Symbolic dynamics of β-expansions in negative base

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Preface

Life isn't always fun and games. Remarkably, this saying applies to this thesis in more than one way.

One way to explain this requires you to go a bit behind the scenes. During my master's (and bachelor's as well) I have always been more focused on practical subjects. Subjects that allow you to quickly think of an example and see how it works out. The subject of this thesis fits this picture as well, as a large part of this thesis has been quite a puzzle to solve.

However, just as the saying goes, a collection of examples does not make a master's thesis. The thesis discusses a lot of theory and, as such, we will see many theorems and proofs come and go. A special mention goes to section 3.2, if you can "survive" this section as a whole then the rest of this thesis will be smooth sailing.

Fortunately, even readers that are less proficient but still interested in math will find much interesting material. To use another cliche, 'follow your intuition', as in this thesis theoretical parts get followed by examples where we rid ourselves (temporarily of course) of as much of the theory as possible. And if you are just like me, you might try a few more examples on the spot. To do so, you will need basic arithmetic skills and the ability to grasp the main ideas of this thesis (once again, the intuition that will play a subtle role in the examples). Balancing the cold yet necessary formalities with the more fun yet fuzzier area of intuitive (at least they should be) examples has been quite a challenge, but I think I did a decent job.

The second problem is that neither β-expansions nor symbolic dynamics are in any standard mathematics curriculum. To overcome this problem, the first section of each chapter is an introductory chapter. You are encouraged to first skim over this section, and then read it as a whole if you find many terms of which you have little knowledge.

I hope you will enjoy reading this thesis, as much as I did during the whole project.

Contents

1

Dynamics of β-expansions

1.1 Introduction to β-expansions

In our decimal numeral system each number in $[0,1)$ has at most two different decimal expansions. For example: $\frac{1}{3} = 0.\overline{3} = 0.33333\cdots$ and $\frac{1}{2} = 0.5$ (but also $\frac{1}{2} = 0.4\overline{9} = 0.49999\cdots$). The notation we use here is in fact shorthand for an infinite sum. For example:

$$\frac{1}{2} = \frac{5}{10},$$

but also

$$\frac{1}{2} = \frac{4}{10} + \sum_{n=2}^{\infty} \frac{9}{10^n}.$$

In 1957, β-expansions were introduced by Rényi in [Ren57]. It is a generalization of the decimal expansion. Let $\beta > 1$ and $x \in [0,1)$ be given then, if we can write

$$x = \sum_{n=1}^{\infty} \frac{x_n}{\beta^n},$$

where x_n is a nonnegative integer not smaller than β, then we will call the series on the right-hand side the β-expansion of x. Instead of writing down the whole series, it suffices to just write down the sequence $(x_1, x_2, \ldots)$ and the base β. Some may write $x = (.x_1x_2\ldots)_\beta$. For $\beta = 10$, we get the usual decimal expansion and we simply write $x = .x_1x_2\ldots$

One particular kind of β-expansion has been researched thoroughly by Parry in [Par60]. The first few digits x_n were defined as

$$\begin{aligned} x_1 &= \lfloor \beta x \rfloor, \\ x_2 &= \lfloor \beta\{\beta x\} \rfloor, \\ x_3 &= \lfloor \beta\{\beta\{\beta x\}\} \rfloor, \end{aligned}$$

where $\{a\}$ is the fractional part of a, i.e., $\{a\} = a - \lfloor a \rfloor$. The digits x_n for $n \geq 4$ were defined similarly to x_1, x_2 and x_3, by following the pattern of these definitions.

Since it was known back then that each number in $[0, 1)$ could have more than one β-expansion, a natural question arose quickly. If $\beta > 1$ is fixed, can we see from any given β-expansion if it is of the kind Parry researched back in 1960?

Example 1.1.1. As an example, let $\beta = \frac{1+\sqrt{5}}{2}$ (the golden mean). We list a few examples of β-expansions:

$$\begin{aligned} 0 &= (.\overline{0})_\beta = (.000000\cdots)_\beta, \\ \beta - 1 &= (.1\overline{0})_\beta = (.100000\cdots)_\beta, \\ 1/2 &= (.\overline{010})_\beta = (.010010\cdots)_\beta. \end{aligned}$$

These β-expansions are of the kind that were originally studied. However, $\beta - 1$ has other β-expansions, for example:

$$\beta - 1 = (.0100\overline{1})_\beta = (.010011\cdots)_\beta.$$

Can you see why this latter is not a Parry-type β-expansion? If so, can you say whether or not

$$(.1001\overline{10})_\beta = (.100110\cdots)_\beta$$

is also of the Parry-type without doing any sort of calculation? □

Since Parry's article got published there has been quite a lot of research done on β-expansions in general and the question posed above got solved. In this thesis we will try to solve this question as well, but for β-expansions with negative base. We will explain this term in the next section.

1.2 Dynamical systems

This thesis is about β-expansions in negative base. They were introduced recently, by Ito and Sadahiro, in their article [IS09]. In this section we will define these expansions and see that we can associate them with a dynamical system.

Before we can start, we need a numeral system. In this thesis, we will make the following assumption:

Assumption 1.2.1. The bases of the numeral systems to be considered will be written as $-\beta$, with $\beta > 1$.

Besides a base, any numeral system also need digits in which to express numbers. These digits together are called the alphabet of the system. While we can use any alphabet as long as its digits are all non-negative and less than β, we will make the following assumption:

Assumption 1.2.2. In this thesis, for given β we only consider the alphabet $\mathcal{A} := \{0, 1, \ldots, \lfloor\beta\rfloor\}$. This alphabet is called the canonical alphabet.

We will now define the β-expansions in negative base:

Definition 1.2.3. Let x be a real number and β be given. If there exists a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in \mathcal{A}$ for all $n \geq 1$ such that

$$x = \sum_{n=1}^{\infty} \frac{x_n}{(-\beta)^n} \tag{1.1}$$

holds, then the series on the right-hand side is called a $(-\beta)$-expansion of x. We may also write $x = (.x_1x_2\ldots)_{-\beta}$.

Definition 1.2.4. Let x be a number with a $(-\beta)$-expansion as in (1.1). Then we define $d_{-\beta}(x) := (x_1, x_2, \ldots)$ and $d_{-\beta}(x, n) := x_n$.

It is clear from Definition 1.2.3 that not every number has a $(-\beta)$-expansion. However, we can determine the lowest and highest number with a $(-\beta)$-expansion easily since there are geometric series hidden in their $(-\beta)$-expansions:

$$\sum_{k=1}^{\infty} \frac{\lfloor\beta\rfloor}{(-\beta)^{2k-1}} = \frac{\lfloor\beta\rfloor}{-\beta} \cdot \sum_{k=0}^{\infty} \frac{1}{\beta^{2k}} = -\frac{\lfloor\beta\rfloor \cdot \beta}{\beta^2 - 1} \tag{1.2}$$

and

$$\sum_{k=1}^{\infty} \frac{\lfloor\beta\rfloor}{(-\beta)^{2k}} = \lfloor\beta\rfloor \cdot \sum_{k=1}^{\infty} \frac{1}{\beta^{2k}} = \frac{\lfloor\beta\rfloor}{\beta^2 - 1}. \tag{1.3}$$

In view of equations (1.2) and (1.3), we define the following:

Definition 1.2.5. Let β be given. We define

$$\ell_{-\beta} := -\frac{\beta \cdot \lfloor \beta \rfloor}{\beta^2 - 1} \quad \text{and} \quad r_{-\beta} := \frac{\lfloor \beta \rfloor}{\beta^2 - 1}.$$

We also define $I_{-\beta} := [\ell_{-\beta}, r_{-\beta}]$. This interval includes all numbers that have a $(-\beta)$-expansion.

Given any number $x \in I_{-\beta}$, how would one go about finding a $(-\beta)$-expansion of this number? Suppose for now that x has a $(-\beta)$-expansion, so that

$$x = \sum_{n=1}^{\infty} \frac{x_n}{(-\beta)^n}.$$

Then we also have

$$-\beta x - x_1 = \sum_{n=1}^{\infty} \frac{x_{n+1}}{(-\beta)^n}.$$

In particular, we have $-\beta x - x_1 \in I_{-\beta}$. By inductively applying this argument we can also find the other digits $x_2, x_3, \ldots$ Hence, it seems worthwhile to investigate the maps $-\beta x - a$ for all $a \in \mathcal{A}$. Below you can see a plot of the maps for one specific choice for β:

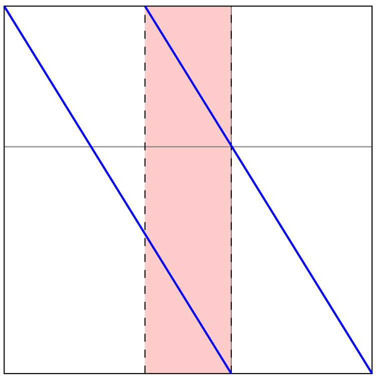


Figure 1.1: The maps $-\beta x - a$ with $a \in \mathcal{A}$ on $I_{-\beta}$ for $\beta = \frac{1+\sqrt{5}}{2}$.

As can be seen in Figure 1.1, the interval $I_{-\beta}$ can be divided in three subintervals for $\beta = \frac{1+\sqrt{5}}{2}$. If $x \in I_{-\beta}$ belongs to the leftmost subinterval, then we have $-\beta x - 1 \in I_{-\beta}$. Likewise, if $x \in I_{-\beta}$ belongs to the rightmost subinterval, then we have $-\beta x \in I_{-\beta}$. The middle subinterval (lightly shaded red in Figure 1.1) is more interesting: for any x in this subinterval we have $-\beta x \in I_{-\beta}$ and $-\beta x - 1 \in I_{-\beta}$. This subinterval will be called a choice region.

Our next step is to define a function $T_{-\beta}$ on $I_{-\beta}$ which turns the above process into a dynamical system on $I_{-\beta}$. Note that this function is of the form $T_{-\beta} = -\beta x - d(x)$, with $d : I_{-\beta} \to \mathcal{A}$ a function. This function will be called the digit function. In order to not make things needlessly complicated, we will make the following assumption on d:

Assumption 1.2.6. Let β be given. The digit function d satisfies the following properties (with $\mathcal{A}$ viewed as subset of $\mathbb{R}$):

- d has $\lfloor \beta \rfloor$ discontinuities,
- d is either left-continuous everywhere or right-continuous everywhere,
- d is nonincreasing on $I_{-\beta}$,
- $d(x)$ is defined such that $x \in I_{-\beta}$ implies $-\beta x - d(x) \in I_{-\beta}$.

Definition 1.2.7. The points at which the digit function is discontinuous are called cut points.

Henceforth, whenever we talk about $T_{-\beta}$ we assume that both β and the cut points of the map are fixed, unless specified otherwise.

We will show that we can indeed find a $(-\beta)$-expansion using these maps. In order to do so, we will first prove the following lemma:

Lemma 1.2.8. For any $x \in I_{-\beta}$, there is at least one $a \in \mathcal{A}$ such that $-\beta x - a \in I_{-\beta}$.

Proof. If $0 \leq x \leq r_{-\beta}$, then $\ell_{-\beta} \leq -\beta x \leq 0$, so 0 suffices for those numbers.

For $\ell_{-\beta} \leq x < 0$, define $S(x) := \{-\beta x - a : \ a \in \mathcal{A}\}$. The smallest number in $S(x)$ cannot be greater than $r_{-\beta}$ (with equality for $x = \ell_{-\beta}$). Similarly, the largest number in $S(x)$ cannot be smaller than 0. It follows that the set $S(x)$ cannot contain numbers which are all greater than $r_{-\beta}$ or all smaller than $\ell_{-\beta}$. Since the distance between two consecutive points in $S(x)$ is 1, it suffices to show that the length of $I_{-\beta}$ cannot be smaller than 1. To this end, notice that the length of $I_{-\beta}$ is given by

$$r_{-\beta} - \ell_{-\beta} \;=\; \frac{\lfloor\beta\rfloor}{\beta^2-1} - \frac{-\beta\cdot\lfloor\beta\rfloor}{\beta^2-1} \;=\; \frac{\lfloor\beta\rfloor}{\beta-1}.$$

Since $\lfloor\beta\rfloor > \beta - 1$, it follows that the length of $I_{-\beta}$ is greater than 1 for any β. Hence, for any $\ell_{-\beta} \leq x < 0$ and any β it follows that $S(x) \cap I_{-\beta} \neq \emptyset$. The lemma has now been proven. □

We now turn to the main theorem of this section:

Theorem 1.2.9. Let $x \in I_{-\beta}$ be arbitrary and let $x_n := d\left(T^{n-1}_{-\beta}(x)\right)$. Then $x = (.x_1x_2\ldots)_{-\beta}$.

Proof. Existence of x_n for all $n \geq 1$ is guaranteed by the previous lemma. We first prove, by induction on n, that $x = \frac{T^n_{-\beta}(x)}{(-\beta)^n} + \sum_{i=1}^n \frac{x_i}{(-\beta)^i}$ holds for all $n \geq 1$.

- Recall that $T_{-\beta}(x) = -\beta x - d(x) = -\beta x - x_1$. This is equivalent with $x = \frac{T_{-\beta}(x)}{-\beta} + \frac{x_1}{-\beta}$. Hence, the statement is true for $n = 1$.
- Now suppose that the statement is true for $n = k - 1$. Since $T^k_{-\beta}(x) = -\beta T^{k-1}_{-\beta}(x) - d\left(T^{k-1}_{-\beta}(x)\right)$ holds, it follows that

$$T^{k-1}_{-\beta}(x) \;=\; \frac{x_k}{-\beta} + \frac{T^k_{-\beta}(x)}{-\beta}$$

  also holds. By the induction hypothesis, we have:

$$\begin{aligned} x &= \frac{T^{k-1}_{-\beta}(x)}{(-\beta)^{k-1}} + \sum_{i=1}^{k} \frac{x_i}{(-\beta)^i} \\ &= \frac{\frac{x_k}{-\beta} + \frac{T^k_{-\beta}(x)}{-\beta}}{(-\beta)^{k-1}} + \sum_{i=1}^{k} \frac{x_i}{(-\beta)^i} \\ &= \frac{T^k_{-\beta}(x)}{(-\beta)^k} + \sum_{i=1}^{k} \frac{x_i}{(-\beta)^i}. \end{aligned}$$

  Hence, the statement also holds for $n = k$.

To complete the proof, note that $\ell_{-\beta} \leq T^n_{-\beta}(x) \leq r_{-\beta}$. It follows that $\frac{T^n_{-\beta}(x)}{(-\beta)^n} \to 0$ as $n \to 0$, which in turn implies $x = \sum_{k=1}^{\infty} \frac{x_k}{(-\beta)^k} = (.x_1x_2\ldots)_{-\beta}$. □

We conclude this chapter with an example.

Example 1.2.10. In this example we will use $\beta = \frac{1+\sqrt{5}}{2}$ (the golden mean). As can be seen in Figure 1.1, there is a choice region and therefore we need to define the digit function d first. In this example we define d as follows:

$$d(x) \quad := \quad \begin{cases} 0 & \text{if } x \geq 0, \\ 1 & \text{if } x < 0. \end{cases}$$

And hence the map $T_{-\beta}$ will look as follows on $I_{-\beta}$:

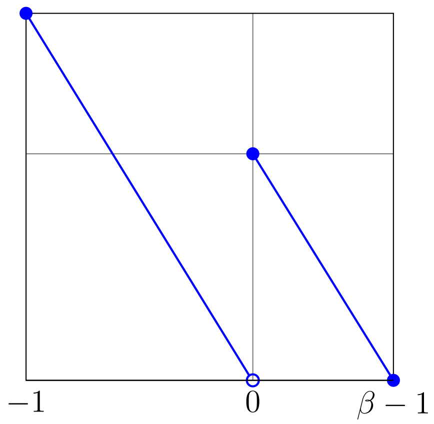


Figure 1.2: The map $T_{-\beta}$ corresponding to Example 1.2.10.

We will demonstrate how to use this map to find the $(-\beta)$-expansion of $-\frac{1}{2}$. The orbit of $-\frac{1}{2}$ under $T_{-\beta}$ is:

$$-\frac{1}{2} \;\to\; \frac{\sqrt{5}-3}{4} \;\to\; \frac{\sqrt{5}-5}{4} \;\to\; \frac{\sqrt{5}-2}{2} \;\to\; \frac{\sqrt{5}-3}{4} \;\to\; \cdots$$

Moreover, we have:

$$d\left(\frac{\sqrt{5}-2}{2}\right) \;=\; 0$$

and

$$d\left(-\frac{1}{2}\right) \;=\; d\left(\frac{\sqrt{5}-3}{4}\right) \;=\; d\left(\frac{\sqrt{5}-5}{4}\right) \;=\; 1.$$

By Theorem 1.2.9, we have $-\frac{1}{2} = (.1110110110110\ldots)_{-\beta}$. □

2
Shift work

2.1 A primer on symbolic dynamics (part 1)

In this section we will introduce certain concepts of symbolic dynamics that will be used in this chapter. For a more thorough introduction, see [LM95].

Let A be a fixed finite set. We will call A the alphabet. In symbolic dynamics the main objects of study are collections of infinite sequences of symbols from our alphabet A. We will only look at one-sided infinite sequences, i.e., sequences from $A^{\mathbb{N}}$, but in literature one may also find $A^{\mathbb{Z}}$.

Consider any sequence $a = (a_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$. The index n can be thought of as indicating time in the sense that it is determined by the index of the first symbol. So, the initial state of a would be $a_1a_2a_3\cdots$ and one time step later it would be $a_2a_3a_4\cdots$ and so on. Hence, the passage of time corresponds to shifting the sequence to the right.

Definition 2.1.1. The shift map $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is the map satisfying $y = \sigma(x)$, where $y_i = x_{i+1}$ for all $i \geq 1$.

Of particular interest are sequences that (almost) repeat itself:

Definition 2.1.2. A sequence $a \in A^{\mathbb{N}}$ is called purely periodic if there exists some $k \geq 1$ for which $\sigma^k(a) = a$. A sequence is called eventually periodic if there exist $i, k \geq 1$ such that $\sigma^{i+k}(a) = \sigma^i(a)$.

The periodic parts of a sequence are marked as such by overlining them. For example, $\overline{01} = 010101010101\cdots$ and the $(-\beta)$-expansion of $-\frac{1}{2}$ as found in Example 1.2.10 can be written as $(1\overline{110})_{-\beta}$.

We would like to have some way to express that sequences on $A^{\mathbb{N}}$ that agree on the first few symbols are "closer to each other" than sequences that do not agree on their first few symbols. To this end, we define the following metric on $A^{\mathbb{N}}$:

$$\rho(x, y) \quad = \quad \inf\left(\left\{2^{-k} : \; k \geq 0, \, x_i = y_i \text{ for all } i \leq k\right\}\right).$$

We can now define the objects of study in symbolic dynamics.

Definition 2.1.3. A shift space is a non-empty subset $X \subseteq A^{\mathbb{N}}$ that is closed with respect to the metric ρ and closed with respect to the shift map σ.

The pair (X, σ) can (and should) be thought of as a discrete-time dynamical system.

Describing shift spaces

There is more than one way to create a shift space. We will discuss two of these methods. Before we continue, we need the following definitions:

Definition 2.1.4. A block of length n (also called an n-block) is a sequence of n symbols, with each symbol belonging to A.

Definition 2.1.5. Let X be a shift space and $a_1 \cdots a_n$ an n-block ($n \geq 1$). If there exists some $x \in X$ such that the n-block $a_1 \cdots a_n$ appears somewhere in x, then $a_1 \cdots a_n$ is called a word.

Definition 2.1.6. Let X be a shift space and $W_n(X)$ the set of all words of length n in X. Then we call $L(X) := \cup_{n=0}^{\infty} W_n(X)$ the language of the shift space.

The following theorem gives us the first way of constructing shift spaces:

Theorem 2.1.7. A subset $X \subseteq A^{\mathbb{N}}$ is a shift space if and only if there exists a set F of finite blocks such that X is the set of all sequences that do not contain any of the blocks in F.

Proof. The proof is split into two parts.

$\Rightarrow$:
Let X be a shift space and suppose that we cannot find a set F satisfying the condition given in the theorem. Let G be a set of blocks such that X contains all sequences that do not contain any of the blocks in G. Then G contains at least one block of infinite length, say $a = a_1 a_2 a_3 \cdots$, and we may assume without loss of generality that each subblock of a is not contained in G. Let $x \in X$ be arbitrary (write $x = x_1 x_2 x_3 \cdots$) and consider the sequence $(y^n)_{n=1}^{\infty}$, defined as:

$$y_i^n \quad := \quad \begin{cases} a_i & \text{for } i \leq n, \\ x_i & \text{for } i > n. \end{cases}$$

The sequence y^n is in X and converges to $a \notin X$. Hence, X is not closed. However, this contradicts the fact that X is a shift space. Therefore, our initial assumption was wrong. We conclude that there must exist a set F as described in the theorem.

$\Leftarrow$:
Suppose that $X \subseteq A^{\mathbb{N}}$ is a subset for which there exists a set F of finite blocks such that X contains all sequences that do not have any finite block of F as a word. Clearly, X is closed with respect to the shift map. We need to show that X is also metrically closed. Suppose that it is not metrically closed. Then there exists some $y \notin X$ and a sequence $(x^n)_{n=1}^{\infty}$ in X such that $x^n \to y$. It follows that there exist $k, m \geq 1$ such that the block $y_k y_{k+1} \cdots y_{k+m} \in F$. However, since convergence with respect to the metric ρ means pointwise convergence, it follows that there is some $p \geq 1$ such that $y_k y_{k+1} \cdots y_{k+m}$ appears in the sequences x^n for all $n \geq p$. This is a contradiction, however, since $x^n \in X$. It follows that X must be metrically closed and hence it is a shift space. □

The blocks in the set F of this theorem are also called forbidden blocks. Hence, one can describe a shift space by giving its forbidden blocks. Note that F may also be empty, as well as an infinite set.

The second way of constructing shift spaces is by (in)directly describing the language of the shift space we want to have. The problem with this is that you cannot expect the resulting space to be a shift space unless the language satisfies certain conditions [LM95, Proposition 1.3.4].

Example 2.1.8. Let $A = \{0, 1\}$ and let X be the space of all sequences that end with an infinite sequence of zeroes. As we cannot directly find a set of forbidden blocks, we might wonder if X might in fact not be a shift space. As a matter of fact, it is quite easy to show that X is not a shift space. Let $a \in A^{\mathbb{N}}$ be arbitrary but fixed. Then define, for $n \geq 1$:

$$x_i^n := \begin{cases} a_i & \text{for } i \leq n, \\ 0 & \text{for } i > n. \end{cases}$$

We see that $(x^n)_{n=1}^{\infty}$ is a sequence in X converging to a. Since a can be arbitrary, it follows that $\overline{X} = A^{\mathbb{N}}$, and hence X is indeed not a shift space. □

The process in the example above is typical for this case: describe the properties of the sequences you would like to study and then consider the closure of this space. It will be a shift space (as long as the described properties are shift-invariant). Of course, we will have to accept the fact that our shift space contains more than the sequences we want to study.

2.2 Alternate lexicographical order

In the previous chapter we developed a dynamical system with which we can obtain another representation of numbers, the $(-\beta)$-expansion. However, $\mathbb{R}$ has an important property which gets lots when translated to sequences of digits: the ordering.

Reconsider Example 1.2.10, where we used the golden mean as the base of the system. The digit function assigns a 1 to negative numbers (and 0 to everything else). We saw that $-\frac{1}{2} = (1\overline{110})_{-\beta}$. By the same procedure, we can determine $0 = (\overline{0})_{-\beta}$. We know that $-\frac{1}{2} < 0$, but this does not seem so apparent when we compare their $(-\beta)$-expansions. Moreover, a number may have more than one $(-\beta)$-expansion. We need to take this into account when we try to find an order on the set of sequences.

Before we continue, we remind ourselves of the following assumption:

Assumption 2.2.1. Whenever we refer to the map $T_{-\beta}$, we assume that β and d are already given. If we wish to explicitly mention whether the map is left-continuous, we use the notation $L_{-\beta}$. Similarly, $R_{-\beta}$ is the right-continuous map.

We want an order on the set of $(-\beta)$-expansions which respects the real ordering. To that end, note that:

$$d_{-\beta}(\ell_{-\beta}) \;=\; (\overline{\lfloor\beta\rfloor, 0}) \quad \text{and} \quad d_{-\beta}(r_{-\beta}) \;=\; (\overline{0, \lfloor\beta\rfloor}).$$

We see that the numbers with even index contribute positively while numbers with odd index contribute negatively (which is not really surprising; odd powers of negative numbers are still negative and even powers of negative numbers are positive). Another thing to note is that both $\ell_{-\beta}$ and $r_{-\beta}$ use the largest nonzero digit in $\mathcal{A}$. These two observations might suggest a lexicographical order which reverses itself at odd indices.

Definition 2.2.2. Let $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ be two sequences in $\mathcal{A}^{\mathbb{N}}$. We write $(x_1, x_2, \ldots) \prec (y_1, y_2, \ldots)$ if and only if there exists an integer $k \geq 1$ such that $x_i = y_i$ for all $i < k$ and $(-1)^k(x_k - y_k) < 0$. We also write $(x_1, x_2, \ldots) \preceq (y_1, y_2, \ldots)$ if and only if either $(x_1, x_2, \ldots) \prec (y_1, y_2, \ldots)$ or $(x_1, x_2, \ldots) = (y_1, y_2, \ldots)$ holds. Similar statements hold for $\succ$ and $\succeq$. We call $\prec$ the alternate lexicographical order.

Example 2.2.3. Revisiting the example at the beginning of this section, we see that $(1, \overline{1, 1, 0}) \prec (\overline{0})$. This is indeed the behaviour we want, since it respects the order on $\mathbb{R}$ (it corresponds to the statement $-\frac{1}{2} < 0$). □

Example 2.2.4. Let $\beta = \frac{1+\sqrt{5}}{2}$. We have $(1,\overline{1,0}) \prec (\overline{0})$, while both sequences correspond to $(-\beta)$-expansions of the same number, namely 0. $\square$

Example 2.2.5. Once again, let $\beta = \frac{1+\sqrt{5}}{2}$. Now, consider the sequences $(1,\overline{1,0})$ and $(\overline{0,0,1,0,0,0})$. While we have $(1,\overline{1,0}) \prec (\overline{0,0,1,0,0,0})$, we also have $(.\overline{001000})_{-\beta} = -\frac{1}{4} < 0 = (.1\overline{10})_{-\beta}$. $\square$

The previous examples show how the alternate lexicographical order works. Of particular interest are the latter two examples, which show that this order does not respect the order $<$ on $\mathbb{R}$. However, it does work the way we intend to if we restrict our view a little.

Theorem 2.2.6. Let $T_{-\beta}$ be given, $x, y \in I_{-\beta}$ with $x \neq y$ and $d_{-\beta}(x)$ and $d_{-\beta}(y)$ be the sequences that follow from Theorem 1.2.9. Then $x < y$ if and only if $d_{-\beta}(x) \prec d_{-\beta}(y)$.

Proof. Write $d_{-\beta}(x) = (x_1, x_2, \ldots)$ and $d_{-\beta}(y) = (y_1, y_2, \ldots)$. Let $n \geq 1$ be the smallest integer such that $x_n \neq y_n$. Then we have

$$x \;=\; \frac{T_{-\beta}^{n-1}(x)}{(-\beta)^{n-1}} + \sum_{k=1}^{n-1} \frac{x_k}{(-\beta)^k} \;<\; \frac{T_{-\beta}^{n-1}(y)}{(-\beta)^{n-1}} + \sum_{k=1}^{n-1} \frac{y_k}{(-\beta)^k} \;=\; y$$

if and only if

$$\frac{T_{-\beta}^{n-1}(x)}{(-\beta)^{n-1}} \;<\; \frac{T_{-\beta}^{n-1}(y)}{(-\beta)^{n-1}}. \tag{2.1}$$

If n is odd, equation (2.1) becomes $T_{-\beta}^{n-1}(x) < T_{-\beta}^{n-1}(y)$ which is, by assumption on n, equivalent with $x_n > y_n$. This in turn is equivalent with $(-1)^n(x_n - y_n) < 0$ and thus equivalent with $d_{-\beta}(x) \prec d_{-\beta}(y)$.
If n is even, equation (2.1) becomes $T_{-\beta}^{n-1}(x) > T_{-\beta}^{n-1}(y)$ which is, once again by assumption on n, equivalent with $x_n < y_n$. This is equivalent with $(-1)^n(x_n - y_n) < 0$ and thus equivalent with $d_{-\beta}(x) \prec d_{-\beta}(y)$.
In short, it follows that $x < y$ is equivalent with $d_{-\beta}(x) \prec d_{-\beta}(y)$. $\square$

Hence the alternate lexicographical order respect the order on $\mathbb{R}$ as long as the sequences involved can be generated by Theorem 1.2.9. Therefore it is interesting to study which sequences can be generated by a given map $T_{-\beta}$. This is basically what we will attempt to do in the remainder of this thesis. Our first step is, for a given map $T_{-\beta}$, constructing the smallest shift space containing all sequences that can be generated by this map.

Definition 2.2.7. An integer sequence $(x_1, x_2, \ldots)$ with $x_i \in A$ is $T_{-\beta}$-admissible if there exists an $x \in I_{-\beta}$ such that the dynamical system with the map $T_{-\beta}$ yields $d_{-\beta}(x) = (x_1, x_2, \ldots)$. A finite block is $T_{-\beta}$-admissible if it appears in a $T_{-\beta}$-admissible sequence.

Definition 2.2.8. Let the cut points $c_1, \ldots, c_n$ be given. For any cut point c_i, we define $d_{L,-\beta}(c_i)$ to be the sequence obtained by the procedure in Theorem 1.2.9 with $T_{-\beta}$ left-continuous. Similarly, $d_{R,-\beta}(c_i)$ is the sequence obtained by the procedure in Theorem 1.2.9 with $T_{-\beta}$ right-continuous.

Recall that the digit function partitions the interval $I_{-\beta}$ into a finite number of subintervals. The limit points of each of these intervals (with the exception of the left-most and the right-most endpoint) are cut points. Hence, we can refer to a cut point by stating whether it is the left or right limit point of such a subinterval.

Definition 2.2.9. Let the digit function d be given. Let I_a be the subinterval on which $d(x) = a$ for all $x \in I_a$. Then the limit points of I_a are called ℓ_a and r_a, with $\ell_a < r_a$.

Example 2.2.10. In Example 1.2.10, we have $\ell_0 = 0$, $r_0 = \frac{\sqrt{5}-1}{2}$, $\ell_1 = -1$ and $r_1 = 0$. The cut point is 0 and, as stated in Example 2.2.4, we have $d_{L,-\beta}(0) = (1, \overline{1,0})$ and $d_{R,-\beta}(0) = (\overline{0})$. We may also write

$$d_{L,-\beta}(\ell_0) \;=\; d_{L,-\beta}(r_1) \;=\; (1, \overline{1,0})$$

and

$$d_{R,-\beta}(\ell_0) \;=\; d_{R,-\beta}(r_1) \;=\; (\overline{0}).$$

Hence only the map determines which sequence we mean, in the sense that we can refer to 0 as either ℓ_0 or r_1, but that the continuity of $T_{-\beta}$ (left- or right-continuous) determines which sequence we mean. □

Using these notations, there is a convenient way to characterize $T_{-\beta}$-admissible sequences. Since $T_{-\beta}$ is either left- or right-continuous, most theorems will have two statements from here on (one for $L_{-\beta}$ and one for $R_{-\beta}$). We will usually only prove $L_{-\beta}$ since the proof for $R_{-\beta}$ is similar.

Theorem 2.2.11. The following hold:

(a) If $(x_1, x_2, \ldots)$ is an $L_{-\beta}$-admissible sequence, then for all $n \geq 1$:

- if $x_n = \lfloor\beta\rfloor$, then $d_{-\beta}(\ell_{\lfloor\beta\rfloor}) \preceq (x_n, x_{n+1}, \ldots) \preceq d_{L,-\beta}(r_{\lfloor\beta\rfloor})$,
- if $x_n = a \neq \lfloor\beta\rfloor$, then $d_{R,-\beta}(\ell_a) \prec (x_n, x_{n+1}, \ldots) \preceq d_{L,-\beta}(r_a)$.

(b) If $(x_1, x_2, \ldots)$ is an $R_{-\beta}$-admissible sequence, then for all $n \geq 1$:

- if $x_n = 0$, then $d_{-\beta}(\ell_0) \preceq (x_n, x_{n+1}, \ldots) \preceq d_{L,-\beta}(r_0)$,
- if $x_n = a \neq 0$, then $d_{R,-\beta}(\ell_a) \preceq (x_n, x_{n+1}, \ldots) \prec d_{L,-\beta}(r_a)$.

Proof. We only prove (a), since the proof of (b) is similar.
If $x_n = \lfloor\beta\rfloor$, then $\ell_{\lfloor\beta\rfloor} \leq L^{n-1}_{-\beta}(x) \leq r_{\lfloor\beta\rfloor}$. By Theorem 2.2.6, this implies $d_{-\beta}(\ell_{\lfloor\beta\rfloor}) \preceq (x_n, x_{n+1}, \ldots) \preceq d_{L,-\beta}(r_{\lfloor\beta\rfloor})$.

If $x_n = a \neq \lfloor\beta\rfloor$, then by the same theorem we know that $(x_n, x_{n+1}, \ldots) \preceq d_{L,-\beta}(r_a)$ and therefore we only need to prove that $d_{R,-\beta}(\ell_a) \prec (x_n, x_{n+1}, \ldots)$. Since we are dealing with a left-continuous map, we know that $d_{L,-\beta}(\ell_a, 1) \neq a$ and hence $\ell_a < L^{n-1}_{-\beta}(x)$. Therefore, there exists a smallest integer $k \geq 1$ such that $d_{R,-\beta}(\ell_a, k) \neq d_{-\beta}(L^{n-1}_{-\beta}(x), k)$ and $d_{R,-\beta}(\ell_a, i) = d_{-\beta}(L^{n-1}_{-\beta}(x), i)$ for all $1 \leq i < k$. If k is odd, then we must have $d_{R,-\beta}(\ell_a, k) > d_{-\beta}(L^{n-1}_{-\beta}(x), k)$ (by definition of k and the fact that $L_{-\beta}$ and $R_{-\beta}$ reverse the order). Similarly, if k is even, we get $d_{R,-\beta}(\ell_a, k) < d_{-\beta}(L^{n-1}_{-\beta}(x), k)$. Either way, we see that

$$(-1)^k \left(d_{R,-\beta}(\ell_a, k) - d_{-\beta}(L^{n-1}_{-\beta}(x), k)\right) \; < \; 0,$$

which shows that $d_{R,-\beta}(\ell_a) \prec (x_n, x_{n+1}, \ldots)$ also holds. □

This theorem allows us to, in theory, check whether a given sequence in $\mathcal{A}^{\mathbb{N}}$ is $T_{-\beta}$-admissible. The converse of this theorem is, however, not true, as the following example will show:

Example 2.2.12. Let $\beta = \frac{1+\sqrt{5}}{2}$ and let the digit function $d(x)$ be given by

$$d(x) \;\; := \;\; \begin{cases} 0 & \text{for } x > 0, \\ 1 & \text{for } x \leq 0. \end{cases}$$

Note that there is a subtle difference between this newly constructed map $L_{-\beta}$ and the map in Example 1.2.10. Since this map is left-continuous, Theorem 2.2.11 tells us that for any admissible sequence $(x_1, x_2, \ldots)$

$$(\overline{1,0}) \preceq (x_n, x_{n+1}, \ldots) \preceq (1, \overline{1,0}) \quad \text{or} \quad (\overline{0}) \prec (x_n, x_{n+1}, \ldots) \preceq (\overline{0,1})$$

holds for all $n \geq 1$. According to this, the sequence $(0, 1, \overline{1, 0})$ should be $L_{-\beta}$-admissible. However, it is not $L_{-\beta}$-admissible. To see why, note that

$$\frac{1}{(-\beta)^2} + \sum_{k=1}^{\infty} \frac{1}{(-\beta)^{2k+1}} \quad = \quad 0.$$

Hence, the sequence corresponds to a $(-\beta)$-expansion of 0. However, the $L_{-\beta}$-admissible $(-\beta)$-expansion of 0 corresponds to the sequence $(1, \overline{1, 0})$. $\square$

2.3 Reaching for the stars

The converse of Theorem 2.2.11 is not true, as shown by the previous example. However, by slightly changing the theorem, we can turn it into an 'if and only if'-theorem. This is what we will do in this section.

Look again at Example 2.2.12. Theorem 2.2.11 is unable to distinguish between $L_{-\beta}$-admissible sequences and sequences that are not $L_{-\beta}$-admissible but do look like they are. In this example, both 01 and $1\overline{10}$ are $L_{-\beta}$-admissible blocks, but the sequence $(0, 1, \overline{1, 0})$ is not $L_{-\beta}$-admissible. We will see shortly that the space of $L_{-\beta}$-admissible sequences need not be closed (the same is true for the space of $R_{-\beta}$-admissible sequences).

Definition 2.3.1. Let β and $T_{-\beta}$ be given and let c be any cut point of the map $T_{-\beta}$. We define $d^*_{L,-\beta}(c) := \lim_{x\uparrow c} d_{-\beta}(x)$ and $d^*_{R,-\beta}(c) := \lim_{x\downarrow c} d_{-\beta}(x)$.

Note that the metric on $\mathcal{A}^{\mathbb{N}}$ implies pointwise convergence. So, by definition, we have the following fact:

Corollary 2.3.2. Let β and $T_{-\beta}$ be given and let c be any cut point of the map $T_{-\beta}$. For any $k \geq 1$ there exist $x, y \in I_{-\beta}$ such that $d^*_{L,-\beta}(c, i) = d_{-\beta}(x, i)$ and $d^*_{R,-\beta}(c, i) = d_{-\beta}(y, i)$ for all $1 \leq i \leq k$.

If β and the cut points are given, and if c is an arbitrary but fixed cut point, is there an easy way to compute $d^*_{L,-\beta}(c)$ and $d^*_{R,-\beta}(c)$? Fortunately there is, and it relies on the fact that $T_{-\beta}$ reverses the order of two points close enough to each other whenever applied. We describe how to do this for $d^*_{L,-\beta}(c)$, since the procudure is similar for $d^*_{R,-\beta}(c)$. Let d^L and d^R be the left- and right-continuous digit function.

Algorithm 2.3.3. Algorithm for determining $d^*_{L,-\beta}(c)$.

1. Start with $k := 1$, $\hat{T}_k := L_{-\beta}$ and $c_0 := c$.

2. Compute $c_k := \hat{T}_k(c_{k-1})$ and set

$$d^*_{L,-\beta}(c, k) \quad := \quad \begin{cases} d^L(c_{k-1}) & \text{if } \hat{T}_k = L_{-\beta}, \\ d^R(c_{k-1}) & \text{if } \hat{T}_k = R_{-\beta}. \end{cases}$$

3. If c_k is a cut point, and if $k - m$ is odd, where $m := \max\{0 \leq i < k : c_i \text{ is a cut point}\}$, then:
   - if you used $\hat{T}_k = L_{-\beta}$ to determine c_k, then $\hat{T}_{k+1} := R_{-\beta}$,
   - if you used $\hat{T}_k = R_{-\beta}$ to determine c_k, then $\hat{T}_{k+1} := L_{-\beta}$.

   Otherwise, $\hat{T}_{k+1} := \hat{T}_k$.
4. Increase k by one and go to step 2. □

If you want to determine $d^*_{R,-\beta}(c)$, then start with $\hat{T} = R_{-\beta}$. Before we prove that this algorithm is correct, we will illustrate it with two examples.

Example 2.3.4. Reconsider Example 2.2.12. The map has only one cut point, namely 0. We also know that $d_{L,-\beta}(0) = (1, \overline{1, 0})$ and $d_{R,-\beta}(0) = (\overline{0})$. We will determine $d^*_{L,-\beta}(0)$ and $d^*_{R,-\beta}(0)$.

We start with $d^*_{L,-\beta}(0)$. The orbit of 0 under $L_{-\beta}$ is periodic and we never hit 0 again. Hence, according to the procedure described a little earlier, we have $d^*_{L,-\beta}(0) = (1, \overline{1, 0}) = d_{L,-\beta}(0)$.

We now determine $d^*_{R,-\beta}(0)$. After one iteration of $R_{-\beta}$ we hit the cut point 0 again (it is a fixed point of $R_{-\beta}$). Therefore, $d^*_{R,-\beta}(0, 1) = 0$ and we will now continue with $L_{-\beta}$. At this point, we know how the sequence will end since 0 never hits 0 under $L_{-\beta}$ again. We conclude that $d^*_{R,-\beta}(0) = (0, 1, \overline{1, 0})$. □

Example 2.3.5. Let $\beta = 2$ and let $-\frac{1}{9}$ and $-\frac{4}{9}$ be the cut points (see the figure below). We will focus on the cut point $-\frac{4}{9}$. In order to find $d_{L,-\beta}(-\frac{4}{9})$ we will

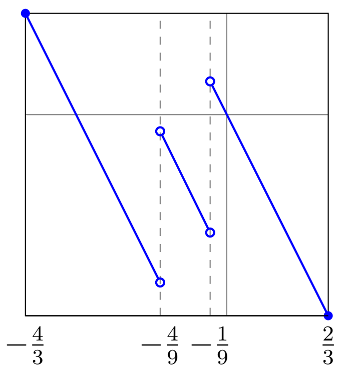


Figure 2.1: The map $T_{-\beta}$ corresponding to Example 2.3.5.

work with the left-continuous version of this map. The orbit of $-\frac{4}{9}$ under $L_{-\beta}$ is:

$$-\frac{4}{9} \to -\frac{10}{9} \to \frac{2}{9} \to -\frac{4}{9} \to \cdots$$

and hence by Theorem 1.2.9 we have $d_{L,-\beta}(-\frac{4}{9}) = (\overline{2,2,0})$.
Similarly, the orbit of $-\frac{4}{9}$ under $R_{-\beta}$ is:

$$-\frac{4}{9} \to -\frac{1}{9} \to \frac{2}{9} \to -\frac{4}{9} \cdots$$

By Theorem 1.2.9, we have $d_{R,-\beta}(-\frac{4}{9}) = (\overline{1,0,0})$.We now determine $d^*_{L,-\beta}(-\frac{4}{9})$. The first cut point $-\frac{4}{9}$ hits under $L_{-\beta}$ is $-\frac{4}{9}$. It does so after three iterations. Since three is an odd number we continue with $R_{-\beta}$. Moreover, we know that $d^*_{L,-\beta}(-\frac{4}{9}, 1) = 2$, $d^*_{L,-\beta}(-\frac{4}{9}, 2) = 2$ and $d^*_{L,-\beta}(-\frac{4}{9}, 3) = 0$.
Continuing by iterating $R_{-\beta}$ we hit another cut point, this time $-\frac{1}{9}$, after one iteration. Since this is an odd number of iterations since last time, we now have $d^*_{L,-\beta}(-\frac{4}{9}, 4) = 1$ and will iterate $L_{-\beta}$ again.
It takes two more iterations for us to return to $-\frac{4}{9}$. From these iterations we conclude that $d^*_{L,-\beta}(-\frac{4}{9}, 5) = 1$ and $d^*_{L,-\beta}(-\frac{4}{9}, 6) = 2$. Moreover, we are now back at where we originally started and hence we conclude that $d^*_{L,-\beta}(-\frac{4}{9})$ is purely periodic, since $d^*_{L,-\beta}(-\frac{4}{9}) = (\overline{2,2,0,1,1,2})$.

Finally, we determine $d^*_{R,-\beta}(-\frac{4}{9})$. It takes one iteration of $R_{-\beta}$ to reach another cut point, namely $-\frac{1}{9}$. Hence, we have $d^*_{R,-\beta}(-\frac{1}{9}, 1) = 1$ and we continue with iterating $L_{-\beta}$.
The orbit of $-\frac{1}{9}$ under $L_{-\beta}$ is: $-\frac{1}{9} \to -\frac{7}{9} \to -\frac{4}{9} \to -\frac{10}{9} \to \frac{2}{9} \to -\frac{4}{9} \to \cdots$, from which we see that we do hit a cut point after an odd amount of iterations (the first time we hit $-\frac{4}{9}$ is not interesting, since we do so after an even number of iterations). Hence, we have $d^*_{R,-\beta}(-\frac{4}{9}, 2) = 1$, $d^*_{R,-\beta}(-\frac{4}{9}, 3) = 2$, $d^*_{R,-\beta}(-\frac{4}{9}, 4) = 2$, $d^*_{R,-\beta}(-\frac{4}{9}, 5) = 2$ and $d^*_{R,-\beta}(-\frac{4}{9}, 6) = 0$.
We now start at $-\frac{4}{9}$ and iterate $R_{-\beta}$. However, this is exactly how we started and thus $d^*_{R,-\beta}(-\frac{4}{9})$ must also be purely periodic. We conclude that
$d^*_{R,-\beta}(-\frac{4}{9}) = (\overline{1,1,2,2,2,0})$. □

We will now prove that Algorithm 2.3.3 correct.

Theorem 2.3.6. Algorithm 2.3.3 is correct.

Proof. We will adopt the notation of Algorithm 2.3.3 but instead call the sequence obtained by this algorithm $s_{L,-\beta}(c)$. Hence, we need to prove that

$s_{L,-\beta}(c) = \lim_{x \uparrow c} d_{-\beta}(x)$. For any integer $n \geq 1$, define

$$\begin{aligned} \varepsilon_n \;:=\; & \frac{1}{\beta^{n-1}} \min\Big(\big\{\big|c_k - \ell_{s_{L,-\beta}(c,k+1)}\big| \;:\; 0 \leq k < n,\; k \text{ even}\big\} \\ & \cup\; \big\{\big|c_k - r_{s_{L,-\beta}(c,k+1)}\big| \;:\; 0 \leq k < n,\; k \text{ odd}\big\}\Big). \end{aligned}$$

We will first show that for all $n \geq 1$ we have $\varepsilon_n \neq 0$. Suppose that $k \geq 1$ is the smallest integer for which we have $\varepsilon_k = 0$.

If k is even, then we have $c_k = \ell_{s_{L,-\beta}(c,k+1)}$ and hence we hit either a cut point or $\ell_{-\beta}$ after an even number of iterations. However, since this implies that $\hat{T}_k = L_{-\beta}$, c cannot hit a cut point of the form $\ell_{s_{L,-\beta}(c,k+1)}$ (otherwise we would have $\hat{T}_k = R_{-\beta}$). Hence, we must have $c_k = \ell_{-\beta}$. However, this implies that c_{k-1} is either a cut point or $r_{-\beta}$. It cannot be a cut point (otherwise we would have $c_{k-1} = r_{s_{L,-\beta}(c,k)}$ and hence $\varepsilon_{k-1} = 0$) and thus it must be equal to $r_{-\beta}$. Repeatedly using this argument implies that either $c = \ell_{-\beta}$ or $c = r_{-\beta}$. Both lead to a contradiction, since c is a cut point. Thus, k cannot be even.

Hence, k must be odd. It follows that $c_k = r_{s_{L,-\beta}(c,k+1)}$ and thus we hit either a cut point or $r_{-\beta}$ after an odd number of iterations. By the same reasoning as in the case of even k, we have $c_k \neq r_{-\beta}$. Hence, we hit a cut point after an odd number of iterations. However, this implies that we continue the process with $R_{-\beta}$ and hence $c_k = \ell_{s_{L,-\beta}(c,k+1)}$ by right-continuity of the map. It follows that k cannot be odd either. We conclude that $\varepsilon_n > 0$ for all $n \geq 1$.

By definition of ε_n, any $x \in (c - \varepsilon_n, c)$ satisfies $d_{L,-\beta}(x,i) = s_{L,-\beta}(c,i)$ for $i = 1, \ldots, n$. It follows that $s_{L,-\beta}(c) = \lim_{x \uparrow c} d_{L,-\beta}(x)$. □

Corollary 2.3.7. Let β be given and let c be any cut point. The following hold:

(a) There exists no $L_{-\beta}$-admissible sequence $(x_1, x_2, \ldots)$ such that $d_{R,-\beta}(c) \prec (x_1, x_2, \ldots) \prec d^*_{R,-\beta}(c)$.

(b) There exists no $R_{-\beta}$-admissible sequence $(x_1, x_2, \ldots)$ such that $d^*_{L,-\beta}(c) \prec (x_1, x_2, \ldots) \prec d_{L,-\beta}(c)$.

Proof. We only prove the first statement, since the proof of the second statement is similar. Suppose that there is some $L_{-\beta}$-admissible sequence such that $d_{R,-\beta}(c) \prec (x_1, x_2, \ldots) \prec d^*_{R,-\beta}(c)$ holds and let $x = \sum_{i=1}^{\infty} \frac{x_i}{(-\beta)^i}$. By assumption, we have $x_1 = d_{R,-\beta}(c,1)$ and hence $x > c$.

Let $(x^n)_{n=1}^{\infty}$ be a decreasing sequence for which $d_{-\beta}(x^n, 1) = d^*_{R,-\beta}(c, 1)$ for all $n \geq 1$. For all $m, n \geq 1$ with $m < n$ we have $d_{-\beta}(x^n) \prec d_{-\beta}(x^m)$ by Theorem 2.2.6. It follows that $d^*_{R,-\beta}(c) \preceq d_{-\beta}(x^n)$ for all $n \geq 1$. Moreover, since this sequence converges to c it follows that there exists some $N \geq 1$ such that $x > x^N$ and hence $d^*_{R,-\beta}(c) \preceq d_{-\beta}(x^N) \prec d_{-\beta}(x)$. We have reached a contradicition and hence the sequence cannot exist. □

We now turn to the main theorem of this section:

Theorem 2.3.8. The following hold:

(a) The sequence $(x_1, x_2, \ldots)$ is $L_{-\beta}$-admissible if and only if for all $n \geq 1$:

- if $x_n = \lfloor\beta\rfloor$, then $d_{-\beta}(\ell_{\lfloor\beta\rfloor}) \preceq (x_n, x_{n+1}, \ldots) \preceq d_{L,-\beta}(r_{\lfloor\beta\rfloor})$,
- if $x_n = a \neq \lfloor\beta\rfloor$, then $d^*_{R,-\beta}(\ell_a) \prec (x_n, x_{n+1}, \ldots) \preceq d_{L,-\beta}(r_a)$.

(b) The sequence $(x_1, x_2, \ldots)$ is $R_{-\beta}$-admissible if and only if for all $n \geq 1$:

- if $x_n = 0$, then $d_{-\beta}(\ell_0) \preceq (x_n, x_{n+1}, \ldots) \preceq d_{L,-\beta}(r_0)$,
- if $x_n = a \neq 0$, then $d_{R,-\beta}(\ell_a) \preceq (x_n, x_{n+1}, \ldots) \prec d^*_{L,-\beta}(r_a)$.

Proof. We will only prove (a).

⇒:
This follows from Theorem 2.2.11 and Corollary 2.3.7.

⇐:
Let $(x_1, x_2, \ldots)$ satisfy the given inequalities and define $y_n := \sum_{k=1}^{n} \frac{x_k}{(-\beta)^k}$. The sequence $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence, since we have (assume $m < n$):

$$|y_n - y_m| = \left| \sum_{k=m+1}^{n} \frac{x_k}{(-\beta)^k} \right| \leq \sum_{k=m+1}^{n} \frac{x_k}{\beta^k} \leq \lfloor\beta\rfloor \cdot \frac{\beta^{n-m} - 1}{\beta^n(\beta - 1)} < \frac{\lfloor\beta\rfloor}{\beta^m(\beta - 1)},$$

and the latter tends to zero as $m \to \infty$. Hence, the sequence converges to some $y := \sum_{k=1}^{\infty} \frac{x_k}{(-\beta)^k}$.

Let us prove that if $(a_1, a_2, \ldots) = d^*_{R,-\beta}(c) \prec (x_n, x_{n+1}, \ldots)$ holds and if $a_1 = x_n$, then also $c < (.x_n x_{n+1} \ldots)_{-\beta}$ holds. Fix $n \geq 1$ and let $r \geq 2$ be an integer. We will show that $(a_1, \ldots, a_r) \prec (x_n, \ldots, x_{n+r-1})$ and $a_1 = x_n$ together imply that $(.a_1 \cdots a_r)_{-\beta} < (.x_n \cdots x_{n+r-1})_{-\beta}$ holds by induction on r.

- For $r = 2$ we have $(x_{n+1}) \prec (a_2)$, which implies $a_2 < x_{n+1}$. It follows that

  $$(.a_1 a_2)_{-\beta} - (.x_n x_{n+1})_{-\beta} \;=\; \frac{a_2 - x_{n+1}}{\beta^2} \;<\; 0.$$

  Hence $(.a_1 a_2)_{-\beta} < (.x_n x_{n+1})_{-\beta}$.

- Now, suppose that the claim holds for $r < k$.
  If $(a_1, \ldots, a_k) \prec (x_n, \ldots, x_{n+k-1})$ and $a_1 = x_n$, then $(x_{n+1}, \ldots, x_{n+k-1}) \prec (a_2, \ldots, a_k)$. By the induction hypothesis, we have $(.x_{n+1} \cdots x_{n+k-1})_{-\beta} < (.a_2 \cdots a_k)_{-\beta}$ and hence:

  $$(.a_1 \cdots a_k)_{-\beta} - (.x_n \cdots x_{n+k-1})_{-\beta} \;=\; \frac{(.a_2 \cdots a_k)_{-\beta} - (.x_{n+1} \cdots x_{n+k-1})_{-\beta}}{-\beta}.$$

  The right-hand side is negative, and hence we have $(.a_1 \cdots a_k)_{-\beta} < (.x_n \cdots x_{n+k-1})_{-\beta}$. Therefore, the claim holds for $r = k$ as well.

The proof of the statement is completed by letting $r \to \infty$.

We will now show that $d_{L,-\beta}(y) = (x_1, x_2, \ldots)$. Suppose that $d_{L,-\beta}(y) = (c_1, c_2, \ldots)$ and that $(c_1, c_2, \ldots) \neq (x_1, x_2, \ldots)$. Then there is a smallest integer $n \geq 1$ for which $x_n \neq c_n$. By assumption, we have $d^*_{R,-\beta}(\ell_{x_n}) \prec (x_n, x_{n+1}, \ldots)$. By the first part of this theorem, we also have $d^*_{R,-\beta}(\ell_{c_n}) \prec (c_n, c_{n+1}, \ldots)$. We now have either

$$\ell_{c_n} < (.c_n c_{n+1} \ldots)_{-\beta} \leq r_{c_n} \leq \ell_{x_n} < (.x_n x_{n+1} \ldots)_{-\beta} \leq r_{x_n}$$

or

$$\ell_{x_n} < (.x_n x_{n+1} \ldots)_{-\beta} \leq r_{x_n} \leq \ell_{c_n} < (.c_n c_{n+1} \ldots)_{-\beta} \leq r_{c_n}.$$

However, since $(.c_n c_{n+1} \ldots)_{-\beta} = (.x_n x_{n+1} \ldots)_{-\beta} = L^{n-1}_{-\beta}(y)$, this implies that $L^{n-1}_{-\beta}(y) < L^{n-1}_{-\beta}(y)$. This is a contradiction, and hence we must conclude that $d_{L,-\beta}(y) = (x_1, x_2, \ldots)$. □

2.4 Closing the gap

Theorem 2.3.8 allows us to create shift-invariant spaces that contain all $T_{-\beta}$-admissible sequences. From here it is only a small step to create a shift space. By definition, we only need to take the closure of this shift-invariant space.

Definition 2.4.1. Let the map $T_{-\beta}$ be given. We define S_T to be the shift-invariant set containing all $T_{-\beta}$-admissible sequences. We may use the notation S_L and S_R if we want to explicitly mention whether $T_{-\beta}$ is left- or right-continuous.

Definition 2.4.2. Let the map $T_{-\beta}$ be given. We define $\mathcal{S}_{-\beta} := \overline{S_T}$ (the closure of S_T). We call $\mathcal{S}_{-\beta}$ the $(-\beta)$-shift. If we want to explicitly mention whether or not the map is left- or right-continuous, we may also use the notation $\mathcal{S}_{L,-\beta}$ and $\mathcal{S}_{R,-\beta}$.

We will now prove two characterizations of the $(-\beta)$-shift.

Theorem 2.4.3. Let $\mathbf{x} = (x_1, x_2, \ldots) \in \mathcal{A}^{\mathbb{N}}$. Then $\mathbf{x} \in \mathcal{S}_{-\beta}$ if and only if every finite subblock of $\mathbf{x}$ is $T_{-\beta}$-admissible.

Proof. The proof is split into two steps.

$\Rightarrow$:
Let $(x^n)_{n=1}^{\infty}$ be a sequence in S^T converging to $\mathbf{x}$. Let $(x_p, \ldots, x_q)$ be any finite subblock of $\mathbf{x}$. Since $x^n \to \mathbf{x}$ as $n \to \infty$, it follows that for some $m \geq 1$, the sequences x^k contain the subblock $(x_p, \ldots, x_q)$ for all $k > m$.

$\Leftarrow$:
If every finite subblock of $\mathbf{x}$ is $T_{-\beta}$-admissible, then construct the following sequence in S^T: $x^n \in S^T$ is such that $x_i^n = \mathbf{x}_i$ for all $i \leq n$. This sequence converges to $\mathbf{x}$ (by construction), and hence $\mathrm{x} \in \mathcal{S}_{-\beta}$. □

The following theorem will be the more important characterization, as we will use it nearly all the time in the rest of this thesis.

Theorem 2.4.4. Let $\mathbf{x} = (x_1, x_2, \ldots) \in \mathcal{A}^{\mathbb{N}}$. The following hold:

(a) We have $\mathbf{x} \in \mathcal{S}_{L,-\beta}$ if and only if $d^*_{R,-\beta}(\ell_{x_n}) \preceq (x_n, x_{n+1}, \ldots) \preceq d_{L,-\beta}(r_{x_n})$ for all $n \geq 1$.

(b) We have $\mathbf{x} \in \mathcal{S}_{R,-\beta}$ if and only if $d_{R,-\beta}(\ell_{x_n}) \preceq (x_n, x_{n+1}, \ldots) \preceq d^*_{L,-\beta}(r_{x_n})$ for all $n \geq 1$.

Proof. We will only prove (a), since the proof of (b) is similar.

$\Rightarrow$:
Let $N \geq 1$ be arbitrary and fixed. If $(x_N, x_{N+1}, \ldots)$ is $L_{-\beta}$-admissible, then the statement is true for $n = N$ by Theorem 2.3.8. If it is not $L_{-\beta}$-admissible, then every finite subblock of this sequence is $L_{-\beta}$-admissible by Theorem 2.4.3. By Theorem 2.3.8, we cannot have $(x_N, x_{N+1}, \ldots) \succ d_{L,-\beta}(r_{x_N})$ as this would imply that for some $k \geq 1$ the subblock $(x_N, \ldots, x_{N+k})$ is not $L_{-\beta}$-admissible. Similarly, we cannot have $(x_N, x_{N+1}, \ldots) \prec d^*_{R,-\beta}(\ell_{x_N})$ either, since this would imply that for some $m \geq 1$ the subblock $(x_N, \ldots, x_{N+m})$ is not $L_{-\beta}$-admissible by Corollary 2.3.7. Hence, we must have $d^*_{R,-\beta}(\ell_{x_N}) \preceq (x_N, x_{N+1}, \ldots) \preceq d_{L,-\beta}(r_{x_N})$.

$\Leftarrow$:
Assume that there is a smallest integer $N \geq 1$ such that we have $(x_N, x_{N+1}, \ldots) = d^*_{R,-\beta}(\ell_{x_N})$ with $\ell_{x_N} \neq \ell_{-\beta}$ (if such an integer does not exist, the sequence is $L_{-\beta}$-admissible by Theorem 2.3.8). It is sufficient to prove that any finite subblock $(x_{N+s}, \ldots, x_{N+t})$ with $s \leq 0 \leq t$ (with s such that $N + s \geq 1$) is $L_{-\beta}$-admissible. Fix s and t. By definition, there exists an $L_{-\beta}$-admissible sequence $(y_1, y_2, \ldots)$ such that $y_i = x_{N+i-1}$ for $i = 1, \ldots, t$. It follows that the sequence $(x_{N+s}, \ldots, x_{N-1}, y_1, y_2, \ldots)$ is $L_{-\beta}$-admissible by Theorem 2.3.8 and hence the subblock $(x_{N+s}, \ldots, x_{N+t})$ is $L_{-\beta}$-admissible. $\square$

3

A new language

3.1 A primer on symbolic dynamics (part 2)

In the previous chapter we focused on constructing shift spaces. In this chapter we try to determine the language of a given shift space. We will not prove the theorems but instead provide references, since the proofs go deeper into symbolic dynamics than we need to in this thesis.

Recall from the previous chapter that any shift space can be given by specifying its forbidden blocks (Theorem 2.1.7). The nicest case is whenever the set of forbidden blocks is finite.

Definition 3.1.1. Let S be a shift space. If S has finitely many forbidden blocks, then we call S a shift of finite type.

Given the forbidden blocks of a shift of finite type S, we can find the language of S in many ways. The first, and most straightforward, method is by specifying the blocks of certain length that are not forbidden. More specifically, let n be the length of the largest forbidden block. Since we have finitely many forbidden blocks of finite length, n exists and is finite.

Example 3.1.2. Let $A = \{0, 1\}$ and let S be the shift of finite type with set of forbidden blocks $F = \{11, 101\}$. The largest forbidden block has length 3, and there are three blocks of length 3 containing the forbidden block 11, namely 011, 110 and 111. The other four blocks of length 3 are not forbidden and hence the language of S contains the blocks 000, 001, 010 and 100. Let $L = \{000, 001, 010, 100\}$, then $(x_1, x_2, \ldots) \in S$ if and only if for all $n \geq 1$ we have $x_n x_{n+1} x_{n+2} \in L$. □

The other method is by constructing a directed graph where the edges ensure that the sequence obtained by an infinite walk on this graph belongs to the shift S.

Definition 3.1.3. A labeled graph $\mathcal{G}$ is a pair $(G, \mathcal{L})$, where $G = (V, E)$ is a directed graph with edge set E, and with $\mathcal{L} : E \to A$ a map which assigns a label to each edge of the graph. The map $\mathcal{L}$ is called the labeling.

In other words, the second method is as follows. Find a labeled graph $\mathcal{G}$ and let $a \in V$ be the starting vertex of the graph. Let $\xi = e_1 e_2 \ldots$ be a path on $\mathcal{G}$, where $e_i \in E$ and e_1 is an outgoing edge of vertex a. Then $\mathcal{L}(e_1)\mathcal{L}(e_2)\cdots$ is in $A^{\mathbb{N}}$.

Definition 3.1.4. Let S be a shift space and $\mathcal{G}$ a labeled graph. If $\mathcal{G}$ is the labeled graph such that any infinite walk on $\mathcal{G}$ belongs to S, then we say that $\mathcal{G}$ recognizes the shift S.

Example 3.1.5. Consider the shift space from Example 3.1.2. The following labeled graph recognizes the shift:

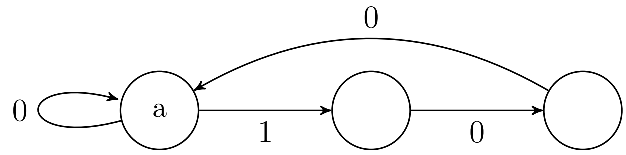


Figure 3.1: A labeled graph recognizing the shift from Example 3.1.2.

It is easy to verify that the shift indeed recognizes the shift. To this end, note that, by starting from the vertex 'a' and choosing three successive edges, the labels on the edges together form a block from the set $L = \{000, 001, 010, 100\}$. Moreover, we cannot form any of the blocks 011, 101, 110 or 111 this way. □

The class of shift spaces that are recognized by labeled graphs is actually larger than the the class of shifts of finite type.

Definition 3.1.6. A shift space S is called a sofic shift if there exists a labeled graph $\mathcal{G}$ that recognizes S.

Theorem 3.1.7. Every shift of finite type is a sofic shift.

Proof. See [LM95, Theorem 3.1.5]. □

The converse is quite easily shown to be false.

Example 3.1.8. Let $A = \{0, 1\}$ and S the shift space of all sequences such that any two consecutive 1s have an even number of 0s between them. This is a shift space, as we can give the set of forbidden blocks:

$$F = \{10^{2n+1}1 : n \geq 0\},$$

where 0^{2n+1} means the block $00\cdots0$ of length $2n+1$. Moreover, since we cannot exclude any of these blocks we see that S is not a shift of finite type. On the other hand, it is a sofic shift, since it is recognized by the following labeled graph.

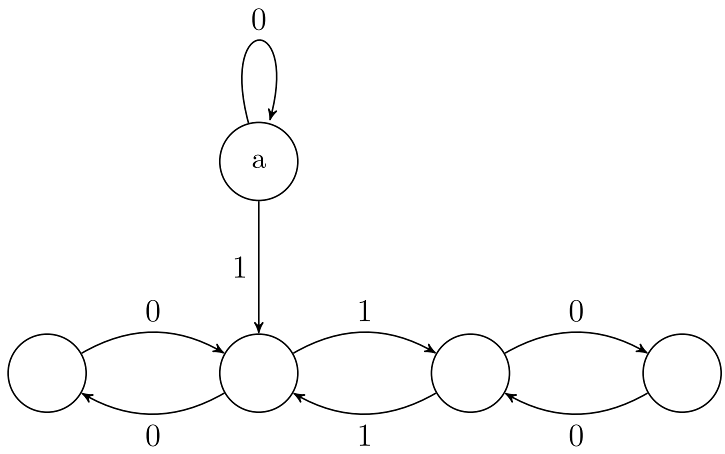


Figure 3.2: A labeled graph recognizing the shift from Example 3.1.8.

Once again, the vertex with an 'a' in it is the vertex from which we start. □

The nice thing about the labeled graphs we have given so far is that for each vertex the outgoing edges all have different labels. This is nice since, given a vertex $v \in \mathcal{G}$ and a finite block $a_1 \cdots a_n$, there is at most one finite path $\xi = e_1 \cdots e_n$ starting at v such that $\mathcal{L}(\xi) = \mathcal{L}(e_1)\mathcal{L}(e_2) \cdots \mathcal{L}(e_n) = a_1 \cdots a_n$. In a way, one might say the labeled graph is "deterministic" because of this property.

Definition 3.1.9. A labeled graph $\mathcal{G}$ is called right-resolving if, for each vertex v, the following holds: if e_1, e_2 are two edges starting at v and $e_1 \neq e_2$, then $\mathcal{L}(e_1) \neq \mathcal{L}(e_2)$.

While the definition of a sofic shift does not require the labeled graph to be right-resolving, one can always find a right-resolving graph representing the shift:

Theorem 3.1.10. Every sofic shift can be recognized by a right-resolving labeled graph.

Proof. See [LM95, Theorem 3.3.2]. □

We conclude this section by giving an example of a shift space which is not sofic.

Example 3.1.11. Let $A = \{1, 2, 3\}$ and let S be the shift space such that the block 12^m3^n1 (where a^k means the block $aa \cdots a$ of length k) may occur if and only if $m = n$. The shift space S is not sofic.

To see why, suppose that it were sofic and let $\mathcal{G}$ be a labeled graph recognizing S. Let p be the number of vertices in $\mathcal{G}$ and let ξ be a path in $\mathcal{G}$

representing the block $12^{p+1}3^{p+1}1$. Let ζ be the subpath of ξ that represents the block 2^{p+1}. Since $\mathcal{G}$ has only p vertices, the subpath ζ visits at least one vertex more than once. It follows that we may write $\zeta = \zeta_1\zeta_2\zeta_3$, where ζ_2 is a loop (a path starting and ending at the same vertex) and ζ_1 and ζ_3 may be empty. Now consider the path $\zeta' = \zeta_1\zeta_2\zeta_2\zeta_3$. Since it appears in the graph $\mathcal{G}$, we can replace the subpath ζ in ξ with ζ' and obtain another block which is in the language of S. However, this new path represents the block $12^{p+q+1}3^{p+1}1$, where q is the length of the path ζ_2. Since $q \neq 0$, this new block is forbidden. It follows that $\mathcal{G}$ cannot exist. □

3.2 Automation

At the end of chapter 2 we proved a very important theorem, Theorem 2.4.4, which characterizes the shift space $\mathcal{S}_{-\beta}$. It is this theorem which allows us to determine the language of the shift space. Note that this theorem says that the shift space $\mathcal{S}_{-\beta}$ is determined by a few sequences. These sequences will play an important role in this chapter.

Definition 3.2.1. Let $\mathcal{S}_{-\beta}$ be given. The boundary sequences are:

- $d^*_{R,-\beta}(\ell_a)$ and $d_{L,-\beta}(r_a)$ for all $a \in \mathcal{A}$ if the underlying map $T_{-\beta}$ is left-continuous,
- $d_{R,-\beta}(\ell_a)$ and $d^*_{L,-\beta}(r_a)$ for all $a \in \mathcal{A}$ if the underlying map $T_{-\beta}$ is right-continuous.

These sequences are called boundary sequences since they are the sequences corresponding to $(-\beta)$-expansions of endpoints of subintervals of $I_{-\beta}$ (the subintervals on which the digit function d is constant).

The following theorem shows the importance of the boundary sequences:

Theorem 3.2.2. If the shift space $\mathcal{S}_{-\beta}$ is sofic, then its boundary sequences are eventually periodic.

Proof. We will only prove the theorem for the shift space $\mathcal{S}_{L,-\beta}$, since the proof for $\mathcal{S}_{R,-\beta}$ is similar.

Let $\mathcal{G}$ be a right-resolving labeled graph representing $\mathcal{S}_{L,-\beta}$ (recall Theorem 3.1.10) with labeling $\mathcal{L}$. Fix $a \in \mathcal{A}$ and consider the sequence $d^*_{R,-\beta}(\ell_a)$. There exists a path $\xi = e_1e_2\cdots$ in $\mathcal{G}$ such that $(\mathcal{L}(e_1), \mathcal{L}(e_2), \ldots) = d^*_{R,-\beta}(\ell_a)$. By definition, $d^*_{R,-\beta}(\ell_a)$ is the smallest sequence with respect to the order $\prec$ starting whose first coordinate is a. Since $\mathcal{G}$ is a finite graph and ξ is an infinite path on $\mathcal{G}$, the path ξ passes some vertex infinitely often, say vertex v. Hence, the path ξ contains a loop of even length (there may also be loops of odd length, but since we are working with the order $\prec$ only loops of even length will work for this proof), i.e., there exist $k, n \geq 1$ such that $e_i = e_{i+2k}$ for all $i \geq n$. It follows that for all $i \geq n$ we have $\mathcal{L}(e_i) = \mathcal{L}(e_{i+2k})$ and hence $d^*_{R,-\beta}(\ell_a)$ is eventually periodic. Similarly, it follows that $d_{L,-\beta}(r_a)$ also must be eventually periodic. □

Is the converse to this theorem also true? If it is, this means that we must be able to construct a labeled graph $\mathcal{G}$ recognizing the shift. Moreover, our construction may only use the boundary sequences of the shift space. The following example illustrates what this may look like.

Example 3.2.3. Recall the map of Example 2.2.12. According to Example 2.3.4 and Theorem 2.4.4, the shift space, whose underlying map is given in Example 2.2.12, has the following characterization: $(x_1, x_2, \ldots) \in \mathcal{S}_{-\beta}$ if and only if for all $n \geq 1$ we have:

$$(\overline{1,0}) \preceq (x_n, x_{n+1}, \ldots) \preceq (1, \overline{1,0}) \quad \text{or} \quad (0, 1, \overline{1,0}) \preceq (x_n, x_{n+1}, \ldots) \preceq (\overline{0,1}).$$

Since $(\overline{1,0})$ and $(\overline{0,1})$ are respectively the smallest and largest possible sequence with respect to the order $\prec$, we may as well ignore those requirements. Hence, $(x_1, x_2, \ldots) \in \mathcal{S}_{-\beta}$ if and only if for all $n \geq 1$ we have:

$$(x_n, x_{n+1}, \ldots) \preceq (1, \overline{1,0}) \quad \text{or} \quad (0, 1, \overline{1,0}) \preceq (x_n, x_{n+1}, \ldots). \tag{3.1}$$

Pretend that we are a machine with a finite amount of memory and that we have to check whether or not any given sequence belongs to this shift space. Moreover, assume that we only read the sequence once from left to right (a plausible assumption, given the length of the sequence). To keep track of what we have read, we use two sets, m_0 and m_1, to remind ourselves what is currently stored in our memory. If, for some fixed $a \in \mathcal{A}$, we have $k \in m_a$ (with $k \geq 1$), then it means that our memory contains a block of length k which equals the prefix of a boundary sequence whose first symbol is a.

Now, suppose we start reading the sequence. We have yet to read anything, so $m_0 = \{0\}$ and $m_1 = \{0\}$. Assume for now that the first symbol is a 1. Then $m_1 = \{1\}$ and m_0 remains unchanged.
Now, let $m_0 = \{0\}$ and $m_1 = \{1\}$ and suppose we read a 0. Since $10 \prec 11$, we know that $(1, 0, x_3, \ldots) \prec (1, \overline{1,0})$ and hence (3.1) is fulfilled. This means that we can forget the very first digit we read and so we get $m_1 = \{0\}$. Moreover, our memory now contains the block 0, which implies $m_0 = \{1\}$.
If, however, we have $m_0 = \{0\}$, $m_1 = \{1\}$ and we read a 1, we cannot conclude anything yet (11 matches the prefix of some boundary sequence) and hence we have $m_1 = \{1, 2\}$ and $m_0 = \{0\}$: our memory contains the blocks 11 and 1.
Next, we assume that $m_0 = \{0\}$, $m_1 = \{1, 2\}$ and that we read a 0. It follows that our memory contains the block 110. Moreover, since $110 \preceq 110$, we still cannot either accept nor reject it based on condition (3.1). However, since $d^*_{R,-\beta}(\ell_1, 4) = d^*_{R,-\beta}(\ell_1, 2)$ holds due to periodicity of the boundary sequence, we might as well check the next digit against the second digit of the boundary sequence. Hence, we set $m_1 = \{1\}$ and $m_0 = \{1\}$ (recall that we have just read a 0).
Finally, if $m_0 = \{0\}$, $m_1 = \{1, 2\}$ and we read a 1, we know that (3.1) holds, since $111 \prec 110$. Therefore, we may discard the first 1 of the block 111, which leaves us with $m_1 = \{1, 2\}$.

We can summarize the above discussion in the table below. Here the notation $m_1(i/d)$ means the set m_1 contains the digit i and we have just read the digit d.

$m_1(i/d)$	0	1
0	0	1
1	0	1,2
2	1	1

In using this table to update m_1 we need to pay attention to the following:

- if the set m_1 contains multiple digits, we take the union of the elements of the table,
- the set m_1 does not contain any digit more than once,
- if m_1 contains a 0 and some nonzero digit, we must discard the zero (otherwise it would imply that we simultaneously do and do not have some prefix of a boundary sequence starting with 1 in our memory).

Similarly, for m_0 we get the following table:

$m_0(i/d)$	0	1
0	1	0
1	F	2
2	1	3
3	1,2	0

One case in this table is worth mentioning. If $m_0 = \{1\}$ and we read a 0, we have the block 00 in our memory. However, $01 \not\preceq 00$, and hence we can reject the sequence as a whole. We denote this with an F for failure.

We will now explain how to construct a graph given these two tables. The vertices of the graph will have the form (m_0, m_1). Start with the vertex $(0, 0)$. From here, draw an edge with label d to the vertex $(m_0/d, m_1/d)$. Repeat this for any other vertex: if the vertex is (m_0, m_1), then we draw an edge with label d to the vertex $(m_0/d, m_1/d)$. For example, if $m_0 = \{1, 2\}$, then $m_0/1 = \{2, 3\}$. Once we are done, remove all vertices containing F and also remove all edges pointing to any of these vertices. The resulting graph is on the next page.

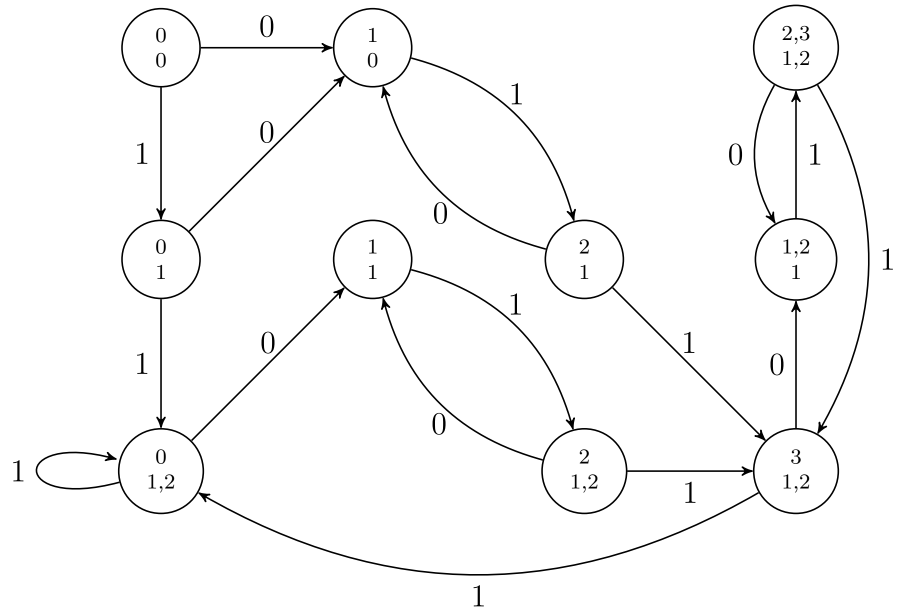


Figure 3.3: A labeled graph recognizing the shift from Example 3.2.3.

In the figure above we omitted the brackets that are usually used to denote sets with and also chose to put the coordinates in a column rather than a row. Both of these choices have been made because of typographical reasons. The labeled graph recognizes the shift by construction (we will prove this later). Any finite admissible block will be recognized by this graph, since it is based on (3.1) and hence on Theorem 2.4.4. Any path representing a forbidden block will end at a vertex with an F. □

Example 3.2.4. Recall Example 1.2.10. One can show, similarly to the previous example, that a sequence $(x_1, x_2, \ldots)$ belongs to the shift space if and only if for all $n \geq 1$:

$$(x_n, x_{n+1}, \ldots) \preceq (1, \overline{1, 0}) \quad \text{or} \quad (\overline{0}) \preceq (x_n, x_{n+1}, \ldots).$$

Just like in the previous example, we determine the tables $m_0(i/d)$ and $m_1(i/d)$:

$m_0(i/d)$	0	1
0	1	0
1	1,2	0
2	1	F

$m_1(i/d)$	0	1
0	0	1
1	0	1,2
2	1	1

And we construct a labeled graph using these tables:

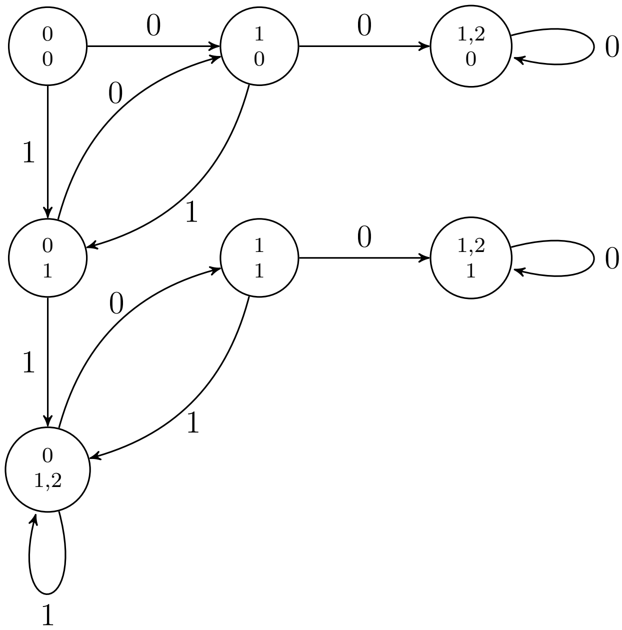


Figure 3.4: A labeled graph recognizing the shift from Example 3.2.4.

The table $m_1(i/d)$ is not different from the previous example. The table $m_0(i/d)$, however, has significantly changed. It is worth pointing out that the table of m_0 contains a 2, even though $(\overline{0})$ is purely periodic with period 1. This is because of one reason: the alternate lexicographical order. There is a difference between comparing $(0, x_2, x_3, \ldots)$ to $(\overline{0})$ and $(0, 0, x_3, \ldots)$ to $(\overline{0})$.

Interestingly, this shift is the right-continuous couterpart to the shift $\mathcal{S}_{L,-\beta}$ from Example 3.2.3. Note how our approach seem to result in getting two graphs that differ from each other quite a bit. We will revisit these two shift spaces in the next section. □

Let us formalize the illustrated procedure. Before we can do so, we need to define quite a few things. Recall that, by Theorem 2.4.4, the shift space is characterized by $2(\lfloor\beta\rfloor+1)$ boundary sequences, two for each $a \in \mathcal{A}$. Regardless of the continuity of the underlying digit function, for each $a \in \mathcal{A}$ one of these boundary sequences correspond to ℓ_a and the other to r_a. Hence, for each $a \in \mathcal{A}$ we define two sets, m_a^ℓ and m_a^r, which are subsets of $\mathbb{Z}_{\geq 0}$. If our memory cuurently is $(b_1, \ldots, b_n)$, then $m_a^\ell = k$ means that our memory contains a subblock $(b_i, \ldots, b_n)$ with $1 \leq i \leq n-k+1$ that equals the prefix of the boundary sequence ℓ_a, and where the next digit needs to be matched against the $(k+1)$-th digit of the boundary sequence ℓ_a (similarly for m_a^r). Since we assume our boundary sequences are all eventually periodic (as we eventually will want to prove the converse to Theorem 3.2.2), we can use the periodicity to ensure that the sets m_a^ℓ and m_a^r cannot contain arbitrary large numbers. We can determine the largest possible number that can appear in the set. For each $a \in \mathcal{A}$ this upper bound will be denoted by U_a^ℓ and U_a^r respectively. Suppose that the boundary sequence is eventually periodic with non-periodic part of length $q \geq 0$ and period p, then this upper bound is equal to

$$\begin{cases} 2p & \text{if } q = 0, \\ 2p + q - 1 & \text{if } q \neq 0 \text{ and } p \text{ is odd}, \\ p + q - 1 & \text{if } q \neq 0 \text{ and } p \text{ is even}. \end{cases} \tag{3.2}$$

In the previous two examples we used tables to help us determine the update rule for our memory sets. These tables will be denoted by $m_a^\ell(i/d)$ and $m_a^r(i/d)$, with $d \in \mathcal{A}$. For now, we assume that the underlying map $T_{-\beta}$ is left-continuous, since the right-continuous case is defined similarly. The number on row i and column d of the table $m_a^\ell(i/d)$ is:

$$m_a^\ell(i/d) \;:=\; \begin{cases} i+1 & \text{if } i < U_a^\ell \text{ and } d = d^*_{R,-\beta}(\ell_a, i+1), \\ p & \text{if } i = U_a^\ell,\ d = d^*_{R,-\beta}(\ell_a, p) \text{ and } q = 0, \\ q & \text{if } i = U_a^\ell,\ d = d^*_{R,-\beta}(\ell_a, q) \text{ and } q \neq 0, \\ 0 & \text{if } i \neq 0 \text{ and } (-1)^{i+1}\left(d^*_{R,-\beta}(\ell_a, i+1) - d\right) < 0, \\ 0 & \text{if } i = 0 \text{ and } d \neq a, \\ 1 & \text{if } i = 0 \text{ and } d = a, \\ F & \text{otherwise}. \end{cases} \tag{3.3}$$

A short explanation on each of these seven cases. If our memory matches the prefix of a boundary sequence but does not contain one complete period of this boundary sequence, we can check the next digit against the next digit of this boundary sequence (case 1). If the boundary sequence is purely periodic and our memory contains two periods, then move back to the beginning of the second period (case 2). If the boundary sequence is eventually periodic and our memory contains one period, then move back to the beginning of this periodic part (case 3). If we can accept the block because of Theorem 2.4.4, forget about this block (case 4). If our memory does not contain the digit a and the digit we read is also not an a, nothing happens (case 5). However, if the digit we read is a we have a block of length 1 (case 6). If none of the above happens, we conclude that we must reject the block (case 7).

Similarly, the number on row i and column d of the table $m_a^r(i/d)$ is:

$$m_a^r(i/d) \;:=\; \begin{cases} i+1 & \text{if } i < U_a^r \text{ and } d = d_{L,-\beta}(r_a, i+1), \\ p & \text{if } i = U_a^r,\ d = d_{L,-\beta}(r_a, p) \text{ and } q = 0, \\ q & \text{if } i = U_a^r,\ d = d_{L,-\beta}(r_a, q) \text{ and } q \neq 0, \\ 0 & \text{if } i \neq 0 \text{ and } (-1)^{i+1}\left(d - d_{L,-\beta}(r_a, i+1)\right) < 0, \\ 0 & \text{if } i = 0 \text{ and } d \neq a, \\ 1 & \text{if } i = 0 \text{ and } d = a, \\ F & \text{otherwise.} \end{cases} \tag{3.4}$$

You might have noticed already that (3.3) and (3.4) differ slightly from the way we filled the tables in the previous two examples. We have done so in order to keep (3.3) and (3.4) as small as possible. The 1's we have excluded (in the case that $d = a$ and $i \neq 0$) will be included in the algorithm below.

Algorithm 3.2.5. Algorithm for constructing a labeled graph recognizing $\mathcal{S}_{L,-\beta}$, whose boundary sequences are all eventually periodic.

1. Let $\mathcal{V}$ be the set of all vertices where each vertex $v \in \mathcal{V}$ is labeled $(m_0^\ell, m_0^r, \ldots, m_{\lfloor\beta\rfloor}^\ell, m_{\lfloor\beta\rfloor}^r)$, such that m_a^ℓ is a nonempty subset of $\{0, 1, \ldots, U_a^\ell\}$ and m_a^r is a nonempty subset of $\{0, 1, \ldots, U_a^r\}$.

2. Start with the vertex $m_0^\ell = m_0^r = \cdots = m_{\lfloor\beta\rfloor}^\ell = m_{\lfloor\beta\rfloor}^r = 0$. For any $d \in \mathcal{A}$:

   - if $F \in \left\{\varphi_{\ell,a}(m_a^\ell, d) : a \in \mathcal{A}\right\}$ or $F \in \{\varphi_{r,a}(m_a^r, d) : a \in \mathcal{A}\}$, do nothing,
   - otherwise, draw an edge with label d from vertex $v = (m_0^\ell, m_0^r, \ldots, m_{\lfloor\beta\rfloor}^\ell, m_{\lfloor\beta\rfloor}^r) \in \mathcal{V}$ to the vertex $w = (w_1, \ldots, w_{2\lfloor\beta\rfloor+2})$ such that:

   $$w_i \;=\; \begin{cases} \varphi_{\ell,(k-1)/2}(m_a^\ell, d) & \text{if } k \text{ is odd,} \\ \varphi_{r,(k-2)/2}(m_a^r, d) & \text{if } k \text{ is even.} \end{cases}$$

   Here the maps $\varphi_{\ell,a}(v,d)$ and $\varphi_{r,a}(v,d)$ are defined as:

   $$\varphi_{\ell,a}(v,d) \;:=\; \begin{cases} \bigcup_{i\in v} m_a^\ell(i/d) \setminus \{0\} & \text{if } \{0\} \subsetneq \bigcup_{i\in v} m_a^\ell(i/d) \\ & \text{and } F \notin \bigcup_{i\in v} m_a^\ell(i/d), \\ \{1\} \cup \bigcup_{i\in v} m_a^\ell(i/d) \setminus \{0\} & \text{if } 1, F \notin \bigcup_{i\in v} m_a^\ell(i/d) \\ & \text{and } d = a, \\ F & \text{if } F \in \bigcup_{i\in v} m_a^\ell(i/d), \\ \bigcup_{i\in v} m_a^\ell(i/d) & \text{otherwise.} \end{cases} \tag{3.5}$$

   The definition of $\varphi_{r,a}(v,d)$ is almost exactly the same: simply replace all instances of ℓ with r.

3. Repeat step 2 for any vertex that has at least one ingoing edge until all such vertices have been done. The resulting graph is $\mathcal{G} = (V, E)$ and it recognizes the shift space $\mathcal{S}_{L,-\beta}$. □

The functions $\varphi_{\ell,a}$ and $\varphi_{r,a}$ are the update rules. We shortly explain the four cases of (3.5). If our memory contains the digit a but the table dictates we must add a 0 to the memory set, then remove it (case 1). If we have read the digit a, but did not have to add a 1 to the memory set according to the tables, then add the 1 now (case 2). If our memory set contains an F, then mark the whole vertex with an F (case 3). Otherwise, nothing special happens and we simply do what the table tells us (case 4). Note that, from the way our graph is constructed, these four cases are disjoint.

We will now prove that this algorithm is indeed correct.

Theorem 3.2.6. Algorithm 3.2.5 is correct.

Proof. The proof of this theorem uses the following result. Let $(m_1, \ldots, m_n)$ be our current memory. Then $m_a^\ell(i/d)$ and/or $m_a^r(i/d)$ will equal F if and only if $(m_1, \ldots, m_n, d)$ contains a forbidden block. This result follows from (3.3) and (3.4), as they were defined with Theorem 2.4.4 in mind.

Let $\mathcal{S}_{L,-\beta}$ be given and let $\mathcal{G}$ be the labeled graph which follows from Algorithm 3.2.5. Let L_S be the language of the shift space and let $L_{\mathcal{G}}$ be the language of the edge shift on $\mathcal{G}$. We need to prove that $L_S = L_{\mathcal{G}}$.

Let $(a_1, \ldots, a_n) \in L_S$. It follows that any subblock $(a_k, \ldots, a_m)$ with $1 \leq k \leq m \leq n$ also belongs to L_S. It follows that there exists a path $\zeta = (e_1, \ldots, e_n)$ on $\mathcal{G}$ such that $\mathcal{L}(e_i) = a_i$ for $i \leq n$. For if not, then any finite path $\xi = (x_1, \ldots, x_j)$ (with $j < n$) such that $x_i = a_i$ for all $i \leq j$ cannot be extended such that $x_{j+1} = a_{j+1}$, which implies that $(a_1, \ldots, a_{j+1})$ contains a forbidden block. This is, however, a contradiction. We conclude that $(a_1, \ldots, a_n) \in L_{\mathcal{G}}$.

Conversely, let $(a_1, \ldots, a_n) \notin L_S$. Suppose that there exists some path $\zeta = (e_1, \ldots, e_n)$ on $\mathcal{G}$ with $\mathcal{L}(e_i) = a_i$ and let v_i be the vertex such that the path $(e_1, \ldots, e_{i-1})$ ends at vertex v_i. Since we have an edge labeled e_n from v_n to v_{n+1} it follows that neither $m^\ell(j/e_n)$ $j \in m_{e_n}^\ell$ at vertex v_n nor $m^r(j/e_n)$ with $j \in m_{e_n}^r$ at vertex v_n equals F. Hence, $(a_1, \ldots, a_n)$ is admissible. However, this is a contradiction and hence $(a_1, \ldots, a_n) \notin \mathcal{G}$. We conclude that $L_S = L_{\mathcal{G}}$. □

Corollary 3.2.7. The shift space $\mathcal{S}_{-\beta}$ is sofic if and only if all its boundary sequences are eventually periodic.

Proof. Theorem 3.2.2 and Theorem 3.2.6. □

This last corollary immediately follows from the two theorems in this section. It is also the main result of this section, which tells us that the language of the shift space $\mathcal{S}_{-\beta}$ indeed depends on its boundary sequences. In the following section we will see another example of this.

Moreover, we gave an algorithm which yields a labeled graph $\mathcal{G}$ recognizing this shift $\mathcal{S}_{-\beta}$. As we will see in the next section, this labeled graph is not necessarily the smallest labeled graph recognizing the shift (small in the sense of number of vertices the graph has).

3.3 Forbidden words

The algorithm given in the previous section works, but the labeled graphs it gives us are sometimes larger than necessary. We will show this by revisiting the two examples of the previous section.

Example 3.3.1. Let $\mathcal{A} = \{0, 1\}$. Then any sequence $(x_1, x_2, \ldots) \in \mathcal{A}^{\mathbb{N}}$ satisfies $(0, 1, \overline{1, 0}) \preceq (0, 1, x_1, x_2, \ldots)$, which in turn implies that any sequence must also satisfy $(1, x_1, x_2, \ldots) \preceq (1, \overline{1, 0})$.

To see what this means to us, consider the shift of Example 3.2.3. If our memory contains the block 01, then we know the sequence $(0, 1, x_{n+2}, \ldots)$ satisfies (3.1) regardless of what $(x_{n+2}, x_{n+3}, \ldots)$ look like. Hence, we might as well forget about this subblock (unlike the algorithm which does not do so). The same holds for the other inequality in (3.1). With these observations, we can modify the tables as follows:

$m_0(i/d)$	0	1
0	1	0
1	F	0

$m_1(i/d)$	0	1
0	0	0

which in turn yield us the following labeled graph that also recognizes the given shift space:

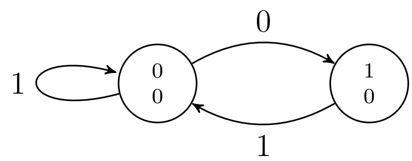


Figure 3.5: Another labeled graph recognizing the shift from Example 3.2.3.

Notice that a simple observation has reduced the our earlier labeled graph of 10 vertices to a much smaller graph with just 2 vertices! From this labeled graph it is clear that any 0 must be followed by a 1, something which is not so obvious from the graph in Figure 3.3. Moreover, this graph shows us that the shift space is in fact a shift of finite type, with 00 being the only forbidden block. □

Example 3.3.2. Now recall Example 3.2.4, where a sequence $(x_1, x_2, \ldots)$ belongs to the shift space if and only if for all $n \geq 1$:

$$(x_n, x_{n+1}, \ldots) \preceq (1, \overline{1,0}) \quad \text{or} \quad (\overline{0}) \preceq (x_n, x_{n+1}, \ldots).$$

By the observations in Example 3.3.1, we have the following tables:

$m_0(i/d)$	0	1
0	1	0
1	1,2	0
2	1	F

$m_1(i/d)$	0	1
0	0	0

Using these tables, we obtain the following labeled graph:

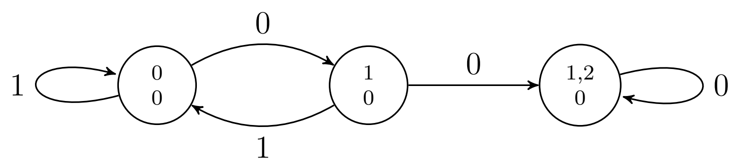


Figure 3.6: Another labeled graph recognizing the shift from Example 3.2.4.

The most important thing to note is that this graph is the graph in Figure 3.5 with just one vertex and two labeled edges added! In particular this implies that this shift space contains the shift space of Example 3.2.3: not only does it contain all sequences where the block 00 does not appear, it also contains many sequences that end with an infinite string of 0s. This shift space is also a shift of finite type, since 001 is the only forbidden block (any 0 is followed by either a 1 or by an infinite string of 0s). □

Unfortunately, being able to reduce at least one table is not sufficient for the shift space to be a shift of finite type, as the following example will show:

Example 3.3.3. Let $\beta = 2$ and let the digit function be given by:

$$d(x) \;:=\; \begin{cases} 0 & \text{if } -\frac{1}{6} < x \leq \frac{2}{3}, \\ 1 & \text{if } -\frac{5}{6} < x \leq -\frac{1}{6}, \\ 2 & \text{if } -\frac{4}{3} \leq x \leq -\frac{5}{6}. \end{cases}$$

By Theorem 2.4.4, we have $(x_1, x_2, \ldots) \in \mathcal{S}_{-\beta}$ if and only if for all $n \geq 1$ we have:

$$\begin{aligned} (\overline{2,0}) &\preceq (x_n, x_{n+1}, \ldots) \preceq (2, \overline{1}), \\ (1, \overline{0,2}) &\preceq (x_n, x_{n+1}, \ldots) \preceq (1, \overline{1,0}), \\ \text{or} \quad (0, \overline{0,1}) &\preceq (x_n, x_{n+1}, \ldots) \preceq (\overline{0,2}). \end{aligned}$$

The sequences $(\overline{2,0})$, $(\overline{0,2})$ and $(1,\overline{0,2})$ are not important to check, since the first two are the smallest respectively largest sequence with respect to $\prec$, and for the third sequence we note that

$$(1,\overline{0,2}) \preceq (1,a_1,a_2,\ldots)$$

holds for any sequence $(a_1,a_2,\ldots)$. Hence, it is sufficient to check that

$$(x_n,x_{n+1},\ldots) \preceq (2,\overline{1}), \qquad (x_n,x_{n+1},\ldots) \preceq (1,\overline{1,0}) \quad \text{or} \quad (0,\overline{0,1}) \preceq (x_n,x_{n+1},\ldots)$$

holds for all $n \geq 1$. Once again, we can create tables:

$m_0(i/d)$	0	1	2
0	1	0	0
1	1,2	0	0
2	1	1	F

$m_1(i/d)$	0	1	2
0	0	1	0
1	0	1,2	F
2	1	1	0

$m_2(i/d)$	0	1	2
0	0	0	1
1	0	2	F
2	F	1	1

We are able to give some forbidden blocks of the shift without constrcuting any labeled graph recognizing the shift. They are the entries with an F in the tables above. The forbidden blocks given by the tables are: 002, 12, 210 and 22. However, there are many more forbidden blocks. For all $n \geq 0$, consider the blocks $w^{(n)} := 1(10)^n 2$, where a^n is the block a repeated n times. These blocks are all forbidden, but the subblocks are all admissible. Hence the blocks $w^{(n)}$ are all forbidden, but there is no finite set of forbidden blocks which cover the set $\{w^{(n)} :\ n \geq 0\}$ as well. It follows that $\mathcal{S}_{-\beta}$ has infinitely many forbidden blocks and hence it is strictly sofic. A labeled graph recognizing the shift can be found on the next page. $\square$

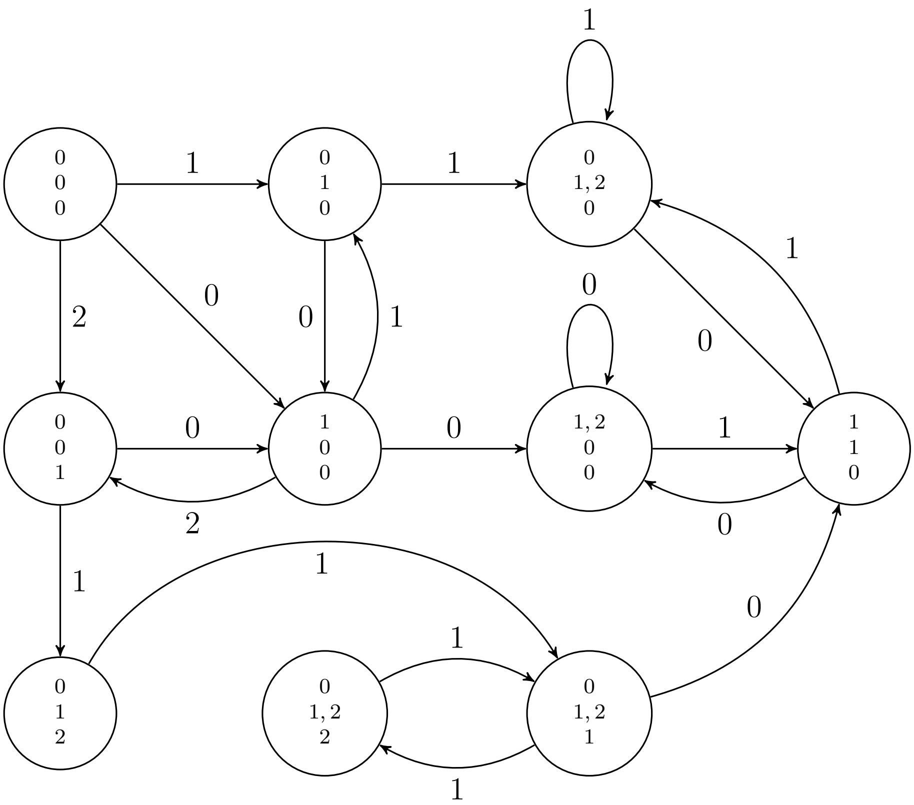


Figure 3.7: A labeled graph recognizing the shift from Example 3.3.3.

Fortunately, we can determine whether or not a shift space is a shift of finite type by only looking at the boundary sequences.

Theorem 3.3.4. The shift space $\mathcal{S}_{-\beta}$ is a shift of finite type if and only if the periodic part of every eventually periodic boundary sequence is a purely periodic boundary sequence.

Proof. The proof is split into two parts.

$\Rightarrow$:
Let $\mathcal{S}_{-\beta}$ be a shift of finite type. By Theorem 3.2.2, all its boundary sequences are eventually periodic. Assume that there exists a left-boundary sequence $(a_1, \ldots, a_k, \overline{a_{k+1}, \ldots, a_{k+p}})$ with $k, p > 0$ whose periodic part does not equal some other boundary sequence (the proof is similar for right-boundary sequences). Moreover, assume that k is odd (the proof is similar for k even).

Let $1 \leq j \leq p$ be the smallest integer such that there exists some $b \in \mathcal{A}$ for which $(-1)^{k+j}(b - a_{k+j}) < 0$ holds. Note that j must exist, otherwise it would imply that $(\overline{a_{k+1}, \ldots, a_{k+p}})$ equals either $d_{-\beta}(\ell_{-\beta})$ or $d_{-\beta}(r_{-\beta})$, which are both boundary sequences. By assumption, we have

$$(a_1, \ldots, a_{k+j-1}, b) \prec (a_1, \ldots, a_{k+j-1}, a_{k+j})$$

and hence the block $a_1 \cdots a_{k+j-1} b$ is forbidden. Define, for $n \geq 0$, the blocks $w^{(n)} := a_1 \cdots a_k (a_{k+1} \cdots a_{k+p})^{2n} a_{k+1} \cdots a_{k+j-1} b$. The blocks $w^{(n)}$ are forbidden for all $n \geq 0$.
Suppose that there exists some integer $1 \leq i \leq j-1$ such that the subblock $a_{k+i} \cdots b$ of length at most j is forbidden. Then the blocks $(a_{k+1} \cdots a_{k+p})^{2n} a_{k+1} \cdots a_{k+j-1} b$ are all forbidden as well. It follows that, for all $n \geq 0$, we have

$$(a_{k+1} \cdots a_{k+p})^{2n} \prec (a_{k+1} \cdots a_{k+p})^{2n} a_{k+1} \cdots a_{k+j-1} b.$$

By letting $n \to \infty$, it follows that $(\overline{a_{k+1}, \ldots, a_{k+p}})$ is a right-boundary sequence. This is a contradiction, since we assume that the periodic part would not equal some other boundary sequence. Hence, for all $1 \leq i \leq j-1$ the subblock $a_{k+i} \cdots b$ of length at most j is admissible.

It follows that there exists a largest integer $1 \leq m \leq k$, such that the blocks $a_m \cdots a_k (a_{k+1} \cdots a_{k+p})^{2n} a_{k+1} \cdots a_{k+j-1} b$ are all forbidden, but any subblock of these blocks is admissible. Hence, any minimal set containing the forbidden blocks of $\mathcal{S}_{-\beta}$ must be infinite. This is a contradiction, since we assumed that $\mathcal{S}_{-\beta}$ is a shift of finite type. It follows that the sequence $(a_1, \ldots, a_k, \overline{a_{k+1}, \ldots, a_{k+p}})$ cannot exist.

$\Leftarrow$:
We give the proof by finding the forbidden blocks. Let $(a_1, a_2, \ldots)$ be any left-boundary sequence (the construction is similar for right-boundary sequences).

If $(a_1, a_2, \ldots)$ is purely periodic with period p, then the forbidden blocks are given by $\{(x_1, \ldots, x_p) \in \mathcal{A}^p : (x_1, \ldots, x_p) \prec (a_1, \ldots, a_p) \ \wedge \ x_1 = a_1\}$ if p is even, and $\{(x_1, \ldots, x_{2p}) \in \mathcal{A}^{2p} : (x_1, \ldots, x_{2p}) \prec (a_1, \ldots, a_p, a_1, \ldots, a_p) \ \wedge \ x_1 = a_1\}$, if p is odd. The number of such forbidden blocks is finite, since it cannot exceed $\lfloor \beta \rfloor^{2p}$.

If $(a_1, a_2, \ldots)$ is eventually periodic with period p and non-periodic part of length $k > 0$, then we claim that the set

$$\{(x_1, \ldots, x_{k+1}) \in \mathcal{A}^{k+1} : (x_1, \ldots, x_{k+1}) \prec (a_1, \ldots, a_{k+1}) \ \wedge \ x_1 = a_1\}$$

is sufficient. To see why, assume that k is odd (the argument is similar for k even) and let $(y_1, y_2, \ldots)$ with $y_1 = a_1$ be a sequence such that $(y_1, y_2, \ldots) \prec (a_1, a_2, \ldots)$

holds. Then there is a smallest integer $m \geq 1$ such that $(-1)^m(y_m - a_m) < 0$ holds. Suppose that $m > k+1$. Then $y_1 \cdots y_{k+1} = a_1 \cdots a_{k+1}$ and $(-1)^{m-k}(y_m - a_m) > 0$, i.e., we have

$$(\overline{a_{k+1}, \ldots, a_{k+p}}) \prec (y_{k+1}, y_{k+2}, \ldots).$$

Since the purely periodic sequence on the left-hand side is a right-boundary sequence, it follows that $(y_{k+1}, y_{k+2}, \ldots)$ is also forbidden. Hence, the forbidden block $(y_{k+1}, \ldots, y_{k+m})$ is also a forbidden block of some purely periodic boundary sequence, which we have already described earlier. Therefore, we do not need to write this sequence down again. This proves the claim.

Since there are finitely many boundary sequences and since each boundary sequence has finitely many forbidden blocks (at most $\max(\lfloor\beta\rfloor^p, \lfloor\beta\rfloor^{k-1})$), it follows that the shift space $\mathcal{S}_{-\beta}$ has finitely many forbidden blocks. Hence, $\mathcal{S}_{-\beta}$ is a shift space. $\square$

3.4 Pisot numbers

So far we have given characterizations of the $(-\beta)$-shift in terms of its boundary sequences. In this section we will give a characterization in terms of the cut points of the underlying map. In this thesis we will only be able to give this characterization for a small class of bases β.

Definition 3.4.1. An algebraic integer is a complex number whose minimal polynomial has coefficients in $\mathbb{Z}$.

Definition 3.4.2. A Pisot number is a positive algebraic integer whose conjugates all have norm less than 1.

We remind the reader of the following facts about algebraic integers:

Lemma 3.4.3. Let β be an algebraic integer of degree d. The following hold:

(a) $\mathbb{Q}(\beta)$ is a vector space over $\mathbb{Q}$ with base $\{\beta^{-1}, \beta^{-2}, \ldots, \beta^{-d}\}$,

(b) if β is a zero of $P(X) \in \mathbb{Z}[X]$, then all the algebraic conjugates of β are also zeroes of $P(X)$,

(c) if the minimal polynomial of β is $M_\beta(X) = X^d - \sum_{n=1}^{d} q_n X^{d-n}$, then the minimal polynomial of $-\beta$ is $M_{-\beta}(X) = X^d - \sum_{n=1}^{d}(-1)^{d-n} q_n X^{d-n}$.

Proof. See [Lai09, Lemma 1.1 & Theorem 1.3]. □

The following theorem shows why this class of numbers is so special.

Theorem 3.4.4. Let β be a Pisot number, $T_{-\beta}$ be a $(-\beta)$-transformation and $x \in I_{-\beta}$. If $d_{-\beta}(x)$ is the $(-\beta)$-expansion of x given by the map $T_{-\beta}$, and if moreover $x \in \mathbb{Q}(\beta)$, then $d_{-\beta}(x)$ is eventually periodic.

Proof. (Based on the proof of [Lai09, Theorem 3.6]).

Let $\beta_1 := \beta$ and let $\beta_2, \ldots, \beta_d$ be the Galois conjugates of β. Finally, we define $B := ((-\beta_j)^{-i})_{1\leq i,j\leq d}$. Let $c_1, \ldots, c_{\lfloor\beta\rfloor}$ be the cut points of $T_{-\beta}$, and let $x \in I_{-\beta} \cap \mathbb{Q}(\beta)$ be fixed. Since $\mathbb{Q}(\beta) = \mathbb{Q}(-\beta)$, we can write (Lemma 3.4.3(a)):

$$x \;=\; b^{-1} \sum_{i=1}^{d} a_i(-\beta)^{-i},$$

with $a_i \in \mathbb{Z}$ for $i = 1, \ldots, d$ and $b \in \mathbb{N}$. Moreover, we may assume that b is as small as possible in order to have uniqueness.

Let $(x_n)_{n=1}^{\infty}$ be the $(-\beta)$-expansion of x generated by $T_{-\beta}$ and define for all $n \geq 0$:

$$r_n^{(1)} \quad := \quad (-\beta)^n \left(x - \sum_{k=1}^{n} x_k (-\beta)^{-k} \right),$$

and for $2 \leq j \leq d$:

$$r_n^{(j)} \quad := \quad (-\beta_j)^n \left(b^{-1} \sum_{i=1}^{d} a_i (-\beta_j)^{-i} - \sum_{k=1}^{n} x_k (-\beta_j)^{-k} \right).$$

Moreover consider the vector $R_n := \left(r_n^{(1)}, \ldots, r_n^{(d)} \right)$ for $n \geq 1$.

We will first show that the sequence $(R_n)_{n=1}^{\infty}$ is uniformly bounded. Note that $r_n^{(1)} = T_{-\beta}^n(x)$ and hence $\left| r_n^{(1)} \right| \leq \frac{\beta \cdot \lfloor \beta \rfloor}{\beta^2 - 1}$ for all $n \geq 1$.
Let $\eta := \max\{|\beta_j| : \ j = 2, \ldots, d\}$, then we have $\eta < 1$, since β is a Pisot number. It follows that:

$$\begin{aligned}
\left| r_n^{(j)} \right| &= \left| \left(b^{-1} \sum_{i=1}^{d} a_i (-\beta_j)^{n-i} - \sum_{k=1}^{n} x_k (-\beta_j)^{n-k} \right) \right| \\
&\leq b^{-1} \sum_{i=1}^{d} |a_i| \eta^{n-i} + \sum_{k=1}^{n} x_k \eta^{n-k} \\
&\leq b^{-1} \max\{|a_i| : \ i = 1, \ldots, d\} \sum_{i=1}^{d} \eta^{n-i} + \lfloor \beta \rfloor \sum_{k=0}^{n-1} \eta^k \\
&< \frac{b^{-1} \max\{|a_i| : \ i = 1, \ldots, d\} + \lfloor \beta \rfloor}{1 - \eta}
\end{aligned}$$

for all $n \geq 1$. We see that $|r_n^{(j)}|$ is bounded for all $n \geq 1$ and all $j = 1, \ldots, d$ by

$$\max \left(\frac{\beta \cdot \lfloor \beta \rfloor}{\beta^2 - 1} + 1, \frac{b^{-1} \max\{|a_i| : \ i = 1, \ldots, d\} + \lfloor \beta \rfloor}{1 - \eta} \right).$$

Hence, $(R_n)_{n=1}^{\infty}$ is uniformly bounded.

Our next step is to show that for all $n \geq 1$, there exists a $Z_n \in \mathbb{Z}^d$ such that $R_n = b^{-1} Z_n B$. If we write $\left(z_n^{(1)}, \ldots, z_n^{(d)} \right) \in \mathbb{Z}^d$ then, by Lemma 3.4.3(b), it is sufficient to prove that

$$r_n^{(1)} \quad = \quad b^{-1} \sum_{k=1}^{d} z_n^{(k)} (-\beta)^{-k} \tag{3.6}$$

holds. The proof is by induction on n.

- First, let $n = 1$. Since $M_\beta(X) = X^d - q_1 X^{d-1} - \ldots - q_d$ is the minimal polynomial of β, we have (Lemma 3.4.3(c)):
$$(-\beta)^d = (-1)^{d-1} q_1 (-\beta)^{d-1} + \cdots + q_d$$
and hence
$$\begin{aligned} 1 &= (-1)^{d-1} q_1 (-\beta)^{-1} + (-1)^{d-2} q_2 (-\beta)^{-2} + \cdots + q_d (-\beta)^{-d} \\ &= \sum_{i=1}^{d} (-1)^{d-i} q_i (-\beta)^{-i}. \end{aligned}$$
It follows that
$$\begin{aligned} r_1^{(1)} &= (-\beta)\left(x - x_1(-\beta)^{-1}\right) \\ &= (-\beta)x - x_1 \\ &= (-\beta) b^{-1} \sum_{i=1}^{d} a_i (-\beta)^{-i} - x_1 \sum_{i=1}^{d} (-1)^{d-i} q_i (-\beta)^{-i} \\ &= b^{-1} \left(a_1 + \sum_{i=1}^{d-1} a_{i+1} (-\beta)^{-i} - b x_1 \sum_{i=1}^{d} (-1)^{d-i} q_i (-\beta)^{-i} \right) \\ &= b^{-1} \left(\sum_{i=1}^{d} (-1)^{d-i} (a_1 - b x_1) q_i (-\beta)^{-i} + \sum_{i=1}^{d-1} a_{i+1} (-\beta)^{-i} \right), \end{aligned}$$
and hence equation (3.6) holds for $n = 1$ if we let:
$$z_1^{(k)} := \begin{cases} (-1)^{d-k} (a_1 - b x_1) q_k + a_{k+1} & \text{if } k \neq d, \\ (a_1 - b x_1) q_d & \text{if } k = d. \end{cases}$$

- Now, suppose that (3.6) holds for $n = i$. Since $r_i^{(1)} = T_{-\beta}^i(x)$, we have $r_{i+1}^{(1)} = -\beta r_i^{(1)} - x_{i+1}$ and hence:
$$\begin{aligned} r_{i+1}^{(1)} &= -\beta r_i^{(1)} - x_{i+1} \\ &= -\beta b^{-1} \sum_{k=1}^{d} z_i^{(k)} (-\beta)^{-k} - x_{i+1} \\ &= -\beta b^{-1} \sum_{k=1}^{d} z_i^{(k)} (-\beta)^{-k} - x_{i+1} \sum_{k=1}^{d} (-1)^{d-k} q_k (-\beta)^{-k} \\ &= b^{-1} \left(z_i^{(1)} + \sum_{k=1}^{d-1} z_i^{(k+1)} (-\beta)^{-k} - b x_{i+1} \sum_{k=1}^{d} (-1)^{d-k} q_k (-\beta)^{-k} \right) \\ &= b^{-1} \left(\sum_{k=1}^{d} (-1)^{d-k} \left(z_i^{(1)} - b x_{i+1} \right) q_k (-\beta)^{-k} + \sum_{k=1}^{d-1} z_i^{(k+1)} (-\beta)^{-k} \right). \end{aligned}$$

Hence, if we let:

$$z_{i+1}^{(k)} := \begin{cases} (-1)^{d-k}\left(z_i^{(1)} - bx_{i+1}\right)q_k + z_i^{(k+1)} & \text{if } k \neq d, \\ \left(z_i^{(1)} - bx_{i+1}\right)q_d & \text{if } k = d, \end{cases}$$

then $z_{i+1}^{(k)} \in \mathbb{Z}$ for $k = 1, \ldots, d$ by the induction hypothesis. Hence, (3.6) holds for $n = i + 1$ as well.

It follows that our claim is true.

We will now finish the proof. We know that $R_n = b^{-1}Z_nB$ holds for some $(Z_n)_{n=1}^{\infty}$. Moreover, $Z_n \in \mathbb{Z}^d$ for all $n \geq 1$. Since the matrix B is invertible and since R_n is uniformly bounded, it follows that Z_n is also uniformly bounded. From the definition of $z_i^{(k)}$ it follows that this implies that there exist $p, q \geq 1$ such that $Z_{p+q} = Z_q$. Hence, $(Z_n)_{n=1}^{\infty}$ is an eventually periodic sequence and in particular $r_n^{(1)}$ is eventually periodic. It follows that $d_{-\beta}(x)$ is eventually periodic. □

Corollary 3.4.5. Let β be a Pisot number and let the map $T_{-\beta}$ be given. If the cut points $c_1, \ldots, c_{\lfloor\beta\rfloor}$ all are in $\mathbb{Q}(\beta)$, then the corresponding shift $\mathcal{S}_{-\beta}$ is sofic.

Proof. By Theorem 3.4.4, the sequences $d_{L,-\beta}(c_a)$ and $d_{R,-\beta}(c_a)$ are eventually periodic for all $a \in \mathcal{A}$. From Algorithm 2.3.3 it follows that for all $a \in \mathcal{A}$ the sequences $d^*_{L,-\beta}(c_a)$ and $d^*_{R,-\beta}(c_a)$ are concatenations of prefixes of $d_{L,-\beta}(c_a)$ and $d_{R,-\beta}(c_a)$. Since there are finitely many cut points, this implies that $d^*_{L,-\beta}(c_a)$ and $d^*_{R,-\beta}(c_a)$ must be eventually periodic as well. By Corollary 3.2.7, the shift space is sofic. □

4 From negative to positive

4.1 A positive look

The β-expansions were introduced by Rényi in 1957. In 1960, Parry has proven some fundamental properties of β-expansions in his paper [Par60]. However, the theory developed back then is slightly different than the general approach we have tried so far. Let us start with a more general approach and see where the original viewpoint comes from.

Let β and $\mathcal{A}$ be as usual. We will look at the power series

$$\sum_{n=1}^{\infty} \frac{x_n}{\beta^n}, \qquad x_n \in \mathcal{A} \text{ for all } n \geq 1.$$

The smallest and largest possible number that can be written as such are 0 respectively $\frac{\lfloor \beta \rfloor}{\beta - 1}$. This largest number is at least 1. However, originally only power series representing number in the interval $[0, 1)$ were studied, presumably for a reason we will see later.

We start with the following definition:

Definition 4.1.1. Let $x \in [0, 1)$ and $(x_n)_{n=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}$. Then, if

$$x \;=\; \sum_{n=1}^{\infty} \frac{x_n}{\beta^n}$$

holds, the right-hand side will be called a β-expansion of x.

As usual, the term β-expansion may also refer to the sequence of digits instead of the formal power series.

The iterative process for finding β-expansions given by Rényi is by iterating the transformation $T_\beta(x) = \beta x \pmod 1$ on $[0, 1)$ and by defining $x_n := \lfloor T_\beta^{n-1}(x) \rfloor$. The proof of convergence of this process is similar to the proof of Theorem 1.2.9 and will not be given.

On the other hand, we could also look at the larger interval $\left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ with the map $T(x) = \beta x - a$, with $a \in \mathcal{A}$ and $\lfloor\beta\rfloor$ cut points. Let us see how the iterative process fits in our more general approach with the following example.

Example 4.1.2. Let $\beta = \frac{1+\sqrt{5}}{2}$. In the picture below one can see exactly how T_β fits in our more general approach:

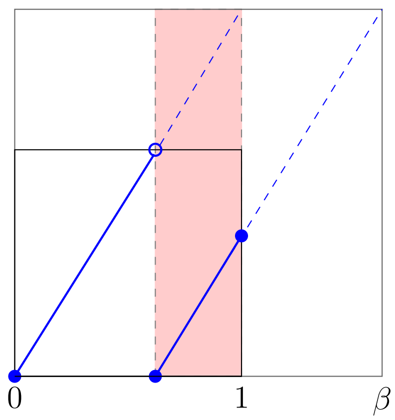


Figure 4.1: The map T_β in the full interval $[0, \beta]$.

We see that T_β is simply the right-continuous map obtained by choosing our cut points as small as possible. Moreover, observe that under these circumstances, the interval $[0, 1)$ is T_β-invariant (in fact, it is an attractor). This final observation might be the reason why originally only numbers in $[0, 1)$ were studied. $\square$

We will continue with our more general approach. Thus, we define $I_\beta := \left[0, \frac{\lfloor\beta\rfloor}{\beta-1}\right]$ and $T_\beta : I_\beta \to I_\beta$ is a piecewise linear map with exactly $\lfloor\beta\rfloor$ discontinuities (the cut points), similar to what we have assumed in the case of negative base. Another similarity between the case of positive base and negative base is the use of an order. In this case, we use the lexicographical order.

Definition 4.1.3. We write $(x_1, x_2, \ldots) \leq_{\text{lex}} (y_1, y_2, \ldots)$ if and only if there exists some $k \geq 1$ such that $x_i = y_i$ for all $i < k$ and $x_k < y_k$. We will call $\leq_{\text{lex}}$ the lexicographical order.

With the lexicographical order we can, as in the case of $T_{-\beta}$, work towards the β-shift. We state the theorems, but will not prove them. The theorems (and their proofs as well) are the positive base equivalents of theorems we saw in earlier chapters.

Theorem 4.1.4. Let β and the cut points $c_1, \ldots, c_{\lfloor\beta\rfloor}$ be given. Let L_β and R_β be the left- respectively right-continuous version of the map with the given cut points on I_β. Moreover let, for any $z \in I_\beta$, the β-expansions obtained by L_β and R_β respectively be denoted with $d_\beta^L(z)$ and $d_\beta^R(z)$. Then the sequence $(x_n)_{n=1}^\infty \in \mathcal{A}^\mathbb{N}$ is L_β-admissible if and only if

$$d_\beta^R(\ell_{x_n}) \leq_{\text{lex}} (x_n, x_{n+1}, \ldots) <_{\text{lex}} d_\beta^L(r_{x_n}) \qquad \text{for all } n \geq 1,$$

and R_β-admissible if and only if

$$d_\beta^R(\ell_{x_n}) <_{\text{lex}} (x_n, x_{n+1}, \ldots) \leq_{\text{lex}} d_\beta^L(r_{x_n}) \qquad \text{for all } n \geq 1.$$

Proof. See [Kal09, Theorem 5.2.1]. □

Corollary 4.1.5. Let $\beta > 1$ and T_β be given. Then $(x_1, x_2, \ldots) \in \mathcal{A}^\mathbb{N}$ belongs to the β-shift $\mathcal{S}_\beta$ (which is the shift space of all sequences of which each finite subblock is T_β-admissible) if and only if

$$d_\beta^R(\ell_{x_n}) \leq_{\text{lex}} (x_n, x_{n+1}, \ldots) \leq_{\text{lex}} d_\beta^L(r_{x_n})$$

holds for all $n \geq 1$.

Proof. Follows directly from Theorem 4.1.4. □

Interestingly, we do not need to distinguish between a left-continuous and right-continuous shift space $\mathcal{S}_\beta$ like we had to in the case of negative bases.

Just like in the case of negative base we can define boundary sequences. If the cut points are $c_1, \ldots, c_{\lfloor\beta\rfloor}$, then the boundary sequences are $d_\beta^L(c_a)$ and $d_\beta^R(c_a)$ for all $a \in \mathcal{A}$.

Theorem 4.1.6. Let $\beta > 1$ and the cut points $c_1, \ldots, c_{\lfloor\beta\rfloor}$ be given.
The following hold:

(a) $\mathcal{S}_\beta$ is sofic if and only if all boundary sequences are eventually periodic,

(b) $\mathcal{S}_\beta$ is a shift of finite type if and only if the periodic parts of all boundary sequences are boundary sequences as well.

Proof. Part (a) follows from [KS10, Proposition 2.14]. The proof of part (b) is similar to the proof of Theorem 3.3.4. □

Just like in the case of negative base, we can find an algorithm which constructs a labeled graph recognizing the β-shift. This algorithm can be found by modifying a few parts of Algorithm 3.2.5, since the positive case relies on the order $<_{\text{lex}}$ instead of $\prec$. The reader is encouraged to do so.

Example 4.1.7. Let $\beta = \frac{1+\sqrt{5}}{2}$ with cut point $c = \beta - 1$. By Corollary 4.1.5, we do not have to specify whether the map is left- or right-continuous. According to this corollary, $(x_1, x_2, \ldots) \in \mathcal{A}^{\mathbb{N}}$ belongs to $\mathcal{S}_\beta$ if and only if

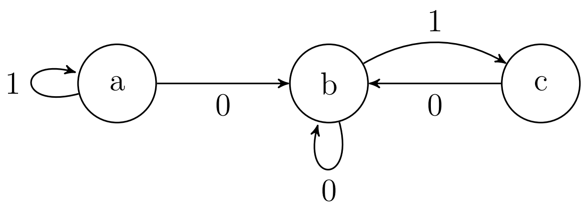
$$(\overline{0}) \leq_{\text{lex}} (x_n, x_{n+1}, \ldots) \leq_{\text{lex}} (\overline{0,1}) \quad \text{or} \quad (1, \overline{0}) \leq_{\text{lex}} (x_n, x_{n+1}, \ldots) \leq_{\text{lex}} (\overline{1})$$

holds for all $n \geq 1$. Since any sequence in $\mathcal{A}^{\mathbb{N}}$ is lexicographically greater than $(\overline{0})$ and since any sequence $(x_1, x_2, \ldots) \in \mathcal{A}^{\mathbb{N}}$ satisfies $(1, \overline{0}) \leq_{\text{lex}} (1, x_1, x_2, \ldots) \leq_{\text{lex}} (\overline{1})$, the above condition is equivalent with the following. The sequence $(x_1, x_2, \ldots) \in \mathcal{A}^{\mathbb{N}}$ belongs to $\mathcal{S}_\beta$ if and only if

$$(x_n, x_{n+1}, \ldots) \leq_{\text{lex}} (\overline{0,1}) \tag{4.1}$$

holds for all $n \geq 1$ for which $x_n = 0$. Note that (4.1) coincides with Parry's characterization in [Par60].

Theorem 4.1.6 states that this shift is a shift of finite type. Using the condition in (4.1) above, it is easy to find a labeled graph recognizing this shift:



To see why this labeled graph recognizes the shift, suppose we are a machine that reads the sequence $(x_1, x_2, \ldots)$. We start at the vertex labeled 'a' and leave this vertex as soon as we read a 0 for the first time (also note that 'a' is a transient state). From this point on, whenever we read a 0 we end at vertex 'b'. If we read the block 00, then we may forget about the first of these two zeroes (since $00 <_{\text{lex}} 01$ and hence (4.1) is immediately fulfilled). If we read the block 01, then the next digit must be a 0 in order to not violate the condition in (4.1). This explains our use of the third vertex 'c'. It also follows from the preceding explanation that 011 is the only forbidden block of this shift space, making it a shift of finite type. □

4.2 Normalization

In Example 4.1.7 we chose our cut point to be as small as possible. As a consequence, if we write $T_\beta(x) = \beta x - d_\beta(x)$, then the digit function $d_\beta(x)$ is (almost) always as large as possible (depending on whether or not the map is left- or right-continuous). The continuity of the map is irrelevant for $\mathcal{S}_\beta$, as we saw in Corollary 4.1.5. This motivates the following definition.

Definition 4.2.1. Let $\beta > 1$ be given. If the cut points $c_1, \ldots, c_{\lfloor \beta \rfloor}$ are chosen to be as small as possible, then the shift space will be called the greedy β-shift.

Given any β-expansion of some $x \in I_\beta$, is there a way to find another β-expansion of x belonging to the greedy β-shift without using any map T_β? In this section we will see that this is indeed possible. The idea is based on adding zero in a clever way.

Let $\beta > 1$ be fixed and let $\mathcal{A}'$ be the extended alphabet, which is the set of integers in $[-\lfloor \beta \rfloor, \lfloor \beta \rfloor]$. We want to find all possible power series satisfying

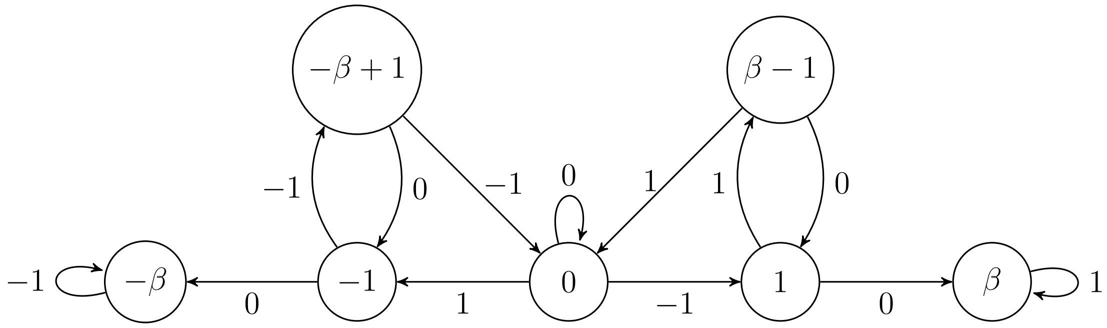
$$\sum_{n=1}^{\infty} \frac{a_n}{\beta^n} = 0, \qquad a_n \in \mathcal{A}' \text{ for all } n \geq 1., \tag{4.2}$$

This can be done by constructing a labeled graph that allows us to generate these power series.

Algorithm 4.2.2. Algorithm for constructing a labeled graph finding all power series satisfying (4.2).

1. Let V be the set of vertices, where each vertex corrsponds to exactly one number of $\mathbb{Z}(\beta) \cap \left[-\frac{\lfloor \beta \rfloor}{\beta - 1}, \frac{\lfloor \beta \rfloor}{\beta - 1}\right]$.
2. For each vertex $x \in V$, draw an edge with label $a \in \mathcal{A}'$ to vertex y if and only if $\beta x - a = y$.
3. The requested labeled graph is the connected component that contains the vertex 0. □

Example 4.2.3. Let $\beta = \frac{1+\sqrt{5}}{2}$, then Algorithm 4.2.2 gives the labeled graph below:



To find any power series satisfying (4.2), consider any infinite walk on this graph starting at the vertex '0'. The labels of this infinite path, in order, are the digits a_n from (4.2). □

Theorem 4.2.4. Algorithm 4.2.2 is correct.

Proof. Let $\mathcal{G}$ be the labeled graph from Algorithm 4.2.2. Let $\zeta = e_1, e_2, \ldots$ be any fixed infinite path on the $\mathcal{G}$ starting at the vertex '0' and let v_n be the vertex where this path ends after n edges, i.e., the subpath $e_1 \cdots e_n$ ends at the vertex v_n. We will prove, by induction on n, that

$$0 = \frac{v_n}{\beta^n} + \sum_{i=1}^{n} \frac{e_i}{\beta^i}$$

holds for all $n \geq 1$.

- First let $n = 1$. Since the path starts at vertex '0', it follows that $v_1 = -e_1$ from step 2 of Algorithm 4.2.2. Hence, we have

$$\frac{v_1}{\beta} + \frac{e_1}{\beta} = \frac{-e_1 + e_1}{\beta} = 0.$$

- Next, assume that the claim is true for $n = k$. From step 2 it follows that $\beta v_k - e_{k+1} = v_{k+1}$ and hence $v_k = \frac{v_{k+1}+e_{k+1}}{\beta}$. Using this and the induction hypothesis, we find:

$$\begin{aligned} 0 &= \frac{v_k}{\beta^k} + \sum_{i=1}^{k} \frac{e_i}{\beta^i} \\ &= \frac{v_{k+1}}{\beta^{k+1}} + \frac{e_{k+1}}{\beta^{k+1}} + \sum_{i=1}^{k} \frac{e_i}{\beta^i} = \frac{v_{k+1}}{\beta^{k+1}} + \sum_{i=1}^{k+1} \frac{e_i}{\beta^i}. \end{aligned}$$

  We see that the claim holds for $n = k + 1$ as well.

We conclude that our claim is true. By letting $n \to \infty$ it follows that any infinite path on $\mathcal{G}$ starting at '0' will yield an power series satisfying (4.2).

Let $\sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$ be a series satisfying (4.2). Since any path on $\mathcal{G}$ starts at '0', we write

$$\beta \cdot 0 - a_1 \quad = \quad \sum_{n=1}^{\infty} \frac{a_{n+1}}{\beta^n}$$

which proves that '0' has an outgoing edge with label a_1 to some vertex, say v_1. By repeating this argument, we see that

$$\beta \cdot \sum_{n=1}^{\infty} \frac{a_{n+1}}{\beta^n} - a_2 \quad = \quad \sum_{n=1}^{\infty} \frac{a_{n+2}}{\beta^n}$$

holds, which implies there is an edge with label a_2 from v_1 to some other vertex v_2. By inductively applying this argument, we can create a path $\zeta = e_1 e_2 \cdots$ on $\mathcal{G}$ such that $\mathcal{L}(e_i) = a_i$. □

From this graph we create another labeled graph. This new graph, however, uses more than one alphabet.

Definition 4.2.5. Let A, B be two alphabets and let $\mathcal{G}$ be a labeled graph where each edge has the label $a|b$, where $a \in A$ and $b \in B$. Then $\mathcal{G}$ is called a transducer (from A to B).

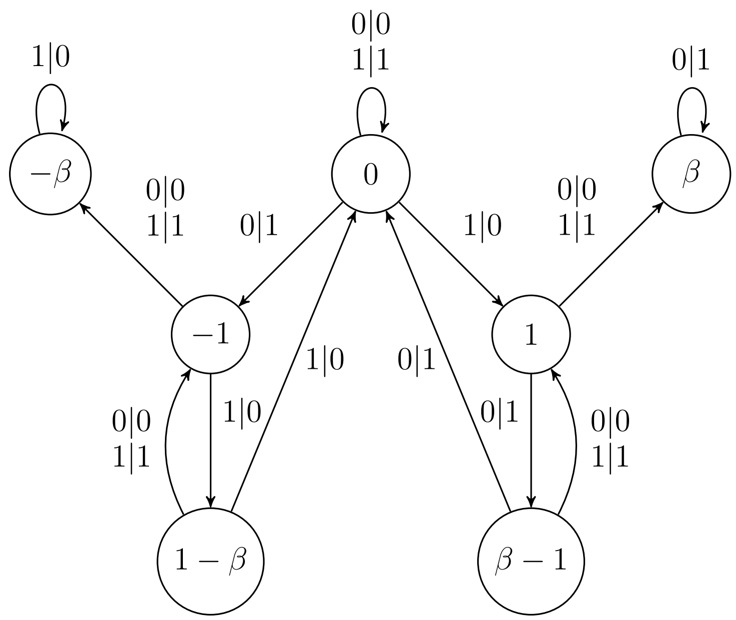
Our next step is to construct a transducer that, for any given $x \in I_\beta$, can convert any β-expansion of x to any other β-expansion of x. To do so, we need the labeled graph from Algorithm 4.2.2.

Algorithm 4.2.6. Algorithm for constructing a general transducer that, for any $x \in I_\beta$, recognizes all β-expansions of x.

1. Obtain the labeled graph $\mathcal{G} = (V, E)$ from Algorithm 4.2.2.

2. For any edge $e \in E$, if its label is $a \in \mathcal{A}'$, replace this edge with $\lfloor \beta \rfloor + 1 - |a|$ different pairs of labels $u|v$, where $u, v \in \mathcal{A}$ such that $u + a = v$. □

Example 4.2.7. In this example we continue Example 4.2.3 and use Algorithm 4.2.6 to construct our transducer. We obtain the following labeled graph:



As in the case of Example 4.2.3, we start at the vertex labeled '0'. Let $(x_1, x_2, \ldots)$ be any β-expansion of some $x \in I_\beta$. Now, find some path $\zeta = e_1, e_2, \ldots$ in this labeled graph such that $\mathcal{L}(e_i) = x_i|y_i$ for all $i \geq 1$ (existence of this path is guaranteed, since the starting vertex has loops $a|a$ for all $a \in \mathcal{A}$), where $\mathcal{L}(e)$ denotes the label of edge e. Then $(y_1, y_2, \ldots)$ is another β-expansion of x. □

Theorem 4.2.8. Algorithm 4.2.6 is correct.

Proof. Let $(a_1, a_2, \ldots)$ be a β-expansion of some number, say x, and let $(b_1, b_2, \ldots)$ an output of the general transducer from Algorithm 4.2.6 with $(a_1, a_2, \ldots)$ as input. Consider the sequence $(c_1, c_2, \ldots)$, where $c_i := b_i - a_i$ for all $i \geq 1$. Since $c_i \in \mathcal{A}'$ for all $i \geq 1$, it follows that $(c_1, c_2, \ldots)$ is a path in the labeled graph from Algorithm 4.2.2. But then we have

$$\sum_{n=1}^{\infty} \frac{b_n}{\beta^n} - \sum_{n=1}^{\infty} \frac{a_n}{\beta^n} = \sum_{n=1}^{\infty} \frac{b_n - a_n}{\beta^n} = \sum_{n=1}^{\infty} \frac{c_n}{\beta^n} = 0,$$

which implies that $(b_1, b_2, \ldots)$ is another β-expansion of x.

Let $(d_1, d_2, \ldots)$ be a β-expansion of x. If we define $f_i := d_i - a_i$ for all $i \geq 1$, then $(f_1, f_2, \ldots)$ is a path in the labeled graph from Algorithm 4.2.2 by Theorem 4.2.4. From step 2 of Algorithm 4.2.6 it follows that the edge with label f_i gets replaced with an edge with label $a_i|d_i$ (and possibly many more labeled edges). Hence, $(d_1, d_2, \ldots)$ is an output of the general transducer from Algorithm 4.2.6 with $(a_1, a_2, \ldots)$ as input. $\square$

Definition 4.2.9. Let $\beta > 1$ be given. A normalizer is a transducer that, for any $x \in I_\beta$, maps any β-expansion of x to some β-expansion of x belonging to the greedy β-shift.

Let $\beta > 1$ be given and fixed. To construct the normalizer we need two things. First of all, we need the general transducer from Algorithm 4.2.6. Lastly, we will also need any labeled graph recognizing the greedy β-shift. The normalizer can be thought of as an intersection of the two labeled graphs.

Algorithm 4.2.10. Algorithm for constructing the normalizer for any given $\beta > 1$.

1. Let $\mathcal{G}_1 = (V_1, E_1)$ be the general transducer from Algorithm 4.2.6 and $\mathcal{G}_2 = (V_2, E_2)$ any labeled graph recognizing the greedy β-shift.
2. Let $\mathcal{V} := V_1 \times V_2$.
3. From any $(v_1, v_2) \in \mathcal{V}$, we draw an edge with label $a|b$ to $(w_1, w_2) \in \mathcal{V}$ if and only if there is an edge with label $a|b$ from v_1 to w_1 in $\mathcal{G}_1$ and an edge with label b from v_2 to w_2 in $\mathcal{G}_2$.
4. Let x_1 and x_2 be the starting vertices of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, then (x_1, x_2) will be the starting vertex of our normalizer.
5. Let V_3 be the largest connected component containing (x_1, x_2) such that any vertex has at least one outgoing edge to some other vertex in V_3. Let E_3 be the set of labeled edges between the vertices in V_3. The normalizer is $\mathcal{G}_3 = (V_3, E_3)$. $\square$

Theorem 4.2.11. Algorithm 4.2.10 is correct.

Proof. We adopt the notation of Algorithm 4.2.10. Let $x \in I_\beta$ be fixed, let $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ be two not necessarily different β-expansions of x, with $(y_1, y_2, \ldots)$ belonging to the greedy β-shift. Suppose that there exists no path $\zeta = (e_1, e_2, \ldots)$ in $\mathcal{G}_3$ such that $\mathcal{L}(e_i) = x_i|y_i$ for all $i \geq 1$, where $\mathcal{L}(e)$ denotes the label of edge e. Since $(y_1, y_2, \ldots)$ is accepted by $\mathcal{G}_2$, step 3 of the algorithm implies that there is no path ζ with the properties described above in the general transducer $\mathcal{G}_1$. However, both $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ are β-expansions of the same number x and hence this path does exist (Theorem 4.2.8). We have reached a contradiction and hence the path ζ does exist. Moreover, since the output of $\mathcal{G}_3$ is accepted by $\mathcal{G}_2$ (by construction), it follows that $\mathcal{G}_3$ maps any β-expansion to a β-expansion of equal value belonging to the greedy β-shift. $\square$

Example 4.2.12. Let $\beta = \frac{1+\sqrt{5}}{2}$. Using Examples 4.1.7 and 4.2.7 and Algorithm 4.2.9, we find the following normalizer:

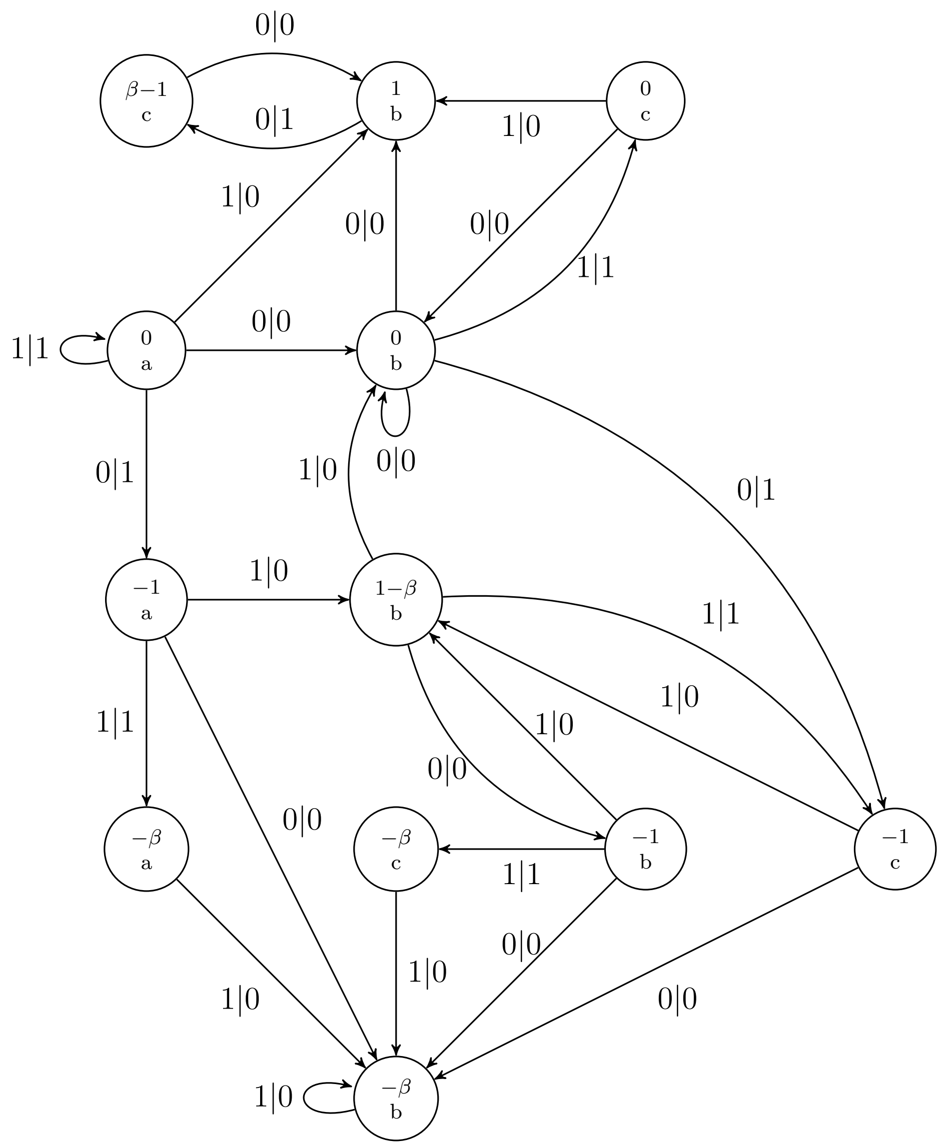


Figure 4.2: The normalizer for $\beta = \frac{1+\sqrt{5}}{2}$.

The starting vertex is vertex $(0, \mathrm{a})$. □

The following theorem shows why having a Pisot number as base is desirable if we want to normalize.

Theorem 4.2.13. If β is a Pisot number then the normalizer is finite.

Proof. (Based on the proof of [FL09, Proposition 6]).

Let β be a Pisot number of rank d. Consider the labeled graph $\mathcal{G}$ from Algorithm 4.2.2. Since β and the integers from $\mathbb{Z}$ are algebraic, it follows that any vertex in this labeled graph represents an algebraic number.

Let v be any vertex on this graph and let $(e_1, \ldots, e_n)$ be the shortest path on this graph from 0 to v. Then, from the proof of Theorem 4.2.4, we see that

$$v \;=\; -\sum_{i=1}^{n} e_i \beta^{n-i} \;=\; P_v(\beta),$$

with $P_v(X) \in \mathbb{Z}[X]$.

Let $\beta_2, \ldots, \beta_d$ be the conjugates of β, M_β the minimal polynomial of β and consider the set $\mathbb{Z}[X]/(M_\beta)$. We define the following norm on this set:

$$\|P(X)\| \;:=\; \max_{i=1,\ldots,d} |P(\beta_i)|,$$

where $\beta_1 = \beta$. For all vertices v on $\mathcal{G}$ we see that $\|P_v(\beta)\| < \frac{\lfloor\beta\rfloor}{\beta-1}$ and for $2 \leq k \leq d$:

$$\begin{aligned} \|P_v(\beta_k)\| \;&=\; \left\| -\sum_{i=1}^{n} e_i \beta_k^{n-i} \right\| \\ &\leq\; \sum_{i=1}^{n} |e_i||\beta_k|^{n-i} \;<\; \frac{\lfloor\beta\rfloor}{1-|\beta_k|}. \end{aligned}$$

It follows that the polynomials $P_v(X)$ are uniformly bounded and hence there can only be finitely many of them (and hence finitely many vertices).

Consider Algorithm 4.2.10. Finiteness of V_1 follows from the above. Similar to the proof of Corollary 3.4.5, one can show that V_2 is finite as well (the cut points are $\frac{a}{\beta}$ with $a \in \mathcal{A}$, which belong to $\mathbb{Q}(\beta)$, and hence [Aki07, Theorem 2] and Theorem 4.1.6 apply). Since the vertex set of the normalizer is a subset of $V_1 \times V_2$, it follows that it must be finite as well. □

4.3 Conversion

Let $\beta > 1$ be given and fixed. Recall the shift map on shift spaces, which we will denote by σ. Given two shift spaces, $\mathcal{S}_{-\beta}$ and $\mathcal{S}_{\beta}$ (with the underlying maps fixed as well), can we find a map $\gamma : \mathcal{S}_{-\beta} \to \mathcal{S}_{\beta}$ such that the diagram

$$\begin{array}{ccc} \mathcal{S}_{-\beta} & \xrightarrow{\gamma} & \mathcal{S}_{\beta} \\ \big\downarrow \sigma & & \big\downarrow \sigma \\ \mathcal{S}_{-\beta} & \xrightarrow{\gamma} & \mathcal{S}_{\beta} \end{array}$$

is commutative? This is still an open problem. In this section we will find such a map γ but, as we shall see, it will come at a cost.

Our first problem is finding a suitable way to work with two different bases, namely β and $-\beta$. Fortunately, if we rewrite $(-\beta)$-expansions as

$$\sum_{n=1}^{\infty} \frac{x_n}{(-\beta)^n} = \sum_{n=1}^{\infty} \frac{-x_{2n-1}}{\beta^{2n-1}} + \sum_{n=1}^{\infty} \frac{x_{2n}}{\beta^{2n}},$$

then we can think of these expansions as β-expansions with alternating alphabets. From here we can easily construct regular β-expansions by adding $\sum_{n=1}^{\infty} \frac{\lfloor \beta \rfloor}{\beta^{2n-1}}$ to it. We can describe this addition as a map $\varphi : \mathcal{S}_{-\beta} \to \mathcal{S}_{\beta}$ such that, for any $\mathbf{x} = (x_1, x_2, \ldots) \in \mathcal{S}_{-\beta}$ we have

$$\varphi(x_n) = \begin{cases} x_n & \text{if } n \text{ is even,} \\ \lfloor \beta \rfloor - x_n & \text{if } n \text{ is odd.} \end{cases}$$

At first sight, this seems like it will work. Indeed $d_{-\beta}(\ell_{-\beta})$ gets mapped to $d_{\beta}(0)$ and $d_{-\beta}(r_{-\beta})$ gets mapped to $d_{\beta}\left(\frac{\lfloor \beta \rfloor}{\beta - 1}\right)$. Unfortunately, this is where the good news ends.

Example 4.3.1. Let $\beta = \frac{1+\sqrt{5}}{2}$ and let $\mathcal{S}_{-\beta}$ be the shift space from Example 3.2.3 (with 11 as forbidden block) and let $\mathcal{S}_{\beta}$ be the shift space from Example 4.1.7 (with 011 as forbidden block). Then $(\overline{0}) \in \mathcal{S}_{-\beta}$ gets mapped to $(\overline{1,0}) \in \mathcal{S}_{\beta}$. On the other hand, $(1,0,0,1,\overline{0}) \in \mathcal{S}_{-\beta}$ gets mapped to $(0,0,1,1,\overline{1,0}) \notin \mathcal{S}_{\beta}$. $\square$

We see that the image of the map φ does not have to be a shift space. However, we can use a general transducer as in Section 4.2 to convert the image so that we will end up with a shift space. To do so, we want $\mathcal{G}_2$ in Algorithm 4.2.10 to be any labeled graph recognizing the shift space we want to convert to (so not necessarily a labeled graph recognizing the greedy β-shift).

Example 4.3.2. Reconsider Example 4.3.1. By using the transducer from Example 4.2.11, we normalize the sequence $(0, 0, 1, 1, \overline{1, 0})$ and end up with $(0, 1, 0, 0, \overline{1, 0})$. This latter sequence is another β-expansion of the same number, but this one does belong to $\mathcal{S}_\beta$. □

We can turn this into a general algorithm.

Algorithm 4.3.3. Algorithm for converting a given shift space $\mathcal{S}_{-\beta}$ into another given shift space $\mathcal{S}_\beta$.

1. For all $\mathbf{x} = (x_1, x_2, \ldots) \in \mathcal{S}_{-\beta}$, replace all digits c in odd positions with $\lfloor \beta \rfloor - c$.
2. Use a transducer recognizing $\mathcal{S}_\beta$ (similar to Algorithm 4.2.10) to convert the sequences obtained after step 1. □

The nice thing about this algorithm is that it works for any combination of shift spaces $\mathcal{S}_{-\beta}$ and $\mathcal{S}_\beta$. On the other hand, it is hardly intuitive: it generally requires the use of a transducer. However, the biggest problem with this algorithm is that it does not commute with the shift map as we originally wanted to.

Example 4.3.4. Reconsider Example 4.1.2 and let $\gamma : \mathcal{S}_{-\beta} \to \mathcal{S}_\beta$ be the map such that if $\mathbf{x} \in \mathcal{S}_{-\beta}$, then $\gamma(\mathbf{x}) \in \mathcal{S}_\beta$ is the output of Algorithm 4.2.3. Then we have

$$\gamma\left(\sigma((\overline{0, 1}))\right) = \gamma\left((\overline{1, 0})\right) = (\overline{0}),$$

but on the other hand

$$\sigma\left(\gamma((\overline{0, 1}))\right) = \sigma\left((\overline{1})\right) = (\overline{1}).$$

We see that $\sigma \circ \gamma \neq \gamma \circ \sigma$. □

Since our algorithm involves changing the digits at odd positions, we might consider using σ^2 (the shift map applied twice) so that digits at odd positions stay at odd positions. Unfortunately, our algorithm does not necessarily commute with σ^2, as it depends on the transducer.

We conclude this section with an example where we can replace the transducer in step 2 with an easier method. This is, however, an exceptional case.

Example 4.3.5. Reconsider Example 4.3.1. For any $k \geq 1$, we have the equality

$$\frac{1}{\beta^k} = \frac{1}{\beta^{k+1}} + \frac{1}{\beta^{k+2}}.$$

Therefore, we can replace the forbidden block 011 with the admissible block 100.

This gives us the following algorithm:

1. Start with any sequence $(x_1, x_2, \ldots) \in \mathcal{S}_{-\beta}$.
2. For all odd $n \geq 1$, replace x_n with $1 - x_n$ (since $\lfloor \beta \rfloor = 1$).
3. (a) Start with $i = 1$.
   (b) Read the block (x_i, x_{i+1}, x_{i+2}).
      - If this block equals $(0, 1, 1)$, then $y_i := 1$ and $y_{i+1} := 0$. Increase i by 2 and go back to step 3(b).
      - Else, $y_i := x_i$. Increase i by 1 and go back to step 3(b).
4. The sequence $(y_1, y_2, \ldots)$ is the converted sequence. □

The conversion algorithm proposed in [FL09, Proposition 8] coincides with our own if we convert to the greedy β-shift.

Summary

The aim of this thesis is to restate as many known theorems on $(-\beta)$-expansions as possible in a more general approach to these expansions.

In Chapter 1 we introduced our more general approach by means of a dynamical system. We quickly had to make a few assumptions as to keep the thesis not too long and complicated; it is after all only a master's thesis. The dynamical system allows us to generate $(-\beta)$-expansions by iterating a fixed map on the interval $I_{-\beta}$.

In Chapter 2 we constructed the shift space containing all $(-\beta)$-expansions that can be generated using a given dynamical system. To do so, we used an invaluable tool: the alternate lexicograhical order. It allowed us to move away from the analytical environment of Chapter 1 to a strictly symbolical environment. Unfortunately, our objects of interest formed a space that is not necessarily closed. So, our next step was to find the closure of this space. We did this by introducing the limits $d^*_{L,-\beta}$ and $d^*_{R,-\beta}$. The closure of this space is a shift space. We concluded the chapter by giving a very important characterization of this shift space by using the two tools introduced in this chapter: Theorem 2.4.4.

In Chapter 3 we looked at the shift space itself. We started by determining when this shift space is sofic. This gave rise to the most important algorithm in this thesis, Algorithm 3.2.5, and subsequently Corollary 3.2.7. A subtle improvement in our algorithm gave us necessary and sufficient conditions for which the shift space is a shift of finite type: Theorem 3.3.4. So far, the conditions given in this chapter were given in our symbolical context. In the remainder of the chapter, we tried to translate these conditions to a more analytical context. We did this for the class of Pisot bases, and managed to translate Corollary 3.2.7 into Corollary 3.4.5.

In Chapter 4 we tried to find a way to convert $(-\beta)$-expansions to β-expansions, preferably in a way that is shift-invariant. We tried to do so by introducing the transducer. Unfortunately, we had to settle with a conversion algorithm that is not shift-invariant (Algorithm 4.3.3).

Bibliography

[Aki07] S. Akiyama – *Pisot number system and its dual tiling*, Physics and theoretical computer science (2007), 133-154.

[FL09] C. Frougny and A.C. Lai – *On negative bases*, Lecture Notes in Computer Science 5583 (2009), 252-263.

[IS09] S. Ito and T. Sadahiro – *Beta-expansions with Negative Bases*, Integers 9 (2009), 239-259.

[Kal09] C. Kalle – *Expansions and extensions*, PhD thesis, 2009.

[KS10] C. Kalle, W. Steiner – *Beta-expansions, natural extensions and multiple tilings associated with Pisot units*, arXiv, 2010.

[Lai09] A.C. Lai – *On expansions in non-integer base*, PhD thesis, 2009.

[LM95] D. Lind and B. Marcus – *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995.

[Par60] W. Parry – *On the* β*-expansions of real numbers*, Acta. Math. Acad. Sci. Hung. 11 (1960), 401-416.

[Ren57] A. Rényi – *Representations for real numbers and their ergodic properties*, Acta. Math. Acad. Sci. Hung. 8 (1957), 477-493.

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