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Bounds on Cohomology by Stratifications

Master's thesis, 24 April 2012

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Contents

Preface

This thesis is the result of my master's research that took place during the past year.

The reader is assumed to have a knowledge of basic algebra and topology and also to be familiar with sheaves and sheaf cohomology. The necessary prerequisites for sheaves and sheaf cohomology are all covered in the paragraphs II.1, III.1 and III.2 of [4].

I like to thank my advisor Dr. Robin de Jong for all his suggestions, help and encouragement during the process. The way he directed me during the research has contributed to it that this has been the part of my mathematical education that I most enjoyed. He is also the person who suggested the topic for this thesis. I also like to thank Prof. Bas Edixhoven and Dr. Lenny Taelman for their corrections and suggestions.

1 Introduction

In 2004, Mike Roth and Ravi Vakil published the article ˝The Affine Stratification Number and the Moduli Space of Curves ˝[11] in which they developed a notion of affine stratifications for separated schemes of finite type over a field. Such an affine stratification for a separated scheme of finite type X is a finite decomposition $X = \cup_{i=0}^{m} Z_i$ into disjoint locally closed affine subschemes Z_i such that for $k \leq m$:

1. $\overline{Z_k} = \bigcup_{i \geq k} Z_i$,
2. Z_k is a dense open affine subset of $\overline{Z_k}$, and
3. $\overline{Z_k}$ is of pure codimension one in $\overline{Z}_{k-1}$.

Roth and Vakil showed that any affine covering for X of cardinality m can be turned into an affine stratification for X of cardinality at most m. The affine stratification number $asn(X)$, defined to be the minimum of the length over all possible affine stratifications of X, is therefore a well-defined invariant for separated schemes of finite type. They proved that it has, among others, the following properties:

- $asn(X) = 0$ if and only if X is affine.
- $cd(X) \leq asn(X)$, where $cd(X)$ is the cohomological dimension of X, i.e. the largest integer n such that $H^n(X, \mathcal{F}) \neq 0$ for some quasicoherent sheaf $\mathcal{F}$.
- $asn(X) \leq dim(X)$.

At some point, Roth and Vakil remarked that the last two properties combined give another proof of Grothendieck´s dimensional vanishing theorem for separated schemes of finite type and quasicoherent sheaves:

Theorem 1.1 (Theorem 3.6.5 in [3] and Theorem III.2.7 in [4]). *Let X be a Noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.*

As we see, Grothendiecks dimensional vanishing theorem applies to a much more general setting than just separated schemes of finite type and quasicoherent sheaves. A similar statement exist for topological manifolds.

Theorem 1.2 (Proposition 3.2.2.IV in [8]). *Let X be an n-dimensional C^0 manifold and let $\mathcal{F}$ be a sheaf of abelian groups on X. Then: $H^j(X, \mathcal{F}) = 0$ for $j > n$.*

In this thesis we will investigate if and how the theory of stratifications as defined by Roth and Vakil can be used to prove vanishing theorems like 1.1 and 1.2 for spaces other than separated schemes of finite type and sheaves other than quasicoherent sheaves. For fixed classes of spaces and sheaves, we will define a notion of simple spaces in such a way that the simple spaces are related to the fixed class of sheaves in the same way as affine schemes are related to quasicoherent sheaves (see [4] Theorem III.3.7). This will lead to more general notions of stratifications, the stratification number and the cohomological dimension.

After discussing the necessary prerequisites in Chapter 2, the actual theory of stratifications will be given in Chapter 3. We will show that stratifications can be used to prove Theorem 1.1 for Alexandrov spaces (Chapter 4) and a weaker version of Theorem 1.1 for Noetherian spaces (Chapter 5). An application for these vanishing theorems is given in Chapter 6 where we use a correspondence between Alexandrov spaces and simplicial complexes to prove a weaker version of Theorem 1.2 for simplicial compexes.

2 Prerequisites

When proving the relation $cd(X) \leq asn(X)$, Roth and Vakil used arguments involving local cohomology and spectral sequences. In order to see how their proof can be generalised, we will discuss the concept and necessary properties of both these topics in this section.

Because different definitions of left-exactness are used in our literature sources, we include the following lemma.

Lemma 2.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant additive functor between abelian categories $\mathcal{C}$ and $\mathcal{D}$. Then the following are equivalent:*

1. *For every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact in $\mathcal{D}$.*
2. *For every exact sequence of the form $0 \to A \to B \to C$ in $\mathcal{C}$, the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact in $\mathcal{D}$.*
3. *For every morphism $\phi : A \to B$ in $\mathcal{C}$ we have $\ker(F(\phi) : F(A) \to F(B)) = F(\ker(\phi))$.*

Proof. (2)$\Rightarrow$ (1). Obvious.
(1) $\Rightarrow$(2). Let $0 \to A \to B \to C$ be exact in $\mathcal{C}$. We denote the map $A \to B$ by f. Then $0 \to A \to B \to coker(f) \to 0$ is exact in $\mathcal{C}$ and hence $0 \to F(A) \to F(B) \to F(coker(f))$ is exact in $\mathcal{D}$. By the universal property of the cokernel, we have an injective map $i : coker(f) \to C$. Therefore the sequence $0 \to coker(f) \to C \to coker(i) \to 0$ is exact in $\mathcal{C}$ and so by hypothesis $0 \to F(coker(f)) \to F(C) \to F(coker(i))$ is exact in $\mathcal{D}$. Because $F(coker(f)) \to F(C)$ is injective, the composition map $F(B) \to F(coker(f)) \to F(C)$ has the same kernel as $F(B) \to F(coker(f))$. Hence the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact.
(2)$\Rightarrow$(3). Let $\phi : A \to B$ be any map in $\mathcal{C}$. Then $0 \to ker(\phi) \to A \to B$ is exact. So by hypothesis $0 \to F(ker(\phi)) \to F(A) \to F(B)$ is exact in $\mathcal{D}$. Hence $\ker(F(\phi) : F(A) \to F(B)) = F(\ker(\phi))$.
(3) $\Rightarrow$ (2). Let $0 \to A \to B \to C$ be exact in $\mathcal{C}$. Denote the map $A \to B$ by ϕ and $B \to C$ by σ. Because $ker(F(\phi) : F(A) \to F(B)) = F(ker(\phi)) = F(0) = 0$ we see that $0 \to F(A) \to F(B)$ is exact. Again, because $F(A) = ker(F(\sigma) : F(B) \to F(C)) = F(ker(\sigma)) = F(im(\phi))$ we see that $F(A) \to F(B) \to F(C)$ is exact. Hence $0 \to F(A) \to F(B) \to F(C)$ is exact. □

Definition 2.2. *A covariant additive functor between abelian categories that satisfies any of the conditions of Lemma 2.1 is said to be* left exact.

2.1 Local cohomology

In this section we will discuss some basic properties of local cohomology groups and local cohomology sheaves. All the results are taken from paragraph 1 of [5].

Let X be any topological space and let $\mathcal{F}$ be any abelian sheaf on X.

Definition 2.3. *For an open subset U of X and a section $s \in \mathcal{F}(U)$ we define the* support *of s to be the subset $\{p \in U : s_p \neq 0\}$ of U. Here s_p is the germ of s in the stalk $\mathcal{F}_p$.*

Note that the support is a closed subset of U, because each point in the complement $\{p \in U : s_p = 0\}$ has an open neighborhood on which s vanishes.

Definition 2.4. *Let X be a topological space with closed subset Z and let $\mathcal{F}$ be an abelian sheaf on X. Then we define the group $\Gamma_Z(X, \mathcal{F})$ to be the subgroup of $\mathcal{F}(X)$ consisting of all sections with support in Z.*

Proposition 2.5. *Let X be a topological space with closed subset Z. Then the map $F : Ab(X) \to Ab$ given by $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$ is a left exact functor.*

Proof. Because F sends sheaves to subgroups of the global sections groups, for F to be a functor it suffices to show that a sheaf morphism maps sections with support in Z to sections with support in Z. But this is the case because zero maps to zero by all the induced stalk maps. For the left exactness we consider any sheaf morphism $\phi : A \to B$. Because the presheaf kernel is already a sheaf, we immediately have $ker(\phi : \Gamma_Z(X, A) \to \Gamma_Z(X, B) = \Gamma_Z(X, ker(\phi))$. This proves the left exactness. □

If $V \subset U \subset X$ are open subsets and $p \in V$, then for a section $s \in \mathcal{F}(U)$ it holds that $s_p = (s|_V)_p \in \mathcal{F}_p$. Therefore a section $s \in \mathcal{F}(U)$ with support in $Z \cap U$ will restrict to a section $s_V \in \mathcal{F}(V)$ with support in $Z \cap V$. Hence the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ induce homomorphisms $\Gamma_{Z\cap U}(U, \mathcal{F}|_U) \to \Gamma_{Z\cap V}(V, \mathcal{F}|_V)$. So $U \mapsto \Gamma_{Z\cap U}(U, \mathcal{F}|_U)$ is a presheaf on X. In fact it is a sheaf, for if we have an open cover $\{U_i\}$ of an open subset U and sections $s_i \in \mathcal{F}(U_i)$ all with support in Z and such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then the unique gluing section $s \in \mathcal{F}(U)$ also has support in Z. We will denote this sheaf by $\Gamma_Z(\mathcal{F})$. Note that $\Gamma_Z(\mathcal{F})$ is a subsheaf of $\mathcal{F}$.

Proposition 2.6. *Let X be a topological space with closed subset Z. Then the functor $G : Ab(X) \to Ab(X)$ given by $\mathcal{F} \mapsto \Gamma_Z(\mathcal{F})$ is left exact.*

Proof. Let $\phi : A \to B$ be any morphism of sheaves in $Ab(X)$. Its kernel is the subsheaf of A-sections that map to zero, so $\Gamma_Z(ker(\phi))$ is the sheaf of A-sections that map to zero and with support in Z. The induced map $\phi : \Gamma_Z(A) \to \Gamma_Z(B)$ is just a restriction of $\phi : A \to B$. Its kernel therefore is also the sheaf of A-sections with support in Z that map to zero. So $ker(\phi : \Gamma_Z(A) \to \Gamma_Z(B)) = \Gamma_Z(ker(\phi))$ and $\Gamma_Z(\cdot)$ is left exact. □

The previous propositions justify the following definitions.

Definition 2.7. *Let X be a topological space, Z a closed subspace and $\mathcal{F}$ an abelian sheaf on X. Then the right derived functors of F respectively G (as in Proposition 2.5 and 2.6) are denoted by $H^i_Z(X, \mathcal{F})$ respectively $\mathcal{H}^i_Z(\mathcal{F})$ and are called the* cohomology groups *respectively* cohomology sheaves *of X with coefficients in $\mathcal{F}$ and support in Z.*

Proposition 2.8. *Let X be a topological space, Z a closed subspace and $\mathcal{F}$ an abelian sheaf on X. Then for each $i \geq 0$, the sheaf $\mathcal{H}^i_Z(\mathcal{F})$ is isomorphic to the sheafification of the presheaf $U \mapsto H^i_{Z\cap U}(U, \mathcal{F}|_U)$.*

Proof. This proof uses the notion of δ-functors as defined in chapter III paragraph 1 of [4]. We denote the sheafification of $U \mapsto H^i_{Z\cap U}(U, \mathcal{F}|_U)$ by T^i. First we will show that the T^i form a universal δ-functor. The presheaves $U \mapsto H^i_{Z\cap U}(U, \mathcal{F}|_U)$ form a δ-functor from the sheaves to the presheaves on X, because the $H^i_{Z\cap U}(U, \mathcal{F}|_U)$ are right derived functors, and right derived functors form a δ-functor. Since the operation of taking a sheaf associated to a presheaf is exact, the collection $(T^i)_{i\geq 0}$ is a δ-functor. To prove that it is universal, we show that T^i is effaceable for $i > 0$ and use Theorem 1.3A in chapter III of [4]. Because every sheaf can be embedded in an injective sheaf, a functor T^i with $i > 0$ is effaceable if it maps injective sheaves to zero. And this is the case for $i > 0$, because for an injective sheaf I it holds that $I|_U$ is injective for any open $U \subseteq X$, so the presheaf $U \mapsto H^i_{Z \cap U}(U, \mathcal{I}|_U)$ is already the zero sheaf. This shows that $(T^i)_{i \geq 0}$ is a universal δ-functor. For $i = 0$ we find $T^0 = \mathcal{H}^0_Z(\mathcal{F}) = \Gamma_Z(\mathcal{F})$. We conclude that both $(T^i)_{i \geq 0}$ and $(\mathcal{H}^i_Z(\mathcal{F}))_{i \geq 0}$ are universal δ-functors both with initial object $\Gamma_Z(\mathcal{F})$. Hence they are isomorphic. □

The local cohomology sheaves $\mathcal{H}^i_Z(\mathcal{F})$ all have support in Z and can therefore be viewed as sheaves on Z. To be precise, $\mathcal{H}^i_Z(\mathcal{F}) \cong j_*(\mathcal{H}^i_Z(\mathcal{F})|_Z)$ where $j : Z \to X$ is the inclusion map. From here on, the notation $\mathcal{H}^i_Z(\mathcal{F})$ will be used for both the sheaf on X and the sheaf on Z.

Recall that flasque sheaves are acyclic for the functor $\Gamma(X, \cdot)$ (see for instance [4] III 2.5) and hence flasque resolutions can be used to compute cohomology.

Proposition 2.9. *Let X be a topological space, Z a closed subspace and $U := X - Z$. Then for any abelian sheaf $\mathcal{F}$ on X there is a long exact sequence*

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to$$

$$\to H^1_Z(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}|_U) \to$$

$$\to H^2_Z(X, \mathcal{F}) \to H^2(X, \mathcal{F}) \to H^2(U, \mathcal{F}|_U) \to \dots \ .$$

Proof. For any flasque sheaf $\mathcal{I}$ on X the injective map $\Gamma_Z(X, \mathcal{I}) \to \Gamma(X, \mathcal{I})$ has cokernel $\Gamma(U, \mathcal{I}|_U)$. It follows that a flasque resolution $\mathcal{I}^\cdot$ for $\mathcal{F}$ gives rise to a short exact sequence of complexes $0 \to \Gamma_Z(X, \mathcal{I}^\cdot) \to \Gamma(X, \mathcal{I}^\cdot) \to \Gamma(U, \mathcal{I}^\cdot|_U) \to 0$. The long exact sequence of this short exact sequence of complexes then gives the desired sequence. □

There is a similar statement for the local cohomology sheaves.

Proposition 2.10. *Let X be a topological space, Z a closed subspace and $U := X - Z$. Let $i : U \to X$ be the inclusion map. Then for any abelian sheaf $\mathcal{F}$ on X there is an exact sequence $0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to i_*(\mathcal{F}|_U) \to \mathcal{H}^1_Z(\mathcal{F}) \to 0$ and isomorphisms $R^j i_*(\mathcal{F}|_U) \cong \mathcal{H}^{j+1}_Z(\mathcal{F})$ for $j > 0$. Here the sheaves $R^j i_*(\mathcal{F}|_U)$ denote the right derived functors of the direct image functor $\mathcal{F} \mapsto i_*(\mathcal{F}|_U)$.*

Proof. For any flasque sheaf $\mathcal{I}$ on X we have a short exact sequence $0 \to \Gamma_Z(\mathcal{I}) \to \Gamma_X(\mathcal{I}) \to i_*(\mathcal{I}|_U) \to 0$. A flasque resolution $\mathcal{I}^\cdot$ for $\mathcal{F}$ therefore gives rise to a short exact sequence of complexes $0 \to \Gamma_Z(\mathcal{I}^\cdot) \to \Gamma_X(\mathcal{I}^\cdot) \to i_*(\mathcal{I}^\cdot|_U) \to 0$. Now taking the long exact sequence and using that the functor $\mathcal{F} \mapsto \Gamma_X(\mathcal{F}) = \mathcal{F}$ is exact gives the result. □

2.2 Spectral sequences

In this section we discuss the notion of spectral sequences and the Grothendieck spectral sequence and show how the local cohomology sheaves and the local cohomology groups are related by a spectral sequence. A good reference for this material is [15] Chapter 5.

First we need some definitions. In each definition, $\mathcal{A}$ is an abelian category.

Definition 2.11. *Let A be an object in $\mathcal{A}$. Then a* decreasing filtration *of A is a finite sequence $F^0A, F^1A, \dots, F^nA$ of objects in $\mathcal{A}$ such that $A = F^0A \supseteq F^1A \supseteq \dots \supseteq F^nA = 0$.*

Definition 2.12. *An* array *in $\mathcal{A}$ is a collection $(A^{p,q})_{p,q \in \mathbb{Z}}$ of objects in $\mathcal{A}$.*

Definition 2.13. *A* page *of* degree $r \in \mathbb{Z}$ *is an array* $(A^{p,q})_{p,q}$ *in an abelian category together with for each* $p, q \in \mathbb{Z}$ *a map* $A^{p,q} \to A^{p+r,q-r+1}$ *such that each composition* $A^{p,q} \to A^{p+r,q-r+1} \to A^{p+2r,q-2r+2}$ *is the zero map.*

Definition 2.14. *Let* A *be a page of degree* r*. Then by* $H(A)$ *we denote the array of homology groups of* A*. To be more specific:*

$$H(A)^{p,q} = ker(A^{p,q} \to A^{p+r,q-r+1})/im(A^{p-r,q-1+r} \to A^{p,q}).$$

All four definitions above are used to define a spectral sequence.

Definition 2.15. *A spectral sequence in* $\mathcal{A}$ *consists of:*

1. *An integer* $r \in \mathbb{Z}$*;*
2. *For each* $s \geq r$ *a page* E_s *of degree* s*. These pages must be such that for each* $p, q \in \mathbb{Z}$ *there exists an* $s_0 \in \mathbb{Z}$ *such that for each* $s \geq s_0$ *the maps* $E_s^{p,q} \to E_s^{p+s,q-s+1}$ *and* $E_s^{p-s,q-1+s} \to E_s^{p,q}$ *are both the zero map.*
3. *For each* $s \geq r$ *and each* $p, q \in \mathbb{Z}$ *an isomorphism* $H(E_s)^{p,q} \to E_{s+1}^{p,q}$*.*

   Note that the last 2 conditions together imply that for each $p, q \in \mathbb{Z}$ *there is an* s_0 *such that there are isomorphisms* $E_{s_0}^{p,q} \simeq E_{s_0+1}^{p,q} \simeq E_{s_0+2}^{p,q} \simeq \ldots$*. So after a certain number of pages, the entry on the position* p, q *is equal for every following page. This way we get limit objects* $E_\infty^{p,q}$ *and a limit array* E_∞*.*
4. *For each* $n \in \mathbb{Z}$ *an object* $E^n \in \mathcal{A}$*.*
5. *For each* $n \in \mathbb{Z}$ *a decreasing filtration* $E^n = F^0E^n \supseteq F^1E^n \supseteq \ldots$*.*
6. *Isomorphisms* $E_\infty^{p,q} \simeq (F^pE^{p+q})/(F^{p+1}E^{p+q})$*.*

We denote these properties by $E_r^{p,q} \Rightarrow E^{p+q}$*.*

The next heorem shows how a composition of covariant functors can give rise to spectral sequences.

Theorem 2.16. *(Grothendieck spectral sequence) Let* $\mathcal{A}$*,* $\mathcal{B}$ *and* $\mathcal{C}$ *be abelian categories such that* $\mathcal{A}$ *and* $\mathcal{B}$ *have enough injectives. Let* $F : \mathcal{A} \to \mathcal{B}$ *and* $G : \mathcal{B} \to \mathcal{C}$ *be left exact covariant functors such that* F *takes injective objects to* G*-acyclic objects. Then for each object* $A \in \mathcal{A}$ *there is a spectral sequence*

$$E_2^{p,q} = (R^pG)(R^qF)(A) \Rightarrow R^{p+q}(G \circ F)(A) = E^{p+q}$$

that is functorial in A*.*

A proof of this theorem can be found in [3].

A special case of the Grothendieck spectral sequence is given in the following proposition.

Proposition 2.17. *Let* X *be a topological space with closed subspace* Z *and* $\mathcal{F}$ *an abelian sheaf on* X*. Then there is a spectral sequence with*

$$H_Z^{p+q}(X, \mathcal{F}) = E^{p+q} \Leftarrow E_2^{p,q} = H^p(X, \mathcal{H}_Z^q(\mathcal{F})).$$

To understand how Grothendieck's spectral sequence is applied here, one must start with the categories $Ab(X)$, $Ab(X)$ and Ab and the functors $\Gamma_Z(\cdot) : Ab(X) \to Ab(X)$ and $\Gamma(X, \cdot) : Ab(X) \to Ab$. The composed functor $\Gamma(X, \Gamma_Z(\cdot))$ then equals the functor $\Gamma_Z(X, \cdot)$ of global sections with support in Z. We already showed in the Propositions 2.5 and 2.6 that these functors are left exact so it remains to show that for an injective sheaf $\mathcal{F} \in Ab(X)$, the sheaf $\Gamma_Z(\mathcal{F})$ is acyclic for the functor $\Gamma(X, \cdot)$. The following lemmas will take care of that.

Lemma 2.18. *Any injective sheaf $\mathcal{I} \in Ab(X)$ is flasque.*

Proof. First we construct the sheaf $\mathcal{F} \in Ab(X)$ of discontinuous sections of $\mathcal{I}$. This sheaf is given by

$$\mathcal{F}(U) = \prod_{x \in U} \mathcal{I}_x,$$

where $\mathcal{I}_x$ denotes the stalk of $\mathcal{I}$ in x. Then with the natural restrictions $\mathcal{F}$ is flasque and we have an injective sheaf morphism $i : \mathcal{I} \to \mathcal{F}$. This gives rise to a short exact sequence:

$$0 \to \mathcal{I} \to \mathcal{F} \to coker(i) \to 0.$$

Because $\mathcal{I}$ is injective, the contravariant functor $Hom(\cdot, \mathcal{I})$ is exact. Hence the map $Hom(i, \mathcal{I}) : Hom(\mathcal{F}, \mathcal{I}) \to Hom(\mathcal{I}, \mathcal{I})$ is surjective. So there is a $f \in Hom(\mathcal{F}, \mathcal{I})$ with $Hom(i, \mathcal{I})(f) = id_{\mathcal{I}}$. Therefore the short exact sequence splits and $\mathcal{I}$ is a direct summand of $\mathcal{F}$. Since $\mathcal{F}$ is flasque, it follows that $\mathcal{I}$ is flasque. □

Lemma 2.19. *Let $\mathcal{F} \in Ab(X)$ be flasque. Then $\Gamma_Z(\mathcal{F})$ is also flasque.*

Proof. Let U and V be open in X such that $V \subseteq U$. We need to show that the restriction map $\Gamma_Z(\mathcal{F})(U) \to \Gamma_Z(\mathcal{F})(V)$ is surjective, so we take any section $s \in \Gamma_Z(\mathcal{F})(V)$. Because s has support inside Z, we can extend s to a section s' on $V \cup Z^c$ that is zero on Z^c. Since $\mathcal{F}$ is flasque, this section again can be extended to a section s'' on $U \cup Z^c$. Finally we restrict s'' to a section s''' on U. Then s''' has support in Z and restricts to s on V. So $\Gamma_Z(\mathcal{F})$ is flasque. □

Flasque sheaves are acyclic for the functor $\Gamma(X, \cdot)$, so these two lemmas show that $\Gamma_Z(\cdot)$ maps injective sheaves to $\Gamma(X, \cdot)$-acyclic sheaves.

3 The concept of stratification

In this chapter we generalise the notion of stratifications that Roth and Vakil used. These stratifications will depend on suitable classes of spaces and abelian sheaves as we will see in the first section. In the second and third section we show how stratifications for a space X relate to the cohomology groups and the dimension of X.

3.1 Suitable classes of spaces and sheaves

From here on, we will work with fixed classes of objects that have topological structure $\mathcal{T}$ and abelian sheaves $\mathcal{S}$. To be precise, we require $\mathcal{T}$ to be a class consisting of objects with topological structure, such that for each $X \in \mathcal{T}$ and each locally closed subset Z of X we have $Z \in \mathcal{T}$. For each $X \in \mathcal{T}$, the set $\mathcal{S}(X)$ has to be a subset of the set of abelian sheaves on X, such that:

1. For each $X \in \mathcal{T}$ with $U \subseteq X$ open and each $\mathcal{F} \in \mathcal{S}(X)$ it holds that $\mathcal{F}|_U \in \mathcal{S}(U)$ and
2. For each $X \in \mathcal{T}$ with $U \subseteq X$ open and each $\mathcal{G} \in \mathcal{S}(U)$ there exists a sheaf $\mathcal{F} \in \mathcal{S}(X)$ such that $\mathcal{F}|_U = \mathcal{G}$.

Examples 3.1.

1. The class of all topological spaces with all abelian sheaves. Most of the properties are obvious. To see that any abelian sheaf $\mathcal{G} \in \mathcal{S}(U)$ can be induced by an abelian sheaf on X, take for example the sheaf $i_!(\mathcal{G}) \in \mathcal{S}(X)$ (See [4] Ex II.1.19b).
2. The class of all topological spaces with the class of constant sheaves (See [4] Example II.1.0.3). This means that for any space X and any abelian group A, the constant sheaf with coefficients in A is included in $\mathcal{S}(X)$. The constant sheaf on X with coefficients in the abelian group A will restrict to the constant sheaf on V with coefficients in A for any open subset $V \subseteq X$. It follows that the class of constant sheaves satisfies the conditions above.
3. The class of Noetherian spaces with the constant sheaves. Here we use the fact that any subset of a Noetherian space is Noetherian (See [4] Ex I.1.7c).
4. The class of algebraic schemes with the quasi coherent sheaves. Locally closed subsets of schemes have a natural structure of scheme so the schemes form a class as defined above. The restriction of a quasicoherent sheaf on X to an open subscheme U gives a quasicoherent sheaf on U and any quasicoherent sheaf $\mathcal{G}$ on U can be induced by the quasicoherent sheaf $i_*(\mathcal{G})$ on X (See [4] Prop II.5.8c).

Remark 3.2. For every $X \in \mathcal{T}$, we can view $\mathcal{S}$ as a contravariant functor on $Op(X)$. The inclusion maps in $Op(X)$ then correspond to restrictions of sheaves and $\mathcal{S}$ is (sort of) a flasque presheaf of abelian sheaves. For a cover $(U_i)_{i \in I}$ of $U \subset X$ with sheaves $\mathcal{F}_i \in \mathcal{S}(U_i)$ such that there are isomorphisms $\theta_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$ for each $i, j \in I$ with $\theta_{ij} \circ \theta_{jk} = \theta_{ik}$ on $U_i \cap U_j \cap U_k$ for each $i, j, k \in I$, there is in fact (Theorem 2.8.1 in [12]) a unique gluing sheaf $\mathcal{F} \in Ab(X)$. But this sheaf $\mathcal{F}$ need not be in $\mathcal{S}(U)$. Therefore $\mathcal{S}$ is not necessarily a "sheaf of abelian sheaves" on X.

Definition 3.3. *Let $\mathcal{T}$ and $\mathcal{S}$ be as above. Then for any $X \in \mathcal{T}$ the* $\mathcal{S}$-cohomological dimension *of X, denoted by $cdim_{\mathcal{S}}(X)$, is the infimum of the set of all integers k for which*

$$H^i(X, \mathcal{F}) = 0 \text{ for all } i > k \text{ and } \mathcal{F} \in \mathcal{S}(X).$$

If this set of integers is empty, then $cdim_{\mathcal{S}}(X) = \infty$. If $cdim_{\mathcal{S}}(X) = 0$ then we say that X is $\mathcal{S}$-simple *or* simple *if it is clear which class of sheaves was assumed.*

In other words, a topological space X is $\mathcal{S}$-simple if each abelian sheaf in $\mathcal{S}(X)$ is acyclic for the functor $\Gamma(X, \cdot)$.

Examples 3.4.

1. In the class of algebraic schemes with the quasi coherent sheaves, the simple spaces are exactly the affine schemes (See [4] Theorem III.3.7).

2. In the class of all topological spaces with the constant sheaves, all contractible spaces are simple because constant sheaves are acyclic on contractible spaces (See [7] Theorem IV.1.1).

3. Any space endowed with the discrete topology is $\mathcal{S}$-simple for any $\mathcal{S}$, because all abelian sheaves on a discrete space are flasque.

With this definition of simple spaces we can define similar notions to the affine stratifications and the affine stratification number as defined in the article of Roth and Vakil [11].

Definition 3.5. *Let $\mathcal{T}$ and $\mathcal{S}$ be as above and let $X \in \mathcal{T}$. An* $\mathcal{S}$-stratification *for X is a finite decomposition $X = \cup_{i=0}^{m} Z_i$ into locally closed, $\mathcal{S}$-simple subspaces such that for any $k \leq m$:*

1. $\overline{Z_k} = \bigcup_{i \geq k} Z_i$,

2. *Z_k is a dense open $\mathcal{S}$-simple subset of $\overline{Z_k}$ with the property that each $p \in \overline{Z_k}$ has an $\mathcal{S}$-simple open neighborhood basis $\mathcal{B}(p)$ in $\overline{Z_k}$ such that for any $V \in \mathcal{B}(p)$ the intersection $Z_k \cap V$ is $\mathcal{S}$-simple,*

3. *For any $\mathcal{F} \in \mathcal{S}(\overline{Z_k})$, it holds that*

$$H^i(Z_k^c, \mathcal{H}^j_{Z_k^c}(\mathcal{F})) = 0 \text{ for } i > cdim_{\mathcal{S}}(Z_k^c) \text{ and } j = 0, 1.$$

Here Z_k^c denotes the complement of Z_k in $\overline{Z_k}$.

Note that in the article of Roth and Vakil the second condition looks easier and this third condition is not there at all. That is because they work with the class of separated schemes and quasicoherent sheaves. Each point on a separated scheme has an affine neighborhood basis, and intersections of affine subschemes are again affine. Also local cohomology sheaves of quasicoherent sheaves are again quasicoherent. Therefore the extra conditions in their case are already part of the class of spaces and sheaves. This means that if we take the class of separated schemes and quasicoherent sheaves, we do get the same stratifications here as in the article of Roth and Vakil. Not every class of spaces and sheaves has these nice properties the class of separated schemes and quasicoherent sheaves has. We will see this in later chapters. Because we want to give a bound on the $\mathcal{S}$-cohomological dimension in a similar way as Roth and Vakil, we had to force these extra conditions in the definition of the stratifications. This will become more clear in the next section.

Also note that a space can only have an $\mathcal{S}$-stratification if at least each of its points has an $\mathcal{S}$-simple neighborhood basis. We will call spaces in which every point has an $\mathcal{S}$-simple neighborhood basis *locally $\mathcal{S}$-simple*.

Definition 3.6. *The* length *of a stratification $\cup_{i=0}^{m} Z_i$ is the largest integer k such that $Z_k \neq \emptyset$.*

Definition 3.7. *Let X be a topological space that admits an $\mathcal{S}$-stratification. Then the* $\mathcal{S}$-stratification number *$sn_{\mathcal{S}}(X)$ of X is the minimal length over all $\mathcal{S}$-stratifications on X.*

Example 3.8. Let $\mathcal{S}$ be the class of constant sheaves and consider the cirkle S^1 with Euclidean topology. For any $p \in S^1$, we get a $\mathcal{S}$-stratification of length 1 by setting $Z_0 = S^1 \setminus \{p\}$ and $Z_1 = \{p\}$. The first property is obvious. For the second property, recall that on a contractible space, constant sheaves are acyclic. That, and the fact that any open subset of $S^1 \setminus \{p\}$ is a disjoint union of contractibles assures the second property. The third property is also not difficult because the restriction of any sheaf to a one-point subset gives a constant sheaf on that point. So for any sheaf $\mathcal{F} \in \mathcal{S}(\overline{Z_0})$, all the sheaves $\mathcal{H}^j_{\overline{Z_1}}(\mathcal{F})$ are in $\mathcal{S}(\overline{Z_1})$. The third property therefore follows by definition of $cdim_{\mathcal{S}}(\overline{Z_1})$. Because $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$, there is no stratification of length 0 and we can conclude that $sn_{\mathcal{S}}(S^1) = 1$.

We conclude with an obvious result.

Proposition 3.9. *Let X be a locally $\mathcal{S}$-simple space. Then*

$$X \text{ is } \mathcal{S}\text{-simple} \Leftrightarrow cdim_{\mathcal{S}}(X) = 0 \Leftrightarrow sn_{\mathcal{S}}(X) = 0.$$

3.2 Stratifications and cohomology

The goal of this section is to show that $cdim_{\mathcal{S}}(X) \leq sn_{\mathcal{S}}(X)$ for spaces $X \in \mathcal{T}$ that admit an $\mathcal{S}$-stratification. The proof will be similar to the one Roth and Vakil gave (See [11] Section 2).

Proposition 3.10. *Let X be an $\mathcal{S}$-simple topological space, $U \subseteq X$ an open subset and $Z = X - U$ the closed complement. Then U is $\mathcal{S}$-simple if and only if $H^i_Z(X, \mathcal{F}) = 0$ for each abelian sheaf $\mathcal{F} \in \mathcal{S}(X)$ and each $i \geq 2$.*

Proof. Consider the long exact excision sequence of cohomology from Proposition 2.9

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}|_U) \to$$
$$\to H^1_Z(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(U, \mathcal{F}|_U) \to$$
$$\to H^2_Z(X, \mathcal{F}) \to H^2(X, \mathcal{F}) \to H^2(U, \mathcal{F}|_U) \to \dots$$

Because X is $\mathcal{S}$-simple, we have that $H^i(X, \mathcal{F}) = 0$ for $i > 0$. Hence $H^i(U, \mathcal{F}|_U) \cong H^{i+1}_Z(X, \mathcal{F})$ for $i \geq 1$. Now because any sheaf in $\mathcal{S}(U)$ can be induced by one in $\mathcal{S}(X)$ we see that $H^i_Z(X, \mathcal{F}) = 0$ for each $\mathcal{F} \in \mathcal{S}(X)$ and $i \geq 2$ is equivalent to $H^i(U, \mathcal{G}) = 0$ for each $\mathcal{G} \in \mathcal{S}(U)$ and $i > 0$ i.e. U is $\mathcal{S}$-simple. That completes the proof. □

Corollary 3.11. *Let X be any topological space. Let U be an $\mathcal{S}$-simple open subset with the property that each $p \in X$ has an $\mathcal{S}$-simple open neighborhood basis $\mathcal{B}(p)$ such that for any $V \in \mathcal{B}(p)$ the intersection $U \cap V$ is $\mathcal{S}$-simple. Let $Z = X - U$. Then $\mathcal{H}^i_Z(\mathcal{F}) = 0$ for each $\mathcal{F} \in \mathcal{S}(X)$ and $i \geq 2$.*

Proof. By Lemma 2.8, the local cohomology sheaf $\mathcal{H}^i_Z(\mathcal{F})$ is the sheafification of the functor $V \mapsto H^i_{Z \cap V}(V, \mathcal{F}|_V)$. We will show that this is the zerosheaf for $i \geq 2$ by showing that for any point $p \in X$ and any $V \in \mathcal{B}(p)$ the group $H^i_{Z \cap V}(V, \mathcal{F}|_V)$ is zero. By hypothesis $U \cap V$ is $\mathcal{S}$-simple, and by definition of $\mathcal{S}$ we have $\mathcal{F}|_V \in \mathcal{S}(V)$. Now because $U \cap V$ is the complement of $Z \cap V$ in V, the previous proposition gives $H^i_{Z \cap V}(V, \mathcal{F}|_V) = 0$ for $i \geq 2$. That completes the proof. □

Corollary 3.12. *Let X, U and Z be as in the previous corollary and additionally assume that for any $\mathcal{F} \in \mathcal{S}(X)$ we have $H^i(Z, \mathcal{H}^j_Z(\mathcal{F})) = 0$ for $i > cdim_{\mathcal{S}}(Z)$ and $j = 0, 1$. Then $H^i_Z(X, \mathcal{F}) = 0$ for all $i > cdim_{\mathcal{S}}(Z) + 1$ and all $\mathcal{F} \in \mathcal{S}(X)$.*

Proof. For any $\mathcal{F} \in \mathcal{S}(X)$, Proposition 2.17 gives a spectral sequence with $H^{p+q}_Z(X, \mathcal{F}) = E^{p+q} \Leftarrow E_2^{p,q} = H^p(X, \mathcal{H}^q_Z(\mathcal{F}))$. For each $i \in \mathbb{Z}$, the group $E^i = H^i_Z(X, \mathcal{F})$ has a decreasing filtration $E^i = F^0E^i \supseteq F^1E^i \supseteq \ldots \supseteq F^rE^i$. We will show that $H^i_Z(X, \mathcal{F}) = 0$ for all $i > cdim_{\mathcal{S}}(Z) + 1$ by showing that each quotient $(F^jE^i)/(F^{j+1}E^i)$ is zero for all $i > cdim_{\mathcal{S}}(Z) + 1$. By definition of a spectral sequence we have $(F^pE^{p+q})/(F^{p+1}E^{p+q}) \simeq E_\infty^{p,q}$ and $E_2^{p,q} = H^p(X, \mathcal{H}^q_Z(\mathcal{F})) = H^p(Z, \mathcal{H}^q_Z(\mathcal{F}))$. Because $\mathcal{H}^q_Z(\mathcal{F}) = 0$ for $q > 1$ by the previous corollary and $H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) = 0$ for $p > cdim_{\mathcal{S}}(Z)$ and $q = 0, 1$ by hypothesis, we see that $H^p(Z, \mathcal{H}^q_Z(\mathcal{F})) = 0$ if $p + q > cdim_{\mathcal{S}}(Z) + 1$. So $E_2^{p,q} = 0$ for $p + q > cdim_{\mathcal{S}}(Z) + 1$. Note that if $E_2^{p,q} = 0$ for certain p and q, in the next page we will have $E_3^{p,q} = 0$ because $E_3^{p,q} \simeq H(E_2)^{p,q}$. Inductively we see that this is the case for any following page. Hence $E_\infty^{p,q} = 0$ for all $p + q > dim_{\mathcal{S}}(Z) + 1$. Now with $i = p + q$ it follows that all the quotients $(F^jE^i)/(F^{j+1}E^i)$ are zero if $i > cdim_{\mathcal{S}}(Z) + 1$, completing the proof. □

Corollary 3.13. *Let X, U and Z be as in the previous corollary. Then $cdim_{\mathcal{S}}(X) \leq cdim_{\mathcal{S}}(Z) + 1$.*

Proof. As in the previous Corollaries we take any sheaf $\mathcal{F} \in \mathcal{S}(X)$. With the excision sequence from Proposition 2.9, and using that U is $\mathcal{S}$-simple, we can conclude that $H^i_Z(X, \mathcal{F}) \cong H^i(X, \mathcal{F})$ for $i \geq 2$ and that $H^1(X, \mathcal{F})$ is a quotient of $H^1_Z(X, \mathcal{F})$. Hence for $i \geq 1$ the statement $H^i_Z(X, \mathcal{F}) = 0$ implies the statement $H^i(X, \mathcal{F}) = 0$. In Corollary 3.12 we got $H^i_Z(X, \mathcal{F}) = 0$ for all $i > dim_{\mathcal{S}}(Z) + 1$ so we conclude that $cdim_{\mathcal{S}}(X) \leq dim_{\mathcal{S}}(Z) + 1$. □

Theorem 3.14. *Let $X \in \mathcal{T}$ be a topological space that admits an $\mathcal{S}$-stratification. Then $cdim_{\mathcal{S}}(X) \leq sn_{\mathcal{S}}(X)$.*

Proof. We prove this by induction on $sn_{\mathcal{S}}(X)$. If $sn_{\mathcal{S}}(X) = 0$ then $X = Z_0$ is $\mathcal{S}$-simple and hence its $\mathcal{S}$-cohomological dimension is zero. Now assume that $m := sn_{\mathcal{S}}(X) > 0$ and that the result is proved for each space Z that admits an $\mathcal{S}$-stratification and with $sn_{\mathcal{S}}(Z) < m$. Let $X = \cup_{i=0}^m Z_i$ be an $\mathcal{S}$-stratification for X and $Z := X - Z_0 = \overline{Z_1} = \cup_{k\geq 1} Z_i$. Then after reindexing $\cup_{k\geq 1} Z_i$, we see that Z has an $\mathcal{S}$-stratification of length $m - 1$. Hence $sn_{\mathcal{S}}(Z) \leq m - 1$. Because Z_0 as subset of X satisfies all the requirements of the U in Corollary 3.13 and it is the complement of Z, we get the inequality $cdim_{\mathcal{S}}(X) \leq cdim_{\mathcal{S}}(Z) + 1$. Furthermore, by the induction hypothesis we get $cdim_{\mathcal{S}}(Z) \leq sn_{\mathcal{S}}(Z)$. Combining these three inequalities gives $cdim_{\mathcal{S}}(X) \leq sn_{\mathcal{S}}(X)$, completing the proof. □

3.3 Stratifications and dimension

In this section we will see that for some $\mathcal{S}$-stratifications on a space X, the length of the stratification is a lower bound for the dimension of the space. If such an $\mathcal{S}$-stratification exists, we immediately have the relations $cdim_{\mathcal{S}}(X) \leq sn_{\mathcal{S}}(X) \leq dim(X)$.

The dimension that we use here is the irreducible dimension. For a non-empty topological space this dimension is defined by the supremum of the set of integers n for which there exists a stricly increasing sequence $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n$ of irreducible closed subsets. Note that the empty set is not irreducible. The dimension is infinite if there exists a stricly increasing sequence $Y_0 \subsetneq Y_1 \subsetneq \ldots \subsetneq Y_n$ of irreducible closed subsets for any integer n. Keep in mind that there is another definition for dimension that is more fitted for topological manifolds. Unless otherwise stated, dimension will always refer to the irreducible dimension.

The irreducible closed subsets of X that are maximal for the inclusion ordering on the irreducible closed subsets of X are called the irreducible components of X.

The next lemma shows that any non-empty space has irreducible components.

Lemma 3.15 ([9], Proposition 2.12a)**.** *Let X be any topological space. Then any irreducible subset of X is contained in an irreducible component of X.*

Proof. Let S be any non-empty set of irreducible subsets of X, partially ordered by inclusion. Let $T \subset S$ be a totally ordered subset and let $Z := \cup_{V \in T} V$. We will show that Z is irreducible, so suppose $Z = X_1 \cup X_2$ for closed subsets X_1 and X_2. Suppose all the elements of T are contained in both X_1 and X_2. Then clearly $Z = X_1 = X_2$. Suppose for some $V \in T$ and say X_1 we have $V \nsubseteq X_1$. Then for any $V' \supseteq V$ in T we get $V' \subseteq X_2$ because $V' \subset X_1 \cup X_2$ and V' is irreducible. It follows that $Z = X_2$, so Z is irreducible. By Zorn's lemma, S will have at least one maximal element and such a maximal element will automatically be closed. Now the lemma follows because for any irreducible subset I we can take S to be the set consisting of the irreducible subsets that contain I. □

Note that for finite-dimensional spaces, or spaces with finitely many irreducible components like Noetherian spaces, we can prove this without the use of Zorn's lemma.

Proposition 3.16. *Let X be a topological space that admits an $\mathcal{S}$-stratification $X = \cup_{i=0}^m Z_i$ with the property that for each $k > 0$ the irreducible components of $\overline{Z_k}$ are not irreducible components in $\overline{Z_{k-1}}$. Then $m \leq dim(X)$.*

Proof. Let Z'_m be an irreducible component of $\overline{Z_m}$. By hypothesis and Lemma 3.15, $\overline{Z_{m-1}}$ has an irreducible component Z'_{m-1} strictly containing Z'_m. Inductively we get a sequence $Z'_m \subsetneq Z'_{m-1} \subsetneq \ldots \subsetneq Z'_0$ where each Z'_i is an irreducible component of $\overline{Z_i}$. Hence $dim(X) \geq dim(Z'_m) + m \geq m$. □

Combined with Theorem 3.14 this gives

Corollary 3.17. *Let X be a topological space that admits a $\mathcal{S}$-stratification $X = \cup_{i=0}^m Z_i$ with the property that for each $k > 0$ the irreducible components of $\overline{Z_k}$ are not irreducible components in $\overline{Z_{k-1}}$. Then $cdim_{\mathcal{S}}(X) \leq sn_{\mathcal{S}}(X) \leq dim(X)$.*

Remark 3.18. Not every space that admits $\mathcal{S}$-stratifications has an $\mathcal{S}$-stratification as in Proposition 3.16. For instance this is the case in Example 3.8. Only single points are irreducible components in S^1, so the point p is an irreducible component of both $\overline{Z_0}$ and $\overline{Z_1}$. This also shows that the condition is necessary for Corollary 3.17 because on S^1 we have $cdim_{\mathcal{S}}(S^1) = sn_{\mathcal{S}}(S^1) = 1$ and $dim(X) = 0$.

The following lemma will show that for certain spaces, including Noetherian spaces, any $\mathcal{S}$-stratification will meet the condition of Proposition 3.16.

Lemma 3.19. *Let X be a non-empty topological space and let U be any subspace of X. Then the irreducible components of U map injectively to irreducible subsets of $\overline{U}$ by the closure operation. Additionally, if U has only finitely many irreducible components, this mapping is a bijection onto the set of irreducible components of $\overline{U}$.*

Proof. Let Z be an irreducible component of U and suppose $\overline{Z} = V_1 \cup V_2$ for V_1 and V_2 closed in $\overline{U}$. Then either $V_1 \cap Z = Z$ or $V_2 \cap Z = Z$, because Z is irreducible. Let's say $V_1 \cap Z = Z$. Then since V_1 is closed we get $\overline{Z} \subseteq V_1$. Hence $\overline{Z} = V_1$. This shows that the closure of an irreducible component of U is irreducible. For the injectivity, take irreducible components Z_1 and Z_2 of U and suppose $\overline{Z_1} = \overline{Z_2}$ in $\overline{U}$. Then $Z_1 = \overline{Z_1} \cap U = \overline{Z_2} \cap U = Z_2$. Now assume that U has only finitely many irreducible components. Because the closures of the irreducible components of U cover $\overline{U}$, any irreducible subset of $\overline{U}$ must be contained in the closure of an irreducible component of U. $\square$

Remark 3.20. In [11], Roth and Vakil proved the relation $asn(X) \leq dim(X)$ for separated schemes and quasicoherent sheaves using the following statement: The complement of a dense affine open subset in any scheme is of pure codimension one (Corollary 2.4 in [11]). The Corollaries 3.17 and 3.19 show it is not necessary to use this specific property for schemes, because the relation $asn(X) \leq dim(X)$ already follows from the fact that X is Noetherian.

4 Alexandrov spaces

The first class of abelian sheaves that we will consider is the class of all abelian sheaves, which we will denote by $\mathcal{A}$. An advantage of this class is that for any space X with abelian sheaf $\mathcal{F} \in \mathcal{A}(X)$ and closed subset Z, all the local cohomology sheaves $\mathcal{H}^j_Z(\mathcal{F})$ are in $\mathcal{A}(Z)$. This means that any finite decomposition $X = \cup_{i=0}^m Z_i$ into locally closed $\mathcal{A}$-simple subspaces that satisfies the first two conditions of a stratification, will also satisfy the third condition. So we do not have to bother about the third condition. However, because of the second condition, a space that admits $\mathcal{A}$-stratifications must at least have the property that each of its points has an $\mathcal{A}$-simple open neighborhood basis. For that reason, we will consider the following class of topological spaces.

Definition 4.1. *A topological space is called an* Alexandrov space *or* A-space *if the intersection of any collection of open subsets is open.*

For example, any finite space is an Alexandrov space.

On Alexandrov spaces, an open neighborhood basis for a point can be given with just one open neighborhood that is the minimal open neighborhood for that point.

Definition 4.2. *Let X be an Alexandrov space and $x \in X$. Then $U_x = \cap\{U \subseteq X : U$ open and $x \in U\}$ is called the* open hull *of x.*

Lemma 4.3. *Let X be an Alexandrov space. Then for any $x \in X$ the open hull U_x is $\mathcal{A}$-simple.*

Proof. Let $\mathcal{F}$ be any abelian sheaf on U_x. Then one can check that $\Gamma(U_x, \mathcal{F}) = \mathcal{F}(U_x)$ equals the stalk $\mathcal{F}_x$. Since taking stalks on sheaves works as an exact functor, we conclude that the global section functor on U_x is exact. It follows that U_x is $\mathcal{A}$-simple. □

So indeed an Alexandrov space has the property that each of its points has an $\mathcal{A}$-simple open neighborhood basis.

Alexandrov spaces can also be viewed as pre-ordered sets and vice versa. Recall that a pre-ordered set is a set with a reflexive and trasitive relation. Given an Alexandrov space X, its preorder $\leq$ is given by $x \leq y$ if $\overline{\{x\}} \subseteq \overline{\{y\}}$ for $x, y \in X$. For a pre-ordered set S, a topology can be generated by the basis $V_x = \{y \in X : y \geq x\}$. To see that this defines an Alexandrov space, we take any subset $Z \subseteq S$ and consider $\cap_{x \in Z} V_x$. If for some $y \in S$ we have $y \in V_x$ for all $x \in Z$, then $V_y \subseteq V_x$ for all $x \in Z$. Hence $\cap_{x\in Z} V_x$ is a union of basis elements and therefore open. So S becomes an Alexandrov space this way.

The open hulls U_x defined above agree with the basis open subsets V_x. To see this we argue as follows. The biggest closed set not containing x is given by $\bigcup\{\overline{\{z\}} : z \in X \text{ and } z \ngeq x\} = \bigcup\{z \in X : z \ngeq x\}$, so for the complement we have $U_x = \{z \in X : z \geq x\} = V_x$.

4.1 Partially ordered sets

In this section we discuss a correspondence between topological spaces and partially ordered sets. We will show that this correspondance preserves $\mathcal{A}$-stratifications for finite dimensional Alexandrov spaces. This will prove useful in the next section when we give explicit $\mathcal{A}$-stratifications for finite dimensional Alexandrov spaces.

First we will describe a procedure to associate a topological space $t(X)$ to any topological space X such that both spaces have isomorphic categories of open sets and $t(X)$ has the property that for each irreducible closed subset Z of $t(X)$ there is a unique $x \in t(X)$ such that $\overline{\{x\}} = Z$. This procedure was also used in the proof of Proposition II.2.6 in [4].

For a topological space X, we denote by $t(X)$ the set of all irreducible closed subsets of X. If $Z \subseteq X$ is closed, then $t(Z) \subseteq t(X)$. For Z_1 and Z_2 closed we have $t(Z_1 \cup Z_2) = t(Z_1) \cup t(Z_2)$, because each irreducible subset of $Z_1 \cup Z_2$ is contained in either Z_1 or Z_2. Furthermore, for a collection Z_i of closed subsets of X we have $t(\cap Z_i) = \cap t(Z_i)$. Hence we can define a topology on $t(X)$ by letting $t(Z)$ be closed in $t(X)$ for Z closed in X.

Lemma 4.4. *Let X be a topological space. Then $Op(X)$ and $Op(t(X))$ are isomorphic categories and $t(X)$ has the property that for each irreducible closed subset Z there is a unique $x \in t(X)$ such that $\overline{\{x\}} = Z$.*

Proof. We will show that the closed subsets of X and $t(X)$ are in a bijective order preserving correspondence. The surjectivity and the order preserving property follow from the construction. As for the injectivity, if we have two closed subsets Z_1 and Z_2 and say $x \in Z_1$ and $x \notin Z_2$, then $\overline{\{x\}} \in t(Z_1)$ and $\overline{\{x\}} \notin t(Z_2)$. So different closed subsets in X give rise to different closed subsets in $t(X)$. This shows that $Op(X)$ and $Op(t(X))$ are isomorphic categories.

Now take any irreducible closed subset Y of $t(X)$ and let Z be the corresponding irreducible closed subset in X. Then $Z \in Y$ and

$$\overline{\{Z\}} = \cap\{t(V) : V \text{ closed in } X \text{ and } Z \subseteq V\} = t(\cap\{V : V \text{ closed in } X \text{ and } Z \subseteq V\}) = t(Z) = Y.$$

Now since this is the case for any irreducible closed subset, and the irreducible closed subsets are in a bijective order preserving correspondence, the uniqueness follows. □

Remark 4.5. Note that for a continuous map $f : X \to Y$ we get a continuous map $t(f) : t(X) \to t(Y)$ sending an irreducible closed subset to the closure of its image. So in fact t defines a functor $Top \to Top$.

Because of the bijective order preserving correspondence between the open and closed sets of X and those of $t(X)$, properties like being Noetherian and the dimension do not change under the t operation. Also t induces an isomorphism $\mathfrak{Ab}(X) \to \mathfrak{Ab}(t(X))$ between the category of abelian sheaves on X and the category of abelian sheaves on $t(X)$. It follows that we get the same cohomology groups for X and $t(X)$. Hence $cdim_{\mathcal{A}}(X) = cdim_{\mathcal{A}}(t(X))$.

Furthermore, the closure of a point is irreducible, so the irreducible closed subsets of $t(X)$ map bijectively to the points of $t(X)$. The inclusion relation on the irreducible closed subsets of $t(X)$ therefore gives rise to a partial ordering on $t(X)$. More explicitly, for $x, y \in t(X)$ we have $x \leq y$ if and only if $\overline{\{x\}} \subseteq \overline{\{y\}}$. The maximal elements for this ordering on $t(X)$ correspond to the irreducible components of X.

The following lemmas are direct consequences of the above observations.

Lemma 4.6. *Let X be a topological space with the property that for each irreducible closed subset Z there is a unique $x \in X$ such that $\overline{\{x\}} = Z$. Then the mapping $x \mapsto \overline{\{x\}}$ defines a homeomorphism between X and $t(X)$.*

Lemma 4.7. *For a finite-dimensional topological space X with $X = t(X)$ we have: $dim(X) = max\{n \in \mathbb{N} : \exists x_0, \ldots, x_n \in X \text{ s.t. } x_0 > \ldots > x_n\}$.*

Example 4.8. Consider the set $\mathbb{Z}$ with topology generated by the open hulls $U_p = \{x \in \mathbb{Z} : x \geq p\}$ for $p \in \mathbb{Z}$. Any non-empty proper closed subset of $\mathbb{Z}$ is irreducible and equals the closure of some point. The space $\mathbb{Z}$ itself is also irreducible but is not the closure of a point. Therefore the space $t(\mathbb{Z})$ as a set will be the disjoint union of $\mathbb{Z}$ with a point q. The closed sets of $t(\mathbb{Z})$ will be the closed sets of $\mathbb{Z}$ together with the whole set $t(\mathbb{Z}) = \mathbb{Z} \coprod \{q\}$.

We now return to the Alexandrov spaces.

Proposition 4.9. *Let X be a finite dimensional Alexandrov space. Then $t(X)$ is a quotient space of X.*

Proof. First we show that any irreducible closed subset Z of X is already the closure of a point in X. Let $\leq$ be the pre-ordering on X defined by $x \leq y$ if $\overline{\{x\}} \subseteq \overline{\{y\}}$ for $x, y \in X$. Then Z has a maximal element p because X is finite dimensional. Suppose $\overline{\{p\}} \subsetneq Z$. Then $\overline{\{p\}}$ and $\cup_{q \in \{Z - \overline{\{p\}}\}} \overline{\{q\}}$ are both non-empty proper closed subsets of Z whose union equals Z. This contradicts with the irreducibility of Z. Hence $\overline{\{p\}} = Z$. It follows that the map $\pi : X \to t(X)$ given by $\pi(x) = \overline{\{x\}} \in t(X)$ is surjective. We now can verify that the quotient topology on $t(X)$ induced by π equals the original topology on $t(X)$ because they both correspond to the same partial ordering. That proves the claim. □

Corollary 4.10. *Let X be a finite dimensional Alexandrov space. Then $t(X)$ is a finite dimensional Alexandrov space.*

Proof. The statement follows directly from the equality

$$\cap \pi^{-1}(U_i) = \pi^{-1}(\cap U_i)$$

where (U_i) is any collection of open subsets in $t(X)$. □

Remark 4.11. The statement above is in general not true for an infinite dimensional Alexandrov space. For example, consider again the spaces $\mathbb{Z}$ and $t(\mathbb{Z})$ from Example 4.8. In $t(\mathbb{Z})$, the union of closed sets not containing q is not closed. So $t(\mathbb{Z})$ is not an Alexandrov space, while $\mathbb{Z}$ is.

Corollary 4.12. *Let X be a finite dimensional Alexandrov space. Then there exists a length-preserving bijection between the set of $\mathcal{A}$-stratifications on X and the set of $\mathcal{A}$-stratifications on $t(X)$.*

Proof. All properties of $\mathcal{A}$-stratifications will be preserved by the bijective order preserving correspondence between $Op(X)$ and $Op(t(X))$ except that maybe not every point in $t(X)$ will get a suitable $\mathcal{A}$-simple neighborhood basis carried over. But that can easily be verified using the quotient map π. □

4.2 A stratification for finite dimensional Alexandrov spaces

The goal of this section is to give an explicit $\mathcal{A}$-stratification for any finite-dimensional Alexandrov space X. As shown in the previous section, we can assume that $X = t(X)$ and that X is a partially ordered set with $x \leq y$ if $\overline{\{x\}} \subseteq \overline{\{y\}}$ for $x, y \in X$.

The stratification will be obtained in the following way:

Let Z_0 be the subset consisting of the maximal elements of X.

Let Z_1 be the subset consisting of the maximal elements of $X - Z_0$.

Let Z_2 be the subset consisting of the maximal elements of $X - Z_0 - Z_1$.
etc.
By Lemma 4.7 we get $X = \bigcup_{i=0}^{n} Z_i$ where $n = dim(X)$.

Proposition 4.13. *For the decomposition $X = \bigcup_{i=0}^{n} Z_i$ as defined above it holds that for $i = 0, \dots, n$:*

1. $\overline{Z_i} = \bigcup_{j \geq i} Z_j$.
2. *Z_i is open in $\overline{Z_i}$, hence locally closed.*
3. *Any subset of Z_i is $\mathcal{A}$-simple.*
4. *For $i > 0 : \overline{Z_i}$ is of pure codimension one in $\overline{Z_{i-1}}$.*

Proof. (1) Trivial.
(2) Follows from:

$$Z_i = (\bigcup_{z \in Z_i} \overline{\{z\}}) \bigcap (\bigcup_{z \in Z_{i-1}} \overline{\{z\}})^c$$

(3) Follows from the fact that the topology induced on the Z_i is the discrete topology.
(4) Trivial. □

This shows that $\bigcup_{i=0}^{n} Z_i$ indeed is a $\mathcal{A}$-stratification for X.

Now we can prove Grothendieck's dimensional vanishing theorem ([4], Theorem III.2.7) for Alexandrov spaces.

Proposition 4.14. *Let X be an Alexandrov space. Then $cdim_{\mathcal{A}}(X) \leq dim(X)$.*

Proof. Assuming that X has finite dimension n and using Corollary 4.12, the construction above gives a $\mathcal{A}$-stratification for X of length n. The result then immediately follows from Corollary 3.17. □

4.3 Properties of $\mathcal{A}$-simple spaces

The $\mathcal{A}$-stratification given in the previous section will in many cases not have minimal length. For instance consider the cases where X is $\mathcal{A}$-simple and has positive dimension. In order to be able to give better $\mathcal{A}$-stratifications, we will discuss some basic properties of $\mathcal{A}$-simple spaces. We will also investigate if $\mathcal{A}$-stratifications can be made for spaces other than Alexandrov spaces.

Affine schemes have the property that any closed subscheme is affine (See [4] Ex II.3.11b). The $\mathcal{A}$-simple spaces have a similar property.

Lemma 4.15. *Let X be any $\mathcal{A}$-simple space (not necessarily Alexandrov). Then any closed subset of X is $\mathcal{A}$-simple.*

Proof. Let Z be a closed subset of X with inclusion map $i : Z \to X$ and let $\mathcal{F}$ be any abelian sheaf on Z. Then with Lemma III.2.10 in [4] we get

$$H^i(Z, \mathcal{F}) = H^i(X, i_*(\mathcal{F})) = 0 \text{ for } i > 0$$

because X is $\mathcal{A}$-simple. Hence Z is $\mathcal{A}$-simple. □

So far, the only $\mathcal{A}$-simple spaces we have seen are spaces that contain a point whose only open neighborhood is the space itself, and disjoint unions of those spaces. A question that one could ask at this point is whether all $\mathcal{A}$-simple spaces are of this type or, equivalently, do there exist $\mathcal{A}$-simple connected spaces with the property that each point has a nontrivial open neighborhood? The next example shows that already the smallest space that is connected and does not contain a point with only the trivial open neighborhood, is not $\mathcal{A}$-simple.

Example 4.16. In this example we show that the three-point space $X = \{a, b, c\}$ with topology $T_X = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$ is not $\mathcal{A}$-simple. Consider the sheaf $\mathcal{F}$ that assigns the group $\mathbb{Z}$ to the subset $\{a\}$ and 0 to each other subset. We can embed this sheaf in the flasque sheaf I^0 that is the constant sheaf with coefficients in $\mathbb{Z}$. The presheaf cokernel $\mathcal{G}$ of the map $\mathcal{F} \to I^0$ then is a presheaf that assigns the group 0 to the subset $\{a\}$ and $\mathbb{Z}$ to each other subset. All non-trivial restriction morphisms of $\mathcal{G}$ are identities. Each non-empty open subset of X, with the exception of X, is the open hull of some point. The sheafification I^1 of $\mathcal{G}$ will therefore not differ on these sets, as the groups on these sets equal the stalk of some point. It follows that I^1 is a flasque sheaf with $I^1(X) = \mathbb{Z} \oplus \mathbb{Z}$ and restricion maps $(I_{\mathbb{Z}}, 0)$ respectively $(0, I_{\mathbb{Z}})$ for the restrictions $I^1(X) \to I^1(\{a, b\})$ respectively $I^1(X) \to I^1(\{a, c\})$. Now $I^0 \to I^1 \to 0$ is a flasque resolution for $\mathcal{F}$ where $I^0(X) \to I^1(X)$ is the diagonal map. Hence we get $H^1(X, \mathcal{F}) = \mathbb{Z}$.

The example also shows, as $\{a, b\} \cup \{a, c\} = X$, that a union of $\mathcal{A}$-simple subspaces need not be $\mathcal{A}$-simple. We will show now that the same is true for intersections. We add two minimal points d and e to the space X and call the new space Y. This means that the only open neighborhoods for d are $\{a, b, c, d\}$ and Y and the only open neighborhoods for e are $\{a, b, c, e\}$ and Y. Now the open hulls U_d and U_e are $\mathcal{A}$-simple but their intersection X is not. Note that the $\mathcal{A}$-simple spaces behave less well in this respect than separated affine schemes with quasicoherent sheaves..

The following proposition will show that the argument in Example 4.16 can be used to characterize all finite dimensional $\mathcal{A}$-simple spaces. The proposition is not only about Alexandrov spaces, therefore we generalise the notion of open hull to any topological space or open subset that is a minimal open subset for some point it contains.

Proposition 4.17. *Let X be a connected topological space. Then for the statements:*

1. *X is an open hull,*
2. *X is $\mathcal{A}$-simple,*
3. *X can not be written as the union of two open proper subsets,*

it holds that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Additionally, if X has a non-empty closed subset that does not contain a smaller non-empty closed subset, then we also have (3) $\Rightarrow$ (1).

Proof. (1)$\Rightarrow$ (2). Similar as the proof of Lemma 4.3.
(2) $\Rightarrow$ (3). Suppose X is the union of the proper open subsets U_1 and U_2. Then $U = U_1 \cap U_2$ is non-empty because X is connected. Let $\mathcal{F}$ be the constant sheaf with coefficients in $\mathbb{Z}$, let $Z = X - U$ and let $i : Z \to X$ and $j : U \to X$ be the inclusion mappings. If $\mathcal{F}$ or $i_*(\mathcal{F}|_Z)$ is not acyclic for the functor $\Gamma(X, \cdot)$ then we are done. Suppose $\mathcal{F}$ and $i_*(\mathcal{F}|_Z)$ are acyclic for the functor $\Gamma(X, \cdot)$. Then the exact sequence

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0$$

from Ex II.1.19 in [4] gives an acyclic resolution for $j_!(\mathcal{F}|_U)$. So we can compute

$$H^1(X, j_!(\mathcal{F}|_U)) = \Gamma(X, i_*(\mathcal{F}|_Z))/\Gamma(X, \mathcal{F}).$$

Now because X is connected and Z is not, it follows that $H^1(X, j_!(\mathcal{F}|_U)) \neq 0$. Hence X is not $\mathcal{A}$-simple.
(3)⇒(1). Let Y be a non-empty closed subset of X that is minimal for these properties. Then for $p \in Y$ we have $\overline{\{p\}} = Y$. Let V be any open neighborhood of p. Then by minimality of Y we have $V^c \cap Y = \emptyset$. Hence $X = V \cup Y^c$ and it follows that $V = X$. So X is the open hull of p. □

Corollary 4.18. *Let X be a finite dimensional space. Then X is $\mathcal{A}$-simple if and only if X is a disjoint union of open hulls.*

Proof. Suppose any connected component V of X does not have a non-empty closed subset that is minimal for these properties. Then we can inductively find an infinite sequence $p_1, p_2, p_3, \dots$ of points in V with $\overline{\{p_1\}} \supsetneq \overline{\{p_2\}} \supsetneq \overline{\{p_3\}} \supsetneq \dots$. This implies that X is infinite dimensional, which contradicts with the hypothesis. Now with Proposition 4.17 we see that a connected component of X is $\mathcal{A}$-simple if and only if it is an open hull. That proves the claim. □

Remark 4.19. It is possible that the same statement is also true for infinite dimensional spaces. However, I was not able to give a proof or a counterexample. We leave it therefore as an open question to the reader.

Now we know that in a finite dimensional space only the open hulls are $\mathcal{A}$-simple, we can ask ourselves if there are finite dimensional spaces other than Alexandrov spaces that admit $\mathcal{A}$-stratifications. The following two lemmas answer this question.

Lemma 4.20. *Let X be a finite dimensional space and let p be a point in X that has a $\mathcal{A}$-simple open neighborhood basis $\mathcal{B}(p)$. Then p has an open hull in X.*

Proof. Suppose $\mathcal{B}(p)$ has no minimal element. Then there exists an infinite decreasing sequence $U_0 \supsetneq U_1 \supsetneq \dots$ of $\mathcal{A}$-simple open subsets in X. By Corollary 4.18, each U_i is an open hull for some point $p_i \in X$. For $i < j$, any open subset containing p_i will also contain p_j. Equivalently, any closed subset containing p_j will also contain p_i. But then we have an infinite increasing sequence $\overline{\{p_0\}} \subsetneq \overline{\{p_1\}} \subsetneq \dots$ of irreducible closed subsets in X, contradicting the fact that X is finite dimensional. Hence $\mathcal{B}(p)$ has a minimal element, the open hull of p. □

Lemma 4.21. *Let X be a topological space with the property that any point $p \in X$ has an open hull. Then X is an Alexandrov space.*

Proof. Let $(V_i)_{i \in I}$ be any collection of open subsets in X. Assume that $\cap_{i \in I} V_i$ is non-empty and let $p \in \cap_{i \in I} V_i$. Because the open hull U_p of p is the minimal open neighborhood of p, we have $U_p \subseteq V_i$ for any $i \in I$. Hence

$$\cap_{i \in I} V_i = \cup_{p \in \cap_{i \in I} V_i} U_p$$

is open. So X is an Alexandrov space. □

Since $\mathcal{A}$-stratifications on X require X to have an $\mathcal{A}$-simple open neighborhood basis for each of its points, it follows that the only finite dimensional spaces that can possibly admit $\mathcal{A}$-stratifications are Alexandrov spaces. We can conclude that our concept of stratifications will not be adequate enough to prove the Theorems 1.1 and 1.2.

5 Noetherian spaces

In this chapter we consider the class of Noetherian spaces with the class of constant sheaves $\mathcal{K}$. Recall that a topological space is called Noetherian if it satisfies the descending chain condition for closed subsets. This means that for any sequence $Y_1 \supset Y_2 \supset \dots$ of closed subsets, there is an integer r such that $Y_r = Y_{r+1} = \dots$.

As in the previous chapter, we first look for $\mathcal{K}$-simple spaces.

Lemma 5.1. *Any irreducible space X is $\mathcal{K}$-simple.*

Proof. Any non-empty open subset of X is connected. Hence any constant sheaf is flasque. □

Of course there are more $\mathcal{K}$-simple spaces. For example, open hulls are $\mathcal{K}$-simple and need not be irreducible. Open hulls are contractible, as we will see later on, and in fact all contractible spaces are $\mathcal{K}$-simple (See [7] Theorem IV.1.1). There also exist $\mathcal{K}$-simple spaces that are not contractible or irreducible. For instance, see Example 4.2.1 in [1].

A space X at least has to be locally $\mathcal{K}$-simple in order to admit $\mathcal{K}$-stratifications. I was not able to classify the Noetherian spaces that satisfy this condition. However, the next section shows that any finite-dimensional Noetherian space that does so, also admits a $\mathcal{K}$-stratification.

5.1 A stratification for finite-dimensional Noetherian spaces

We will construct a $\mathcal{K}$-stratification using the interiors of the irreducible components of a finite dimensional Noetherian space. Recall that the interior of a subset Z in a space X is the maximal open subset U of X that is contained in Z.

Lemma 5.2. *For any space X that has only finitely many irreducible components $Z_1, Z_2, \dots, Z_n$ it holds that each irreducible component has non-empty interior.*

Proof. For any $i = 1, \dots, n$ we can define the closed set $Z := \cup\{Z_j : j \neq i\}$. Then $U = Z^c$ is a non-empty subset of Z_i. Hence Z_i has non-empty interior. □

Let X be any finite dimensional Noetherian space. We define Z_0 to be the union of the interiors of the irreducible components of X.

Lemma 5.3. *Z_0 is an $\mathcal{K}$-simple, dense, open subset of X.*

Proof. By Lemma 3.19, the interior of an irreducible component is irreducible. We will show that these interiors are pairwise disjoint. The result then follows from Lemma 5.1. Let Y_1 and Y_2 be any two irreducible components of X with interiors U_1 and U_2 and suppose $U_1 \cap U_2 \neq \emptyset$. Then $U_1 \cap U_2$ is dense in both Y_1 and Y_2. It follows that $Y_1 = Y_2$. □

Lemma 5.4. *Any open subset of Z_0 is $\mathcal{K}$-simple.*

Proof. Any open subset of Z_0 will be a disjoint union of open subsets of the interiors of the irreducible components. One by one, these will be irreducible by Lemma 3.19 and therefore $\mathcal{K}$-simple. □

By Lemma 3.19, the complement of Z_0 will have lower dimension than X. Repeating this process on the complement of Z_0 gives Z_1. And so on we obtain $Z_0, \dots, Z_n$, already looking like a $\mathcal{K}$-stratification.

Lemma 5.5. *The subsets $Z_0, \dots, Z_n$ as defined above satisfy condition 3 of a $\mathcal{K}$-stratification.*

Proof. Let $\mathcal{G}$ be the constant sheaf on $\overline{Z_i}$ with coefficients in the abelian group G. Because Z_i is dense in $\overline{Z_i}$, there are no open subsets of $\overline{Z_i}$ completely contained in Z_i^c. For a constant sheaf, any non-zero section will have support on an open set. Therefore the sheaf $\Gamma_{Z_i^c}(\mathcal{G})$ is just the zero sheaf on $\overline{Z_i}$. With Proposition 2.10 we get a short exact sequence

$$0 \to \mathcal{G} \to i_*(\mathcal{G}|_{Z_i}) \to \mathcal{H}^1_{Z_i^c}(\mathcal{G}) \to 0$$

of abelian sheaves on $\overline{Z_i}$. We will restrict this sequence to an exact sequence of sheaves on Z_i^c. Restrictions of sheaves are defined at the end of section II.1 in [4]. When restricted to Z_i^c, the sheaf $\mathcal{G}$ will be just the constant sheaf on Z_i^c with coefficients in G. The restriction of $i_*(\mathcal{G}|_{Z_i})$ to Z_i^c is a bit more difficult. Suppose an open subset V of Z_i^c has non-empty intersection with certain irreducible components of $\overline{Z_i}$. Then any open subset U of $\overline{Z_i}$ that induces V on Z_i^c will have non-empty intersection with these irreducible components. By intersecting with the complemement of the union of the other irreducible components, we see that such an U always contains a V inducing open subset U' of $\overline{Z_i}$ that has no intersection with the other irreducible components of $\overline{Z_i}$. For these U' we have that $i_*(\mathcal{G}|_{Z_i})(U') = \oplus G$, where the sum ranges over the irreducible components of $\overline{Z_i}$ that have non-empty intersection with U'. Hence, the restriction of $i_*(\mathcal{G}|_{Z_i})$ to Z_i^c is given by $i_*(\mathcal{G}|_{Z_i})(V) = \oplus G$ where the sum ranges over the irreducible components of $\overline{Z_i}$ that have non-empty intersection with V. It follows that the sheaf $i_*(\mathcal{G}|_{Z_i})$ restricted to Z_i^c is flasque. Now with the long exact sequence of the short exact sequence $0 \to \mathcal{G} \to i_*(\mathcal{G}|_{Z_i}) \to \mathcal{H}^1_{Z_i^c}(\mathcal{G}) \to 0$ restricted to Z_i^c we get $H^j(Z_i^c, \mathcal{H}^1_{Z_i^c}(\mathcal{G})) \cong H^{j+1}(Z_i^c, \mathcal{G})$ for $j > 0$. Hence $H^j(Z_i^c, \mathcal{H}^1_{Z_i^c}(\mathcal{G})) = 0$ for $j > dim_{\mathcal{K}}(Z_i^c)$. That completes the proof. □

The only thing left now that could stop $Z_0, \dots, Z_n$ from being a $\mathcal{K}$-stratification is that it is not sure if each point in $\overline{Z_i}$ has a $\mathcal{K}$-simple open neighborhood basis. Hence, if a finite dimensional Noetherian space X has the property that for any closed set Z any point $p \in Z$ has a $\mathcal{K}$-simple open neighborhood basis in Z, then $Z_0, \dots, Z_n$ is a $\mathcal{K}$-stratification for X. Note that this is at least the case for all finite spaces, where we have the open hulls.

The following propositions follow directly.

Proposition 5.6. *Let X be a Noetherian space of dimension n for which any closed subset is locally $\mathcal{K}$-simple. Then X admits a $\mathcal{K}$-stratification of lenth $\leq n$.*

Proposition 5.7. *Let X be a Noetherian space for which any closed subset is locally $\mathcal{K}$-simple. Then $cdim_{\mathcal{K}}(X) \leq dim(X)$.*

Remark 5.8. One can check that the process of repeatedly taking the union of the interiors of the irreducible components can also be used to make $\mathcal{K}$-stratifications for finite-dimensional Alexandrov spaces. We only need another proof to show that the interior of an irreducible component is non-empty. But this is easy, because on a finite dimensional Alexandrov space $X = t(X)$, any irreducible component is given by $\overline{\{p\}}$ for some $p \in X$ and has open subset $\{p\}$.

5.2 Stratifications for the finite two-dimensional sphere

In this section we consider a number of stratification candidates for the finite two-dimensional sphere $\mathbb{S}^2$. The examples will show some interesting aspects of stratifications and the class of constant sheaves. For all the stratification candidates we will check if it is an $\mathcal{A}$-stratification and if it is a $\mathcal{K}$-stratification.

Recall that the finite spheres are defined in the following recursive way. It starts with the two point space $\mathbb{S}^0 = \{p_{0,0}, p_{0,1}\}$ with discrete topology. Then $\mathbb{S}^1$ is obtained by adding two minimal points $p_{1,0}$ and $p_{1,1}$ to $\mathbb{S}^0$. So $\mathbb{S}^1$ is a four-point space with topological basis

$$\{\{p_{0,0}\}, \{p_{0,1}\}, \{p_{0,0}, p_{0,1}, p_{1,0}\}, \{p_{0,0}, p_{0,1}, p_{1,1}\}\}.$$

Similarly the space $\mathbb{S}^n$ is obtained by adding two minimal points $p_{n,0}$ and $p_{n,1}$ to S^{n-1}.

In [2], Michael C. McCord shows that there exists a weak homotopy equivalence between the finite sphere $\mathbb{S}^n$ and the real Euclidean sphere S^n. This implies that finite spheres and real Euclidean spheres have isomorphic cohomology groups for constant sheaves. So for the finite sphere $\mathbb{S}^2$ we have $H^2(\mathbb{S}^2, \mathbb{Z}) = \mathbb{Z}$. It follows that $cdim_{\mathcal{A}}(\mathbb{S}^2) = cdim_{\mathcal{K}}(\mathbb{S}^2) = 2$. More details on weak homotopy equivalences are given in the next chapter.

All the stratification candidates will at least satisfy the first property of a stratification, so we will not discuss that.

Stratification 1. $Z_0 = \{p_{0,0}, p_{0,1}\}$, $Z_1 = \{p_{1,0}, p_{1,1}\}$, $Z_2 = \{p_{2,0}, p_{2,1}\}$.
Both the processes in section 4.2 and in section 5.1 give this stratification. So this is both an $\mathcal{A}$-stratification and a $\mathcal{K}$-stratification.

Stratification 2. $Z_0 = \{p_{0,0}, p_{0,1}, p_{1,0}\}$, $Z_1 = \{p_{1,1}, p_{2,0}, p_{2,1}\}$.
A priori we can already say that this can neither be an $\mathcal{A}$-stratification or a $\mathcal{K}$-stratification because then its length should be at least equal to the cohomological dimension by Theorem 3.14. Nevertheless it is interesting to see where it fails to be a stratification. For the class $\mathcal{A}$ this is because Z_1 is not an open hull and therefore not $\mathcal{A}$-simple. Because Z_1 is irreducible, this argument does not hold for $\mathcal{K}$. Also all the subsets of Z_0 and Z_1 are $\mathcal{K}$-simple, so the second property for a $\mathcal{K}$-stratification is satisfied. The problem therefore has to do with the third condition. Indeed the sheaf $\mathcal{H}^1_{Z_0^c}(\mathcal{F})$ in not acyclic on Z_0^c if $\mathcal{F}$ is a constant sheaf on $\mathbb{S}^2$ with coefficients in a nontrivial abelian group. In fact this sheaf equals the sheaf from Example 4.16 if $\mathcal{F}$ has coefficients in $\mathbb{Z}$.

This example shows that the class of constant sheaves does not behave as nicely as the class with all abelian sheaves and the class of quasicoherent sheaves in the sense that not all local cohomology sheaves are part of the class. An option would be to enlarge $\mathcal{K}$ with these local cohomology sheaves, but then we loose the property that all contractible subsets are simple. For this reason we added the third condition in the definition of a stratification.

Stratification 3. $Z_0 = \{p_{0,0}, p_{0,1}, p_{1,0}, p_{1,1}, p_{2,0}\}$, $Z_1 = \{p_{2,1}\}$.
Again, because of its length, this can not be an $\mathcal{A}$-stratification or a $\mathcal{K}$-stratification. However Z_0 and Z_1 are open hulls and the third condition is also satisfied because Z_1 is a one-point space. The problem here is that $p_{2,1}$ does not have a simple open neighborhood basis $\mathcal{B}$ in $\overline{Z_0}$ such that for any $V \in \mathcal{B}$ the intersection $Z_0 \cap V$ is simple. The open hull U of $p_{2,1}$ intersected with Z_0 gives $\mathbb{S}^1$ and $\mathbb{S}^1$ is not simple because $H^1(\mathbb{S}^1, \mathbb{Z}) = \mathbb{Z}$.

Stratification 4. $Z_0 = \{p_{0,0}, p_{0,1}, p_{1,0}\}$, $Z_1 = \{p_{1,1}\}$, $Z_2 = \{p_{2,0}, p_{2,1}\}$.
The first two conditions are easily checked for both $\mathcal{A}$ and $\mathcal{K}$. For the third condition, note that we have the same Z_0 is in Stratification 2. This means that we also get the same local cohomology sheaves $\mathcal{H}^1_{Z_0^c}(\mathcal{F})$ on Z_0^c. Now because $cdim_{\mathcal{A}}(Z_0^c) = 1$ and $cdim_{\mathcal{K}}(Z_0^c) = 0$, we do have an $\mathcal{A}$-stratification here but not a $\mathcal{K}$-stratification.

We can conclude from this example that an $\mathcal{S}$-stratification for a space X need not be an $\mathcal{S}'$-stratification, even when the class of sheaves $\mathcal{S}$ is bigger than the class of sheaves $\mathcal{S}'$.

Stratification 5. $Z_0 = \{p_{0,0}, p_{0,1}\}$, $Z_1 = \{p_{1,0}, p_{1,1}, p_{2,0}\}$, $Z_2 = \{p_{2,1}\}$.
This one is both an $\mathcal{A}$-stratification and a $\mathcal{K}$-stratification. The details are left to the reader.

6 Topological manifolds

Another class of spaces where $\mathcal{K}$-stratifications ($\mathcal{K}$ again refering to the class of constant sheaves) can be found is the class of topological manifolds, along with their closed subsets. In this class all spaces are locally contractible, so there will always exist $\mathcal{K}$-simple neighborhoodbases. One should be careful though that intersections of contractible subsets need not be contractible or $\mathcal{K}$-simple.

For topological manifolds, there is already a different concept of stratifications called CW-complexes (See [10] section 4). The CW-complexes are comparable to our $\mathcal{K}$-stratifications in the sense that they are also built up by dense open $\mathcal{K}$-simple subsets. Another notion of stratification that we will study in this chapter and that is stronger than a CW-complex is the following.

Definition 6.1 (Section 3.1 in [13])**.** *A* simplicial complex *K consists of a set V of* vertices *and a set S of finite non-empty subsets of V called* simplexes *such that:*

1. *Any set consisting of exactly one vertex is a simplex.*
2. *Any non-empty subset of a simplex is a simplex.*

For a simplicial complex $K = (V, S)$, the topological space $|K|$ is the union of the convex hulls of the simplexes in S in the real vector space with basis V.

For K with non-empty V, the dimension of K, which we will refer to as Euclidean dimension*, is the maximal cardinality of the simplices minus one.*

6.1 Stratifications for the spheres

In this section we will give a $\mathcal{K}$-stratification for the real sphere $S^n = \{(a_0, \dots, a_n) \in \mathbb{R}^{n+1} : \sum a_i^2 = 1\}$. For $m < n$, we will identify subspheres $S^m \subseteq S^n$ by letting the last $n - m$ coordinates of the points in S^n be equal to zero.

Define $Z_0 = S^n \setminus S^{n-1}$, $Z_1 = S^{n-1} \setminus S^{n-2}$, etcetera up to $Z_n = S^0$. Then condition one of a stratification is satisfied. Condition two is also not hard to see. The space $S^n \setminus S^{n-1}$ is just a disjoint union of two open subsets, each homeomorphic to an open, contractible subset of $\mathbb{R}^n$. So it is $\mathcal{K}$-simple. For any point $p \in S^{n-1}$ with contractible open neighborhood U, the intersection $U \cap (S^n \setminus S^{n-1})$ is also a disjoint union of two open subsets, each homeomorphic to an open, contractible subset of $\mathbb{R}^n$.

For the third condition, as in the previous chapter we get $\Gamma_{Z_i^c}(G) = 0$ and we have the exact sequence $0 \to \mathcal{G} \to i_*(\mathcal{G}|_{Z_i}) \to \mathcal{H}^1_{Z_i^c}(\mathcal{G}) \to 0$ of sheaves on $\overline{Z_i}$. Again we will restrict the sequence to Z_i^c. Let V be open in Z_i^c and assume that V is connected. Then any V inducing open subset U of $\overline{Z_i}$ contains a V inducing open subset U' that is connected. Because $i_*(\mathcal{G}|_{Z_i})(U') = G \oplus G$, we can conclude that the sheaf $i_*(\mathcal{G}|_{Z_i})$ when restricted to Z_i^c is constant with coefficients in $G \oplus G$. The third condition now follows from the long exact sequence of the short exact sequence $0 \to \mathcal{G} \to i_*(\mathcal{G}|_{Z_i}) \to \mathcal{H}^1_{Z_i^c}(\mathcal{G}) \to 0$ restricted to Z_i^c.

Remark 6.2. For any simplicial complex K of Euclidean dimension n we can make a subdivision $Z_0, \dots, Z_n$ for $|K|$ by defining Z_0 to be the union of the interiors of the maximal cells, Z_1 to be the union of the interiors of the maximal cells of $|K| - Z_0$, etc. It is not hard to see that this subdivision satisfies the first two conditions of a $\mathcal{K}$-stratification. However, I was not able to prove the third property, or give a counterexample.

6.2 A vanishing theorem for simplicial complexes

In this section we use Proposition 4.14 to prove a dimensional vanishing theorem for simplicial complexes. We will use the article [2] by Michael C. McCord. In this article, McCord describes a correspondence between simplicial complexes and T_0 Alexandrov spaces. Before giving the theorem we first recall some terminology.

The T_0 separation axiom reads: for each pair of distinct points, there exists an open set containing exactly one of those points. Note that for any space X, the space $t(X)$ is a T_0 space.

A locally finite space is a space in which every point has a finite neighborhood. On locally finite spaces each point has an open hull. Any intersection of open sets therefore equals a union of open hulls. So a locally finite space is an Alexandrov space.

A weak homotopy equivalence is a map $f : X \to Y$ for which all the induced maps $f_* : \pi_i(X, x) \to \pi_i(Y, f(x))$ are isomorphisms. By a theorem of J.H.C. Whitehead (see [6] pp. 167), a weak homotopy equivalence induces isomorphisms on singular homology groups and hence on singular cohomology groups by the universal coefficient theorem for cohomology.

Theorem 6.3 (Theorem 3 in [2])**.** *There exists a correspondence that assigns to each simplicial complex K a locally finite T_0 space $\mathcal{X}(K)$ whose points are the barycenters of simplexes of K, and a weak homotopy equivalence $f_K : |K| \to \mathcal{X}(K)$. Furthermore, to each simplicial map $\psi : K \to L$ is assigned a map $\psi' : \mathcal{X}(K) \to \mathcal{X}(L)$ such that $\psi' f_K = f_L|\psi|$.*

In his article, McCord also describes the topology for the locally finite T_0 space $\mathcal{X}(K)$. This topology is defined by the ordering on the barycenters induced by the reversed inclusion ordering on the cells of K. To be precise, if C_1 and C_2 are cells of K with barycenters b_1 and b_2, then $b_1 \leq b_2$ in $\mathcal{X}(K)$ if and only if $C_1 \supseteq C_2$. The reversed relation is not in Mc Cord, but this is due to the fact that he defined his relation on an Alexandrov space by $x \leq y$ if $U_x \subseteq U_y$ instead of $x \leq y$ if $\overline{\{x\}} \subseteq \overline{\{y\}}$. The next lemma follows immediately.

Lemma 6.4. *Let K be a simplicial complex. Then the Euclidean dimension of K equals the dimension of $\mathcal{X}(K)$.*

Example 6.5. Suppose we start with a simplicial complex K given by an n-cell and all its faces and we give the locally finite T_0 space $\mathcal{X}(K)$ the stratification $(Z_i)_{i=0}^n$ as in the previous section. Then the zero stratum Z_0 will contain $n+1$ points, the vertices of K. The one stratum Z_1 will contain $\binom{n+1}{2}$ points, the barycenters of the edges of K. In general, the Z_i stratum will contain $\binom{n+1}{i+1}$ points.

Another theorem we need is the following.

Theorem 6.6 (Theorem 4.47 in [14])**.** *Let X be a locally contractible space. Then for any abelian group G we have a canonical isomorphism*

$$H^q_{sing}(X, G) \cong H^q(X, \mathcal{G}).$$

Here $\mathcal{G}$ denotes the constant sheaf with coefficients in G, and $H^q_{sing}(X, G)$ denotes the q-th singular cohomology group with coefficients in G.

The following lemma shows that Alexandrov spaces are locally contractible.

Lemma 6.7. *Let X be an Alexandrov space. Then for any $p \in X$, the open hull U_p is contractible.*

Proof. We can define a homotopy $\pi : U_p \times [0,1] \to U_p$ by $\pi(x,t) = x$ if $t < 1$ and $\pi(x,t) = p$ if $t = 1$. To show that π is continuous, we take an open subset $U \subseteq X$. If $p \notin U$ then $\pi^{-1}(U) = U \times [0,1)$ is open in $X \times [0,1]$. If $p \in U$ then $U = X$ and $\pi^{-1}(U) = X \times [0,1]$ is also open. So π is a continuous deformation from U_p to the constant space $\{p\}$. Hence U_p is contractible. □

So we can use Theorem 6.3 and our stratifications for Alexandrov spaces to bound the cohomology of simplicial complexes. This leads to another prove for the following standard result on simplicial complexes.

Theorem 6.8. *Let K be a simplicial complex of Euclidean dimension n and G an abelian group. Then*

$$H^i(|K|, G) = 0 \text{ for } i > n.$$

Proof. By Theorem 6.3 there is a locally finite T_0 space $\mathcal{X}(K)$ of dimension n (Lemma 6.4) with isomorphic singular cohomology groups. Because both $|K|$ and $\mathcal{X}(K)$ are locally contractible, their singular cohomology groups with coefficients in G equal their cohomology groups for the constant sheaf G by Theorem 6.6. Now the dimension bound on $\mathcal{X}(K)$ from Proposition 4.14 carries over to $|K|$. □

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