informational Sciences 17(3): 369-396.

5.1 Introduction

The sample path large deviation problem for random walks with boundaries has been studied in [15] and [28].

For linear paths

$$\varphi : [0,1] \to \mathbb{R}_+^2 \text{ with } \varphi(t) = x + vt,$$

explicit expressions for the local rate function $L(x, v)$ for the case that either $x \neq (0,0)$ or $v \neq (0,0)$ have been derived in [15] and [28]. Clearly, the local rate function for $x = (0,0)$ and $v = (0,0)$ equals 0 if the process is ergodic. The determination of $L(0,0)$ was left as an open problem in [15] for the transient random walk. In this chapter we derive an explicit expression for a lower bound for $L(\mathsf{0},\mathsf{0})$, and under an extra condition it holds that the lower bound is equal to an upper bound. Hence, in this case the lower bound gives the local rate function.

In [6] it has been shown for a large class of queueing networks including the model of this chapter, that the large deviations principle holds with local rate function expressed in terms of a stochastic optimal control problem. In the recent paper [13] the local rate function is connected with the convergence parameter of associated local transform matrices. Both approaches are quite general and hold for a large class of models. The specific model we consider in this chapter allows for an expression in terms of the cumulant generating functions on the different faces of $\mathbb{R}_+^2$, this expression makes an easy numerical calculation possible. For specific queueing networks explicit expressions are obtained.

We provide a selfcontained proof based on the change of measure and the analysis of the cumulant generating functions. Almost closed sets play an essential role in the proof. In this chapter we restrict ourselves to the local rate function for a path identically equal to zero. The proof for linear paths of the type $\varphi(t) = x + vt$ with $v \neq (0,0)$ can be done with the same type of analysis except that the analysis of almost closed sets is not needed. The analysis for the paths not identically to zero can be found in [15] and [28]. The combination of these results with ours gives a complete solution of the sample path large deviations problem in the positive quadrant.

Also in the recent paper [4] an extensive study has been made for the asymptotic behavior of large deviations for Markov chains in the positive quadrant. Precise asymptotics are obtained for the logarithm of the transition probabilities.

The outline of this chapter is as follows. In section 5.2 we give the model description and we state our main result, including the expression for the local rate function for a linear path identically equal to zero. In section 5.3 we first give a complete description of the local rate function for all linear paths. We also summarize in that section the classification of ergodicity, null recurrence and transience and the results on almost closed sets we need for our analysis. For completeness we briefly introduce the twisted process and the change of measure lemma that we use in section 5.4. The main part of section 5.4 is devoted to the analysis of the cumulant generating functions on the different faces of $\mathbb{R}^2_+$, and to the proof of the large deviations lower bound. With our condition the proof of the large deviations upper bound is straightforward. In section 5.5 we illustrate how the expression for the local rate function can be used to compute the large deviations bounds for specific random walk in the quarter plane. In Chapter 6 applications will be made to stochastic networks that model queueing networks with coupled processors.

5.2 Model description and main result

5.2.1 Model description

We consider an irreducible, aperiodic Markov chain $\mathcal{M} = \{\xi_n\}$ on the state space $\mathbf{Z}^2_+$.

We assume that the following conditions are satisfied for $\mathcal{M}$.

Condition A The transition probabilities are

$$p_{i,j} = \mathsf{P}\{i \to j\} = \begin{cases} p^0_{j-i} & \text{if } i_1 = 0, i_2 = 0, \\ p^1_{j-i} & \text{if } i_1 > 0, i_2 = 0, \\ p^2_{j-i} & \text{if } i_1 = 0, i_2 > 0, \\ p^3_{j-i} & \text{if } i_1 > 0, i_2 > 0. \end{cases} \tag{5.2.1}$$

So there are four faces of homogeneity

$$\mathbf{\Lambda}^1 = \{i \in \mathbf{Z}^2 : i_1 > 0, i_2 = 0\},\ \mathbf{\Lambda}^2 = \{i \in \mathbf{Z}^2 : i_1 = 0, i_2 > 0\},$$

$$\mathbf{\Lambda}^3 = \{i \in \mathbf{Z}^2 : i_1 > 0, i_2 > 0\} \text{ and } \mathbf{\Lambda}^0 = \{0\}.$$

Recall that by $\mathbf{\Lambda}(i)$ we denote the face, to which state i belongs. Then we can write

$$p_{i,j} = p^{\mathbf{\Lambda}(i)}_{j-i}.$$

Condition B (lower boundedness of jumps)

$$\begin{cases} p^0_{j-i} = 0 & \text{if } j_1 - i_1 < 0 \text{ or } j_2 - i_2 < 0, \\ p^1_{j-i} = 0 & \text{if } j_1 - i_1 < -1 \text{ or } j_2 - i_2 < 0, \\ p^2_{j-i} = 0 & \text{if } j_1 - i_1 < 0 \text{ or } j_2 - i_2 < -1, \\ p^3_{j-i} = 0 & \text{if } j_1 - i_1 < -1 \text{ or } j_2 - i_2 < -1. \end{cases} \tag{5.2.2}$$

Condition C (upper boundedness of jumps)

For a face $\mathbf{\Lambda}$ we have

$$p^{\mathbf{\Lambda}}_{j-i} = 0 \text{ if } j_1 - i_1 > d_1^+ \text{ or } j_2 - i_2 > d_2^+ \text{ for some integer } d_1^+ \geq 1, d_2^+ \geq 1.$$

The special case where $d^+ = d^- = 1$ is depicted in Figure 1.5.

Condition D (local irreducibility)

Let $\mathcal{M}^3$ be a homogeneous Markov chain on $\mathbf{Z}^2$ with transition probabilities equal to p^3_{j-i}. We suppose that $\mathcal{M}^3$ is irreducible. This condition will play an essential role in lemma 5.4.1

Condition E

$$\sum_{k \in \mathbf{Z}^2 \,:\, k_2 > 0} p^1_k > 0 \text{ and } \sum_{k \in \mathbf{Z}^2 \,:\, k_1 > 0} p^2_k > 0. \tag{5.2.3}$$

5.2.2 Main result

To any face $\mathbf{\Lambda}$ there corresponds a jump variable $y^{\mathbf{\Lambda}} = (y_1^{\mathbf{\Lambda}}, y_2^{\mathbf{\Lambda}}) \in \mathbf{Z}^2$ having the distribution $\mathsf{P}\{y^{\mathbf{\Lambda}} = k\} = p_k^{\mathbf{\Lambda}}, k \in \mathbf{Z}^2$ (see (5.2.1)). For any face $\mathbf{\Lambda}$ we define the cumulant generating function $H^{\mathbf{\Lambda}} : \mathbf{R}^2 \to \mathbf{R}$:

$$H^{\mathbf{\Lambda}}(\alpha) = \log\big(\mathsf{E}\exp\{\alpha_1 y_1^{\mathbf{\Lambda}} + \alpha_2 y_2^{\mathbf{\Lambda}}\}\big),$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbf{R}^2$ and E denotes the expectation. We have that

$$H^{\mathbf{\Lambda}}(\alpha) = \log\big(\sum_{k \in \mathbf{Z}^2} p_k^{\mathbf{\Lambda}} \exp\{\alpha_1 k_1 + \alpha_2 k_2\}\big).$$

We define two important points in $\mathbf{R}^2$:

$$\begin{aligned} \alpha^1 &= \arg\min_{\alpha}\{H^1(\alpha) \vee H^3(\alpha)\}, \\ \alpha^2 &= \arg\min_{\alpha}\{H^2(\alpha) \vee H^3(\alpha)\}. \end{aligned} \tag{5.2.4}$$

In section 5.4 we show that α^1 and α^2 exist and they are finite. Let

$$\hat{\alpha} = \arg\max\{H^3(\alpha^1), H^3(\alpha^2)\}.$$

By $\|\cdot\|$ we denote the euclidean norm in $\mathbf{Z}_+^2$, i.e. $\|i\| = \sqrt{i_1^2 + i_2^2}$.

Theorem 5.1 *(LD theorem)*
If for any $\mathbf{\Lambda} = 0, 1, 2$, we have

$$H^{\mathbf{\Lambda}}(\hat{\alpha}) \leq H^3(\hat{\alpha}) \tag{5.2.5}$$

then the following LD bounds are satisfied with local rate function $L^0 = H^3(\hat{\alpha})$:

(LD upper bound) *for any $\delta > 0$ there exists $N(\delta)$ such that for all $N > N(\delta)$*

$$\mathsf{P}\big\{\xi_0 = 0, \sup_{t=0,\cdots,[\tau N]} \|\xi_t\| < \delta N\big\} \leq \exp\{+\delta N + N\tau L^0\}, \tag{5.2.6}$$

(LD lower bound) *for any $\delta > 0, \delta' > 0$ there exists $N(\delta, \delta')$ such that for all $N > N(\delta, \delta')$*

$$\mathsf{P}\big\{\xi_0 = 0, \sup_{t=0,\cdots,[\tau N]} \|\xi_t\| < \delta N\big\} \geq \exp\{-\delta' N + N\tau L^0\}. \tag{5.2.7}$$

Remark 5.2.1 *In fact we prove that the LD lower bound always holds with $L^0 = H^3(\hat{\alpha})$. But for proof of the LD upper bound we need condition (5.2.5). In section 5.5.2 we will give an example, where condition (5.2.5) does not hold.*

Remark 5.2.2 *As we point out in section 5.4, condition (5.2.5) is always satisfied for null recurrent MC, but it is not always fulfilled for transient MC. For an ergodic MC condition (5.2.5) does not hold, but for ergodic MC the local rate function is known to be equal to zero.*

5.3 Related results and main definitions

5.3.1 Large deviations principle

In this section we recall some results on the LD principle for our model and we give a complete description of the rate function.

In section 1.4 the LD principle for the random walk ξ_t has been written in the same form as it was given in [15]. In that section we also recall theorem 3.1.1 from [15], which states sufficient conditions for ξ_t to satisfy the LD principle. The LD principle for general random walks can conveniently be found in the literature (see e.g. [6],[15],[28]).

In [6] and [28] it has been shown that the random walk $\{\xi_n\}$ satisfies the LD principle with a good rate function

$$\mathcal{L}_\tau : C\big([0;\tau], \mathbf{R}^2_+\big) \to [0, +\infty],\ \tau > 0,$$

where $C\big([0;\tau], \mathbf{R}^2_+\big)$ denotes the set of all continuous functions

$$\varphi : [0;\tau] \to \mathbf{R}^2_+.$$

In [15] it has been proved under the assumption that ξ_t is ergodic, that the good rate function has the folowing form

$$\mathcal{L}_\tau(\varphi) = \begin{cases} \int_0^\tau L\big(\varphi(t), \varphi'(t)\big)dt & \text{if the path } \varphi \text{ is absolutely continuous,} \\ +\infty & \text{otherwise ,} \end{cases} \tag{5.3.1}$$

where the local rate function

$$L(\cdot,\cdot) : \mathbf{Z}_+^2 \times \mathbf{R}^2 \to \mathbf{R}_+ \cup \{+\infty\}$$

is defined by

$$L(i,v) = \begin{cases} L^3(v_1,v_2) & \text{if } i_1 > 0, i_2 > 0, \\ L^2(v_2) & \text{if } i_1 = 0, i_2 > 0, \\ L^1(v_1) & \text{if } i_1 > 0, i_2 = 0, \\ L^0 & \text{if } i_1 = 0, i_2 = 0, \end{cases} \tag{5.3.2}$$

where $L^0 = 0$ and L^1, L^2, L^3 are the following Legendre transforms:

$$L^3(v) = \sup_{\alpha}\{(\alpha, v) - H^3(\alpha)\},$$
$$L^2(v_2) = \sup_{\alpha_2}\big\{\alpha_2 v_2 - H^3(\alpha_1(\alpha_2), \alpha_2)\big\},$$
$$L^1(v_1) = \sup_{\alpha_1}\big\{\alpha_1 v_1 - H^3(\alpha_2(\alpha_1), \alpha_1)\big\},$$

and the functions $\alpha_1(\cdot), \alpha_2(\cdot) : \mathbf{R} \to \mathbf{R}$ are defined as

$$\begin{aligned} \alpha_1(\alpha_2) &\triangleq \arg\min_{\alpha_1}\{H^3(\alpha_1,\alpha_2) \vee H^2(\alpha_1,\alpha_2)\}, \\ \alpha_2(\alpha_1) &\triangleq \arg\min_{\alpha_2}\{H^3(\alpha_1,\alpha_2) \vee H^1(\alpha_1,\alpha_2)\}. \end{aligned} \tag{5.3.3}$$

In section 5.4 we prove that $\alpha_1(\alpha_2)$ and $\alpha_2(\alpha_1)$ are finite. As a consequence of the results in [15] or in [28] and the analysis of this chapter we have the following theorem.

Theorem 5.2

If the random walk ξ_n is nonergodic and condition (5.2.5) is satisfied then the LD principle remains true with the rate function $\mathcal{L}_\tau(\cdot)$, where the local rate function L^0 as in theorem 5.1.

In this chapter we focus on the derivation of L^0, and the proof of the bounds (5.2.6),(5.2.7) under the condition (5.2.5).

5.3.2 Classification : ergodicity, null recurrence and transience.

Here we recall the criteria for our random walk to be ergodic, null recurrent or transient in terms of the mean drift on the faces. We shall especially use these criteria for the twisted processes.

Define the vector

$$m^{\mathbf{\Lambda}} = (m_1^{\mathbf{\Lambda}}, m_2^{\mathbf{\Lambda}}) \in \mathbf{R}^2, \ \mathbf{\Lambda} = 0,1,2,3, \tag{5.3.4}$$

as the one step mean drift from a point, which is an element of $\mathbf{\Lambda}$, by

$$m_1^{\mathbf{\Lambda}} = \mathsf{E}\{y_1^{\mathbf{\Lambda}}\} = \sum_{k_1} k_1 \mathsf{P}\{y_1^{\mathbf{\Lambda}} = k_1\} = \sum_{k\in\mathbf{Z}^2} k_1 p_k^{\mathbf{\Lambda}},$$
$$m_2^{\mathbf{\Lambda}} = \mathsf{E}\{y_2^{\mathbf{\Lambda}}\} = \sum_{k_2} k_2 \mathsf{P}\{y_2^{\mathbf{\Lambda}} = k_2\} = \sum_{k\in\mathbf{Z}^2} k_2 p_k^{\mathbf{\Lambda}}.$$

The following lemma is a consequence of theorems 3.3.1 and 3.3.2 in [7].

Lemma 5.3.1 *Assume for the MC $\mathcal{M}$ the conditions* **A**, **B** *and* **C** *are satisfied. With the mean drift vectors (5.3.4) we define two constants :*

$$V^1 = m_1^3 m_2^1 - m_2^3 m_1^1 \text{ and } V^2 = m_2^3 m_1^2 - m_1^3 m_2^2 \,. \tag{5.3.5}$$

Then

a) *if* $m_1^3 < 0, m_2^3 < 0$ *then the MC* $\mathcal{M}$ *is*

(i) ergodic iff $V^1 < 0$ *and* $V^2 < 0$*;*

(ii) null recurrent iff

$$\text{either} \begin{cases} V^2 = 0 \\ V^1 \le 0 \end{cases} \text{or} \begin{cases} V^1 = 0 \\ V^2 \le 0; \end{cases} \tag{5.3.6}$$

(iii) transient iff $V^1 > 0$ *or* $V^2 > 0$*;*

b) *if* $m_1^3 \ge 0, m_2^3 < 0$ *then the MC* $\mathcal{M}$ *is*

ergodic iff $V^1 < 0,$ *null recurrent iff* $V^1 = 0,$ *transient iff* $V^1 > 0;$

c) *if* $m_1^3 < 0, m_2^3 \ge 0$ *then the MC* $\mathcal{M}$ *is*

ergodic iff $V^2 < 0,$ *null recurrent iff* $V^2 = 0,$ *transient iff* $V^2 > 0;$

d) *if* $m_1^3 \ge 0, m_2^3 \ge 0$ *and* $m_1^3 + m_2^3 > 0$ *then the MC* $\mathcal{M}$ *is transient.*

Remark 5.3.1 *If* $m_1^3 = m_2^3 = 0$ *then the random walk can be ergodic, null recurrent or transient (see [7]).*

5.4 Large deviations bounds for the zero path

This section is devoted to the proof of our main result, i.e. the LD bounds as stated in theorem 5.1. First we introduce the twisted process and the change of measure and we analyse the H-functions. Then in subsections 5.4.4 and 5.4.5 we prove the LD lower and upper bound.

5.4.1 Twisted process

First we recall the well-known twisted MC. For any $\alpha \in \mathbf{R}^2$ we define a MC

$$\mathcal{M}(\alpha) = \{\xi_n^\alpha,\ n = 0, 1, 2, \cdots\}$$

on the state space $\mathbf{Z}_+^2$ with transition probabilities

$$p_{i,j}(\alpha) \triangleq \frac{p_{i,j} \exp\{(\alpha, j) - (\alpha, i)\}}{\sum_j p_{i,j} \exp\{(\alpha, j) - (\alpha, i)\}}, \tag{5.4.1}$$

where $(\cdot,\cdot)$ is the scalar product in $\mathbf{R}^2$. The MC $\mathcal{M}(\alpha)$ is said to be a $twisted\ MC$.

Note that $\mathcal{M}(0) = \mathcal{M}$. Clearly, the conditions **A, B, C, D** of section 5.2.1 hold for the MC $\mathcal{M}(\alpha)$ iff they are satisfied for $\mathcal{M}$.

By P_α we denote the probability measure for the twisted MC $\mathcal{M}(\alpha)$. Recall that in subsection 5.2.2 we defined the jumps variable $y^{\mathbf{\Lambda}}$. Hence, we have that

$$\mathsf{P}_\alpha\{y^{\mathbf{\Lambda}} = k\} = p_k^{\mathbf{\Lambda}}(\alpha).$$

By E_α we denote the expectation corresponding to P_α. Similarly to (5.3.4) define the vector

$$m^{\mathbf{\Lambda}}(\alpha) = \big(m_1^{\mathbf{\Lambda}}(\alpha), m_2^{\mathbf{\Lambda}}(\alpha)\big) \triangleq \mathsf{E}_\alpha\{y^{\mathbf{\Lambda}}\} = \sum_{k\in\mathbf{Z}^2} k p_k^{\mathbf{\Lambda}}(\alpha). \tag{5.4.2}$$

By $Cov_\alpha\{y_1^{\mathbf{\Lambda}}, y_2^{\mathbf{\Lambda}}\}$ we denote the covariance of the random variables $y_1^{\mathbf{\Lambda}}, y_2^{\mathbf{\Lambda}}$ with respect to P_α, i.e.

$$Cov_\alpha\{y_1^{\mathbf{\Lambda}}, y_2^{\mathbf{\Lambda}}\} \triangleq \mathsf{E}_\alpha\big(y_1^{\mathbf{\Lambda}} y_2^{\mathbf{\Lambda}}\big) - \mathsf{E}_\alpha y_1^{\mathbf{\Lambda}}\, \mathsf{E}_\alpha y_2^{\mathbf{\Lambda}}.$$

By $Var_\alpha\{y_1^{\mathbf{\Lambda}}\}$ and $Var_\alpha\{y_2^{\mathbf{\Lambda}}\}$ we denote the variance of $y_1^{\mathbf{\Lambda}}, y_2^{\mathbf{\Lambda}}$ with respect to P_α. For completeness we include the following lemma.

Lemma 5.4.1 *For any face* $\mathbf{\Lambda}$ *we have*

$$m^{\mathbf{\Lambda}}(\alpha) = \big(\frac{\partial}{\partial\alpha_1}H^{\mathbf{\Lambda}}(\alpha), \frac{\partial}{\partial\alpha_2}H^{\mathbf{\Lambda}}(\alpha)\big),\ Cov_\alpha\{y_1^{\mathbf{\Lambda}}, y_2^{\mathbf{\Lambda}}\} = \frac{\partial^2}{\partial\alpha_1\partial\alpha_2}H^{\mathbf{\Lambda}}(\alpha), \tag{5.4.3}$$

$$Var_\alpha\{y_1^{\mathbf{\Lambda}}\} = \frac{\partial^2}{\partial^2\alpha_1}H^{\mathbf{\Lambda}}(\alpha) \text{ and } Var_\alpha\{y_2^{\mathbf{\Lambda}}\} = \frac{\partial^2}{\partial^2\alpha_2}H^{\mathbf{\Lambda}}(\alpha). \tag{5.4.4}$$

Proof. First note that from (5.4.1) it follows that

$$p_k^{\mathbf{\Lambda}}(\alpha) = p_k^{\mathbf{\Lambda}} \exp\{(\alpha,k) - H^{\mathbf{\Lambda}}(\alpha)\}. \tag{5.4.5}$$

Let us prove first that $m_2^{\mathbf{\Lambda}}(\alpha) = \frac{\partial}{\partial\alpha_2}H^{\mathbf{\Lambda}}(\alpha)$. Clearly, the function

$$H^{\mathbf{\Lambda}}(\alpha) = \log\big(\sum_{k\in\mathbf{Z}^2} p_k^{\mathbf{\Lambda}} \exp\{\alpha_1 k_1 + \alpha_2 k_2\}\big)$$

is differentiable at any point $\alpha \in \mathbf{R}^2$, and

$$\frac{\partial}{\partial\alpha_2}H^{\mathbf{\Lambda}}(\alpha) = \sum_{k\in\mathbf{Z}^2} k_2 \frac{p_k^{\mathbf{\Lambda}} \exp\{\alpha_1 k_1 + \alpha_2 k_2\}}{\sum_k p_k^{\mathbf{\Lambda}} \exp\{\alpha_1 k_1 + \alpha_2 k_2\}} = \sum_{k\in\mathbf{Z}^2} k_2 p_k^{\mathbf{\Lambda}}(\alpha) = m_2^{\mathbf{\Lambda}}(\alpha).$$

Now we calculate $Cov_\alpha\{y_1^{\mathbf{\Lambda}}, y_2^{\mathbf{\Lambda}}\}$. We have that

$$\frac{\partial}{\partial\alpha_1}p_k^{\mathbf{\Lambda}}(\alpha) = p_k^{\mathbf{\Lambda}} \exp\{(\alpha,k) - H^{\mathbf{\Lambda}}(\alpha)\}\big(k_1 - \frac{\partial}{\partial\alpha_1}H^{\mathbf{\Lambda}}(\alpha)\big) = k_1 p_k^{\mathbf{\Lambda}}(\alpha) - p_k^{\mathbf{\Lambda}}(\alpha) m_1^{\mathbf{\Lambda}}(\alpha).$$

Hence

$$\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}H^{\mathbf{\Lambda}}(\alpha) = \frac{\partial}{\partial\alpha_1}m_2^{\mathbf{\Lambda}}(\alpha) = \sum_{k\in\mathbf{Z}^2} k_2\frac{\partial}{\partial\alpha_1}p_k^{\mathbf{\Lambda}}(\alpha)$$
$$= \sum_{k\in\mathbf{Z}^2} k_1k_2p_k^{\mathbf{\Lambda}}(\alpha) - m_1^{\mathbf{\Lambda}}(\alpha)\sum_{k\in\mathbf{Z}^2} k_2p_k^{\mathbf{\Lambda}}(\alpha) = \mathsf{E}_\alpha(y_1^{\mathbf{\Lambda}}y_2^{\mathbf{\Lambda}}) - m_1^{\mathbf{\Lambda}}(\alpha)m_2^{\mathbf{\Lambda}}(\alpha).$$

The assertion (5.4.4) is a consequence of assertion (5.4.3). □

5.4.2 The change of measure

Let $\delta > 0, \tau > 0$. By $\mathbf{1}_{\{A_N\}}$ we denote the indicator of the event

$$A_N = \{\omega : \sup_{0\le t\le[\tau N]} \|\xi_t(\omega)\| < \delta N\}.$$

The following lemma is well-known, for completeness we include a proof. First recall that by $\mathbf{\Lambda}(i)$ we denote the face, which i belongs to.

Lemma 5.4.2 *For any $\alpha \in \mathbf{R}^2$ and any Borel set Ω we have*

$$\mathsf{P}\{A_N \cap \Omega\} = \mathsf{E}_\alpha\big\{\exp\{-(\alpha, \xi_{[\tau N]}) + (\alpha, \xi_0) + \sum_{t=0}^{[\tau N]-1} H^{\mathbf{\Lambda}(\xi_t)}(\alpha)\}\mathbf{1}_{\{A_N\cap\Omega\}}\big\}. \quad (5.4.6)$$

Proof. The relation (5.4.5) in a different notation is,

$$\mathsf{P}\{i \to j\} = \mathsf{P}_\alpha\{i \to j\}\exp\{-(\alpha, j) + (\alpha, i) + H^{\mathbf{\Lambda}(i)}(\alpha)\}. \quad (5.4.7)$$

Let $n = [\tau N]$. Then for any $\omega \in A_N \cap \Omega$ we have

$$-\big(\alpha, \xi_n(\omega)\big) + \big(\alpha, \xi_0(\omega)\big) = -\sum_{t=0}^{n-1}\big(\alpha, \xi_{t+1}(\omega)\big) - \big(\alpha, \xi_t(\omega)\big).$$

Taking (5.4.7) in account we get

$$\mathsf{P}\{A_N \cap \Omega\} = \sum_{\omega\in A_N\cap\Omega}\prod_{t=0}^{n-1}\mathsf{P}\{\xi_t(\omega) \to \xi_{t+1}(\omega)\}$$
$$= \sum_{\omega\in A_N\cap\Omega}\prod_{t=0}^{n-1}\exp\{-(\alpha, \xi_{t+1}(\omega)) - (\alpha, \xi_t(\omega)) + H^{\mathbf{\Lambda}(\xi_t)}(\alpha)\}\mathsf{P}_\alpha\{\xi_t(\omega) \to \xi_{t+1}(\omega)\}$$
$$= \sum_{\omega\in A_N\cap\Omega}\exp\{-\big(\alpha, \xi_n(\omega)\big) + \big(\alpha, \xi_0(\omega)\big) + \sum_{t=0}^{n-1} H^{\mathbf{\Lambda}(\xi_t)}(\alpha)\}\mathsf{P}_\alpha\{\xi_t(\omega) \to \xi_{t+1}(\omega)\}$$
$$= \mathsf{E}_\alpha\{\exp\{-(\alpha, \xi_{[\tau N]}) + (\alpha, \xi_0) + \sum_{t=0}^{[\tau N]-1} H^{\mathbf{\Lambda}(\xi_t)}(\alpha)\}\mathbf{1}_{\{A_N\cap\Omega\}}\}.$$

□

5.4.3 Analysis of the H-functions

Lemma 5.4.3 *The functions $H^{\mathbf{\Lambda}}$ are convex. The functions $H^{\mathbf{\Lambda}}$ are strictly convex iff the probability mass of $y^{\mathbf{\Lambda}} = \big(y_1^{\mathbf{\Lambda}}, y_2^{\mathbf{\Lambda}}\big)$ is not concentrated on a line.*

Proof. For any $t_1, t_2 \in \mathbf{R}$ consider the stochastic variable $t_1 y_1^{\mathbf{\Lambda}} + t_2 y_2^{\mathbf{\Lambda}}$.

For any α we have that

$$Var_\alpha\{t_1 y_1^{\mathbf{\Lambda}} + t_2 y_2^{\mathbf{\Lambda}}\} = t_1^2 Var_\alpha\{y_1^{\mathbf{\Lambda}}\} + 2t_1t_2 Cov_\alpha\{y_1^{\mathbf{\Lambda}}, y_2^{\mathbf{\Lambda}}\} + t_2^2 Var_\alpha\{y_2^{\mathbf{\Lambda}}\}.$$

Then from Lemma 5.4.1 it follows for any α that

$$Var_\alpha\{t_1 y_1^{\mathbf{\Lambda}} + t_2 y_2^{\mathbf{\Lambda}}\} = t_1^2 \frac{\partial^2}{\partial^2\alpha_1} H^{\mathbf{\Lambda}}(\alpha) + 2t_1t_2 \frac{\partial^2}{\partial\alpha_1\partial\alpha_2} H^{\mathbf{\Lambda}}(\alpha) + t_2^2 \frac{\partial^2}{\partial^2\alpha_2} H^{\mathbf{\Lambda}}(\alpha).$$

Since the variance is nonnegative, it follows that the Hessian matrix of $H^{\mathbf{\Lambda}}(\alpha)$, i.e.

$$\begin{pmatrix} \frac{\partial^2}{\partial\alpha_1\partial\alpha_1} H^{\mathbf{\Lambda}}(\alpha) & \frac{\partial^2}{\partial\alpha_1\partial\alpha_2} H^{\mathbf{\Lambda}}(\alpha) \\ \frac{\partial^2}{\partial\alpha_2\partial\alpha_1} H^{\mathbf{\Lambda}}(\alpha) & \frac{\partial^2}{\partial\alpha_2\partial\alpha_2} H^{\mathbf{\Lambda}}(\alpha) \end{pmatrix}$$

is positive semi-definite. Hence the function $H^{\mathbf{\Lambda}}(\alpha)$ is convex (see e.g. [21] p.448).

The variance of $t_1 y_1^{\mathbf{\Lambda}} + t_2 y_2^{\mathbf{\Lambda}}$ is equal to 0 if and only if for some constant c

$$\mathsf{P}\{t_1 y_1^{\mathbf{\Lambda}} + t_2 y_2^{\mathbf{\Lambda}} = c\} = 1.$$

This means that the probability mass of $t_1 y_1^{\mathbf{\Lambda}} + t_2 y_2^{\mathbf{\Lambda}}$ is concentrated on the line $t_1 x_1 + t_2 x_2 = c$. Since the Hessian matrix of $H^{\mathbf{\Lambda}}$ is positive definite if and only if $H^{\mathbf{\Lambda}}(\alpha)$ is strictly convex (see e.g. [21] p.448), the assumption follows. □

Corollary 5.4.1 *The function $H^3(\alpha)$ is stricly convex.*

Proof. By the condition D the MC $\mathcal{M}^3$ is irreducible. Therefore the probability mass of y^3 is not concentrated on a line. Hence, by Lemma 5.4.3 the function $H^3(\alpha)$ is stricly convex. □

By α^3 we denote the point, where the function $H^3(\alpha)$ has its global minimum, i.e.

$$\alpha^3 = \arg\min_\alpha H^3(\alpha).$$

Lemma 5.4.4 *The point α^3 is finite iff the condition D holds.*

Proof. By corollary 5.4.1 the function $H^3(\alpha)$ is stricly convex. Then it has its minimum at a finite point iff for any fixed $\alpha \neq 0$ the function $f_\alpha(t) \triangleq H^3(t\alpha)$ has a minimum at a finite point. We have that f_α is stricly convex and

$$f_\alpha(t) = \log\big(\sum_{k:\,(\alpha,k)<0} p_k^3 \exp\{t(\alpha,k)\} + \sum_{k:\,(\alpha,k)\geq 0} p_k^3 \exp\{t(\alpha,k)\}\big).$$

Clearly, the function $f_\alpha(t)$ has its minimum at a finite point iff

$$\sum_{k:(\alpha,k)<0} p_k^3 > 0 \text{ and } \sum_{k:(\alpha,k)\geq 0} p_k^3 > 0. \tag{5.4.8}$$

By $\cos(\alpha, k)$ we denote the cosine between vectors $\alpha, k \in \mathbf{R}^2$, then

$$(\alpha, k) = \|\alpha\|\|k\| \cos(\alpha, k).$$

Hence the condition (5.4.8) holds iff there exist $k, l \in \mathbf{R}^2$ such that

$$p_k^3 > 0, \cos(\alpha, k) > 0 \text{ and } p_l^3 > 0, \cos(\alpha, l) < 0. \tag{5.4.9}$$

Now we will show that $\mathcal{M}^3$ is irreducible iff for any $\alpha \neq 0$ there exist $k, l \in \mathbf{Z}^2$ such that condition (5.4.9) is satisfied. Clearly, $\mathcal{M}^3$ is not irreducible if the probability mass is concentrated only in two points. Let us consider the case where the probability mass is concentrated only at three points (vectors k^1, k^2, k^3). One can easily check that in Figure 5.1a and Figure 5.1b the vectors k^1, k^2, k^3 correspond to the irreducible MC $\mathcal{M}^3$ and for any $\alpha \neq 0$ condition (5.4.9) is satisfied.

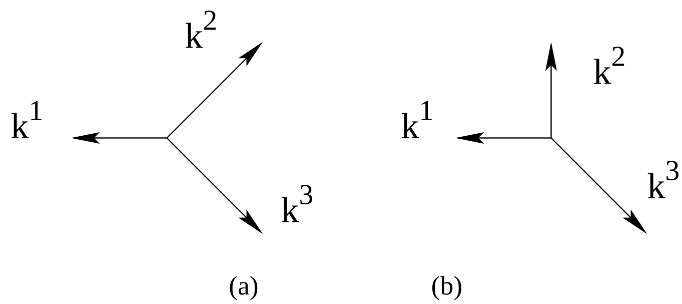

Figure 5.1: the probability mass related to the irreducible Markov chain $\mathcal{M}^3$

By rotating Figure 5.1a and Figure 5.1b over $90, 180, 270$ degrees one can get all possible cases, which correspond to irreducible $\mathcal{M}^3$ with probability mass concentrated only in three points.

Similarly one can consider the other cases, where the probability mass is concentrated in more than three points. □

In analogy to (5.3.5) we define the functions $V^1(\alpha), V^2(\alpha)$ as follows :

$$V^1(\alpha) \triangleq m_1^3(\alpha)m_2^1(\alpha) - m_2^3(\alpha)m_1^1(\alpha),$$

$$V^2(\alpha) \triangleq m_2^3(\alpha)m_1^2(\alpha) - m_1^3(\alpha)m_2^2(\alpha).$$

Note that $V^1(0) = V^1$ and $V^2(0) = V^2$.

Lemma 5.4.5 *The points α^1, α^2 are finite. Either*

$$H^1(\alpha^1) \le H^3(\alpha^1), m^3(\alpha^1) = 0$$

or α^1 is the unique solution of the system

$$H^1(\alpha) = H^3(\alpha),\ V^1(\alpha) = 0,\ m_2^3(\alpha) < 0. \tag{5.4.10}$$

Either

$$H^2(\alpha^2) \le H^3(\alpha^2), m^3(\alpha^2) = 0$$

or α^2 is the unique solution of system

$$H^2(\alpha) = H^3(\alpha),\ V^2(\alpha) = 0,\ m_1^3(\alpha) < 0. \tag{5.4.11}$$

Proof. By Corollary 5.4.1 the function $H^3(\alpha)$ is strictly convex and by Lemma 5.4.4 it has its minimum at a finite point. Hence, the set $\{\alpha : H^3(\alpha) \le C\}$ with $C > H^3(\alpha^3)$ is a compact set. Since $H^1(0) = H^2(0) = H^3(0) = 0$, we have that

$$\alpha^1, \alpha^2 \in \{\alpha : H^3(\alpha) \le 0\}.$$

Hence α^1, α^2 are finite.

Let us analyse the point α^2. For any α_2 define

$$\alpha_1^0 = \alpha_1^0(\alpha_2) \triangleq \arg\min_{\alpha_1} H^3(\alpha_1, \alpha_2).$$

From condition D of section 5.2 it follows that

$$\sum_{k_1<0}\sum_{k_2} p_k^3 > 0 \text{ and } \sum_{k_1>0}\sum_{k_2} p_k^3 > 0.$$

Then for any fixed α_2 the function

$$H^3(\alpha_1, \alpha_2) = \log\Big\{\sum_{k_1<0} e^{\alpha_1 k_1}\Big(\sum_{k_2} p_k^3 e^{\alpha_2 k_2}\Big) + \sum_{k_1\ge 0} e^{\alpha_1 k_1}\Big(\sum_{k_2} p_k^3 e^{\alpha_2 k_2}\Big)\Big\}$$

is strictly monotone decreasing in α_1 on the interval $(-\infty, \alpha_1^0)$ and it is strictly monotone increasing in α_1 on the interval $(\alpha_1^0, +\infty)$. From condition E of section 5.2 it follows for any fixed α_2 that the function

$$H^2(\alpha_1, \alpha_2) = \log\Big\{\sum_{k_1\ge 0} e^{\alpha_1 k_1}\sum_{k_2} p_k^2 e^{\alpha_2 k_2}\Big\},$$

is strictly monotone increasing in α_1. Hence for each fixed α_2 we have two cases.

Case 1

$$H^2\big(\alpha_1^0(\alpha_2), \alpha_2\big) > H^3\big(\alpha_1^0(\alpha_2), \alpha_2\big).$$

Then H^2, H^3 as functions of α_1 intersect on $(-\infty, \alpha_1^0(\alpha_2))$ at

$$\alpha_1(\alpha_2) = \arg\min_{\alpha_1}\{H^2(\alpha_1, \alpha_2) \vee H^3(\alpha_1, \alpha_2)\}.$$

This case is depicted in Figure 5.2. Since $H^3(\alpha_1, \alpha_2)$ is strictly monotone decreasing in α_1 on $(-\infty, \alpha_1^0]$, we have that $m_1^3\big(\alpha_1(\alpha_2), \alpha_2\big) < 0$.

Case 2

$$H^2\big(\alpha_1^0(\alpha_2), \alpha_2\big) \leq H^3\big(\alpha_1^0(\alpha_2), \alpha_2\big).$$

Then $\alpha_1^0(\alpha_2) = \alpha_1(\alpha_2)$, and we have that $m_1^3\big(\alpha_1^0(\alpha_2), \alpha_2\big) = 0$. This case is depicted in Figure 5.3.

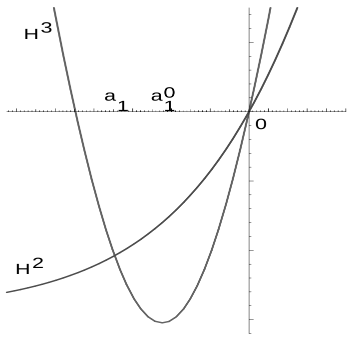

Figure 5.2: Case 1 for each fixed α_2

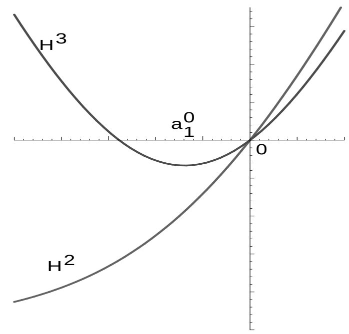

Figure 5.3: Case 2 for each fixed α_2

Since for each α_2 either Case 1 or Case 2 holds, we have that

$$H^2(\alpha^2) = H^3(\alpha^2), m_1^3(\alpha^2) < 0 \text{ or } H^2(\alpha^2) \leq H^3(\alpha^2), m_1^3(\alpha^2) = 0.$$

From Case 1 and Case 2 it also follows that

$$H^3(\alpha^2) = \min_{\alpha_2} H^3\big(\alpha_1(\alpha_2), \alpha_2\big). \tag{5.4.12}$$

Now let us show that either $V^2(\alpha^2) = 0$ or $m_2^3(\alpha^2) = 0$.

Since (5.4.12) holds, it follows that

$$\frac{d}{d\alpha_2} H^3\big(\alpha_1(\alpha_2), \alpha_2\big) = 0 \text{ iff } V^2\big(\alpha_1(\alpha_2), \alpha_2\big) = 0 \text{ or } m^3\big(\alpha_1(\alpha_2), \alpha_2\big) = 0.$$

Here we give a geometrical proof.

If $H^3(\alpha^3) \geq H^2(\alpha^3)$ then $\alpha^2 = \alpha^3$ and clearly, $m^3(\alpha^3) = 0$.

Suppose that $H^3(\alpha^3) < H^2(\alpha^3)$. Then for any $C > 0$ let us consider the set

$$K_C = \{\alpha : H^3(\alpha) \leq C\} \cap \{\alpha : H^2(\alpha) \leq C\}. \tag{5.4.13}$$

Since the H-functions are convex, we have that the sets

$$\{\alpha : H^3(\alpha) \leq C\} \text{ and } \{\alpha : H^2(\alpha) \leq C\}$$

are convex as well. If K_C is not empty then K_C is a compact convex set and $\alpha^2 \in K_C$. Hence,

$$\alpha^2 \in \bigcap_{C\,:\,K_C \neq \emptyset} K_C.$$

Moreover, for any two levels $C_1 < C_2$ we have $K_{C_1} \subset K_{C_2}$. Therefore the intersection of all nonempty K_C sets is compact. Since $\max\{H^2(\alpha), H^3(\alpha)\}$ is stricly convex, this intersection is a single point. Hence α^2 is uniquely defined. Moreover it is the point where the functions H^2, H^3 have a point of contact.

Note that $m^2(\alpha^2)$ is a normal to the level line $\{\alpha : H^2(\alpha) = H^2(\alpha^2)\}$ and $m^3(\alpha^2)$ is a normal to the level line $\{\alpha : H^3(\alpha) = H^3(\alpha^2)\}$. It means that m^2, m^3 have opposite directions at α^2, i.e.

$$\frac{m_1^2(\alpha^2)}{m_2^2(\alpha^2)} = \frac{m_1^3(\alpha^2)}{m_2^3(\alpha^2)} \quad \text{and so} \quad V^2(\alpha^2) = 0.$$

This completes the proof of (5.4.11). The proof of (5.4.10) goes similarly. □

Lemma 5.4.6 *If $m^3(\hat{\alpha}) \neq 0$ then the twisted MC $\mathcal{M}(\hat{\alpha})$ is null recurrent or transient.*

Proof. Suppose that $\hat{\alpha} = \alpha^1$. If $M^3(\alpha^1) \neq 0$ then by Lemma 5.4.5 we obtain

$$m_2^3(\alpha^1) < 0 \text{ and } V^1(\alpha^1) = 0. \tag{5.4.14}$$

If additionally to (5.4.14) we have that $m_1^3(\alpha^1) \geq 0$ then by Lemma 5.3.1 the twisted MC $\mathcal{M}(\alpha^1)$ is null recurrent. Suppose that $m_1^3(\alpha^1) < 0$ besides (5.4.14), then by the same lemma $\mathcal{M}(\alpha^1)$ is null recurrent if $V^2(\alpha^1) \leq 0$; it is transient if $V^2(\alpha^1) > 0$.

The same analysis holds for the case $\hat{\alpha} = \alpha^2$. □

Lemma 5.4.7 *If the MC $\mathcal{M}$ is null recurrent then $\hat{\alpha} = 0$.*

Proof. Let the MC $\mathcal{M}$ be null recurrent.

Suppose that $m^3(0) = 0$, then $\alpha^3 = 0$. Since $H^{\mathbf{\Lambda}}(0) = 0$ for any face $\mathbf{\Lambda}$, we get $\alpha^3 = \hat{\alpha}$, i.e. $\hat{\alpha} = 0$.

Suppose that $m^3(0) \neq 0$. From Lemma 5.3.1 it follows that one of the following cases is satisfied :

$$V^1(0) = 0, m_2^3(0) < 0 \quad \text{(see case a, b in Lemma 5.3.1)}, \tag{5.4.15}$$

$$V^2(0) = 0, m_1^3(0) < 0 \quad \text{(see case a, c in Lemma 5.3.1)}. \tag{5.4.16}$$

Then by Lemma 5.4.5 we have that $\alpha^1 = 0$ if (5.4.15) holds and $\alpha^2 = 0$ if (5.4.16) holds. Since $\hat{\alpha} = \alpha^1$ or $\hat{\alpha} = \alpha^2$, we get that $\hat{\alpha} = 0$. □

Remark 5.4.1 *From Lemma 5.4.7 it follows for a null recurrent $\mathcal{M}$ that $H^{\mathbf{\Lambda}}(\hat{\alpha}) = H^{\mathbf{\Lambda}}(0) = 0$ for any $\mathbf{\Lambda}$. Hence in this case the condition (5.2.5) is fulfilled. In section 5.5 we present some transient models for which condition (5.2.5) is satisfied, but also a transient MC for which it is not true. In Chapter 6 we show that coupled processor models do satisfy the condition. In general an ergodic MC does not satisfy it, but for these models the local rate function $L^0 = 0$.*

5.4.4 Proof of the LD lower bound

To prove the LD lower bound we need results on almost closed sets for our model. Recall that a subset A of the states is called *almost closed* if

$$\mathsf{P}\big(\cup_{m>0}\cap_{t>m}\{\xi_t\in A\}\big)=\mathsf{P}\big(\cap_{m>0}\cup_{t>m}\{\xi_t\in A\}\big)>0.$$

We refer to subsection 2.3 for an introduction to the theory of almost closed sets.

The following lemma follows from Theorem 3 in [18].

Lemma 5.4.8 *Let* $m_1^3>0,\ m_2^3>0$ *then the set* $\mathbf{\Lambda}^3$ *is almost closed.*

Let

$$m_2^3<0, V^1>0$$

then there exists a set $A^1\subseteq\mathbf{\Lambda}^3$ *such that the set* $\mathbf{\Lambda}^1\cup A^1$ *is almost closed.*

Let

$$m_1^3<0, V^2>0$$

then there exists a set $A^2\subseteq\mathbf{\Lambda}^3$ *such that the set* $\mathbf{\Lambda}^2\cup A^2$ *is almost closed.*

If

$$m_1^3<0, m_2^3<0, V^1>0, V^2>0.$$

Then the sets A^1, A^2 *can be taken disjoint.*

□

The proof of the LD lower bound is rather involved. Let

$$A_N=\{\omega:\ \xi_0(\omega)=0,\sup_{0\le t\le[\tau N]}\|\xi_t(\omega)\|<\delta N\}.$$

The assertion of lemma 5.4.9 is the relation (5.2.7) in adequate notation. For the proof of lemma 5.4.9 we need two more lemmas (5.4.10 and 5.4.11).

Lemma 5.4.9 *For any* $\delta>0,\delta'>0$ *we have that*

$$\mathsf{P}\{A_N\}\ge\exp\{N\tau H^3(\hat\alpha)-\delta' N\} \tag{5.4.17}$$

for all sufficiently large N.

Proof. **Case 1.** Let $m^3(\hat\alpha)=0$. It means that

$$H^3(\hat\alpha)=\min_\alpha H^3(\alpha). \tag{5.4.18}$$

We have that

$$\mathsf{P}\{A_N\}>\mathsf{P}\{A_N\cap\Omega_m^3\},\ \text{where } \Omega_m^3=\{\omega:\xi_t(\omega)\in\mathbf{\Lambda}^3 \text{ for all } t>m\}.$$

From Lemma 5.4.2 by the change of measure it follows for any α that

$$\mathsf{P}\{A_N\ \cap\ \Omega_m^3\}=\mathsf{E}_\alpha\exp\{-(\alpha,\xi_{[\tau N]})+(\alpha,\xi_0)+\sum_{t=0}^{[\tau N]-1}H^{\mathbf{\Lambda}(\xi_t)}(\alpha)\}\mathbf{1}_{\{A_N\cap\Omega_m^3\}} \tag{5.4.19}$$

First note that $\|\xi_{[\tau N]}(\omega)\| \leq \delta N$ for any $\omega \in A_N$. Hence, $|(\alpha, \xi_{[\tau N]})| \leq \|\alpha\|\delta N$. Secondly, $\xi_t(\omega) \in \mathbf{\Lambda}^3, t > m$, for any $\omega \in \Omega^3_m$. Then for any α we get

$$\sum_{t=0}^{[\tau N]-1} H^{\mathbf{\Lambda}(\xi_t(\omega))}(\alpha) = [\tau N] H^3(\alpha) + \sum_{t=0}^{m} \left(H^{\mathbf{\Lambda}(\xi_t(\omega))}(\alpha) - H^3(\alpha)\right). \tag{5.4.20}$$

Clearly, for any given α and any $m > 0, \delta' > 0$ there exists $N(\alpha, m, \delta')$ such that

$$\sum_{t=0}^{m} \left(H^{\mathbf{\Lambda}(\xi_t(\omega))}(\alpha) - H^3(\alpha)\right) > -\delta' N \tag{5.4.21}$$

for any ω and for all $N > N(\alpha, m, \delta')$. Recall that $H^3(\alpha) \geq H^3(\hat{\alpha})$ by (5.4.18). Then from (5.4.20) and (5.4.21) it follows for any $m > 0, \omega \in \Omega^3_m$ and all large N that

$$\sum_{t=0}^{[\tau N]-1} H^{\mathbf{\Lambda}(\xi_t(\omega))}(\alpha) \geq [\tau N] H^3(\hat{\alpha}) - \delta' N.$$

Now using (5.4.19) we obtain that

$$\mathsf{P}\{A_N \cap \Omega^3_m\} \geq \exp\{N\tau H^3(\alpha^3) - \|\alpha\|\delta N - \delta' N\} \mathsf{P}_\alpha\{A_N \cap \Omega^3_m\} \tag{5.4.22}$$

for any α, δ, δ' and for all sufficiently large N.

Recall that $m^3(\hat{\alpha}) = 0$. Since the function $H^3(\alpha)$ is strictly convex, for any $\delta > 0$ we can find α_δ close to $\hat{\alpha}$ such that

$$m^3_1(\alpha_\delta) > 0,\ m^3_2(\alpha_\delta) > 0 \text{ and } \tau\|m^3(\alpha_\delta)\| < \delta. \tag{5.4.23}$$

We conclude the proof of the Lemma for the Case 1 by proving the following Lemma. Indeed, the relation (5.4.22) together with (5.4.24) implies (5.4.17).

Lemma 5.4.10 *There exists a positive constant $q > 0$ such that*

$$\mathsf{P}_{\alpha_\delta}\{A_N \cap \Omega^3_m\} > q. \tag{5.4.24}$$

for all sufficiently large m and N.

Proof of Lemma 5.4.10.
First note that

$$\mathsf{P}_{\alpha_\delta}\{A_N \cap \Omega^3_m\} = \mathsf{P}_{\alpha_\delta}\{A_N \,|\, \Omega^3_m\} \mathsf{P}_{\alpha_\delta}\{\Omega^3_m\}.$$

Since $m^3_1(\alpha_\delta) > 0, m^3_2(\alpha_\delta) > 0$ by relation (5.4.23), by lemma 5.4.8 the set $\mathbf{\Lambda}^3$ is almost closed for the twisted process $\mathcal{M}(\alpha^\delta)$. Hence, $\mathsf{P}_{\alpha_\delta}\{\cup_{m>0}\Omega^3_m\} > 0$. Recall that

$$\Omega^3_m \subset \Omega^3_{m+1} \subset \cup_{m>0}\Omega^3_m$$

and therefore there exists a positive q such that

$$\mathsf{P}_{\alpha_\delta}\{\cup_{m>0}\Omega^3\} > \mathsf{P}_{\alpha_\delta}\{\Omega^3_m\} > q$$

for all sufficiently large m.

On other hand

$$m^3(\alpha_\delta) = \mathsf{E}_{\alpha_\delta}\{\xi_{t+1} - \xi_t \,|\, \xi_t \in \mathbf{\Lambda}^3\}$$

and therefore it follows from Kolmogorov's inequality that for any $\epsilon > 0$

$$\mathsf{P}_{\alpha_\delta}\{A_N \,|\, \Omega^3_m\} = \mathsf{P}_{\alpha_\delta}\{\sup_{0\le t\le[\tau N]} \|\xi_t - tm^3(\alpha_\delta)\| < \epsilon N \,|\, \Omega^3_m\} \to 1.$$

By relation (5.4.23) we can take $\epsilon > 0$ small enough such that $\tau\|m^3(\alpha_\delta)\| < \delta - \epsilon$ Then

$$\{\sup_{0\le t\le[\tau N]} \|\xi^{\alpha_\delta}_t - tm^3(\alpha_\delta)\| < \epsilon N\} \subset \{\sup_{0\le t\le[\tau N]} \|\xi^{\alpha_\delta}_t\| < \delta N\}.$$

It implies that

$$\mathsf{P}_{\alpha_\delta}\{A_N \,|\, \Omega^3_m\} \to 1,$$

which completes the proof of (5.4.24). □

Case 2. We continue the proof of lemma 5.4.9 with the case $\hat{\alpha} = \alpha^1$ and $m^3(\hat{\alpha}) \neq 0$. We have that

$$\mathsf{P}\{A_N\} > \mathsf{P}\{A_N \cap \Omega^1_m\}, \text{ where } \Omega^1_m = \{\omega : \xi_t(\omega) \in \mathbf{\Lambda}^1 \cup \mathbf{\Lambda}^3 \text{ for all } t > m\}.$$

From Lemma 5.4.2 by the change of measure it follows for any α that

$$\mathsf{P}\{A_N \cap \Omega^1_m\} = \mathsf{E}_\alpha \exp\{-(\alpha, \xi_{[\tau N]}) + (\alpha, \xi_0) + \sum_{t=0}^{[\tau N]-1} H^{\mathbf{\Lambda}(\xi_t)}(\alpha)\}\mathbf{1}_{\{A_N \cap \Omega^1_m\}}. \quad (5.4.25)$$

By Lemma 5.4.5 we have

$$H^1(\alpha^1) = H^3(\alpha^1),\ V^1(\alpha^1) = 0,\ m^3_2(\alpha^1) < 0.$$

Recall that $m^1_2(\alpha) > 0$ for all α by condition **E**. Hence, for any $\delta > 0, \delta' > 0$ we can find α_δ close to α^1 such that

$$m^3_2(\alpha_\delta) < 0,\ \ 0 < \frac{V^1(\alpha_\delta)}{m^1_2(\alpha_\delta) - m^3_2(\alpha_\delta)} < \delta \quad (5.4.26)$$

and

$$H^1(\alpha_\delta) > H^3(\alpha^1) - \delta'\ ,\ \ H^3(\alpha_\delta) > H^3(\alpha^1) - \delta'. \quad (5.4.27)$$

Note that our conditions **D** and **E** exclude the case that $V^1 \equiv 0$.

We have $\xi_t(\omega) \in \mathbf{\Lambda}^1 \cup \mathbf{\Lambda}^3$ for any $\omega \in \Omega^1_m$ and all $t > m$. It means that for any $t > m$

$$\text{either } H^{\mathbf{\Lambda}(\xi_t(\omega))}(\alpha_\delta) = H^1(\alpha_\delta) \text{ or } H^{\mathbf{\Lambda}(\xi_t(\omega))}(\alpha_\delta) = H^3(\alpha_\delta).$$

Then using the condition (5.4.27) we get

$$\sum_{t>m}^{[\tau N]-1} H^{\mathbf{\Lambda}(\xi_t(\omega))}(\alpha_\delta) > \sum_{t>m}^{[\tau N]-1} (H^3(\alpha^1) - \delta').$$

Therefore for any given $m > 0, \delta > 0, \delta' > 0$ there exists $N(m, \delta, \delta')$ such that

$$\sum_{t=0}^{[\tau N]-1} H^{\mathbf{\Lambda}(\xi_t(\omega))}(\alpha_\delta) \geq [\tau N] H^3(\alpha^1) - \delta' N$$

for all $N > N(m, \delta, \delta')$. Then using (5.4.25) we get

$$\mathsf{P}\{A_N \cap \Omega_m^1\} \geq \exp\{N\tau H^3(\alpha^1) - \|\alpha_\delta\| \delta N - \delta' N\} \mathsf{P}_{\alpha_\delta}\{A_N \cap \Omega_m^1\}.$$

In order to complete the proof of lemma 5.4.9 for the case 2 we need to show the following lemma.

Lemma 5.4.11 *There exists a positive constant q such that*

$$\mathsf{P}_{\alpha_\delta}\{A_N \cap \Omega_m^1\} \geq q$$

for all sufficiently large m and N.

Proof. Since $V^1(\alpha_\delta) > 0$ and $M_2^3(\alpha_\delta) < 0$ by (5.4.26), from Lemma 5.4.8 we find that the set $\mathbf{\Lambda}^1 \cup \mathbf{\Lambda}^3$ is almost closed. Hence, $\mathsf{P}_{\alpha_\delta}\{\cup_{m>0}\Omega_m^1\} > 0$, and since

$$\Omega_m^1 \subset \Omega_{m+1}^1 \subset \cup_{m>0}\Omega_m^1$$

there exists a positive q such that

$$\mathsf{P}_{\alpha_\delta}\{\cup_{m>0}\Omega_m^1\} > \mathsf{P}_{\alpha_\delta}\{\Omega_m^1\} > q \tag{5.4.28}$$

for all sufficiently large m.

Suppose for simplicity of the notations that $\tau = 1$. Let

$$v(\alpha) = \Big(\frac{V^1(\alpha)}{m_2^1(\alpha) - m_2^3(\alpha)}, 0\Big).$$

It follows from (5.4.26) that $\|v(\alpha_\delta)\| < \delta$. Then for any positive ϵ with

$$2\epsilon < \delta - \|v(\alpha_\delta)\|$$

we have that

$$A_N = \{\xi_0 = 0, \sup_{0 \leq t \leq N} \|\xi_t\| < \delta N\} \supset \{\sup_{0 \leq t \leq N} \|\xi_t - tv(\alpha_\delta)\| < 2\epsilon N\}. \tag{5.4.29}$$

By $M_t(\alpha)$ we denote the mean drift of ξ_t^α at time t, i.e.

$$M_t(\alpha) = \mathsf{E}_\alpha\{\xi_{t+1} - \xi_t \,|\, \xi_l, 0 \leq l \leq t\}.$$

Introduce two events :

$$A_N^1 = \{\sup_{0 \leq t \leq N} \|\xi_t - \sum_{l=0}^{t-1} M_l(\alpha_\delta)\| < \epsilon N\},$$

$$A_N^2 = \{\sup_{0 \leq t \leq N} \|tv(\alpha_\delta) - \sum_{l=0}^{t-1} M_l(\alpha_\delta)\| < \epsilon N\}.$$

Then from (5.4.29) it follows for any N that $A_N \supset A_N^1 \cap A_N^2$, and so for any m and N

$$\mathsf{P}_\alpha\{A_N \cap \Omega_m^1\} \geq \mathsf{P}_\alpha\{A_N^1 \cap A_N^2 \cap \Omega_m^1\}. \tag{5.4.30}$$

First we will a give lower bound for the probability $\mathsf{P}_{\alpha_\delta}\{A_N^1\}$. Let

$$\eta_t \triangleq \xi_t - \xi_0 - \sum_{l=0}^{t-1} M_l,\ \eta_0 = 0.$$

We have that

$$\eta_{t+1} - \eta_t = \xi_{t+1} - \xi_t - M_t \text{ for all } t > 0.$$

Since ξ_t has bounded jumps, also η_t has bounded jumps, i.e. for some constant $D > 0$ we have $\|\eta_{t+1} - \eta_t\| \leq D$. Moreover, for any α

$$\begin{aligned}&\mathsf{E}_\alpha\{\eta_{t+1} - \eta_t\,|\eta_l, 0 \leq l \leq t\} = \\ &\quad = \mathsf{E}_\alpha\{\xi_{t+1} - \xi_t\,|\xi_l, 0 \leq l \leq t\} - \mathsf{E}_\alpha\{M_t\,|\xi_l, 0 \leq l \leq t\} = 0.\end{aligned}$$

It means that η_t is a zero-mean martingale with $\eta_0 = 0$. From the Azuma-Hoeffding inequality (see lemma 2.2.2 or [29],p.237) it follows for any $\epsilon > 0$ and α that

$$\mathsf{P}_\alpha\{A_N^1\} = \mathsf{P}_\alpha\{\sup_{0\leq t\leq N} \|\eta_t\| \leq \epsilon N\} \geq 1 - \exp\{-\frac{\epsilon N}{2D^2}\}.$$

Since $\mathsf{P}_\alpha\{A_N^1\} \to 1$ by the previous inequality, we conclude that

$$\mathsf{P}_\alpha\{A_N^1 \cap A_N^2 \cap \Omega_m^1\} \geq \mathsf{P}_\alpha\{A_N^2 \cap \Omega_m^1\} - \exp\{-\frac{\epsilon N}{2D^2}\}. \tag{5.4.31}$$

Now we estimate the probability $\mathsf{P}_{\alpha_\delta}\{A_N^2 \cap \Omega_m^1\} = \mathsf{P}_{\alpha_\delta}\{A_N^2 \,|\, \Omega_m^1\}\mathsf{P}\{\Omega_m^1\}$. Let

$$B_t = \{\|tv(\alpha) - \sum_{l=0}^{t-1} M_l(\alpha)\| > \epsilon t\},\ \epsilon > 0.$$

Then for any α, m, N we have

$$\mathsf{P}_\alpha\{A_N^2 \,|\, \Omega_m^1\} \geq 1 - \mathsf{P}_\alpha\{\cup_{t>0}^N B_t \,|\, \Omega_m^1\}.$$

Clearly, for any sufficiently large N there exists $t_N = o(N)$ such that for any ω

$$\sup_{0\leq t\leq t_N} \|tv(\alpha) - \sum_{l=0}^{t-1} M_l(\alpha)\| < \epsilon N.$$

Then

$$\mathsf{P}_\alpha\{A_N^2 \,|\, \Omega_m^1\} > 1 - \sum_{t>t_N}^{N} \mathsf{P}_\alpha\{B_t\}. \tag{5.4.32}$$

For any $\omega \in \Omega^1_m$ either $M_t(\alpha) = m^1(\alpha)$ or $M_t(\alpha) = m^3(\alpha)$ for all $t > m$. Note that

$$v(\alpha) = \pi^0(\alpha)m^1(\alpha) + (1 - \pi^0(\alpha))m^3(\alpha)\ , \ \ \pi^0(\alpha) = \frac{-m^3_2(\alpha)}{m^1_2(\alpha) - m^3_2(\alpha)},$$

where $\pi^0(\alpha)$ is the stationary probability that the first component of the twisted process ξ^α_t is equal to zero, conditioned on the sample path set Ω^1, where $\Omega^1 = \bigcup_m \Omega^1_m$.

If $m^3_2(\alpha) < 0$ then $0 < \pi^0(\alpha) < 1$ and from results of [1] it follows that for any $\epsilon > 0$ there exists $C > 0$ and $b > 0$ such that

$$\mathsf{P}_\alpha\{B_t \,|\, \Omega^1_m\} < C\exp\{-bt\}, \tag{5.4.33}$$

for all sufficiently large t. We have that $m^3_2(\alpha_\delta) < 0$. So (5.4.33) holds for $\alpha = \alpha_\delta$ as well. Then it follows from (5.4.32) and (5.4.33) that

$$\mathsf{P}_{\alpha_\delta}\{A^2_N \,|\, \Omega^1_m\} > 1 - C\sum_{t>t_N}^{+\infty} \exp\{-bt\}$$

Since the series $\sum_t \exp\{-bt\}$ is convergent and $t_N \to +\infty$, we get that

$$\mathsf{P}_{\alpha_\delta}\{A^2_N \,|\, \Omega^1_m\} \to 1 \text{ as } N \to +\infty. \tag{5.4.34}$$

Now the assertion of our Lemma follows from (5.4.28),(5.4.30),(5.4.31) and (5.4.34). □

5.4.5 Proof of the LD upper bound

The proof of the LD upper bound is a straightforward consequence of our condition (5.2.5). It is proved in the following lemma.

Lemma 5.4.12 *Let condition (5.2.5) be satisfied then for any $\delta > 0$ we have*

$$\mathsf{P}\{A_N\} \le \exp\{N\tau H^3(\hat\alpha) + \delta N\}$$

for all sufficiently large N.

Proof. By the condition (5.2.5) we have that

$$\sum_{t=0}^{[\tau N]-1} H^{\mathbf{\Lambda}(\xi_t(\omega))}(\hat\alpha) \le [\tau N]H^3(\hat\alpha).$$

Therefore from (5.4.6) it follows that

$$\mathsf{P}\{A_N\} \le \mathsf{E}_{\hat\alpha}\exp\{-(\hat\alpha, \xi_{[\tau N]}) + (\hat\alpha, \xi_0) + [\tau N]H^3(\hat\alpha)\}. \tag{5.4.35}$$

Since $\|\xi_t(\omega)\| \le \delta N$ for any $\omega \in A_N$ and $t \le [\tau N]$, we get that $\|(\hat\alpha, \xi_{[\tau N]})\| \le \|\hat\alpha\|\delta N$. It implies immediately that

$$\mathsf{P}\{A_N\} \le \exp\{\|\hat\alpha\|\delta N + \tau N H^3(\hat\alpha)\},$$

which proves the lemma. □

5.5 Applications and numerical examples

In this section we demonstrate how the local rate function L^0 can be computed for specific models (the computations have been done with Maple). In Chapter 6 we show how the large deviations theorems 5.1 and 5.2 can be applied in a coupled processors system.

5.5.1 Numerical examples

In this subsection we present a couple of models, where condition (5.2.5) is satisfied, i.e. the LD bounds hold.

Model 1. Let the transition probabilities $p_k^{\mathbf{\Lambda}}$ have the values as shown in Figure 5.4.

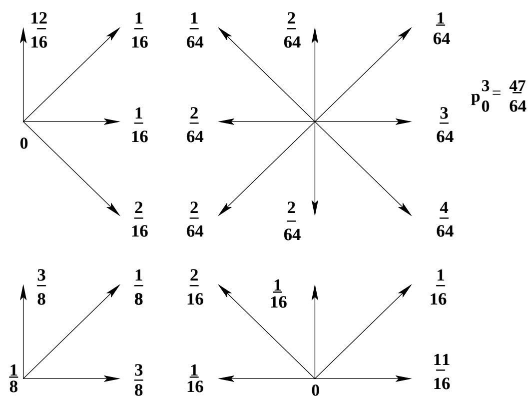

Figure 5.4: Model 1 transition probabilities

The level lines $\{\alpha \in \mathbf{R}^2 : H^{\mathbf{\Lambda}}(\alpha) = 0\}$, $\mathbf{\Lambda} = 0, 1, 2, 3$, are depicted in Figure 5.5.

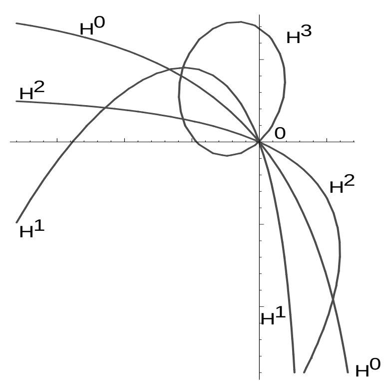

Figure 5.5: the H-functions level lines at level 0

Here we have

$$m_1^3 > 0, m_2^3 < 0 \text{ and } m_1^1 > 0, m_2^1 > 0.$$

Then $V^1 > 0$ and therefore by Lemma 5.3.1 (see **b**) the MC $\mathcal{M}$ (at $\alpha = 0$) is transient.

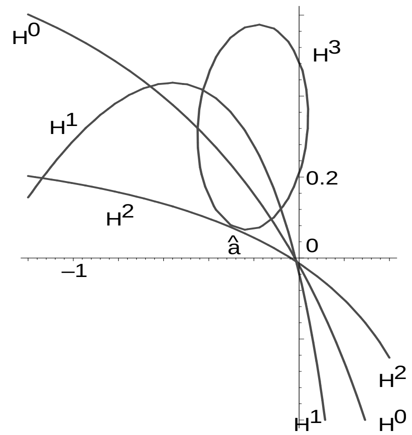

Figure 5.6: the H-functions lines at level $H^3(\hat{\alpha})$

Figure 5.6 shows the level lines $\{\alpha : H^{\mathbf{\Lambda}}(\alpha) = H^3(\hat{\alpha})\}, \mathbf{\Lambda} = 0, 1, 2, 3$. Here

$$H^3(\hat{\alpha}) = H^2(\hat{\alpha}) \text{ with } \hat{\alpha} = \alpha^2 \approx (-0.27, 0.1).$$

We have

$$m_1^3(\hat{\alpha}) < 0, m_2^3(\hat{\alpha}) < 0 \text{ and } V^1(\hat{\alpha}) > 0, V^2(\hat{\alpha}) = 0.$$

Then $\mathcal{M}(\hat{\alpha})$ is transient by Lemma 5.3.1.

Note that $H^0(\hat{\alpha}) \leq H^3(\hat{\alpha})$ and $H^1(\hat{\alpha}) \leq H^3(\hat{\alpha})$, i.e. condition (5.2.5) is satisfied. Hence, by Theorem 5.1 the LD bounds hold with $L^0 = H^3(\hat{\alpha}) \approx -0.009$.

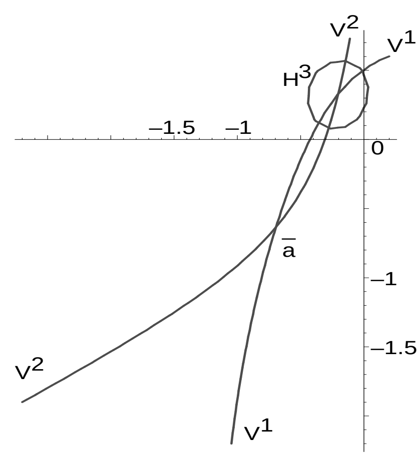

Figure 5.7: the lines $V^1 = 0, V^2 = 0, H^3 = 0$

In Figure 5.7 the lines $\{\alpha : V^1(\alpha) = 0\}$ and $\{\alpha : V^2(\alpha) = 0\}$ intersect each other in two points

$$\alpha^3 \approx (-0.21, 0.3) \text{ and } \bar{\alpha} \approx (-0.7, -0.7).$$

Starting from the point $\bar{\alpha}$ and forwards to south-west direction these lines form a cone-shaped region. We call this region the $ergodic$ region. The interior of this region contains the points α for which $\mathcal{M}(\alpha)$ is ergodic. Outside the ergodic region we have the points for which $\mathcal{M}(\alpha)$ is transient. The boundary (the lines $V^1(\alpha) = 0, V^2(\alpha) = 0$) correspond to null recurrent MC $\mathcal{M}(\alpha)$. In Figure 5.7 also the level line $\{\alpha : H^3(\alpha) = H^3(\hat{\alpha})\}$ is depicted, the intersection with $V^2(\alpha) = 0$ gives $\hat{\alpha}$.

Model 2. Next we give a model with $\hat{\alpha} = \alpha^2$ and MC $\mathcal{M}(\hat{\alpha})$ is null recurrent.

Let the transition probabilities have the following values as shown in Figure 5.8.

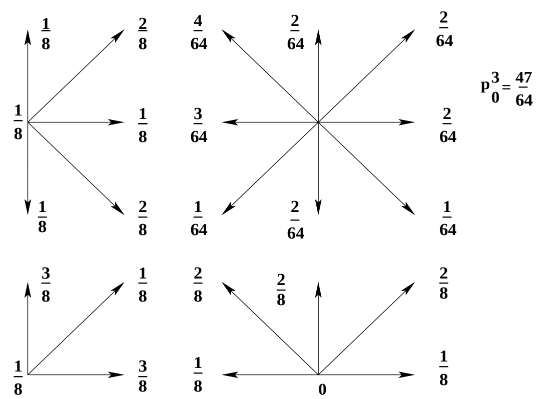

Figure 5.8: Model 2 transition probabilities

Figure 5.9 shows the level lines of the H-functions at level 0.

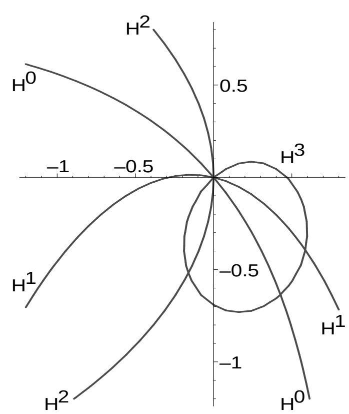

Figure 5.9: the H-functions lines at level 0

Here we have that

$$m_1^3 < 0, m_2^3 > 0 \text{ and } m_2^2 = 0, m_1^2 > 0.$$

Then $V^2 > 0$ and therefore by Lemma 5.3.1 the MC $\mathcal{M}$ (at $\alpha = 0$) is transient.

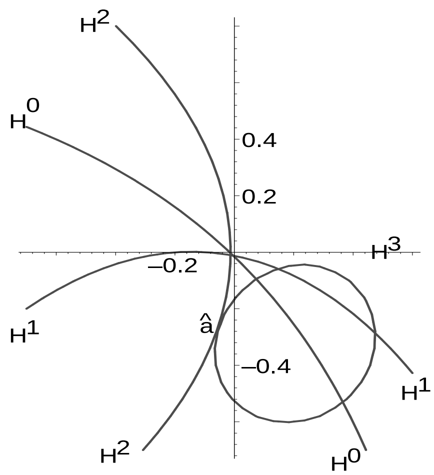

Figure 5.10: the H-functions lines at level $H^3(\hat{\alpha})$

In Figure 5.10 we have the H-functions level lines at the level equal to $H^3(\hat{\alpha})$ with $\hat{\alpha} = \alpha^2 \approx (-0.05, -0.27)$. Here

$$m_1^3(\hat{\alpha}) < 0, m_2^3(\hat{\alpha}) > 0 \text{ and } V^2(\hat{\alpha}) = 0.$$

By Lemma 5.3.1 the MC $\mathcal{M}(\hat{\alpha})$ is null recurrent.

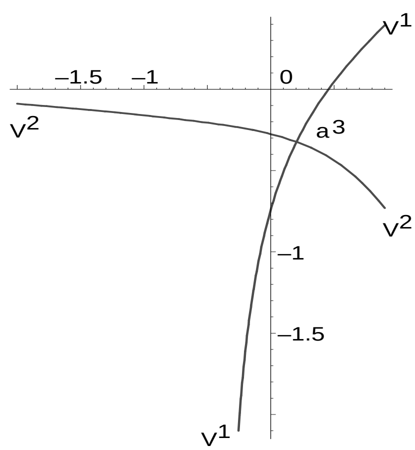

Figure 5.11: the level lines $V^1 = 0, V^2 = 0$

It is easily checked that

$$H^0(\hat{\alpha}) < H^3(\hat{\alpha}) \text{ and } H^1(\hat{\alpha}) \leq H^3(\hat{\alpha}),$$

i.e. condition (5.2.5) of Theorem 5.1 is satisfied. Hence, also for this model the LD bounds hold with $L^0 = H^3(\hat{\alpha}) \approx -0.0078$.

Figure 5.11 shows the level lines

$$\{\alpha \in \mathbf{R}^2 : V^1(\alpha) = 0\} \text{ and } \{\alpha \in \mathbf{R}^2 : V^2(\alpha) = 0\}$$

intersect at

$$\alpha^3 \approx (0.21, -0.32).$$

Starting from the point α^3 and forwards to south-west direction these lines form the $ergodic$ region.

5.5.2 Open problem

The numerical examples of section 5.5.1 satisfy the condition (5.2.5) of Theorem 5.1, and hence the local rate function L^0 is found. Now we present a model, where $\mathcal{M}$ is transient and the condition (5.2.5) is not satisfied. It is our conjecture that also in this case the LD lower bound is tight, i.e. that the local rate function is equal to $H^3(\hat{\alpha})$.

Model 3. Let the transition probabilities $p_k^{\mathbf{\Lambda}}$ have the values as shown in Figure 5.12.

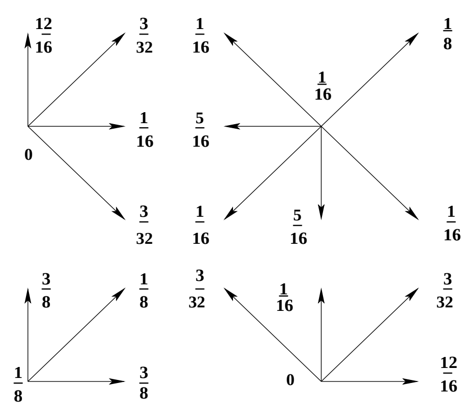

Figure 5.12: Model 3 transition probabilities

This model is interesting by the following property : for each $\alpha \neq 0$

$$\text{either } H^1(\alpha) \leq H^3(\alpha) < H^2(\alpha) \text{ or } H^2(\alpha) \leq H^3(\alpha) < H^1(\alpha). \tag{5.5.1}$$

Hence, the condition (5.2.5) is not satisfied.

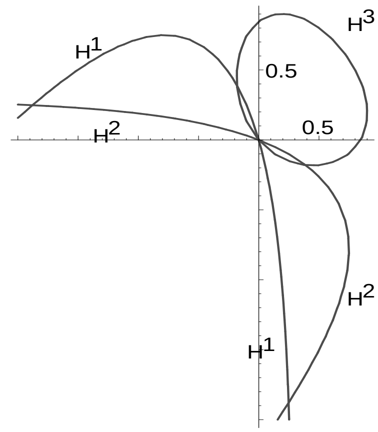

Figure 5.13: the lines $H^1 = H^2 = H^3 = 0$

The level lines $\{\alpha : H^1(\alpha) = 0\}$, $\{\alpha : H^2(\alpha) = 0\}$ and $\{\alpha : H^3(\alpha) = 0\}$ are depicted in Figure 5.13. In this model

$$\alpha^1 \approx (-0.1, 0.2) \text{ and } \alpha^2 \approx (0.1, -0.2).$$

Moreover, $H^3(\alpha^1) = H^3(\alpha^2)$ since the transition probabilities are symmetric. So we can take $\hat{\alpha} = \alpha^1$ or $\hat{\alpha} = \alpha^2$.

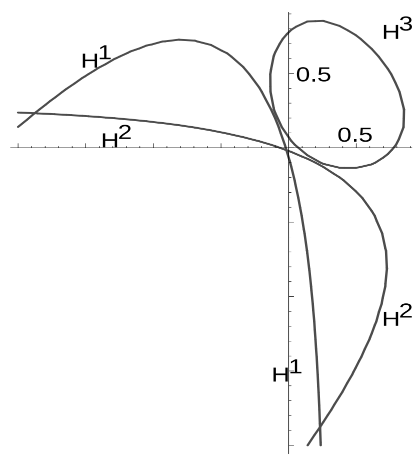

Figure 5.14: the H-functions lines at level $H^3(\hat{\alpha})$

Figure 5.14 shows the level lines of the functions $H^1(\alpha), H^2(\alpha), H^3(\alpha)$ at level

$$H^3(\hat{\alpha}) \approx -0.015.$$

In Figure 5.15 the level lines $\{\alpha : V^1(\alpha) = 0\}$ and $\{\alpha : V^2(\alpha) = 0\}$ intersect at two points

$$\bar{\alpha} \approx (-1.3, -1.3) \text{ and } \alpha^3 \approx (0.3, 0.3).$$

Starting from the point $\bar{\alpha}$ these lines give the ergodic region. Clearly, the point α^1, α^2 do not belong to the ergodic region. Hence, the twisted MC $\mathcal{M}(\hat{\alpha})$ is transient, and from relation (5.5.1) we have that the condition (5.2.5) is not satisfied.

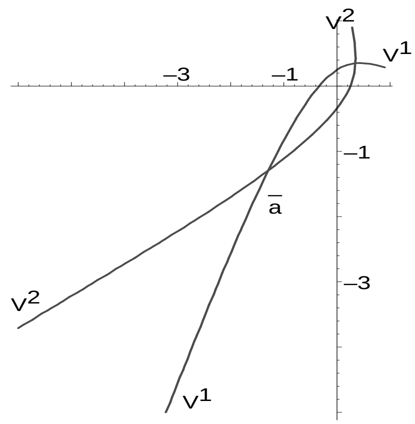

Figure 5.15: the level lines $V^1 = 0, V^2 = 0$

Chapter 6

Large deviations for a coupled processors system

This chapter has appeared as A. Hordijk, N. Popov (2003). *Large deviations analysis of a coupled processors system.* Probability in the Engineering and Informational Sciences 17(3): 397-409.

6.1 Introduction

In our previous Chapter 5 we were able to find an expression for the local rate for the path identically equal zero. However, we needed, besides some mild regularity conditions, an extra condition which seems to be unnecessary. In the recent paper [13] the local rate function is related to the convergence parameter of associated local transform matrices. Our expressions make an easy numerical calculation of the local rate possible and also determine this convergence radius for the models we studied. Moreover for specific queueing networks closed-form expressions can be derived, as we do in this chapter. Indeed, here we focus on the queueing network in $\mathbb{R}^2_+$, which is the mathematical model for two processors which handle two types of jobs. The arrival processes are supposed to be Markovian and the service times are assumed to be exponentially distributed. If there are jobs of both types then both processors proceed independently but if the buffer of one job-type is empty then the processor of the other job-type changes its service speed. This kind of coupled processors model allows for various specifications as: ”the full capacity given to the nonempty buffer” and ”switched off processors if one of the buffers is empty”. It turns out that for these special coupled processors models we can give closed-form expressions for the local rate function for the identically zero path in terms of the arrival and service rates. These expressions turn out to be surprisingly simple. As a by-product we show that the extra condition of Chapter 5 (see relation (6.2.3) of theorem 6.1), is indeed not necessary in these specific models. Also we characterize the condition for the general coupled processors model.

The outline of this chapter is as follows. In section 6.2 we give the mathematical model, i.e. the class of rw in $\mathbb{R}^2_+$ corresponding to the induced queueing networks. Also in section 6.2

we formulate the main result of Chapter 5 with the extra condition and moreover we recall some lemmas needed for our analysis. The section 6.3 we start with formulating the lemma of Chapter 5 which is the key to the analysis in this chapter. In subsection 6.3.1 we derive the characterization of the extra condition for the coupled processors model, and we derive the closed form expression for the local rate in case it is satisfied. In subsection 6.3.2 we treat the ”full capacity model with full capacity given to the nonempty buffer”. In the case that the queueing network is nonergodic there is one expression for the local rate for all parameter instances. In subsection 6.3.3 the ”system with switched off processor if one of the buffers is empty” is analyzed, it turns out that in this model we have two expressions for different regions in the space of service and arrival rates. Also we provide some numerical results for a model which has only twisted processes which are either null recurrent or transient. To conclude we add in subsection 6.3.4 as special case the one processor model and we show that its local rate is a special case of the expressions derived in subsections 6.3.2 and 6.3.3.

6.2 Model and LD bounds

6.2.1 Model description

Consider a system of two processors indexed by 1 and 2. By $\lambda_1, \lambda_2 > 0$ we denote their input rates respectively. If both queues are not empty then the processor $1, 2$ works with service rate $\mu_1, \mu_2 > 0$ respectively. If queue 1 becomes empty the service rate of processor 2 switches to $\mu_2^0 \geq 0$, and if queue 2 becomes empty the service rate of processor 1 switches to $\mu_1^0 \geq 0$.

We consider the time-discretized version obtained by uniformization, with transition probabilities $p_{i,j}, i, j \in \mathbf{Z}_+^2$, depicted in Figure 6.1, where we assume that relation (6.2.1) holds.

$$\lambda_1 + \lambda_2 + \mu_1 + \mu_2 \leq 1 \text{ and } \lambda_1 + \lambda_2 + \mu_i^0 \leq 1, i = 1, 2. \tag{6.2.1}$$

This model is a Markov chain, with state space $\mathbf{Z}_+^2$ and with four faces of homogeneity:

$$\mathbf{\Lambda}^1 = \{i \in \mathbf{Z}^2 : i_1 > 0, i_2 = 0\}, \mathbf{\Lambda}^2 = \{i \in \mathbf{Z}^2 : i_1 = 0, i_2 > 0\},$$
$$\mathbf{\Lambda}^3 = \{i \in \mathbf{Z}^2 : i_1 > 0, i_2 > 0\} \text{ and } \mathbf{\Lambda}^0 = \{0\}.$$

In Chapter 5 a general face-homogeneous random walk has been analysed under certain conditions (see conditions A-E of section 5.2). It is easily checked that these conditions hold for the model of this chapter.

6.2.2 LD bounds

Let us introduce the cumulant generating functions :

$$H^0(\alpha) = \log\left\{1 - \lambda_1 - \lambda_2 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2}\right\},$$
$$H^1(\alpha) = \log\left\{1 - \lambda_1 - \lambda_2 - \mu_1^0 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2} + \mu_1^0 e^{-\alpha_1}\right\},$$

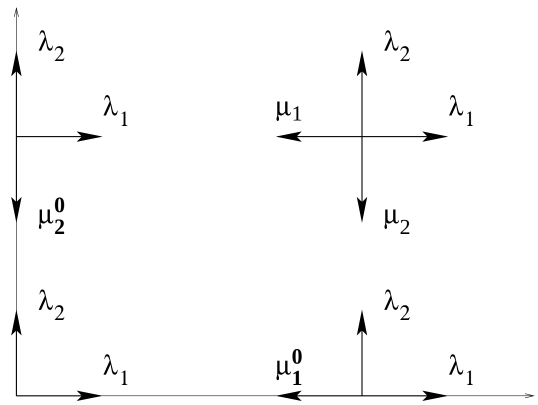

Figure 6.1: transition probabilities of a coupled processors system

$$H^2(\alpha) = \log\left\{1 - \lambda_1 - \lambda_2 - \mu_2^0 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2} + \mu_2^0 e^{-\alpha_2}\right\},$$
$$H^3(\alpha) = \log\left\{1 - \lambda_1 - \lambda_2 - \mu_1 - \mu_2 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2} + \mu_1 e^{-\alpha_1} + \mu_2 e^{-\alpha_2}\right\}.$$

We use the following points in $\mathbf{R}^2$:

$$\alpha^1 = \arg\min_\alpha \{H^1(\alpha) \vee H^3(\alpha)\},$$
$$\alpha^2 = \arg\min_\alpha \{H^2(\alpha) \vee H^3(\alpha)\}.$$

It has been proved there that α^1 and α^2 exist and are finite. Let

$$\hat{\alpha} = \arg\max\{H^3(\alpha^1), H^3(\alpha^2)\}. \tag{6.2.2}$$

In Chapter 5 the following LD theorem has been proved for face-homogeneous rw on $\mathbf{Z}_+^2$, with the euclidean norm $\|i\| = \sqrt{i_1^2 + i_2^2}$.

Theorem 6.1 *If for* $\mathbf{\Lambda} = 0, 1, 2$, *we have*

$$H^{\mathbf{\Lambda}}(\hat{\alpha}) \leq H^3(\hat{\alpha}). \tag{6.2.3}$$

then the following LD bounds are satisfied with local rate function $L^0 = H^3(\hat{\alpha})$ *:*

(LD upper bound) *for any* $\delta > 0$ *there exists* $N(\delta)$ *such that for all* $N > N(\delta)$

$$\mathsf{P}\Big\{\xi_0 = 0, \sup_{t=0,\cdots,[\tau N]} \|\xi_t\| < \delta N\Big\} \leq \exp\{+\delta N + N\tau L^0\},$$

(LD lower bound) *for any* $\delta > 0, \delta' > 0$ *there exists* $N(\delta, \delta')$ *such that for all* $N > N(\delta, \delta')$

$$\mathsf{P}\Big\{\xi_0 = 0, \sup_{t=0,\cdots,[\tau N]} \|\xi_t\| < \delta N\Big\} \geq \exp\{-\delta' N + N\tau L^0\}.$$

6.2.3 Twisted process

In this subsection we will recall necessary notations and results obtained in Chapter 5.

For any $\alpha \in \mathbf{R}^2$ we define a MC

$$\mathcal{M}(\alpha) = \{\xi_n^\alpha,\ n = 0, 1, 2, \cdots\}$$

on the state space $\mathbf{Z}_+^2$ with transition probabilities

$$p_{i,j}(\alpha) \triangleq \frac{p_{i,j} \exp\{(\alpha, j) - (\alpha, i)\}}{\sum_j p_{i,j} \exp\{(\alpha, j) - (\alpha, i)\}},$$

where $(\cdot, \cdot)$ is the scalar product in $\mathbf{R}^2$. The MC $\mathcal{M}(\alpha)$ is said to be a $twisted\ MC$.

For any α and any face $\mathbf{\Lambda}$ we have that

$$p_{i,j}(\alpha) = p_{j-i}^{\mathbf{\Lambda}}(\alpha) \text{ for any } i \in \mathbf{\Lambda} \text{ and } j \in \mathbf{Z}_+^2.$$

Hence, for any α the twisted process $\mathcal{M}(\alpha)$ is a face-homogeneous random walk on $\mathbf{Z}_+^2$ with the same homogeneity faces.

Now we define the vector $m^{\mathbf{\Lambda}}(\alpha) \in \mathbf{R}^2, \mathbf{\Lambda} = 0, 1, 2, 3,$ as the mean drift from state $i \in \mathbf{\Lambda}$ for the twisted process $\mathcal{M}(\alpha)$. We have

$$m^{\mathbf{\Lambda}}(\alpha) \triangleq \sum_{k \in \mathbf{Z}_+^2} k p_k^{\mathbf{\Lambda}}(\alpha).$$

One can easily verify that the functions $H^{\mathbf{\Lambda}}(\alpha), \mathbf{\Lambda} = 0, 1, 2, 3,$ are continuously differentiable. It has been proved in lemma 5.4.1 that

$$m^{\mathbf{\Lambda}}(\alpha) = \big(\frac{\partial}{\partial \alpha_1} H^{\mathbf{\Lambda}}(\alpha), \frac{\partial}{\partial \alpha_2} H^{\mathbf{\Lambda}}(\alpha)\big).$$

Let

$$V^1(\alpha) \triangleq m_1^3(\alpha) m_2^1(\alpha) - m_2^3(\alpha) m_1^1(\alpha),$$
$$V^2(\alpha) \triangleq m_2^3(\alpha) m_1^2(\alpha) - m_1^3(\alpha) m_2^2(\alpha).$$

The ergodicity, null recurrence and transience conditions for the twisted process are given in Lemma 5.3.1, where we omit α.

In the following, we will write for simplicity $m^{\mathbf{\Lambda}}, V^1, V^2$ instead of $m^{\mathbf{\Lambda}}(0)$, $V^1(0)$, $V^2(0)$.

6.3 The local rate function for the zero path

In the following subsections we derive explicit expressions for the local rate L^0 in terms of the input and service rates. The analysis is based on lemma 5.4.5, which has been proved in Chapter 5.

6.3.1 The case that the local rate function is equal to $\min_\alpha H^3(\alpha)$

In this subsection we give criteria in terms of $\mu_l^0, \mu_l, \lambda_l, l = 1, 2$ for $\hat{\alpha}$ to be equal to α^3, where

$$\alpha^3 = \arg\min_\alpha H^3(\alpha).$$

It has been proved in Chapter 5 that α^3 is finite. For $\lambda_l \neq \mu_l, l = 1, 2$, we define the constants:

$$a_i = \frac{\sqrt{\mu_i}}{\sqrt{\lambda_i} - \sqrt{\mu_i}} \big(\sqrt{\lambda_1 \mu_1} + \sqrt{\lambda_2 \mu_2} - \mu_1 - \mu_1 \big).$$

Lemma 6.3.1 *The condition*

$$H^{\mathbf{\Lambda}}(\alpha^3) \leq H^3(\alpha^3) \text{ for } \mathbf{\Lambda} = 0, 1, 2, \tag{6.3.1}$$

holds if and only if one of the following conditions is satisfied :

(a) $\mu_1 < \lambda_1, \mu_2 < \lambda_2$ *and* $\mu_1^0 < a_1, \mu_2^0 < a_2$*;*

(b) $\mu_1 = \lambda_1,\ \mu_2^0 \leq \mu_2 < \lambda_2$ *or* $\mu_2 = \lambda_2,\ \mu_1^0 \leq \mu_1 < \lambda_1;$

(c) $\mu_1 > \lambda_1, \mu_2 < \lambda_2$ *and* $\mu_2^0 \leq a_2$*;*

(d) $\mu_1 < \lambda_1, \mu_2 > \lambda_2$ *and* $\mu_1^0 \leq a_1$.

(e) $\mu_1 = \lambda_1, \mu_2 = \lambda_2$.

We have that $\mu_1 < a_1, \mu_2 < \lambda_2$ *in case* **(a)***;* $a_2 < \mu_2$ *in case* **(c)***;* $a_1 < \mu_1$ *in case* **(d)**.

Proof. Recall that $m^3(\alpha^3) = 0$. It is easily checked that

$$m_1^3(\alpha) = 0 \text{ iff } \lambda_1 e^{\alpha_1} - \mu_1 e^{-\alpha_1} = 0 \text{ iff } \alpha_1 = \log \sqrt{\mu_1/\lambda_1},$$

$$m_2^3(\alpha) = 0 \text{ iff } \lambda_2 e^{\alpha_2} - \mu_2 e^{-\alpha_2} = 0 \text{ iff } \alpha_2 = \log \sqrt{\mu_2/\lambda_2},$$

i.e.

$$\alpha^3 = \big(\log \sqrt{\mu_1/\lambda_1}, \log \sqrt{\mu_2/\lambda_2} \big).$$

Then

$$H^0(\alpha^3) = \log\{1 - \lambda_1 - \lambda_2 + \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2}\},$$
$$H^1(\alpha^3) = \log\{1 - \lambda_1 - \lambda_2 - \mu_1^0 + \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} + \mu_1^0\sqrt{\lambda_1/\mu_1}\},$$
$$H^2(\alpha^3) = \log\{1 - \lambda_1 - \lambda_2 - \mu_2^0 + \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} + \mu_2^0\sqrt{\lambda_2/\mu_2}\},$$

$$H^3(\alpha^3) = \log\{1 - \lambda_1 - \lambda_2 - \mu_1 - \mu_2 + 2\sqrt{\lambda_1\mu_1} + 2\sqrt{\lambda_2\mu_2}\}. \tag{6.3.2}$$

Now we obtain that

$$H^0(\alpha^3) \le H^3(\alpha^3) \text{ iff } 0 \le \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} - \mu_1 - \mu_2. \tag{6.3.3}$$

$$H^1(\alpha^3) \le H^3(\alpha^3) \text{ iff } \mu_1^0 \frac{\sqrt{\lambda_1} - \sqrt{\mu_1}}{\sqrt{\mu_1}} \le \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} - \mu_1 - \mu_2. \tag{6.3.4}$$

$$H^2(\alpha^3) \le H^3(\alpha^3) \text{ iff } \mu_2^0 \frac{\sqrt{\lambda_2} - \sqrt{\mu_2}}{\sqrt{\mu_2}} \le \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2} - \mu_1 - \mu_2. \tag{6.3.5}$$

Case a. Let $\mu_1 < \lambda_1, \mu_2 < \lambda_2$. Then (6.3.3) follows immediately. The conditions (6.3.4) and (6.3.5) hold iff

$$\mu_1^0 \le a_1 \text{ and } \mu_2^0 \le a_2.$$

Case b. Let $\mu_1 = \lambda_1$. Then (6.3.3) and (6.3.4) hold iff $\mu_2 \le \lambda_2$. If $\mu_2 < \lambda_2$ then (6.3.5) holds iff $\mu_2^0 \le \mu_2$. Therefore (6.3.1) follows from the condition $\mu_1 = \lambda_1, \mu_2^0 \le \mu_2 < \lambda_2$.

Similarly we prove the case $\mu_2 = \lambda_2, \mu_1^0 \le \mu_1 < \lambda_1$.

Case c. Let $\mu_1 > \lambda_1, \mu_2 < \lambda_2$.

Since $\mu_2 < \lambda_2$, we have that (6.3.3) holds iff $a_2 \ge 0$. Clearly, if (6.3.3) is true then (6.3.4) holds since $\mu_1 > \lambda_1$. Now one can easily see that (6.3.5) is satisfied iff $\mu_2^0 \le a_2$.

The **Case d** can be proved similarly to the **Case c**. The **Case e** is obvious.

One can easily see that

$$a_i = \mu_i + \sqrt{\mu_1\mu_2} \frac{\sqrt{\lambda_j} - \sqrt{\mu_j}}{\sqrt{\lambda_i} - \sqrt{\mu_i}}, \; i, j \in \{1, 2\}, i \ne j.$$

Now if $\mu_1 < \lambda_1$, $\mu_2 < \lambda_2$ then $\mu_1 < a_1$, $\mu_2 < a_2$. If $\mu_1 < \lambda_1$, $\mu_2 > \lambda_2$ then $a_1 < \mu_1$. If $\mu_2 < \lambda_2$, $\mu_1 > \lambda_1$ then $a_2 < \mu_2$. □

Theorem 6.2 *Let one of the conditions of lemma 6.3.1 be satisfied. Then the LD bounds hold with the local rate function*

$$L^0 = H^3(\alpha^3) = \log\{1 - (\sqrt{\lambda_1} - \sqrt{\mu_1})^2 - (\sqrt{\lambda_2} - \sqrt{\mu_2})^2\} \tag{6.3.6}$$

Proof. If one of the conditions **(a)-(e)** of lemma 6.3.1 is satisfied, then $H^{\mathbf{\Lambda}}(\alpha^3) \le H^3(\alpha^3)$ for $\mathbf{\Lambda} = 0, 1, 2$, by lemma 6.3.1. Hence, $\alpha^1 = \alpha^3$ and $\alpha^2 = \alpha^3$ by lemma 5.4.5. It means that $\hat{\alpha} = \alpha^3$ and so $L^0 = H^3(\alpha^3)$. Note that relation (6.3.2) is a slightly different expression of (6.3.6). □

6.3.2 Full capacity given to the nonempty buffer

In this subsection we consider the system with $\mu_1^0 = \mu_2^0 = \mu_1 + \mu_2$.

Lemma 6.3.2 *Let $\mu_1 \ne \lambda_1$ and $\mu_2 \ne \lambda_2$. Then the system is transient iff $\mu_1 + \mu_2 < \lambda_1 + \lambda_2$. It is null recurrent iff $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$ and it is ergodic iff $\mu_1 + \mu_2 > \lambda_1 + \lambda_2$.*

Proof. We have that

$$V^1 = (\lambda_1 - \mu_1)\lambda_2 - (\lambda_2 - \mu_2)(\lambda_1 - \mu_1 - \mu_2) = \mu_2(\lambda_1 + \lambda_2 - \mu_1 - \mu_2),$$
$$V^2 = (\lambda_2 - \mu_2)\lambda_1 - (\lambda_1 - \mu_1)(\lambda_2 - \mu_1 - \mu_2) = \mu_1(\lambda_1 + \lambda_2 - \mu_1 - \mu_2).$$

Now the criteria for the system to be transient, null recurrent or ergodic follow immediately from lemma 5.3.1, within paratheses we indicate the case of lemma 5.3.1, which is applicable.

(a) if $\mu_1 > \lambda_1, \mu_2 > \lambda_2$ (case a) then $V^1 < 0, V^2 < 0$ and so the system is ergodic;

(b,c) if $\lambda_1 \geq \mu_1, \lambda_2 < \mu_2$ (case b) or $\lambda_1 < \mu_1, \lambda_2 \geq \mu_2$ (case c) then the system is
ergodic if $\lambda_1 + \lambda_2 < \mu_1 + \mu_2$,
null recurrent if $\lambda_1 + \lambda_2 = \mu_1 + \mu_2$,
transient if $\lambda_1 + \lambda_2 > \mu_1 + \mu_2$;

(d) if $\lambda_1 \geq \mu_1, \lambda_2 \geq \mu_2$ and $\lambda_1 + \lambda_2 > \mu_1 + \mu_2$ (case d) then the system is transient.

□

In the next theorem we determine the local rate function L^0 for the nonergodic random walk. Since $L^0 = 0$ for the ergodic case, we have obtained a complete solution for this model.

Theorem 6.3 *Let the system be nonergodic. Then the LD bounds hold with the local rate function*

$$L^0 = \log\left\{1 - \left(\sqrt{\lambda_1 + \lambda_2} - \sqrt{\mu_1 + \mu_2}\right)^2\right\}. \tag{6.3.7}$$

Proof. Let us calculate $\hat{\alpha}$ explicitly. Recall that either $\hat{\alpha} = \alpha^1$ or $\hat{\alpha} = \alpha^2$ by definition (6.2.2).

Our calculations are based on lemma 5.4.5. First we will solve the system

$$H^1(\alpha) = H^3(\alpha),\ V^1(\alpha) = 0.$$

From the equation $H^1(\alpha) = H^3(\alpha)$ it follows that

$$(\mu_1 + \mu_2)e^{-\alpha_1} = \mu_1 e^{-\alpha_1} + \mu_2 e^{-\alpha_2}$$

and so $\alpha_1 = \alpha_2$. The equation $V^1(\alpha) = 0$ leads to

$$(\lambda_1 e^{\alpha_1} - \mu_1 e^{-\alpha_1})\lambda_2 e^{\alpha_2} - (\lambda_2 e^{\alpha_2} - \mu_2 e^{-\alpha_2})(\lambda_1 e^{\alpha_1} - (\mu_1 + \mu_2)e^{-\alpha_1}) = 0.$$

Now taking in account that $\alpha_1 = \alpha_2$ we get that

$$\mu_2\lambda_1 + \mu_2\lambda_2 - \mu_2(\mu_1 + \mu_2)e^{-2\alpha_1} = 0,$$

and therefore

$$\alpha_1 = \frac{1}{2}\log\frac{\mu_1 + \mu_2}{\lambda_1 + \lambda_2}.$$

Thus the system $H^1(\alpha) = H^3(\alpha), V^1(\alpha) = 0$ has the unique solution

$$\tilde{\alpha} = \Big(\frac{1}{2}\log\frac{\mu_1+\mu_2}{\lambda_1+\lambda_2}, \frac{1}{2}\log\frac{\mu_1+\mu_2}{\lambda_1+\lambda_2}\Big). \tag{6.3.8}$$

Let us show that this solution is exactly $\hat{\alpha}$, i.e. we will prove that $\tilde{\alpha} = \hat{\alpha}$. First note that from similar calculations it follows that (6.3.8) is the unique solution of $H^2(\alpha) = H^3(\alpha), V^2(\alpha) = 0$ as well. Now we need to check the signs of $m_1^3(\tilde{\alpha})$ and $m_2^3(\tilde{\alpha})$. We have that

$$\begin{aligned}\mathrm{sgn}\{m_1^3(\tilde{\alpha})\} &= \mathrm{sgn}\{\lambda_1 e^{\tilde{\alpha}_1} - \mu_1 e^{-\tilde{\alpha}_1}\} = \mathrm{sgn}\{\lambda_1\mu_2 - \lambda_2\mu_1\},\\ \mathrm{sgn}\{m_2^3(\tilde{\alpha})\} &= \mathrm{sgn}\{\lambda_2 e^{\tilde{\alpha}_2} - \mu_2 e^{-\tilde{\alpha}_2}\} = \mathrm{sgn}\{\lambda_2\mu_1 - \lambda_1\mu_2\}.\end{aligned}$$

Therefore one of the following cases holds :

(1) $\lambda_1\mu_2 > \lambda_2\mu_1$.

Then $m_2^3(\tilde{\alpha}) < 0$ and so $\tilde{\alpha}$ is the unique solution of (5.4.10), i.e. $\tilde{\alpha} = \alpha^1$. Since $m_1^3(\tilde{\alpha}) > 0$, the system (5.4.11) has no solution at all. It means that $\alpha^2 = \alpha^3$ by lemma 5.4.5. Note that $H^3(\alpha^3) \le H^3(\alpha^1)$ always. Hence, in this case $\hat{\alpha} = \alpha^1 = \tilde{\alpha}$.

(2) $\lambda_1\mu_2 < \lambda_2\mu_1$.

Then $m_1^3(\tilde{\alpha}) < 0$ and so $\tilde{\alpha}$ is the unique solution of (5.4.11), i.e. $\tilde{\alpha} = \alpha^2$. Since $m_2^3(\tilde{\alpha}) > 0$, the system (5.4.10) has no solution at all. It means that $\alpha^1 = \alpha^3$ by lemma 5.4.5. Note that $H^3(\alpha^3) \le H^3(\alpha^2)$ always. Hence, in this case $\hat{\alpha} = \alpha^2 = \tilde{\alpha}$.

(3) $\lambda_1\mu_2 = \lambda_2\mu_1$.

Then $m_1^3(\tilde{\alpha}) = 0 = m_2^3(\tilde{\alpha})$. It means that the systems (5.4.10) and (5.4.11) have no solution. Hence, by lemma 5.4.5 we have that $\alpha^1 = \alpha^3 = \alpha^2$ and so $\hat{\alpha} = \alpha^3 = \tilde{\alpha}$.

From all the three cases it follows that $\hat{\alpha}$ is equal to (6.3.8).

Finally we check that the condition $H^{\mathbf{\Lambda}}(\hat{\alpha}) \le H^3(\hat{\alpha}), \mathbf{\Lambda} = 0, 1, 2$, is satisfied. In our calculations above we proved that $H^1(\hat{\alpha}) \le H^3(\hat{\alpha})$ and $H^2(\hat{\alpha}) \le H^3(\hat{\alpha})$. Let us verify that $H^0(\hat{\alpha}) \le H^3(\hat{\alpha})$ holds as well. We have that

$$H^3(\hat{\alpha}) = \log\big\{1 - \lambda_1 - \lambda_2 - \mu_1 - \mu_2 + 2\sqrt{(\lambda_1+\lambda_2)(\mu_1+\mu_2)}\big\} \tag{6.3.9}$$

and

$$H^0(\hat{\alpha}) = \log\big\{1 - \lambda_1 - \lambda_2 + \sqrt{(\lambda_1+\lambda_2)(\mu_1+\mu_2)}\big\}.$$

Since we consider the nonergodic MC, $\lambda_1 + \lambda_2 \ge \mu_1 + \mu_2$ by lemma 6.3.2. Taking this inequality into account one can easily see that $H^0(\hat{\alpha}) \le H^3(\hat{\alpha})$, i.e. the condition of theorem 6.1 is satisfied. Hence, the LD bounds hold with the local rate function $L^0 = H^3(\hat{\alpha})$. To conclude, it is easily seen that (6.3.7) follows from (6.3.9). □

Remark 6.3.1 *Close reading of the proofs of lemma 6.3.1 and theorem 6.3 gives that only if $\lambda_1\mu_2 = \lambda_2\mu_1$ we have that the system of this section is one of the cases of lemma 6.3.1 (the case $\mu_1 = \lambda_1, \mu_2 = \lambda_2$ or the case $\mu_1 < \lambda_1, \mu_2 < \lambda_2$). One can easily check that (6.3.6) and (6.3.7) coincide if $\lambda_1\mu_2 = \lambda_2\mu_1$. Moreover, if $\lambda_1\mu_2 \ne \lambda_2\mu_1$ then from lemma 5.3.1 it follows that the twisted process $\mathcal{M}(\hat{\alpha})$ is null recurrent.*

6.3.3 Switched off processors whenever a queue is empty

In this subsection we consider the system with $\mu_1^0 = \mu_2^0 = 0$. In the next lemma we show that the MC of this model is always nonergodic.

Lemma 6.3.3 *The system is always nonergodic. If $\mu_1 \neq \lambda_1$ or $\mu_2 \neq \lambda_2$ then the system is null recurrent iff*

$$\lambda_1\mu_2 = \lambda_2\mu_1 \text{ and } \mu_1 > \lambda_1, \mu_2 > \lambda_2,$$

and it is transient otherwise.

Proof. We have that

$$V^1 = (\lambda_1 - \mu_1)\lambda_2 - (\lambda_2 - \mu_2)\lambda_1 = \mu_2\lambda_1 - \mu_1\lambda_2 = -V^2.$$

From lemma 5.3.1 it follows that one of the following cases holds :

a) if $\mu_1 > \lambda_1, \mu_2 > \lambda_2$ then the system is null recurrent iff $\lambda_1\mu_2 = \lambda_2\mu_1$, and it is transient otherwise;

b) if $\mu_1 \leq \lambda_1, \mu_2 > \lambda_2$ then $0 < V^1$ and so the system is transient;

c) if $\mu_1 > \lambda_1, \mu_2 \leq \lambda_2$ then $0 < V^2$ and so the system is transient;

d) if $\mu_1 \leq \lambda_1, \mu_2 \leq \lambda_2$ and $\lambda_1 + \lambda_2 > \mu_1 + \mu_2$ then the system is transient, if $\mu_1 = \lambda_1, \mu_2 = \lambda_2$ then from the analysis in [7] it follows that the system is nonergodic. □

Theorem 6.4 *The LD bounds hold with the local rate function*

$$L^0 = \log\{1 - (\sqrt{\lambda_1} - \sqrt{\mu_1})^2 - (\sqrt{\lambda_2} - \sqrt{\mu_2})^2\} \tag{6.3.10}$$

if

$$\mu_1 + \mu_2 \leq \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2}. \tag{6.3.11}$$

If (6.3.11) does not hold then the local rate function is

$$L^0 = \log\big\{1 - \frac{(\sqrt{\lambda_1\mu_2} - \sqrt{\lambda_2\mu_1})^2}{\mu_1 + \mu_2}\big\}. \tag{6.3.12}$$

Proof. If $\mu_1^0 = \mu_2^0 = 0$ then

$$H^0(\alpha) = H^1(\alpha) = H^2(\alpha) = \log\big(1 - \lambda_1 - \lambda_2 + \lambda_1 e^{\alpha_1} + \lambda_2 e^{\alpha_2}\big). \tag{6.3.13}$$

Note that the functions $H^0(\alpha), H^3(\alpha)$ coincide with those of lemma 6.3. Then from (6.3.3) it follows that

$$H^0(\alpha^3) \leq H^3(\alpha^3) \text{ if and only if } \mu_1 + \mu_2 \leq \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2}.$$

Hence, if (6.3.11) holds, then the condition of theorem 6.1 is satisfied and the LD bounds hold with the local rate function L^0 as in (6.3.6).

Suppose that (6.3.11) does not hold. Then $H^0(\alpha^3) > H^3(\alpha^3)$ and so $\hat{\alpha} \neq \alpha^3$. Therefore by lemma 5.4.5 the point α^1 is the solution of the system:

$$H^1(\alpha) = H^3(\alpha),\ V^1(\alpha) = 0,\ m_2^3(\alpha) < 0,$$

and α^2 is the solution of the system:

$$H^2(\alpha) = H^3(\alpha),\ V^2(\alpha) = 0,\ m_1^3(\alpha) < 0.$$

The equations $H^1(\alpha) = H^3(\alpha)$ and $V^1(\alpha) = 0$ give the following system

$$\begin{cases} \mu_1 e^{-\alpha_1} + \mu_2 e^{-\alpha_2} = \mu_1 + \mu_2 \\ \mu_1 \lambda_2 e^{\alpha_2 - \alpha_1} = \mu_2 \lambda_1 e^{\alpha_1 - \alpha_2}. \end{cases}$$

The solution of this system is

$$\alpha_1 = \log\Big\{\frac{\mu_1 + \mu_2\sqrt{\frac{\mu_1\lambda_2}{\mu_2\lambda_1}}}{\mu_1 + \mu_2}\Big\} \text{ and } \alpha_2 = \log\Big\{\frac{\mu_2 + \mu_1\sqrt{\frac{\mu_2\lambda_1}{\mu_1\lambda_2}}}{\mu_1 + \mu_2}\Big\}. \tag{6.3.14}$$

Since $H^1(\alpha) = H^2(\alpha)$, then $V^1(\alpha) = -V^2(\alpha)$. Hence, (6.3.14) is the solution of $H^2(\alpha) = H^3(\alpha), V^2(\alpha) = 0$ as well, and so $\hat{\alpha}$ is equal to (6.3.14). Substituting (6.3.14) in (6.3.13) we find with the same calculation that

$$H^3(\hat{\alpha}) = \log\Big\{1 - \lambda_1 - \lambda_2 + \frac{(\sqrt{\mu_1\lambda_1} + \sqrt{\mu_2\lambda_2})^2}{\mu_1 + \mu_2}\Big\} = \log\Big\{1 - \frac{(\sqrt{\lambda_1\mu_2} - \sqrt{\lambda_2\mu_1})^2}{\mu_1 + \mu_2}\Big\}.$$

Since $H^0(\hat{\alpha}) = H^1(\hat{\alpha}) = H^2(\hat{\alpha}) = H^3(\hat{\alpha})$, the condition (6.2.3) of theorem 6.1 is satisfied and the local rate function L^0 is as in (6.3.12).

Note that (6.3.10) and (6.3.12) coincide if $\mu_1 + \mu_2 = \sqrt{\lambda_1\mu_1} + \sqrt{\lambda_2\mu_2}$. □

Remark 6.3.2 *Since $V^1(\hat{\alpha}) = 0 = V^2(\hat{\alpha})$, it follows from lemma 5.3.1 that the twisted MC $\mathcal{M}(\hat{\alpha})$ is null recurrent in all cases of this subsection except the case (6.3.11), where the MC $\mathcal{M}(\hat{\alpha})$ can be transient as well. Note that if (6.3.11) holds, then $\hat{\alpha} = \alpha^3$ and this is one of the cases of lemma 6.3.1.*

Now we give some numerical computations for our model. Let

$$\lambda_1 = 0.2, \lambda_2 = 0.5, \mu_1 = 0.5, \mu_2 = 0.2.$$

In this case the condition (6.3.11) is not satisfied. Hence, the local rate function L^0 is equal to (6.3.12). By simple calculations we find from (6.3.12) and (6.3.14) that

$$L^0 = \log\frac{61}{70} \approx -0.1376 \text{ and } \hat{\alpha} = (\log\frac{10}{7}, \log\frac{4}{7}) \approx (0.356, -0.56).$$

Figure 6.2 shows the lines $\{\alpha \in \mathbf{R} \,|\, H^3(\alpha) = 0\}$ (the circle) and $\{\alpha \in \mathbf{R} \,|\, H^0(\alpha) = 0\}$.

Figure 6.3 shows the level lines $\{\alpha \in \mathbf{R} \,|\, V^1(\alpha) = 0\}$ (the straight line) and the level lines of H^0 and H^3 at the level equal to L^0. Starting from the point $\alpha^3 \approx (0.458, -0.458)$ and forward to south-west direction the line $V^1(\alpha) = 0$ contains the points α for which the MC $\mathcal{M}(\alpha)$ is null recurrent. All other points α correspond to the transient MC $\mathcal{M}(\alpha)$. Note that in Figure 6.3 all the lines intersect at $\hat{\alpha}$, and the MC $\mathcal{M}(\hat{\alpha})$ is null recurrent.

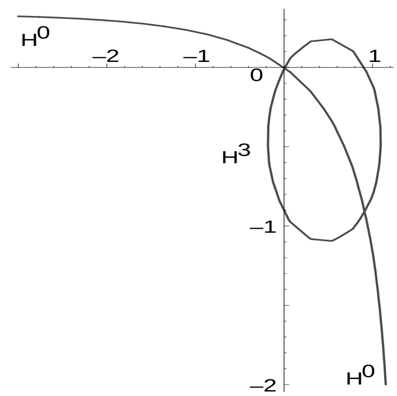

Figure 6.2: the lines $H^0(\alpha) = 0,\ H^3(\alpha) = 0$

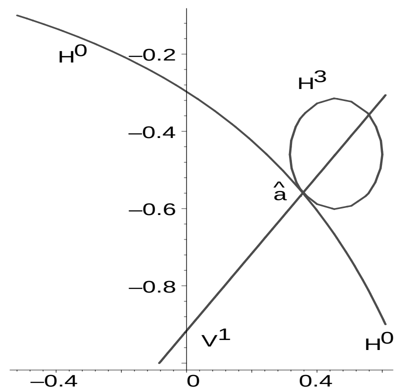

Figure 6.3: the lines $V^1 = 0, H^0 = L^0, H^3 = L^0$

6.3.4 One processor system

For completeness we also derive the local rate function for the model with one processor and one buffer. Clearly (6.3.15) is the special case of (6.3.6) with $\lambda_2 = \mu_2 = 0$. Also it can be argued that by considering the total number of customers at both processors that (6.3.7) is equal to (6.3.15) with $\lambda = \lambda_1 + \lambda_2$ and $\mu = \mu_1 + \mu_2$.

Let $\lambda > 0, \mu > 0$ and $\lambda + \mu \leq 1$. The system is nonergodic iff $\mu \leq \lambda$. We suppose this below. Here we have the following cumulant generating functions :

$$H^0(\alpha) = \log(1 - \lambda + \lambda e^{\alpha}) \text{ and } H^1(\alpha) = \log(1 - \lambda - \mu + \lambda e^{\alpha} + \mu e^{-\alpha}).$$

Then $H^0(\alpha) = H^1(\alpha)$ iff $\alpha = 0$. It means that $\hat{\alpha} = \arg\min_\alpha H^1(\alpha)$. We have that $m^1(\alpha) = 0$ iff $\lambda e^{\alpha} - \mu e^{-\alpha} = 0$, i.e. $\hat{\alpha} = \log\{\sqrt{\mu/\lambda}\}$, and hence

$$L^0 = H^1(\hat{\alpha}) = \log\{1 - \lambda - \mu + 2\sqrt{\mu\lambda}\} = \log\{1 - (\sqrt{\lambda} - \sqrt{\mu})^2\}. \qquad (6.3.15)$$

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Samenvatting

In dit proefschrift bestuderen we een speciale klasse van irreducibele stochastische wandelingen op het rooster in $\mathbf{Z}^p$. Deze klasse bestaat uit de zogenaamde fase-homogene stochastische wandelingen. Dit is een natuurlijke uitbreiding van de klasse van homogene stochastische wandelingen.

Het is bekend dat homogeniteit betekent dat de sprongen van een stochastische wandeling onafhankelijke, identiek verdeelde stochastische variabelen zijn. Homogene stochastische wandelingen hebben een relatief simpele structuur. Het is echter niet altijd mogelijk om een gegeven model te formuleren als een irreducibele, homogene stochastische wandeling op een gewenste toestandsruimte. De reden is dat grenzen of beperkingen de homogeniteit verstoren. Veel interessante modellen hebben wel de eigenschap dat de toestandsruimte in een relatief klein en eindig aantal deelgebieden kan worden opgesplitst, waarop het te bestuderen stochastische proces zich als een homogene stochastische wandeling gedraagt. Deze deelgebieden worden maximaal gekozen en ze worden dan *fases* genoemd. De corresponderende stochastische wandeling noemen we *fase-homogeen*.

In dit proefschrift concentreren we ons op de volgende aspecten van fase-homogene stochastische wandelingen: de vloeistof limiet, grote afwijkingen en de bijna gesloten verzamelingen (van de toestandsruimte). Deze drie onderwerpen vormen afzonderlijke deelgebieden van de kansrekening. Eén van onze doelen is, om het verband tussen deze drie aspecten te bestuderen in de context van fase-homogene stochastische wandelingen. In het bijzonder zullen we laten zien hoe de structuur van de collectie bijna gesloten verzamelingen kan worden gebruikt voor de analyse van de vloeistof limiet en de grote afwijkingen.

Als een stochastische wandeling homogeen is, dan hebben de vloeistof limiet, de grote-afwijkingen kansen en de bijna gesloten verzamelingen simpele structuren die uitvoerig in de literatuur beschreven zijn. In Hoofdstuk 1 worden de relevante resultaten hierover ǵerecapituleerd. Deze structuren worden complexer naarmate de stochastische wandeling meer fasen bevat.

De bijna gesloten verzamelingen bestuderen we met behulp van het begrip verblijfsverzameling. We leiden voldoende condities af, opdat een verzameling toestanden een verblijfsverzameling is of dat deze bijna gesloten is. In Hoofdstuk 3 concentreren we ons op modellen die slechts atomische bijna gesloten verzamelingen bevatten. Voor deze modellen analyseren we de vloeistof limiet. In het algemeen is de vloeistof limiet stochastisch. Het is een trekking uit een kansverdeling op de ruimte van alle mogelijk banen die zowel een discreet als een continu gedeelte kan bevatten. We vermoeden dat realisaties van het discrete gedeelte van deze kansverdeling ieder corresponderen met precies één atoom. We tonen aan dat dit vermoeden juist is voor een aantal modellen van fase-homogene stochastische wandelingen. Tevens geven we een expliciete uitdrukking voor de vloeistoflimiet van deze modellen. Daarnaast beschrijven we in Hoofdstuk 3 een methode om de vloeistof limiet voor sommige speciale modellen in hogere dimensies te berekenen.

De twee speciale modellen die we in dit proefschrift bestuderen, zijn fase-homogene stochastische wandelingen in twee dimensies. Het eerste model wordt behandeld in Hoofdstuk 4 en betreft een fase-homogene stochastische wandeling op het twee-dimensionale rooster. De assen verdelen het rooster in 9 homogeniteitsgebieden. Voor dit model bestuderen we de vloeistof limiet en de verblijfsverzamelingen. De vloeistofbanen worden expliciet bepaald en berekend. We bewijzen dat de vloeistof limiet bestaat voor ieder lineair geschaald startpunt ongelijk aan de oorsprong en dat het een stuksgewijs lineaire functie van de tijdsparameter is. De condities voor transiëntie en ergodiciteit zijn in dit geval eenvoudige functies van de gemiddelde stapgrootten. In het geval van transiëntie, bewijzen we dat het gebied tussen twee vloeistofbanen een verblijfsverzameling is. Dat is het startpunt voor een complete classificatie van alle bijna gesloten verzamelingen van dit model. We bewijzen ook dat het limietproces bijna zeker geconcentreerd is op een gesloten curve die we de *isochroon* noemen. Er wordt een expliciete uitdrukking voor deze curve afgeleid. Een belangrijke vraag is of het geschaalde proces een limietverdeling op de isochrone heeft voor een gegeven (vast) startpunt. Als het startpunt ver genoeg van de oorsprong verwijderd is dan convergeert het proces niet.

Een tweede speciaal model behandelen we in Hoofdstuk 5. Nu is de toestandsruimte het eerste kwadrant in $\mathbf{Z}^2$ en zijn er derhalve 4 homogeniteitsgebieden. De gemiddelde stapgroottes bepalen de structuur van de bijna gesloten verzamelingen. Voor dit model zijn we geïnteresseerd in de grote-afwijkingen kansen, d.w.z. in het asymptotisch gedrag van de logaritme van de waarschijnlijkheid dat het geschaalde proces dichtbij een gegeven pad blijft. Met name wanneer dit gegeven pad het *nul pad* is, was dit nog een open probleem voor de transiënte wandeling.

Het is gelukt dit probleem onder een extra conditie op te lossen, waarvan we vermoeden dat deze niet nodig zal blijken te zijn. Met deze conditie volgt een bovengrens voor de gevraagde waarschijnlijkheid gemakkelijk. Het bewijs van de ondergrens is onafhankelijk van de extra conditie. Om dit laatste te bewijzen hebben we een nieuwe methode ontwikkeld die gebaseerd is op de eigenschappen van de bijna gesloten verzamelingen van dit model. We vermoeden dat deze goed toepasbaar is op andere modellen.

De kracht van onze methode is dat we het asymptotisch gedrag van de logaritme van de gevraagde kans expliciet hebben kunnen formuleren als functie van de overgangswaarschijnlijkheden. Deze functie is voor een aantal gegeven modellen die aan de extra voorwaarde voldoen, eenvoudig te berekenen. Hiervan geven we een aantal voorbeelden.

In Hoofdstuk 6 passen we de resultaten van Hoofdstuk 5 toe op een systeem bestaande uit twee processoren. Klanten van twee typen komen aan in het systeem, ieder type heeft zijn eigen buffer en processor. We veronderstellen dat het aankomstproces van elke type Markov is en dat de bedieningstijden exponentieel zijn verdeeld. Beide processoren werken onafhankelijk van elkaar zolang er klanten van beide typen aanwezig zijn. Echter, zodra één van de buffers leeg raakt, past de processor die de klanten uit de andere buffer bedient, zijn bedieningssnelheid aan. Met name worden de volgende bedieningsdisciplines bestudeerd: ”alle capaciteit wordt gegeven aan de klanten in de niet-lege buffer” en ”beide processoren worden uitgeschakeld als er één buffer leeg is”.

De condities voor transiëntie en ergodiciteit zijn eenvoudig te bepalen uit de aankomst- en bedieningsparameters. Dankzij de speciale structuur van dit systeem, geldt de extra conditie uit Hoofdstuk 5. Hierdoor kunnen we het asymptotisch gedrag van de grote-afwijkingen kans op het nulpad expliciet berekenen voor alle aankomst- en bedieningsparameters.

Curriculum Vitae

Nicolai Popov werd op 6 januari 1973 te Rivne in de Ukraine geboren. Na het behalen van het einddiploma aan de middelbare school te Rivne in 1990, begon hij in hetzelfde jaar zijn studie aan de faculteit voor Wiskunde en Mechanica van de Universiteit van Moskou. In 1996 studeerde hij daar af met de scriptie getiteld ”Een model van een stochastische wandeling in een stochastische omgeving” die onder begeleiding van prof. dr. R.A. Minlos geschreven werd. Van 1 juni 1997 tot 30 september 2001 was hij assistent in opleiding aan de Universiteit Leiden. Zijn onderzoek, waarvan de resultaten in dit proefschrift te vinden zijn, werd begeleid door prof.dr. A. Hordijk en dr. F.M. Spieksma. Van 1998 tot 2000 heeft hij de cursussen van het Landelijk Netwerk voor Mathematische Besliskunde gevolgd.