Representations of Lie algebras and the $\mathfrak{su}(5)$ Grand Unified Theory

Bachelor Thesis

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Introduction

At the core of particle physics theory lies the Standard Model, which is widely accepted as a good model for elementary particles and forces. But even though this model it is in accordance with experimental data, there are reasons to search for a different theory.

Firstly, the Standard Model does not explain why the electric charge of the electron and the proton are equal in magnitude.

Secondly, theoretical physicists were inspired to think that the four fundamental forces, namely gravity, the electromagnetic interaction, and the weak and strong interactions could be manifestations of an encompassing force. This idea stems from the already existent unification of the electromagnetic and weak interaction. This electroweak unification has proven to be very successful in providing predictions for experimental data.

Hence the rise of grand unified theories (GUT's), which unify the electroweak and the strong interaction, and theories of everything (TOE's), which unify all the fundamental interactions into one force. Apart from the aesthetic appeal of such theories, it turns out that they also correctly predict the connection between the electric charge of the proton and the electron.

In this thesis we shall elaborate the simplest of the grand unified theories, namely the $\mathfrak{su}(5)$-GUT, initially developed by Howard Georgi and Sheldon Glashow in 1973. Firstly we need to have the mathematical tools for this unification, which are essentially contained in the theory of Lie algebras and their representations. This will be the subject of the first part of this thesis. In the second part we shall show how the $\mathfrak{su}(5)$ unification works, and we will try to see if it is a good model for the physical world.

Part I

Lie algebras and their representations

1 Lie algebras

In this section k is a field, and V is a k-vectorspace.

Definition 1.1. A *Lie bracket* $[.,.]$ on V is a map $[.,.] : V \times V \to V$ that satisfies the following properties:

1. *Right linearity*: $\forall x, y, z \in V$, $\forall \lambda, \mu \in k, [\lambda x + \mu y, z] = \lambda[x, z] + \mu[y, z]$,
2. *Antisymmetry*: $\forall x, y \in V, [x, y] = -[y, x]$,
3. *Jacobi identity*: $\forall x, y, z \in V, [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$.

Remark 1.2. Properties 1 and 2 of a Lie bracket give us left-linearity:

$$\forall x, y, z \in V, \forall \lambda, \mu \in k, [x, \lambda y + \mu z] = \lambda[x, y] + \mu[x, z].$$

So the Lie bracket is bilinear.

Definition 1.3. A *k-Lie algebra* $\mathfrak{g}$ is a k-vectorspace equipped with a Lie bracket. The *dimension of* $\mathfrak{g}$ is the dimension over k of its underlying vectorspace.

Definition 1.4. Let $(A, +, \cdot)$ be a k-algebra, and let $x, y \in A$. We define on $(A, +)$ the *commutator* of x and y as:

$$[x, y] := (x \cdot y) - (y \cdot x). \tag{1.1}$$

It is easy to see that the commutator is a Lie bracket for the k-vectorspace $(A, +)$ (the notation $[.,.]$ is thus justified). From now on any k-algebra inherits a natural structure of a k-Lie algebra, where the Lie bracket is just the commutator.

Definition 1.5. A k-Lie algebra $\mathfrak{g}$ is *commutative* (or *abelian*) if for any $x, y \in \mathfrak{g}$, $[x, y] = 0$.

Remark 1.6. If $(A, +, \cdot)$ is a k-algebra, the associated k-Lie algebra is commutative iff the algebra $(A, +, \cdot)$ is commutative.

Definition 1.7. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two k-Lie algebras. A k-linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ is called a *Lie algebra morphism* if for all $x, y \in \mathfrak{g}$ we have:

$$\phi([x, y]) = [\phi(x), \phi(y)]. \tag{1.2}$$

Example 1.8. By definition, $\mathfrak{gl}(V)$ is the k-Lie algebra associated to the k-algebra $\operatorname{End}(V)$ as constructed in 1.4. If $\dim(V) = n$ (for a positive integer n), then $\dim(\mathfrak{gl}(V)) = n^2$. We also define $\mathfrak{gl}(n,k) := \mathfrak{gl}(k^n)$, which is the k-Lie algebra that has $M_n(k)$ (the set of $n \times n$ matrices with entries in k) as its underlying vectorspace.

Definitions 1.9. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ a sub vectorspace of $\mathfrak{g}$. If $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{a}$ then $\mathfrak{a}$ is called a *Lie subalgebra* of $\mathfrak{g}$. A Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is called an *ideal* if $[\mathfrak{g},\mathfrak{a}] \subset \mathfrak{a}$. Note that due to the bilinearity of the Lie bracket this is equivalent to $[\mathfrak{a},\mathfrak{g}] \subset \mathfrak{a}$.

Remarks 1.10.

1. A Lie subalgebra is also a Lie algebra, when equipped with the induced Lie bracket.
2. If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of a Lie algebra $\mathfrak{g}$, then $[\mathfrak{g}_1,\mathfrak{g}_2] \subset \mathfrak{g}_1 \cap \mathfrak{g}_2$.
3. If $\mathfrak{g}$ is a Lie algebra over $\mathbf{R}$ (resp. over $\mathbf{C}$), then $\mathfrak{g}$ is called a *real* (resp. a *complex*) Lie algebra.

Examples 1.11. Here are some more examples of Lie algebras.

1. A *trivial k-Lie algebra* consists of the zero dimensional k-vectorspace with the trivial Lie bracket. It is denoted by 0.
2. Let $n \in \mathbf{Z}_{\geq 1}$. The (n^2-1)-dimensional k-Lie algebra $\mathfrak{sl}(n,k) = \{x \in \mathfrak{gl}(n,k) : \operatorname{Tr}(x) = 0\}$ is an ideal of $\mathfrak{gl}(n,k)$. This is because for $x, y \in \mathfrak{gl}(n,k)$ we have that $\operatorname{Tr}([x,y]) = \operatorname{Tr}(xy) - \operatorname{Tr}(yx) = 0$, since the trace is linear and cyclic in its argument. We shall also use the notation $\mathfrak{sl}(n)$ to denote $\mathfrak{sl}(n,\mathbf{C})$.
3. Let $i, j \in \{1,2,3\}$, and let $E_{ij} \in M_3(\mathbf{R})$ denote the matrix with a 1 in row i and column j, and with all other entries 0. The *Heisenberg algebra* is the 3-dimensional $\mathbf{R}$-Lie algebra with basis $\{E_{12}, E_{23}, E_{13}\}$. The Lie brackets are:
$$[E_{12}, E_{23}] = E_{13}, \quad [E_{23}, E_{13}] = [E_{13}, E_{12}] = 0. \tag{1.3}$$
It is the space of upper triangular matrices in $\mathfrak{gl}(3,\mathbf{R})$.
4. Let $n \in \mathbf{Z}_{\geq 1}$. The $\mathbf{R}$-Lie algebra $\mathfrak{su}(n)$ is (n^2-1)-dimensional, and consists of traceless anti-hermitian matrices in $M_n(\mathbf{C})$.
5. The 1-dimensional $\mathbf{R}$-Lie algebra $\mathfrak{u}(1)$ consists of all the elements of $i\mathbf{R}$ (imaginary numbers). Its Lie bracket is trivial.

Remark 1.12. There is a deep reason why we use the notation $\mathfrak{sl}(n)$, $\mathfrak{su}(n)$, and $\mathfrak{u}(1)$. It stems from the fact that these are the tangent spaces of the Lie groups $\mathrm{SL}(n)$, resp. $\mathrm{SU}(n)$, resp. $\mathrm{U}(1)$. We have not defined these concepts here, but it is nice to keep this in mind when we encounter Lie groups.

Definition 1.13. Let $\mathfrak{g}$ be a Lie algebra and let $\{\mathfrak{g}_1, \ldots, \mathfrak{g}_m\}$ be a collection of finite dimensional Lie subalgebras of $\mathfrak{g}$. We say that $\mathfrak{g}$ is a *direct sum* of the $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ (notation $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$) if the underlying vectorspace of $\mathfrak{g}$ is a direct sum $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ of the underlying vectorspaces of $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$. So $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ if every $x \in \mathfrak{g}$ can be uniquely written as a sum $x = x_1 + \ldots + x_m$, where $x_i \in \mathfrak{g}_i$ for all $i \in \{1, \ldots, m\}$.
If in addition the $\mathfrak{g}_1, \ldots, \mathfrak{g}_m$ are ideals of $\mathfrak{g}$, then we write $\mathfrak{g}_1 \times \ldots \times \mathfrak{g}_m$.

Remark 1.14. Suppose that $\mathfrak{g} = \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_m$. If $x \in \mathfrak{g}_i, y \in \mathfrak{g}_j$ for $i, j \in \{1, \ldots, m\}, i \neq j$, then in particular $[x, y] = 0$.

2 Representations of Lie algebras

In this section k is a field, $\mathfrak{g}$ is a k-Lie algebra, and V is a k-vectorspace.

Definition 2.1. A *representation* of $\mathfrak{g}$ is a Lie algebra morphism $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$. The *dimension of the representation* is the dimension of the vectorspace V over k.

Example 2.2. For any real or complex Lie algebra with elements in $\mathfrak{gl}(n, \mathbf{C})$ (for any given n), the *defining representation* is the canonical morphism $\mathfrak{g} \to \mathfrak{gl}(n, \mathbf{C})$.

Example 2.3 (Adjoint representation)**.** Using the notation $\mathfrak{g}$ for the underlying vectorspace of $\mathfrak{g}$, we can consider $\mathfrak{gl}(\mathfrak{g})$ as a k-Lie algebra. For all $x \in \mathfrak{g}$ we define a map

$$\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}, y \mapsto [x, y]. \tag{2.1}$$

The map $\mathrm{ad}(x)$ is linear for every $x \in \mathfrak{g}$. The assignment $x \mapsto \mathrm{ad}(x)$ gives us a linear map

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}). \tag{2.2}$$

We will now show that for any $x, y \in \mathfrak{g}$ we have $[\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}([x, y])$, so that ad is a representation of $\mathfrak{g}$. For $x, y, z \in \mathfrak{g}$ we have:

$$\begin{aligned}[\mathrm{ad}(x), \mathrm{ad}(y)](z) &= \mathrm{ad}(x)\,\mathrm{ad}(y)(z) - \mathrm{ad}(y)\,\mathrm{ad}(x)(z), \\ &= [x, [y, z]] - [y, [x, z]].\end{aligned}$$

And the Jacobi identity gives

$$\begin{aligned}[x,[y,z]]-[y,[x,z]] &= [[x,y],z],\\ &= \mathrm{ad}([x,y])(z).\end{aligned}$$

Thus $[\mathrm{ad}(x),\mathrm{ad}(y)](z) = \mathrm{ad}([x,y])(z)$. The map ad is called the *adjoint representation* of $\mathfrak{g}$. The dimension of the adjoint representation is equal to the dimension of $\mathfrak{g}$.

Examples 2.4. Let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$.

1. If $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$, then *the restriction of* ϕ *to* $\mathfrak{a}$, $\phi|_{\mathfrak{a}}$, is a representation of $\mathfrak{a}$. Note that $\dim(\phi) = \dim(\phi|_{\mathfrak{a}})$.

2. If there is a linear subspace $V' \subset V$ such that $\phi(\mathfrak{g})(V') \subset V'$ (we say that V' is invariant under ϕ), then ϕ induces a representation $\phi' : \mathfrak{g} \to \mathfrak{gl}(V')$, defined as $\phi'(x)v := \phi(x)v$ for all $x \in \mathfrak{g}; v \in V'$. We say that ϕ' is a *subrepresentation* of ϕ.

Definition 2.5. Let V' be another k-vectorspace. Two representations $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\phi' : \mathfrak{g} \to \mathfrak{gl}(V')$ of $\mathfrak{g}$ are called *equivalent* if there exists a vectorspace isomorphism $f : V \to V'$ such that for all $x \in \mathfrak{g}$ we have:

$$\phi(x) = f^{-1} \circ \phi'(x) \circ f. \tag{2.3}$$

3 Direct sum of representations and semisimple Lie algebras

Let k be a field, V a finite-dimensional k-vectorspace, and let $\mathfrak{g}$ be a finite dimensional k-Lie algebra.

Definition 3.1. Given two finite dimensional representations $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\phi' : \mathfrak{g} \to \mathfrak{gl}(V')$ we shall define a new representation $\phi \oplus \phi' : \mathfrak{g} \to \mathfrak{gl}(V \oplus V')$, called the *direct sum* of ϕ and ϕ'. Let $x \in \mathfrak{g}, v \in V, v' \in V'$. We define $(\phi \oplus \phi')(x)$ as:

$$(\phi \oplus \phi')(x)(v + v') := \phi(x)v + \phi'(x)v'. \tag{3.1}$$

Remark 3.2. Note that $\phi \oplus \phi'$ is well defined. It is a linear map, because ϕ and ϕ' are linear. And it respects the Lie bracket, since for all $x, y \in \mathfrak{g}$;

$v \in V, v' \in V'$:

$$\begin{aligned}(\phi \oplus \phi')([x,y])(v+v') &= \phi([x,y])v + \phi'([x,y])v', \\ &= [\phi(x),\phi(y)]v + [\phi'(x),\phi'(y)]v', \\ &= \phi(x)\phi(y)v - \phi(y)\phi(x)v, \\ &+ \phi'(x)\phi'(y)v' - \phi'(y)\phi'(x)v', \\ &= [(\phi \oplus \phi')(x), (\phi \oplus \phi')(y)](v+v').\end{aligned}$$

We can consider $\oplus$ to be an operation on the set of finite dimensional representations of $\mathfrak{g}$. This operation is commutative, since $V \oplus V' = V' \oplus V$ gives us that

$$\phi \oplus \phi' = \phi' \oplus \phi. \tag{3.2}$$

We shall denote by 0 the zero dimensional representation. Then 0 is the identity element for the operation $\oplus$:

$$\phi \oplus 0 = \phi. \tag{3.3}$$

Finally $\oplus$ is associative, for if we have another representation $\psi : \mathfrak{g} \to \mathfrak{gl}(W)$ (where W is a finite dimensional k-vectorspace), then it is easy to see that:

$$\phi \oplus (\phi' \oplus \psi) = (\phi \oplus \phi') \oplus \psi. \tag{3.4}$$

We have now proved the following lemma.

Lemma 3.3. *The set of finite dimensional representations of $\mathfrak{g}$ together with the operation $\oplus$ forms a commutative monoid.* □

Definitions 3.4. A Lie algebra $\mathfrak{g}$ is called *simple* if it is non-abelian and has no nontrivial ideals. If $\mathfrak{g}$ has no nonzero abelian ideals, then it is called *semisimple*. Note that a simple Lie algebra is also semisimple.

Example 3.5. For all $n \in \mathbf{Z}_{\geq 1}$, the Lie algebra $\mathfrak{sl}(n, \mathbf{C})$ from example 1.11 is simple. We shall prove this in section 4.

Definitions 3.6. If $\mathfrak{g} \neq 0$, then a representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is called *irreducible* if V has exactly two invariant subspaces under the action of ϕ ($\{0\}$ and V). Otherwise it is called *reducible*. We say that ϕ is *completely reducible* or *semisimple* if it is a direct sum of irreducible representations.

The following theorem is very important in the theory of semisimple Lie algebras. For the proof see [9, paragraph 10.2].

Theorem 3.7 (H. Weyl)**.** *Every (finite-dimensional) linear representation of a semisimple Lie algebra is completely reducible.* □

Definition 3.8. Let $k = \mathbf{C}$, suppose that $\mathfrak{g}$ is semisimple, and let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is called a *Cartan subalgebra* if it is maximal with respect to the following two conditions:

1. The subalgebra $\mathfrak{h}$ is abelian;

2. There exists a basis (of the underlying vectorspace) of $\mathfrak{g}$ with respect to which for all $h \in \mathfrak{h}$ the matrix $\mathrm{ad}(h)$ is diagonal.

Remarks 3.9. 1. The second part of definition 3.8 tells us that the elements of $\{\mathrm{ad}(h) : h \in \mathfrak{h}\}$ all have the same eigenvectors, which span the underlying vectorspace of $\mathfrak{g}$.

2. Actually, we can define a Cartan subalgebra for any Lie algebra, see [8, chapter 3].

We will state the next theorem without proof. For the proof see [8, chapter 3].

Theorem 3.10. *Every semisimple Lie algebra $\mathfrak{g}$ has a Cartan subalgebra, and all Cartan subalgebras of $\mathfrak{g}$ have the same dimension. This dimension is called the* rank *of $\mathfrak{g}$.* □

Remarks 3.11. Because all Cartan subalgebras of a semisimple Lie algebra have the same dimension, they are isomorphic as vectorspaces. Also, because they are abelian, they are actually isomorphic as Lie algebras. Furthermore, the theorem 3.10 is actually also true for any Lie algebra (see [8, chapter 3]).

The *dual* of V, denoted V^* is by definition the k-vectorspace of linear maps $V \to k$. We have a natural pairing

$$V \times V^* \to k, \quad (v, f) \mapsto f(v).$$

If $b : V \times V \to k$ is a bilinear form on V we define its associated morphism:

$$\chi_b : V \to V^*, \quad v \mapsto (w \mapsto b(v, w)).$$

And any morphism $\chi : V \to V^*$ yields a bilinear map

$$V \times V \to k, \quad (v, w) \mapsto (\chi(v))w.$$

If V^{**} denotes the dual of V^*, usually called the *bidual* of V, we have a canonical map

$$V \to V^{**}, \quad v \mapsto (f \mapsto f(v)).$$

Note that k, equipped with the commutator, is an abelian Lie algebra. So any $x \in \mathfrak{g}^*$ (here we view $\mathfrak{g}$ as its underlying vectorspace) is actually a Lie algebra morphism $\mathfrak{g} \to k$.

For the rest of this section let $k = \mathbf{C}$, and $\mathfrak{g}$ a semisimple Lie algebra, and let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$.

Definition 3.12. Let $\alpha \in \mathfrak{h}^*$. An element $x \in \mathfrak{g}$ is said *to have weight* α if for all $h \in \mathfrak{h}$ we have:

$$\mathrm{ad}(h)(x) = \alpha(h)x. \tag{3.5}$$

The subspace of $\mathfrak{g}$ spanned by all $x \in \mathfrak{g}$ with weight α is called the *eigenspace* corresponding to α, notation $\mathfrak{g}^\alpha$. If $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq 0$, then α is called a *root* of $\mathfrak{h}$. The set of roots of $\mathfrak{h}$ of will be denoted R.

Remarks 3.13.

1. Note that the map α in def. 3.12 is really a linear map, because $\mathrm{ad}(h)$ is a linear map for all $h \in \mathfrak{h}$.
2. We immediately see that if $\alpha \in R$, then $-\alpha \in R$.
3. Note that $\mathfrak{g}^0 = \mathfrak{h}$, because $\mathfrak{h}$ is a maximal abelian Lie subalgebra of $\mathfrak{g}$.

Theorem 3.14 (Cartan decomposition of $\mathfrak{g}$)**.** *Let R be the set of roots of $\mathfrak{h}$. We can write $\mathfrak{g}$ as a direct sum:*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}^\alpha \tag{3.6}$$

proof. Let $\alpha, \beta \in \mathfrak{h}^*$, $\alpha \neq \beta$. Then there is a $h \in \mathfrak{h}$ such that $\alpha(h) \neq \beta(h)$. Suppose that there is a nonzero $x \in \mathfrak{g}^\alpha \cap \mathfrak{g}^\beta$. That would mean that for all $h \in \mathfrak{h}$ we have that $\mathrm{ad}(h)x = \alpha(h)x = \beta(h)x$. Then, because $x \neq 0$, we see that $\alpha(h) = \beta(h)$ for all $h \in \mathfrak{h}$. This is a contradiction, so we see that $\mathfrak{g}^\alpha \cap \mathfrak{g}^\beta = 0$. And we have seen in remark 3.9 that the eigenvectors of $\{\mathrm{ad}(h) : h \in \mathfrak{h}\}$ span the space $\mathfrak{g}$, so the elements of all the $\mathfrak{g}^\alpha$ span $\mathfrak{g}$. Now, we have seen in remark 3.13 that $\mathfrak{g}^0 = \mathfrak{h}$. Because of the way we defined R, we see now that $\mathfrak{g}^\alpha \neq 0$ precisely when $\alpha \in R \cup \{0\}$. So $\mathfrak{g} = \bigoplus_{\alpha \in R \cup \{0\}} \mathfrak{g}^\alpha$. $\square$

4 The Lie algebra $\mathfrak{sl}(n)$

Let $n \in \mathbf{Z}_{\geq 1}$, and define $I := \{1, \dots, n\}$. We have defined the complex Lie algebra $\mathfrak{sl}(n)$ in example 1.11. It consists of all the matrices of $M_n(\mathbf{C})$ with trace zero, and has dimension $n^2 - 1$. In this section we shall explore properties of $\mathfrak{sl}(n)$.

Definitions 4.1. We shall define $H_{\lambda_1 \dots \lambda_n} \in M_n(\mathbf{C})$ to be the traceless diagonal matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_n) \in M_n(\mathbf{C})$. Let the matrix $H_{ij} \in M_n(\mathbf{C})$ be the matrix with 1 on its ith diagonal entry, -1 on its jth diagonal entry, and everywhere else 0. And by E_{ij} $(i, j \in I)$ we shall denote the matrix in $M_n(\mathbf{C})$ with entry $(E_{ij})_{ij} = 1$ and with all other entries 0.

Remark 4.2. Note that the set $\{E_{ij} \in M_n(\mathbf{C}); i \neq j\}$ consists of $n^2 - n$ independent elements of $\mathfrak{sl}(n)$, and that

$$\mathfrak{h} := \{H_{\lambda_1 \dots \lambda_n} \in \mathfrak{sl}(n)\} \tag{4.1}$$

is a $(n-1)$-dimensional *abelian* Lie subalgebra of $\mathfrak{sl}(n)$. Now we can see that $\{E_{ij} \in M_n(\mathbf{C}); i \neq j\}$ and $\mathfrak{h}$ together generate a subvectorspace of $\mathfrak{sl}(n)$ of dimension $(n^2 - n) + (n - 1) = n^2 - 1$. We know that $\dim(\mathfrak{sl}(n)) = n^2 - 1$, so this subvectorspace must be $\mathfrak{sl}(n)$ itself.
Note that the set $\{H_{ij} \in M_n(\mathbf{C}) : i < j\}$ is a basis of $\mathfrak{h}$.

We would like to derive the Lie brackets for the generators of $\mathfrak{sl}(n)$ that we have found in remark 4.2. Let $H_{\lambda_1 \dots \lambda_n} \in \mathfrak{h}$, let $i, j, k, l \in I, i \neq j$, and let $\delta_{kl} \in \mathbf{C}$ be Kronecker symbols. Then:

$$\begin{aligned}[H_{\lambda_1 \dots \lambda_n}, E_{ij}] &= \sum_{k=1}^{n} \lambda_k [E_{kk}, E_{ij}] = \sum_{k=1}^{n} \lambda_k (E_{kk}E_{ij} - E_{ij}E_{kk}), \\ &= \sum_{k=1}^{n} \lambda_k (\delta_{ki}E_{kj} - \delta_{jk}E_{ik}) = (\lambda_i - \lambda_j)E_{ij}, \qquad (4.2) \\ [E_{ij}, E_{kl}] &= E_{ij}E_{kl} - E_{kl}E_{ij} = \delta_{jk}E_{il} - \delta_{li}E_{kj}. \qquad (4.3)\end{aligned}$$

Now that we know all the Lie brackets for the generators of $\mathfrak{sl}(n)$, we can prove that $\mathfrak{sl}(n)$ is simple for all $n \in \mathbf{Z}_{\geq 1}$.

Proposition 4.3. *The Lie algebra $\mathfrak{sl}(n)$ is simple.*

proof. Suppose that $\mathfrak{a}$ is a nonzero ideal of $\mathfrak{sl}(n)$. If $E_{ij} \in \mathfrak{a}$ for some $i, j \in I, i \neq j$, then $\mathfrak{a} = \mathfrak{sl}(n)$, because of the following:

$$[E_{ij}, E_{ji}] = H_{ij} \in \mathfrak{a}, \ \text{ so: } \ [H_{ij}, E_{ji}] = -2E_{ji} \in \mathfrak{a}.$$

So:

$$E_{ij} \in \mathfrak{a} \implies H_{ij}, E_{ji} \in \mathfrak{a}. \tag{4.4}$$

For $n = 2$ we are now finished, because we now have that the basis of $\mathfrak{sl}(2)$ is in $\mathfrak{a}$. If $n > 2$ then there is a $k \in I$, such that i, j and k are pairwise different. Then for all such k we have:

$$[E_{ij}, E_{jk}] = E_{ik} \in \mathfrak{a}, \text{ so: } [E_{ki}, E_{ij}] = E_{kj} \in \mathfrak{a},$$

Now we are done for $n = 3$, because with equation 4.4 we see that the basis of $\mathfrak{sl}(3)$ is in $\mathfrak{a}$. If $n > 3$ then there is a $l \in I$, such that i, j, k and l are pairwise different. Then for all such l we have:

$$[E_{kj}, E_{jl}] = E_{kj}E_{jl} - E_{jl}E_{kj} = E_{kl} \in \mathfrak{a}.$$

Now we are also done for $n > 3$, because with equation 4.4 we see that the basis of $\mathfrak{sl}(n)$ is in $\mathfrak{a}$. So for all $n \in \mathbf{Z}_{\geq 1}$ we have: $E_{ij} \in \mathfrak{a} \implies \mathfrak{a} = \mathfrak{sl}(n)$.
We shall now show that there is an element of the form E_{ij} in $\mathfrak{a}$.

Let $A \in \mathfrak{a}, A \neq 0$. If $A \in \mathfrak{h}$, then there exist some $a \in \mathbf{C}^*, i, j \in I, i \neq j$, such that $[A, E_{ij}] = aE_{ij} \neq 0$, which means that $E_{ij} \in \mathfrak{a}$ and we are finished. So without loss of generality we can assume that

$$A = H + \sum_{k,l \in I, k \neq l} a_{kl}E_{kl}, \tag{4.5}$$

where $H \in \mathfrak{h}, a_{kl} \in \mathbf{C}$, and there exist $i, j \in I, i \neq j$ such that $a_{ij} \neq 0$. If

$$A = H + a_{ij}E_{ij} + a_{ji}E_{ji}, \tag{4.6}$$

then

$$\begin{aligned}[H_{ij}, A] + \frac{1}{2}[H_{ij}, [H_{ij}, A]] &= (2a_{ij}E_{ij} - 2a_{ji}E_{ji}) + (2a_{ij}E_{ij} + 2a_{ji}E_{ji}), \\ &= 4a_{ij}E_{ij},\end{aligned}$$

so in this case $E_{ij} \in \mathfrak{a}$, and we are finished. If there are nonzero terms in 4.5, other than the E_{ij}, E_{ji} and H terms, then the element

$$B := [E_{ij}, [H_{ij}, A]] \tag{4.7}$$

is of the form 4.6, but without the E_{ij} and E_{ji} terms. Now if there are some $k, l \in I, k \neq l$ such that the E_{kl} term of B is nonzero, then we can repeat 4.7, only this time we replace ij by kl. And we can keep doing this (removing terms E_{kl} and E_{lk} in this way) until we have an element in $\mathfrak{a}$ that is of the form 4.6 (but maybe with ij replaced with some other index). And we know how to construct an element in $\{E_{ij} \in M_n(\mathbf{C}); i \neq j\}$ from this.
So indeed $\mathfrak{a} = \mathfrak{sl}(n)$. □

Now we can use the results from the previous section for $\mathfrak{sl}(n)$. First we shall show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(n)$.

Proposition 4.4. *The Lie subalgebra $\mathfrak{h} \subset \mathfrak{sl}(n)$ as defined in remark 4.2 is a Cartan subalgebra of $\mathfrak{sl}(n)$.*

proof. Equation 4.2 tells us that for any $H \in \mathfrak{h}, E_{ij} \in \{E_{ij} \in M_n(\mathbf{C}); i \neq j\}$ we have that $\mathrm{ad}(H)(E_{ij}) = [H, E_{ij}] = \alpha_{ij} E_{ij}$ for some $\alpha_{ij} \in \mathbf{C}$. Also, since $\mathfrak{h}$ is abelian we now see that $\mathrm{ad}(H)$ is diagonal in the basis of $\mathfrak{sl}(n)$ consisting of the elements in $\{H_{ij} \in M_n(\mathbf{C}) : i < j\}$ and $\{E_{ij} \in M_n(\mathbf{C}); i \neq j\}$.
Finally, it suffices to prove that $\mathfrak{h}$ is maximal with respect to its abelian property. Suppose that it is not maximal abelian. Then there is an element $A \notin \mathfrak{h}$ such that $[\mathfrak{h}, A] = 0$. Now, A is of the form 4.5, where there exist $i, j \in I, i \neq j$ such that the E_{ij} term is not zero. But then $[H_{ij}, A] \neq 0$, which gives a contradiction. So $\mathfrak{h}$ is a maximal abelian Lie subalgebra of $\mathfrak{sl}(n)$. □

Corollary 4.5. *The rank of $\mathfrak{sl}(n)$ is $n-1$.* □

For all $i, j \in I, i \neq j$ we define a linear map $\alpha_{ij} : \mathfrak{h} \to \mathbf{C}$ as:

$$\forall H_{\lambda_1 \dots \lambda_n} \in \mathfrak{h} : \quad \alpha_{ij}(H_{\lambda_1 \dots \lambda_n}) := \lambda_i - \lambda_j. \tag{4.8}$$

From equation 4.2 we can see that an element $E_{ij} \in \{E_{ij} \in M_n(\mathbf{C}); i \neq j\}$ has weight α_{ij}. And we see that the set of roots R corresponding to $\mathfrak{h}$ is $R = \{\alpha_{ij} : i, j \in I, i \neq j\}$, and $\#R = n^2 - n$.

The elements from $\{E_{ij} \in M_n(\mathbf{C}); i \neq j\}$ are linearly independent, so $\mathfrak{g}^{\alpha_{ij}}$ is 1-dimensional for all $\alpha_{ij} \in R$. A Cartan decomposition of $\mathfrak{sl}(n)$ is:

$$\mathfrak{sl}(n) = \mathfrak{h} \oplus \bigoplus_{\alpha_{ij} \in R} \mathbf{C} E_{ij}. \tag{4.9}$$

Lemma 4.6. *(Properties of roots) Let $\alpha_{ij}, \alpha_{kl} \in R$. Then:*

1. *$\alpha_{ji} = -\alpha_{ij}$,*
2. *$\alpha_{ij} + \alpha_{kl}$ is a root iff $i = l, j \neq k$ or $j = k, i \neq l$. In particular $2\alpha_{ij}$ is* not *a root.*

proof. 1. This is clear from equation 4.8. See also remark 3.13.2.
2. For all $H_{\lambda_1 \dots \lambda_n} \in \mathfrak{h}$:

$$(\alpha_{ij} + \alpha_{kl})(H_{\lambda_1 \dots \lambda_n}) = \lambda_i - \lambda_j + \lambda_k - \lambda_l. \tag{4.10}$$

It is easy to see that $\alpha_{ij} + \alpha_{kl}$ is a root if $i = l, j \neq k$ or if $j = k, i \neq l$. In the case that $i = l$ and $j = k$, we have that $\alpha_{ij} + \alpha_{kl} = \alpha_{ij} + \alpha_{ji} = 0$. But $0 \notin R$, so in this case $\alpha_{ij} + \alpha_{kl}$ is not a root. If $i \neq l$ and $j \neq k$ then equation 4.10 can never be of the form 4.8, since both equations are true for all $H_{\lambda_1 \ldots \lambda_n} \in \mathfrak{h}$. So also in this case $\alpha_{ij} + \alpha_{kl}$ is no root. The last claim follows from this. □

We now know the structure of $\mathfrak{sl}(n)$, but we know little about its representations. In the next section we shall derive all the finite dimensional representations of $\mathfrak{sl}(2)$.

5 Representations of $\mathfrak{sl}(2)$

In the previous section we have seen the structure of $\mathfrak{sl}(2)$, and we know that $\mathfrak{sl}(2)$ is simple. In this section we shall derive all the finite dimensional representations of $\mathfrak{sl}(2)$.

Let V be a finite dimensional complex vectorspace, and let $\phi : \mathfrak{sl}(2) \to \mathfrak{gl}(V)$ be a representation. If $V = 0$, then the representation is trivial. Now take $V \neq 0$. Note that we know that there exists a nontrivial representation of $\mathfrak{sl}(2)$, namely its two dimensional defining representation.

We define in $\mathfrak{sl}(2)$ the following matrices:

$$H := \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \quad X^+ := \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad X^- := \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{5.1}$$

The set $\{H, X^+, X^-\}$ is a basis for $\mathfrak{sl}(2)$. The commutator relations are:

$$[H, X^+] = X^+, \qquad [H, X^-] = -X^-, \qquad [X^+, X^-] = H. \tag{5.2}$$

Definitions 5.1. Let $\lambda \in \mathbf{C}$. $v \in V$ is said *to have weight* λ, if:

$$\phi(H)v = \lambda v. \tag{5.3}$$

The subspace of V spanned by all $v \in V$ with weight λ is called the *eigenspace* corresponding to λ, notation V^λ. Let E be the set of eigenvalues of $\phi(H)$. A nonzero element $v \in V$ is called *primitive of weight* λ if $v \in V^\lambda$ and $X^+v = 0$.

Remark 5.2. If we compare 5.1 with the definition 3.12 of roots, we see that these are closely related.

Proposition 5.3. *The element $\phi(H)$ is diagonalizable.*

We can see from equation 5.2 that in the basis $\{H, X^+, X^-\}$ we have $\mathrm{ad}(H) = \mathrm{diag}(0, 1, -1)$. Then the proposition 5.3 is a corollary of the following theorem that can be found in [8, page 7].

Theorem 5.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra, let $\psi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation, and let $x \in \mathfrak{g}$. If $\mathrm{ad}(x)$ is diagonalizable, then $\psi(x)$ is diagonalizable.* □

Notation 5.5. We shall be using the following notation for $X \in \mathfrak{sl}(2), v \in V$:

$$Xv := \phi(X)v. \tag{5.4}$$

Proposition 5.6. *1. We have a decomposition $V = \bigoplus_{\lambda \in E} V^\lambda$,*

2. If $v \in V^\lambda$, then $X^+v \in V^{\lambda+1}$ and $X^-v \in V^{\lambda-1}$,

3. There exists a $\lambda \in E$, such that V contains a primitive element of weight λ.

proof. 1. We know from proposition 5.3 that $\phi(H)$ is diagonalizable, so the set of eigenspaces of $\phi(H)$ spans V. The sum of the V^λ is direct, because eigenvectors corresponding to different eigenvalues are linearly independent.
2. We know that $\phi([H, X^\pm]) = [\phi(H), \phi(X^\pm)] = \phi(H)\phi(X^\pm) - \phi(X^\pm)\phi(H)$. So, for all $v \in V^\lambda$:

$$HX^\pm v = X^\pm Hv + [H, X^\pm]v = (\lambda \pm 1)X^\pm v. \tag{5.5}$$

3. Since $\phi(H)$ is diagonalizable, we know that $E \neq \emptyset$. Let $\lambda' \in E, v \in V^{\lambda'}$, $v \neq 0$. Since $\dim(V) < \infty$, we can see from part 1 and 2 of this theorem that there must be a smallest positive integer k such that $(X^+)^k v = 0$. Then $(X^+)^{k-1}v$ is a primitive element of weight $\lambda = \lambda' + k - 1$. □

Remark 5.7. Part two of proposition 5.6 is the reason why we use the notation X^+ and X^-. The matrix X^+ is called the *raising operator*, and X^- is called the *lowering operator*.

Lemma 5.8. *Let $e \in V$ be a primitive element of weight λ. Then define $e_{-1} = 0$ and $e_k := (X^-)^k e/k!$ for $k \in \mathbf{Z}_{\geq 0}$. We then have for all $k \geq 0$:*

1. $e_k \in V^{\lambda-k}$,

2. $X^-e_k = (k+1)e_{k+1}$,

3. $X^+e_k = (\lambda - \frac{k-1}{2})e_{k-1}$.

proof. 1: This follows from proposition 5.6.2.
2: This is clear from the definition of e_k.
3. We will prove this with induction on k. Because e is a primitive element, we have $X^+e_0 = X^+e = 0 = (\lambda+1)e_{-1}$, so the formula is true for $k = 0$. Now suppose that the formula is true for $k-1$, with $k > 1$. Using the results from formulas 1 and 2, and remembering that $\phi([X^+, X^-]) = [\phi(X^+), \phi(X^-)]$, we have:

$$\begin{aligned} kX^+e_k &= X^+X^-e_{k-1} = [X^+, X^-]e_{k-1} + X^-X^+e_{k-1}, \\ &= He_{k-1} + (\lambda - \frac{k-2}{2})X^-e_{k-2}, \\ &= ((\lambda - k + 1) + (\lambda - \frac{k}{2} + 1)(k-1))e_{k-1}, \\ &= k(\lambda - \frac{k-1}{2})e_{k-1}. \end{aligned}$$

In the second line we have used the induction assumption. The formula 3 is proved if we divide by k. □

Proposition 5.9. *Let $e \in V$ be a primitive element of weight λ, and let $W \subset V$ be the subspace spanned by the e_k's.*

1. *There is a unique positive integer n such that $e_i = 0$ for any $i \geq n$, and $e_{n-1} \neq 0$. So W is spanned by the set $\{e_0, \ldots, e_{n-1}\}$.*
2. *We have $\lambda = (n-1)/2$.*
3. *If ϕ is irreducible then $V^\lambda = \mathbf{C}e$, $\dim(V) = n$.*

proof. We know that $e_k \in V^{\lambda-k}$. Also: $e_k = 0$ if $k \geq \dim(V)$, since eigenvectors corresponding to different eigenvalues are linearly independent. So there must be a smallest positive integer $n \leq \dim(V)$ such that $e_n = 0$.
For $k \geq n$:

$$e_k = \frac{1}{k!}(X^-)^k e = \frac{n!}{n!k!}(X^-)^{k-n}(X^-)^n e = \frac{n!}{k!}(X^-)^{k-n}e_n = 0,$$

so W is spanned by the set $\{e_0, \ldots, e_{n-1}\}$. Also: $0 = X^+e_n = (\lambda - \frac{n-1}{2})e_{n-1}$ and $e_{n-1} \neq 0$, so $\lambda = (n-1)/2$. Now, because $\{H, X^+, X^-\}$ is a basis for $\mathfrak{sl}(2)$, we can see with proposition 5.6 that $\phi(\mathfrak{sl}(2))(W) \subset W$. And for every $k \in \{0, \ldots, n-1\}$ we have $(X^-)^k e \in \mathbf{C}e_k$, so there is no nontrivial subspace of W that is invariant under ϕ. The last two claims follow from this. □

We can now classify all the finite dimensional representations of $\mathfrak{sl}(2)$. Namely for every positive integer n, *if* there exists an irreducible n-dimensional representation ψ of $\mathfrak{sl}(2)$, then this is the only irreducible n-dimensional representation of $\mathfrak{sl}(2)$ (up to equivalence). In particular we have that $\bar{\psi} = \psi$. We shall now show that there exists such a representation ψ. Firstly, we know the eigenvalues of $\psi(H)$:

$$\{\lambda, \lambda - 1, \dots, -\lambda + 1, -\lambda\}, \tag{5.6}$$

where $\lambda = (n-1)/2$. We define a linear map $\psi' : \mathfrak{sl}(2) \to \mathfrak{gl}(n, \mathbf{C})$:

$$H \mapsto \operatorname{diag}(\lambda, \lambda - 1, \dots, -\lambda + 1, -\lambda), \tag{5.7}$$

$$X^+ \mapsto \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}, \quad X^- \mapsto \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}. \tag{5.8}$$

It is a simple computation to show that ψ' respects the Lie brackets 5.2, so it is a representation of $\mathfrak{sl}(2)$. Furthermore it is easy to show that it is irreducible. Then we see that for any positive integer n there indeed exists an irreducible n-dimensional representation of $\mathfrak{sl}(2)$.

We have now found every finite dimensional representation of $\mathfrak{sl}(2)$, since it is a direct sum of irreducible representations, by theorem 3.7.

6 Complexification

In this section $\mathfrak{g}$ is a $\mathbf{R}$-Lie algebra. We shall denote by $V_{\mathfrak{g}}$ its underlying vectorspace.

If V is a real vectorspace, we know how to extend the scalars to $\mathbf{C}$ and thus construct a complex vectorspace $V \otimes_{\mathbf{R}} \mathbf{C}$, the *complexification of* V. If $\dim_{\mathbf{R}}(V)$ is finite, we have $\dim_{\mathbf{C}}(V \otimes_{\mathbf{R}} \mathbf{C}) = \dim_{\mathbf{R}}(V)$. Note that we can view

$$V \otimes_{\mathbf{R}} \mathbf{C} = \{v_1 + iv_2 : v_1, v_2 \in V\}, \tag{6.1}$$

with scalar multiplication $i(v_1 + iv_2) = (-v_2 + iv_1)$.

It is straightforward to show that there is a unique complex Lie algebra $\mathfrak{f}$ such that $\mathfrak{g} \hookrightarrow \mathfrak{f}$ is a Lie algebra morphism, and such that $V_{\mathfrak{f}} = V_{\mathfrak{g}} \otimes_{\mathbf{R}} \mathbf{C}$. We shall denote $\mathfrak{f}$ by $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, the *complexification of* $\mathfrak{g}$.

Examples 6.1. Recall the Lie algebras from example 1.11. We have:

1. $\mathfrak{gl}(n, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{gl}(n, \mathbf{C})$,
2. $\mathfrak{sl}(n, \mathbf{R}) \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{sl}(n)$,
3. $\mathfrak{su}(n) \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{sl}(n)$.

Let V be a complex vectorspace. If $\phi : \mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} \to \mathfrak{gl}(V)$ is a representation, we can restrict ϕ to $\mathfrak{g}$ and find a representation of $\mathfrak{g}$. This so called *complex representation* of $\mathfrak{g}$ is $\mathbf{R}$-linear, *not* $\mathbf{C}$-linear. Conversely, if $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a complex representation, we can construct a canonical representation $\phi \otimes_{\mathbf{R}} \mathbf{C}$ of $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$, namely for $x \in \mathfrak{g}, \lambda \in \mathbf{C}$:

$$(\phi \otimes_{\mathbf{R}} \mathbf{C})(x \otimes_{\mathbf{R}} \lambda) = \lambda(\phi \otimes_{\mathbf{R}} \mathbf{C})(x \otimes_{\mathbf{R}} 1) = \lambda\phi(x). \tag{6.2}$$

Example 6.2. The defining representation from example 2.2 is a complex representation of dimension n.

Example 6.3. (Complex conjugate representation) Suppose that $V = \mathbf{C}^n$ (where n is a positive integer), $V_{\mathfrak{g}} \subset M_n(\mathbf{C})$, and let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex representation of $\mathfrak{g}$. We define a map $\bar{\phi} : \mathfrak{g} \to \mathfrak{gl}(V)$ by $\bar{\phi}(x) := -\phi(x)^*$, where $\phi(X)^*$ is the conjugate transpose of $\phi(x)$. For all $x, y \in \mathfrak{g}$ we have $\bar{\phi}([x, y]) = [\bar{\phi}(x), \bar{\phi}(y)]$, since

$$\bar{\phi}([x, y]) = -\phi([x, y])^* = -[\phi(x), \phi(y)]^*,$$

and

$$[\bar{\phi}(x), \bar{\phi}(y)] = [-\phi(x)^*, -\phi(y)^*] = -[\phi(x), \phi(y)]^*.$$

Note that $\bar{\phi}$ is $\mathbf{R}$-linear, so it is actually a complex representation of $\mathfrak{g}$, and it is called the *complex conjugate of* ϕ.

Proposition 6.4. *Let V be a complex vectorspace, and let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a complex representation of $\mathfrak{g}$. Then ϕ is irreducible iff $\phi \otimes_{\mathbf{R}} \mathbf{C}$ is irreducible.*

proof. Suppose ϕ is irreducible, and suppose $W \subset V$ is invariant under $\phi \otimes_{\mathbf{R}} \mathbf{C}$. Then W is invariant under ϕ (as a restriction of $\phi \otimes_{\mathbf{R}} \mathbf{C}$ to $\mathfrak{g}$). So $W = V$ or $W = \{0\}$. Conversely, suppose $\phi \otimes_{\mathbf{R}} \mathbf{C}$ is irreducible, and suppose $W \subset V$ is invariant under ϕ. Now, $(\phi \otimes_{\mathbf{R}} \mathbf{C})(\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C})(W) \subset \mathbf{C}\phi(\mathfrak{g})(W) + \mathbf{C}\phi(\mathfrak{g})(W) \subset W$, since W is invariant under ϕ. But $\phi \otimes_{\mathbf{R}} \mathbf{C}$ is irreducible, so $W = V$ or $W = \{0\}$. □

Remark 6.5. In particular, we see that there is a one to one correspondence between representations of $\mathfrak{sl}(n)$ (for any $n \in \mathbf{Z}_{\geq 1}$) and complex representations of $\mathfrak{su}(n)$.

7 More operations on representations

In section 3 we defined a direct sum of representations. In this section we shall define some more operations on representations, and explore their properties. This shall prove to be very useful in the second part of this thesis, when we discuss the $\mathfrak{su}(5)$ unification theory.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field k, let V and V' be finite dimensional k-vectorspaces of dimension n resp. m, and let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\phi' : \mathfrak{g} \to \mathfrak{gl}(V')$ be two representations of $\mathfrak{g}$.

7.1 Tensor product of representations

We know how to construct a tensor product of vectorspaces. Let us now define a tensor product of representations.

Definition 7.1. Given the two representations $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ and $\phi' : \mathfrak{g} \to \mathfrak{gl}(V')$ we shall define a new representation $\phi \otimes \phi' : \mathfrak{g} \to \mathfrak{gl}(V \otimes V')$, called the *tensor product* of ϕ and ϕ'. Let $x \in \mathfrak{g}, v \in V, v' \in V'$. We define $(\phi \otimes \phi')(x)$ as the linear extension of:

$$(\phi \otimes \phi')(x)(v \otimes v') := \phi(x)v \otimes v' + v \otimes \phi'(x)v'. \tag{7.1}$$

Remark 7.2. Note that $\phi \otimes \phi'$ is well defined, because it is linear and it respects the Lie bracket, since for all $x, y \in \mathfrak{g}, v \in V, v' \in V'$ we have that:

$$\begin{aligned}
(\phi \otimes \phi')([x,y])(v \otimes v') &= \phi([x,y])v \otimes v' + v \otimes \phi'([x,y])v', \\
&= [\phi(x), \phi(y)]v \otimes v' + v \otimes [\phi'(x), \phi'(y)]v', \\
&= \phi(x)\phi(y)v \otimes v' - \phi(y)\phi(x)v \otimes v', \\
&+ v \otimes \phi'(x)\phi'(y)v' - v \otimes \phi'(y)\phi'(x)v',
\end{aligned}$$

and

$$\begin{aligned}
[(\phi \otimes \phi')(x), (\phi \otimes \phi')(y)](v \otimes v') &= (\phi \otimes \phi')(x)(\phi(y)v \otimes v' + v \otimes \phi'(y)v'), \\
&- (\phi \otimes \phi')(y)(\phi(x)v \otimes v' + v \otimes \phi'(x)v'), \\
&= \phi(x)\phi(y)v \otimes v' + \cancel{\phi(y)v \otimes \phi'(x)v'}, \\
&+ \cancel{\phi(x)v \otimes \phi'(y)v'} + v \otimes \phi'(x)\phi'(y)v', \\
&- \phi(y)\phi(x)v \otimes v' - \cancel{\phi(x)v \otimes \phi'(y)v'}, \\
&- \cancel{\phi(y)v \otimes \phi'(x)v'} - v \otimes \phi'(y)\phi'(x)v'.
\end{aligned}$$

Remark 7.3 (Matrix notation)**.** Let $v \otimes v' \in V \otimes V'$. If we write out the vectors v and v' in some basis of V resp. V', say $v^T = (v_1, \ldots, v_n)$ and $v'^T = (v'_1, \ldots, v'_m)$, then we can identify $v \otimes v'$ as vv'^T, which is just the usual matrix product:

$$v \otimes v' \simeq vw^T := \begin{pmatrix} v_1v'_1 & \ldots & v_1v'_m \\ \vdots & \ddots & \vdots \\ v_nv'_1 & \ldots & v_nv'_m \end{pmatrix} \tag{7.2}$$

We can now easily see that we can identify $V \otimes V'$ as the vectorspace $M_{n\times m}(k)$ of $n \times m$ matrices over k. Then, for $M \in M_{n\times m}(k)$ we have

$$(\phi \otimes \phi')(x)(M) = \phi(x)M + M\phi'(x)^T, \tag{7.3}$$

since for all $v \in V, v' \in V'$:

$$\begin{aligned}(\phi \otimes \phi')(x)(v(v')^T) &= \phi(x)v(v')^T + v(\phi'(x)v')^T, \\ &= \phi(x)v(v')^T + v(v')^T\phi'(x)^T.\end{aligned}$$

Lemma 7.4. *We shall denote by* 1 *the one dimensional trivial representation. Let $\psi : \mathfrak{g} \to \mathfrak{gl}(W)$ be another finite dimensional representation of $\mathfrak{g}$. Then:*

1. $\phi \otimes 1 \simeq \phi$,
2. $\phi \otimes \phi' \simeq \phi' \otimes \phi$,
3. $\phi \otimes (\phi' \otimes \psi) = (\phi \otimes \phi') \otimes \psi$,
4. $\phi \otimes 0 = 0 \otimes \phi = 0$,
5. *Distribution over* $\oplus$*:* $\psi \otimes (\phi \oplus \phi') = (\psi \otimes \phi) \oplus (\psi \otimes \phi')$, *and* $(\phi \oplus \phi') \otimes \psi = (\phi \otimes \psi) \oplus (\phi' \otimes \psi)$.

proof. Let $x \in \mathfrak{g}, v \in V, v' \in V', w \in W$.

1. Note that $V \otimes V'$ is isomorphic to $V' \otimes V$ via the linear map $f : V \otimes V' \to V' \otimes V : v \otimes v' \mapsto v' \otimes v$. Then:

$$\begin{aligned}(\phi' \otimes \phi)(x)(f(v \otimes v')) &= (\phi' \otimes \phi)(x)(v' \otimes v), \\ &= \phi'(x)v' \otimes v + v' \otimes \phi(x)v, \\ &= f(v \otimes \phi'(x)v' + \phi(x)v \otimes v'), \\ &= f((\phi \otimes \phi')(x)(v \otimes v')).\end{aligned}$$

2. Let $\phi' = 1$, and let e'_1 be the basis of V'. Then $v' = \lambda e'_1$ for some $\lambda \in k$, and the linear map $g : V \otimes V' \to V : v \otimes v' \mapsto \lambda v$ is a vectorspace isomorphism. Then:

$$\begin{aligned} g((\phi \otimes 1)(x)(v \otimes v')) &= g(\phi(x)v \otimes v' + v \otimes 0), \\ &= g(\phi(x)v \otimes v'), \\ &= \lambda\phi(x)v = \phi(x)(g(v \otimes v')). \end{aligned}$$

3. This is a straightforward computation.

4. $(\phi \otimes 0)(x)(v \otimes 0) = 0 = (0 \otimes \phi)(x)(0 \otimes v)$.

5. Finally, we shall prove the distributive property:

$$\begin{aligned} &(\psi \otimes (\phi \oplus \phi'))(x)(w \otimes (v + v')) \\ &= \psi(x)w \otimes (v + v') + w \otimes (\phi \oplus \phi')(x)(v + v'), \\ &= \psi(x)w \otimes v + \psi(x)w \otimes v' + w \otimes \phi(x)v + w \otimes \phi'(x)v', \\ &= (\psi \otimes \phi)(x)(w \otimes v) + (\psi \otimes \phi')(x)(w \otimes v'), \\ &= ((\psi \otimes \phi) \oplus (\psi \otimes \phi'))(w \otimes (v + v')). \end{aligned}$$

The other distribution property is proved similarly.

□

We can define operations $\oplus$ and $\otimes$ in a natural way on the space of equivalence classes of finite dimensional representations of $\mathfrak{g}$. Let's call this space $\mathrm{EQR}(\mathfrak{g})$. For a finite dimensional representation ϕ of $\mathfrak{g}$, the corresponding element in $\mathrm{EQR}(\mathfrak{g})$ is $[\phi]$. The previous lemma will be important to prove that $\mathrm{EQR}(\mathfrak{g})$ together with the operations $\oplus$ and $\otimes$ has a natural structure of a semiring. The precise definition of a semiring will follow shortly, but for now we can imagine it to be a "ring without inverse elements for the summation".

Definition 7.5. For elements $[\phi], [\phi'] \in \mathrm{EQR}(\mathfrak{g})$ we define:

$$[\phi] \oplus [\phi'] := [\phi \oplus \phi'], \tag{7.4}$$

$$[\phi] \otimes [\phi'] := [\phi \otimes \phi']. \tag{7.5}$$

Remark 7.6. We need to check that these operations are well defined. Let $\psi : \mathfrak{g} \to \mathfrak{gl}(W)$, and $\psi' : \mathfrak{g} \to \mathfrak{gl}(W')$ be two other finite dimensional representations of $\mathfrak{g}$. We need to show that if $[\phi] = [\psi]$ and $[\phi'] = [\psi']$, then

$[\phi \oplus \phi'] = [\psi \oplus \psi']$ and $[\phi \otimes \phi'] = [\psi \otimes \psi']$. We know that there are vectorspace isomorphisms $f : V \to W$ and $f' : V' \to W'$, such that for all $x \in \mathfrak{g}$: $f \circ \phi(x) = \psi(x) \circ f$, $f' \circ \phi'(x) = \psi'(x) \circ f'$. We define two new vectorspace isomorphisms g and h as follows:

$$g : V \oplus V' \to W \oplus W' : g(v + v') := f(v) + f'(v'),$$
$$h : V \otimes V' \to W \otimes W' : h(v \otimes v') := f(v) \otimes f'(v').$$

Then:

$$\begin{aligned} g((\psi \oplus \psi')(x)(v + v')) &= g(\phi(x)v + \phi'(x)v'), \\ &= f(\phi(x)v) + f'(\phi'(x)v'), \\ &= \psi(x)(f(v)) + \psi'(x)(f'(v')), \\ &= (\psi \oplus \psi')(x)(f(v) + f'(v')), \\ &= (\psi \oplus \psi')(x)(g(v + v')), \end{aligned}$$

and

$$\begin{aligned} h((\psi \otimes \psi')(x)(v \otimes v')) &= h(\phi(x)v \otimes v' + v \otimes \phi'(x)v'), \\ &= f(\phi(x)v) \otimes f'(v') + f(v) \otimes f'(\phi'(x)v'), \\ &= \psi(x)(f(v)) \otimes f'(v') + f(v) \otimes \psi'(x)(f'(v')), \\ &= (\psi \otimes \psi')(x)(f(v) \otimes f'(v')), \\ &= (\psi \otimes \psi')(x)(h(v \otimes v')). \end{aligned}$$

So indeed the operations $\oplus$ and $\otimes$ are well defined on $\mathrm{EQR}(\mathfrak{g})$.

Definition 7.7. Let R be a set, and let $+, \cdot$ be two operations on R. Then $(R, +, \cdot)$ is called a *semiring* if:

1. $(R, +)$ and $(R, \cdot)$ are monoids with identity elements 0 resp. 1, and $(R, +)$ is commutative.
2. Distribution over $+$: For all $x, x', y \in R$ we have $y \cdot (x + x') = y \cdot x + y \cdot x'$, and $(x + x') \cdot y = x \cdot y + x' \cdot y$.
3. The element 0 annihilates R: For all $x \in R$ we have $0 \cdot x = x \cdot 0 = 0$.

A semiring is called *commutative* if $(R, \cdot)$ is commutative.

With lemmas 3.3 and 7.4 we have now proved the following proposition.

Proposition 7.8. *The set* $\mathrm{EQR}(\mathfrak{g})$ *equipped with operations* $\oplus$ *and* $\otimes$ *is a commutative semiring, with distribution over* $\oplus$. □

7.2 Symmetric and antisymmetric tensor product

Definitions 7.9. We shall now define a linear map $S : V \otimes V \to V \otimes V$, called *symmetrization map*. For all $v_1, v_2 \in V$:

$$S(v_1 \otimes v_2) := v_1 \otimes v_2 + v_2 \otimes v_1. \tag{7.6}$$

The linear subspace $S(V \otimes V)$ of $V \otimes V$ is called the *symmetrization of* $V \otimes V$. In a similar way we define a linear map $A : V \otimes V \to V \otimes V$, called the *antisymmetrization map*. For all $v_1, v_2 \in V$:

$$A(v_1 \otimes v_2) := v_1 \otimes v_2 - v_2 \otimes v_1. \tag{7.7}$$

The linear subspace $A(V \otimes V)$ of $V \otimes V$ is called the *antisymmetrization of* $V \otimes V$.

Remark 7.10 (Matrix notation)**.** If we identify $V \otimes V$ as $M_n(k)$, then for all $v_1, v_2 \in V$ we have that:

$$S(v_1 v_2^T) = v_1 v_2^T + v_2 v_1^T = v_1 v_2^T + (v_1 v_2^T)^T,$$

$$A(v_1 v_2^T) = v_1 v_2^T - v_2 v_1^T = v_1 v_2^T - (v_1 v_2^T)^T.$$

So for all $M \in M_n(k)$: $S(M) = M + M^T$, and $A(M) = M - M^T$. We now see that we can identify $S(V \otimes V)$ and $A(V \otimes V)$ as the subspace of symmetric resp. antisymmetric matrices in $M_n(k)$.

Remarks 7.11. Let $\{e_1, \ldots, e_n\}$ be a basis of V. Then

$$\{S(e_{i_1} \otimes e_{i_2}) | 1 \leq i_1 \leq i_2 \leq n\} \tag{7.8}$$

is a basis of $S(V \otimes V)$. Furthermore, if $n \leq 1$, then $A(V \otimes V) = 0$. Otherwise, if $n \geq 2$, then $A(V \otimes V)$ has a basis

$$\{A(e_{i_1} \otimes e_{i_2}) | 1 < i_1 < i_2 < n\}. \tag{7.9}$$

We can now calculate the dimensions of $S(V \otimes V)$ and $A(V \otimes V)$ with standard combinatorics:

$$\dim(S(V \otimes V)) = \binom{n+1}{2} = n(n+1)/2, \tag{7.10}$$

$$\dim(A(V \otimes V)) = \binom{n}{2} = n(n-1)/2. \tag{7.11}$$

In the case that $n \geq 2$ we get:

$$\dim(S(V \otimes V)) + \dim(A(V \otimes V))) = \binom{n+1}{2} + \binom{n}{2} = n^2 = \dim(V \otimes V).$$

Also, we can easily see that $S \circ A \equiv 0 \equiv A \circ S$, so for $v_1, v_2 \in V$ we have:

$$V \otimes V = S(V \otimes V) \oplus A(V \otimes V). \tag{7.12}$$

In particular, for $v_1, v_2 \in V$ we have:

$$v_1 \otimes v_2 = 1/2(S(v_1 \otimes v_2) + A(v_1 \otimes v_2)). \tag{7.13}$$

If we have a representation $\phi \otimes \phi : \mathfrak{g} \to \mathfrak{gl}(V \otimes V)$, then this induces representations on $S(V \otimes V)$ and on $A(V \otimes V)$, because $(\phi \otimes \phi)(S(V \otimes V)) \subset S(V \otimes V)$ and $(\phi \otimes \phi)(A(V \otimes V)) \subset A(V \otimes V)$, since for all $x \in \mathfrak{g}; v_1, v_2 \in V$ we have:

$$\begin{aligned}
(\phi \otimes \phi)(x)(S(v_1 \otimes v_2)) &= (\phi \otimes \phi)(x)(v_1 \otimes v_2 + v_2 \otimes v_1), \\
&= S(\phi(x)v_1 \otimes v_2 + v_1 \otimes \phi(x)v_2), \\
&= S((\phi \otimes \phi)(x)(v_1 \otimes v_2)), \\
(\phi \otimes \phi)(x)(A(v_1 \otimes v_2)) &= (\phi \otimes \phi)(x)(v_1 \otimes v_2 - v_2 \otimes v_1), \\
&= A(\phi(x)v_1 \otimes v_2 + v_1 \otimes \phi(x)v_2), \\
&= A((\phi \otimes \phi)(x)(v_1 \otimes v_2)).
\end{aligned}$$

We will denote these induced representations as $S(\phi \otimes \phi) : \mathfrak{g} \to \mathfrak{gl}(S(V \otimes V))$ and $A(\phi \otimes \phi) : \mathfrak{g} \to \mathfrak{gl}(A(V \otimes V))$.

Example 7.12. For $n \geq 2$ we can see from equation 7.13 that

$$\phi \otimes \phi = \frac{1}{2}(S(\phi \otimes \phi) \oplus A(\phi \otimes \phi)). \tag{7.14}$$

7.3 Direct product of representations

In this subsection let $\mathfrak{g}_1, \mathfrak{g}_2$ be finite dimensional k-Lie algebras, and let $\phi_1 : \mathfrak{g}_1 \to \mathfrak{gl}(V_1)$ and $\phi_2 : \mathfrak{g}_2 \to \mathfrak{gl}(V_2)$ be finite dimensional representations.

Definition 7.13. Suppose that $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$. We shall define a representation $\phi_1 \times \phi_2 : \mathfrak{g} \to \mathfrak{gl}(V_1 \otimes V_2)$ of $\mathfrak{g}$, which we shall call a *direct product representation* of $\mathfrak{g}$. Let $x \in \mathfrak{g}, x_1 \in \mathfrak{g}_1, x_2 \in \mathfrak{g}_2, v_1 \in V_1, v_2 \in V_2$ such that $x = x_1 + x_2$. Then:

$$(\phi_1 \times \phi_2)(x)(v_1 \otimes v_2) := \phi_1(x_1)v_1 \otimes v_2 + v_1 \otimes \phi_2(x_2)v_2. \tag{7.15}$$

Remark 7.14. Note that $\phi_1 \times \phi_2$ really is a representation: It is a linear map because ϕ_1 and ϕ_2 are linear. We shall show that it also respects the Lie bracket. Let $x, y \in \mathfrak{g}$ such that $x = x_1 + x_2, y = y_1 + y_2$, where $x_1, y_1 \in \mathfrak{g}_1, x_2, y_2 \in \mathfrak{g}_2$. From remark 1.14 we can see that $[x, y] = [x_1, y_1] + [x_2, y_2]$. Then we have:

$$\begin{aligned}
(\phi_1 \times \phi_2)([x,y])(v_1 \otimes v_2) &= \phi_1([x_1,y_1])v_1 \otimes v_2 + v_1 \otimes \phi_2([x_2,y_2])v_2, \\
&= [\phi_1(x_1), \phi_1(y_1)]v_1 \otimes v_2 + v_1 \otimes [\phi_2(x_2), \phi_2(y_2)]v_2, \\
&= \phi_1(x_1)\phi_1(y_1)v_1 \otimes v_2 - \phi_1(y_1)\phi_1(x_1)v_1 \otimes v_2, \\
&+ v_1 \otimes \phi_2(x_2)\phi_2(y_2)v_2 - v_1 \otimes \phi_2(y_2)\phi_2(x_2)v_2,
\end{aligned}$$

and

$$\begin{aligned}
&[(\phi_1 \times \phi_2)(x), (\phi_1 \times \phi_2)(y)](v_1 \otimes v_2), \\
&= (\phi_1 \times \phi_2)(x)(\phi_1 \times \phi_2)(y)(v_1 \otimes v_2) - (\phi_1 \times \phi_2)(y)(\phi_1 \times \phi_2)(x)(v_1 \otimes v_2), \\
&= (\phi_1 \times \phi_2)(x)(\phi_1(y_1)v_1 \otimes v_2) + (\phi_1 \times \phi_2)(x)(v_1 \otimes \phi_2(y_2)v_2), \\
&- (\phi_1 \times \phi_2)(y)(\phi_1(x_1)v_1 \otimes v_2) - (\phi_1 \times \phi_2)(y)(v_1 \otimes \phi_2(x_2)v_2), \\
&= \phi_1(x_1)\phi_1(y_1)v_1 \otimes v_2 + \cancel{\phi_1(y_1)v_1 \otimes \phi_2(x_2)v_2} + \cancel{\phi_1(x_1)v_1 \otimes \phi_2(y_2)v_2}, \\
&+ v_1 \otimes \phi_2(x_2)\phi_2(y_2)v_2 - \phi_1(y_1)\phi_1(x_1)v_1 \otimes v_2 - \cancel{\phi_1(x_1)v_1 \otimes \phi_2(y_2)v_2}, \\
&- \cancel{\phi_1(y_1)v_1 \otimes \phi_2(x_2)v_2} - v_1 \otimes \phi_2(y_2)\phi_2(x_2)v_2.
\end{aligned}$$

Lemma 7.15. *Let $\psi_1 : \mathfrak{g}_1 \to \mathfrak{gl}(W_1)$ and $\psi_2 : \mathfrak{g}_2 \to \mathfrak{gl}(W_2)$ be representations, where W_1 and W_2 are finite dimensional k-vectorspaces. Then:*

$$(\phi_1 \times \phi_2) \otimes (\psi_1 \times \psi_2) \simeq (\phi_1 \otimes \psi_1) \times (\phi_2 \otimes \psi_2). \tag{7.16}$$

proof. Let $x \in \mathfrak{g}, x_1 \in \mathfrak{g}_1, x_2 \in \mathfrak{g}_2, v_1 \in V_1, v_2 \in V_2, w_1 \in W_1, w_2 \in W_2$, such that $x = x_1 + x_2$. Then:

$$\begin{aligned}
&((\phi_1 \times \phi_2) \otimes (\psi_1 \times \psi_2))(x)((v_1 \otimes v_2) \otimes (w_1 \otimes w_2)) \\
&= (\phi_1 \times \phi_2)(x)(v_1 \otimes v_2) \otimes (w_1 \otimes w_2) + (v_1 \otimes v_2) \otimes (\psi_1 \times \psi_2)(x)(w_1 \otimes w_2), \\
&= (\phi_1(x_1)v_1 \otimes v_2) \otimes (w_1 \otimes w_2) + (v_1 \otimes \phi_2(x_2)v_2) \otimes (w_1 \otimes w_2), \\
&+ (v_1 \otimes v_2) \otimes (\psi_1(x_1)w_1 \otimes w_2) + (v_1 \otimes v_2) \otimes (w_1 \otimes \psi_2(x_2)w_2),
\end{aligned}$$

and

$$\begin{aligned}
&((\phi_1 \otimes \psi_1) \times (\phi_2 \otimes \psi_2))(x)((v_1 \otimes w_1) \otimes (v_2 \otimes w_2)) \\
&= (\phi_1 \otimes \psi_1)(x_1)(v_1 \otimes w_1) \otimes (v_2 \otimes w_2) + (v_1 \otimes w_1) \otimes (\phi_2 \otimes \psi_2)(x_2)(v_2 \otimes w_2), \\
&= (\phi_1(x_1)v_1 \otimes w_1) \otimes (v_2 \otimes w_2) + (v_1 \otimes \psi_1(x_1)w_1) \otimes (v_2 \otimes w_2) \\
&+ (v_1 \otimes w_1) \otimes (\phi_2(x_2)v_2 \otimes w_2) + (v_1 \otimes w_1) \otimes (v_2 \otimes \psi_2(x_2)w_2).
\end{aligned}$$

Now note that $(V_1 \otimes V_2) \otimes (W_1 \otimes W_2)$ is isomorphic to $(V_1 \otimes W_1) \otimes (V_2 \otimes W_2)$, via the isomorphism $(v_1 \otimes v_2) \otimes (w_1 \otimes w_2) \mapsto (v_1 \otimes w_1) \otimes (v_2 \otimes w_2)$. □

Remarks 7.16. Let $\mathfrak{g}_3$ be another finite dimensional k-Lie algebra with finite dimensional representation $\phi_3 : \mathfrak{g}_3 \to \mathfrak{gl}(V_3)$. Now suppose that $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \times \mathfrak{g}_3$. Then, it can easily be checked that

$$(\phi_1 \times \phi_2) \times \phi_3 = \phi_1 \times (\phi_2 \times \phi_3),$$

so we can use the notation $\phi_1 \times \phi_2 \times \phi_3$ for this representation. We shall introduce another notation which we shall use in the second part of this thesis, because it is used in particle physics: (ϕ_1, ϕ_2, ϕ_3).
Now, let ψ_1, ψ_2 be as in lemma 7.15, and let $\psi_3 : \mathfrak{g}_3 \to \mathfrak{gl}(W_3)$ be another finite dimensional representation. It is easy to see that the lemma can be extended:

$$(\phi_1, \phi_2, \phi_3) \otimes (\psi_1, \psi_2, \psi_3) \simeq (\phi_1 \otimes \psi_1, \phi_2 \otimes \psi_2, \phi_3 \otimes \psi_3). \tag{7.17}$$

Part II

Application: the $\mathfrak{su}(5)$ grand unification

8 Particle physics and the Standard Model

With the mathematical theory from the previous sections we shall model interactions between elementary particles. In this section we will present the results of the *Standard Model*, which describes the basic principles of particle physics theory.

We distinguish *fermions* and *forces (interactions)*: forces act on fermions in some way. We shall discuss the relation between *fundamental or elementary* forces and fermions. These are without substructure, which means that all the other forces and particles are composites of the fundamental ones.

8.1 Fermions

Fermions consist of *matter* and *antimatter*. The fundamental fermions of matter resp. antimatter are called elementary particles resp. antiparticles. For every particle there is an antiparticle that has same mass as the particle, but opposite electric charge. All the stable fermions that exists in the universe are particles, which are divided in two groups: *leptons* and *quarks*.

Quarks and leptons consist of six types of particles, also called *flavors*: Lepton flavors: e (electron), ν (electron-neutrino), μ (muon), ν_μ (muon-neutrino), τ (tauon) and ν_τ (tauon-neutrino). Quark flavors: u (up), d (down), s (strange), c (charm), t (top), b (bottom). Of course all of these flavors have their antiparticle counterpart, which is denoted with a bar, for example $\bar{e}$ (antielectron) and $\bar{u}$ (anti up quark). The antielectron is also called the *positron*. We should note that the three neutrinos have negligible mass, and they also have electric charge zero.

There are three *generations (or families)* of particles/antiparticles, each selected by the mass of the fermions (except for the neutrinos, which are selected differently): the first generation consists of the lightest, and the third of the heaviest fermions.

	Leptons	Quarks
1st generation	e, ν	u, d
2nd generation	μ, ν_μ	s, c
3rd generation	τ, ν_τ	t, b

Figure 8.1: Elementary particles

8.2 Fundamental forces

In particle physics there are four fundamental forces: the electromagnetic, the weak, the strong, and the gravitational force. The first three of these act on fermions in a way that can be modeled by representations of Lie algebras. These forces are mediated by particles called *bosons*, which are identified by a basis of the Lie algebra in question.

Model 8.1 (Relation between Lie algebra theory and particle physics)**.**
A force is modeled as a real Lie algebra $\mathfrak{g}$, particles/antiparticles are modeled as elements of a complex vector space V. And the action of the force on a particle resp. antiparticle is modeled as a complex representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ resp. complex conjugate representation $\bar{\phi} : \mathfrak{g} \to \mathfrak{gl}(V)$. There are $\dim(\mathfrak{g})$ independent bosons that are mediators of the force.

In figure 8.2 we can see which Lie algebra is associated with which fundamental force. Note that $\dim(\mathfrak{su}(3)) = 8$, and we know from physics theory that there are eight independent bosons for the strong force, the *gluons*. There are three independent bosons for the weak interaction, the W^+, W^-, Z^0 bosons, and we see that $\dim(\mathfrak{su}(2)) = 3$. There is one independent photon, $\dim(\mathfrak{u}(1)) = 1$.

The way forces act on a fermion depends on the properties of the fermion in question. The physical property connected to the strong force is called *color*. The only fermions with color are quarks and antiquarks. Furthermore, *electric charge* is connected to the electromagnetic interaction, and *weak isospin* is connected to the weak interaction. Whether or not a fermion has weak isospin depends on another physical property, the so called *helicity*.

Definition 8.2. Every fermion has a property called *spin*, which is a vector in $\mathbf{R}^3$. The component of spin in the direction of motion is called *helicity*. A particle is said to be *right-handed (R)* (resp. *left-handed (L)*) if it has positive (resp. negative) helicity.

Remark 8.3. If a particle has helicity h, then its antiparticle has helicity $-h$. Note that helicity is not an intrinsic property of fermions (like mass and electric charge), except when the fermion is massless. Because of relativity theory, a massive fermion can be right- or left-handed depending on its frame of reference. A massless fermion however must be either right- or left-handed, because its velocity is the same is all frames of reference. For example, we have never seen right-handed neutrinos or left-handed antineutrinos, but only left-handed neutrinos and their antiparticles the right-handed anti-neutrinos. We will treat neutrinos and antineutrinos as particles with only one possible helicity, which means that we will treat them as massless particles.

We know from experiments that the weak force acts differently on particles with different handedness, it only acts on left handed particles and right handed antiparticles. See figure 8.2 for an overview of the fundamental interactions.

Interaction	Acts nontrivially on	Lie alg.	Bosons
Strong	quarks, antiquarks	$\mathfrak{su}(3)$	gluons
Weak	L-particles, R-antiparticles	$\mathfrak{su}(2)$	Z,W bosons
Electromagnetic	electrically charged fermions	$\mathfrak{u}(1)$	photon
Gravitational	all fermions	-	graviton*

*The graviton is a postulated particle, its existence has not yet been verified

Figure 8.2: Fundamental interactions.

As it turns out, the electromagnetic and the weak interaction can be unified into the so called *electroweak interaction*, which is nicer to work with than the two separate interactions when we want to make a grand unification model. This force is modeled as $\mathfrak{su}(2) \times \mathfrak{u}(1)$, and has four independent bosons. These are the W^+, W^-, W^0 bosons, which are a basis for $\mathfrak{su}(2)$, and the weak hypercharge boson Y, which generates $\mathfrak{u}(1)$. The photon and the Z^0 boson are linear combinations of the W^0 and Y bosons.

8.3 The Standard Model

In this subsection we will give the mathematical formulation of the *Standard Model*, which tells us how the strong and the electroweak interaction act on fermions. In essence this model is a collection of representations of the Lie algebra $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$, where $\mathfrak{su}(3)$ denotes the strong force and $\mathfrak{su}(2) \times \mathfrak{u}(1)$ denotes the electroweak force.

It turns out that we can treat the three generations of fermions separately. All three are modeled in the same way, so we can just make the model for the first generation without loss of generality. Let's introduce some notation before we present the Standard Model.

Notations 8.4. We shall denote by

$$(\phi_3, \phi_2, \phi_1) \tag{8.1}$$

a direct product representation of $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$, see subsection 7.3. Let n be a positive integer. For the defining representation of $\mathfrak{su}(n)$ we shall use the notation n. Also, if ϕ is an m-dimensional complex representation of $\mathfrak{su}(n)$, $m \neq n$, then (if there is no confusion) we shall use the notation m for this representation. Note that the 1-dimensional representation of $\mathfrak{su}(n)$ is trivial, since $\mathfrak{su}(n)$ is a simple Lie algebra.

Representations of $\mathfrak{u}(1)$ will be denoted differently, since we will be using only 1-dimensional complex representations of $\mathfrak{u}(1)$. Let $s \in \mathbf{C}$, and recall that the underlying vectorspace of $\mathfrak{u}(1)$ is $i\mathbf{R}$. Then the s-representation of $\mathfrak{u}(1)$ is:

$$s : \mathfrak{u}(1) \to \mathfrak{gl}(\mathbf{C}) : \quad x \mapsto sx, \quad x \in \mathfrak{u}(1). \tag{8.2}$$

In figure 8.3 we present the Standard Model for the first generation of particles (see [2, chapter 18]). It consists of representations of Lie algebras $\mathfrak{su}(3), \mathfrak{su}(2)$ and $\mathfrak{u}(1)$ (using notations 8.4) corresponding to the fundamental interactions. When we compare this model to figure 8.2, we see that indeed the weak force acts nontrivially only on left-handed particles (and right-handed antiparticles), and that the strong force acts nontrivially only on quarks (and antiquarks).

	u_R	d_R	e_R	$\begin{pmatrix} d_L \\ u_L \end{pmatrix}$	$\begin{pmatrix} e_L \\ \nu_L \end{pmatrix}$
$\mathfrak{su}(3)$ strong (color)	3	3	1	3	1
$\mathfrak{su}(2)$ weak isospin	1	1	1	2	2
$\mathfrak{u}(1)$ weak hypercharge	$\frac{2}{3}$	$-\frac{1}{3}$	-1	$\frac{1}{6}$	$-\frac{1}{2}$

Figure 8.3: The Standard Model

For the 2 representation of $\mathfrak{su}(2)$ we have $\bar{2} = 2$, and for the 3 representation of $\mathfrak{su}(3)$ we have $\bar{3} \neq 3$. And it is easy to see that the $\bar{s}$ representation of $\mathfrak{u}(1)$ is equivalent to the $-s^*$ representation (here * denotes complex conjugation in $\mathbf{C}$). We can now deduce from figure 8.3 that together the right-handed fermions u_R, d_R, e_R, $\begin{pmatrix} \bar{d}_R \\ \bar{u}_R \end{pmatrix}$ and $\begin{pmatrix} \bar{e}_R \\ \bar{\nu}_R \end{pmatrix}$ transform according to the complex representation:

$$(3, 1, 2/3) \oplus (3, 1, -1/3) \oplus (1, 1, -1) \oplus (\bar{3}, 2, -1/6) \oplus (1, 2, 1/2). \tag{8.3}$$

9 The $\mathfrak{su}(5)$ grand unification

Now that we know the structure of the Standard Model, we can try to unify the fundamental forces into an encompassing force. If we look at our model

8.1, we see that this can be done by finding a suitable Lie algebra $\mathfrak{g}$ and a representation ϕ of $\mathfrak{g}$, which contain all the information of the fundamental forces that we know from the Standard Model. Note that since gravity cannot be modeled by a Lie algebra, we cannot include it in this unification. But we can try to find a force that will unify the strong and the electroweak interaction. The Lie algebra $\mathfrak{g}$ corresponding to this force must then have $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ as a Lie subalgebra, such that ϕ is equivalent to the representation 8.3 when restricted to $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$.

What properties do we know that $\mathfrak{g}$ must have? Firstly, the dimension of $\mathfrak{g}$ should be at least $\dim(\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)) = 8 + 3 + 1 = 12$.

Just like $\mathfrak{sl}(n)$, it turns out that $\mathfrak{su}(n)$ is simple, with rank $n-1$ (see [5, chapter 9] resp. [8, chapter 3.6]). So $\mathrm{rank}(\mathfrak{su}(3)) = 2$ and $\mathrm{rank}(\mathfrak{su}(2)) = 1$. and $\mathfrak{u}(1)$ is a 1-dimensional abelian Lie algebra. This means that $\mathfrak{g}$ should have an abelian subalgebra of dimension at least $2 + 1 + 1 = 4$. Also, if a Cartan subalgebra of $\mathfrak{su}(3)$ resp. $\mathfrak{su}(2)$ is diagonalizable in a basis B resp. B', then this abelian subalgebra of $\mathfrak{g}$ is diagonalizable in the basis $B \cup B' \cup \{x\}$, where $x \in \mathfrak{u}(1), x \neq 0$. If $\mathfrak{g}$ would be a simple Lie algebra, this condition could be stated as: $\mathrm{rank}(\mathfrak{g}) \geq 4$.

Now, we know a simple Lie algebra $\mathfrak{g}$, such that $\dim(\mathfrak{g}) \geq 12$ and $\mathrm{rank}(\mathfrak{g}) = 4$, namely $\mathfrak{su}(5)$. Recall that $\dim(\mathfrak{su}(5)) = 5^2 - 1 = 24$. Furthermore, we can easily see that $\mathfrak{su}(5)$ contains $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$ as a Lie subalgebra via the following linear map ι. For all $x \in \mathfrak{su}(3), y \in \mathfrak{su}(2), z \in \mathfrak{u}(1)$:

$$\iota : \mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1) \to \mathfrak{su}(5) : \tag{9.1}$$

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \qquad z \mapsto z \begin{pmatrix} -\frac{1}{3} I_3 & 0 \\ 0 & \frac{1}{2} I_2 \end{pmatrix}, \tag{9.2}$$

where I_2 and I_3 are identity matrices of dimension 2 resp 3, and the 0's denote zero matrices of the corresponding size.
For ease of notation, we define:

$$\mathfrak{a} := \iota(\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)). \tag{9.3}$$

Clearly $\mathfrak{a} \simeq \mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$. We shall prove that $\mathfrak{su}(5)$ is a good Lie algebra for the unification.

Let us now look at what should be the dimension of the representation ϕ. It should be the same as the dimension of the representation 8.3, since the last representation would be a restriction of ϕ to $\mathfrak{a}$. From the definition of a representation (ϕ_3, ϕ_2, ϕ_1), we know that it has dimension $\dim(\phi_3)\dim(\phi_2)\dim(\phi_1)$. So, the representation 8.3 has dimension:

$$(3 \cdot 1 \cdot 1) + (3 \cdot 1 \cdot 1) + (1 \cdot 1 \cdot 1) + (3 \cdot 2 \cdot 1) + (1 \cdot 2 \cdot 1) = 15. \tag{9.4}$$

Proposition 9.1. *For the defining representation* 5 *of* $\mathfrak{su}(5)$ *we have:*

$$5|_{\mathfrak{a}} \simeq (3, 1, -1/3) \oplus (1, 2, 1/2). \tag{9.5}$$

proof. An element in $\mathfrak{a}$ is of the form $x + y + z$, where $x \in \iota(\mathfrak{su}(3)), y \in \iota(\mathfrak{su}(2)), z \in \iota(\mathfrak{u}(1))$. Consider $\mathbf{C}^5$ to be the direct sum $V \oplus W$, where $\dim(V) = 3, \dim(W) = 2$. Then:

$$5(x+y+z)(v+w) = (x+y+z)(v+w) = (x - z\frac{1}{3}I_3)v + (y + z\frac{1}{2}I_2)w.$$

Let us define vectorspaces $V_1, V_2, V_3, W_1, W_2, W_3$, such that $\dim(V_1) = 3$, $\dim(V_2) = 1, \dim(V_3) = 1, \dim(W_1) = 1, \dim(W_2) = 2, \dim(W_3) = 1$.
And let $(3, 1, -1/3) : \mathfrak{a} \to \mathfrak{gl}(V_1 \otimes V_2 \otimes V_3)$, $(1, 2, 1/2) : \mathfrak{a} \to \mathfrak{gl}(W_1 \otimes W_2 \otimes W_3)$.
Let $v_1 \in V_1, v_2 \in V_2, v_3 \in V_3$. Then:

$$\begin{aligned}
&(3, 1, -1/3)(x+y+z)(v_1 \otimes v_2 \otimes v_3) \\
&= (3(x)v_1) \otimes v_2 \otimes v_3 + v_1 \otimes v_2 \otimes (-1/3zv_3) \\
&= (3(x)v_1) \otimes v_2 \otimes v_3 + (-1/3zv_1) \otimes v_2 \otimes v_3 \\
&= ((3(x) - 1/3zI_3)v_1) \otimes v_2 \otimes v_3.
\end{aligned}$$

Let $w_1 \in W_1, w_2 \in W_2, w_3 \in W_3$. Then:

$$\begin{aligned}
&(1, 2, 1/2)(x+y+z)(w_1 \otimes w_2 \otimes w_3) \\
&= w_1 \otimes (2(y)w_2) \otimes w_3 + w_1 \otimes w_2 \otimes (1/2zw_3) \\
&= w_1 \otimes (2(y)w_2) \otimes w_3 + w_1 \otimes (1/2zw_2) \otimes w_3 \\
&= w_1 \otimes ((2(y) + 1/2zI_2)w_2) \otimes w_3.
\end{aligned}$$

And $V \simeq V_1$, $W \simeq W_2$. We are finished when we see that there are natural identifications: $V_1 \otimes V_2 \otimes V_3 \to V_1 : v_1 \otimes v_2 \otimes v_3 \mapsto v_1$, and
$W_1 \otimes W_2 \otimes W_3 \to W_2 : w_1 \otimes w_2 \otimes w_3 \mapsto w_2$. □

Now if we can find a representation ψ of $\mathfrak{su}(5)$ that is equivalent to $(3, 1, 2/3) \oplus (1, 1, -1) \oplus (\bar{3}, 2, -1/6)$ when we restrict it to $\mathfrak{a}$, then the desired representation is $5 \oplus \psi$ and we are done. The dimension of ψ must be $15 - 5 = 10$. We already know one representation of $\mathfrak{su}(5)$ that has dimension 10, namely $A(5 \otimes 5)$ (see equation 7.11). To simplify notation we shall define the 10-representation of $\mathfrak{su}(5)$ as

$$10 := A(5 \otimes 5). \tag{9.6}$$

Proposition 9.2. *For the* 10 *representation of* $\mathfrak{su}(5)$ *we have:*

$$10|_{\mathfrak{a}} \simeq (\bar{3}, 1, -2/3) \oplus (1, 1, 1) \oplus (3, 2, 1/6). \qquad (9.7)$$

proof. We first explore $(5 \otimes 5)|_{\mathfrak{a}}$. It is easy to see that $(5 \otimes 5)|_{\mathfrak{a}} = 5_{\mathfrak{a}} \otimes 5_{\mathfrak{a}}$. Let us write $5 : \mathfrak{su}(5) \to \mathfrak{gl}(V)$, $V = V_1 \oplus V_2$ where $\dim(V_1) = 3, \dim(V_2) = 2$, such that $(3, 1, -1/3) : \mathfrak{a} \to \mathfrak{gl}(V_1)$, $(1, 2, 1/2) : \mathfrak{a} \to \mathfrak{gl}(V_2)$.

Let $v_1, v_1' \in V_1, v_2, v_2' \in V_2$. It is instructive to use matrix notation for $(v_1 + v_2) \otimes (v_1' + v_2')$ (see remark 7.3). Let us construct a matrix $M \in M_5(\mathbf{C})$ as follows:

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

such that

$$M_{11} = \begin{pmatrix} v_{11}v_{11}' & \dots & v_{11}v_{12}' \\ \vdots & \ddots & \vdots \\ v_{13}v_{11}' & \dots & v_{13}v_{12}' \end{pmatrix} M_{12} = \begin{pmatrix} v_{11}v_{21}' & \dots & v_{11}v_{22}' \\ \vdots & \ddots & \vdots \\ v_{13}v_{21}' & \dots & v_{13}v_{22}' \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} v_{21}v_{11}' & \dots & v_{21}v_{12}' \\ \vdots & \ddots & \vdots \\ v_{23}v_{11}' & \dots & v_{23}v_{12}' \end{pmatrix} M_{22} = \begin{pmatrix} v_{21}v_{21}' & \dots & v_{21}v_{22}' \\ \vdots & \ddots & \vdots \\ v_{23}v_{21}' & \dots & v_{23}v_{22}' \end{pmatrix}.$$

Then we see that

$$\begin{gathered} M \simeq (v_1 + v_2) \otimes (v_1' + v_2'); \\ M_{11} \simeq v_1 \otimes v_1'; M_{12} \simeq v_1 \otimes v_2'; \\ M_{21} \simeq v_2 \otimes v_1'; M_{22} \simeq v_2 \otimes v_2'; \end{gathered}$$

Let $x \in \mathfrak{a}$, and let us write out what happens when we let $(5 \otimes 5)|_{\mathfrak{a}}(x)$ work on M.

$$\begin{aligned} (5 \otimes 5)|_{\mathfrak{a}}(x)M \simeq\ & ((3, 1, -1/3) \oplus (1, 2, 1/2))(x)(v_1 + v_2) \\ & \otimes ((3, 1, -1/3) \oplus (1, 2, 1/2))(x)(v_1' + v_2'), \\ & = ((3, 1, -1/3) \otimes (3, 1, -1/3))(x)M_{11} \\ & \oplus ((3, 1, -1/3) \otimes (1, 2, 1/2))(x)M_{12} \\ & \oplus ((1, 2, 1/2) \otimes (3, 1, -1/3))(x)M_{21} \\ & \oplus ((1, 2, 1/2) \otimes (1, 2, 1/2))(x)M_{22}. \end{aligned}$$

Now, if s and s' are representations of $\mathfrak{u}(1)$, then it is easy to see from the definition of the tensor product of representations 7.1 that $s \otimes s' \simeq s + s'$. Then, with remark 7.16 and lemma 7.4.1 we deduce the following:

$$\begin{aligned}(3,1,-1/3)\otimes(3,1,-1/3) &\simeq ((3\otimes 3),1,-2/3),\\ (3,1,-1/3)\otimes(1,2,1/2) &\simeq (3,2,1/6),\\ (1,2,1/2)\otimes(3,1,-1/3) &\simeq (3,2,1/6),\\ (1,2,1/2)\otimes(1,2,1/2) &\simeq (1,(2\otimes 2),1),\end{aligned}$$

But we are actually interested in $10_{\mathfrak{a}} = A(5\otimes 5)|_{\mathfrak{a}}$. Let's see what happens when we antisymmetrize $(5\otimes 5)|_{\mathfrak{a}}$. From remark 7.10 we can see that we should just take an antisymmetric matrix M in $(5\otimes 5)|_{\mathfrak{a}}(x)M$. We have the following:

$$M = -M^T \implies M_{11} = -M_{11}^T, M_{22} = -M_{22}^T, M_{21} = -M_{12}^T. \tag{9.8}$$

We see that after the antisymmetrization of M the matrices M_{11}, M_{22}, M_{12} are independent, but M_{21} is completely determined by M_{12}. We are now able to construct $10_{\mathfrak{a}}$ from $(5\otimes 5)|_{\mathfrak{a}}$. We should antisymmetrize $(3,1,-1/3)\otimes(3,1,-1/3)$ and $(1,2,1/2)\otimes(1,2,1/2)$. And from the two $(3,2,1/6)$ representations we should keep only one. So:

$$\begin{aligned}10|_{\mathfrak{a}} &\simeq A((3,1,-1/3)\otimes(3,1,-1/3)) \oplus A((1,2,1/2)\otimes(1,2,1/2)) \oplus (3,2,1/6)\\ &\simeq (A(3\otimes 3),1,-2/3)\oplus(1,A(2\otimes 2),1)\oplus(3,2,1/6).\end{aligned}$$

We know from equation 7.11 that $\dim(A(3\otimes 3)) = 3$, $\dim(A(2\otimes 2)) = 1$. We can work out that $A(3\otimes 3) = \bar{3}$ by straightforward computation (or see [2, chapter 18]). This concludes the proof. □

Since for $\mathfrak{su}(2)$ we have $\bar{2} = 2$, we see that:

$$\bar{10}|_{\mathfrak{a}} \simeq (3,1,2/3)\oplus(1,1,-1)\oplus(\bar{3},2,-1/6). \tag{9.9}$$

So $\bar{10}$ is the representation ψ that we are looking for. Now we see that

$$5 \oplus \bar{10}, \tag{9.10}$$

when restricted to $\mathfrak{a}$, is equivalent to the representation 8.3 of the right handed fermions. It can be shown that both the 5 and $\bar{10}$ representations are irreducible (see [2]), but we will not do that here since it is a tedious computation.

9.1 Implications of the $\mathfrak{su}(5)$ grand unification

In the previous section we have found that $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1) \simeq \mathfrak{a} \subset \mathfrak{su}(5)$, such that $(5 \oplus \bar{10})|_{\mathfrak{a}}$ is the representation 8.3. This means that the strong and the electroweak force could be realizations of one force that is modeled by $\mathfrak{su}(5)$, but that this force for some reason has broken down into three distinct interactions: the electromagnetic, weak and strong interaction. This process is called *spontaneous symmetry breaking*. For the details see [4, chapter 18].

Since $\dim(\mathfrak{su}(5)) = 5^2 - 1 = 24$, this encompassing force would have 24 force mediating bosons, instead of the 12 bosons corresponding to $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$. This arises the immediate question: If the $\mathfrak{su}(5)$ unification is correct, why have we never seen the 12 remaining bosons? The absence of these bosons in experiments could lead us to think that the unification is not physical, but on the other hand there could be some reason why we have never seen them.

Now there is another implication of the $\mathfrak{su}(5)$ unification that could indicate that this theory is possibly correct, namely it explains the relation between quark and electron charges.

Because the hydrogen atom is found to be electrically neutral to any degree of experimental accuracy so far, there is every reason to believe that the electric charges of the proton (which consists of three quarks) and the electron are equal, but with opposite sign. Only there is no reason in physics theory why these two charges should be linked. For if we look at figure 8.3, we see that the fundamental forces can transform quarks into each other, and they can transform leptons into each other, but there is no relation whatsoever linking quarks to leptons in the Standard Model. It turns out that the $\mathfrak{su}(5)$ grand unification gives a very elegant way of explaining this link between electric charge.

Let us look at the 5 representation of $\mathfrak{su}(5)$. Proposition 9.1 indicates that the 5 representation of $\mathfrak{su}(5)$ works on the five dimensional space corresponding to vectors

$$((d_R)_1, (d_R)_2, (d_R)_3, \bar{e}_R, \bar{\nu}_R)^T. \tag{9.11}$$

So d_R forms a triplet, and $\bar{e}_R$ and $\bar{\nu}_R$ are singlets in this representation.

We know that $\bar{\nu}_R$ is electrically neutral, since is does not interact with the electromagnetic force. Then the electric charge generator Q corresponding to the 5 representation is:

$$Q = \mathrm{diag}(q_d, q_d, q_d, q_{\bar{e},0}), \tag{9.12}$$

where q_d is the electric charge of d_R, and $q_{\bar{e}}$ is the electric charge of $\bar{e}_R$. Since $Q \in \mathfrak{su}(5)$, we have that $\mathrm{Tr}(Q) = 0 = 3q_d + q_{\bar{e}}$, so

$$3q_d = -q_{\bar{e}}. \tag{9.13}$$

This is the correct relation, since $q_{\bar{e}} = +1$, and $q_d = -1/3$. So the 5 representation would explain the link between the charges of the quarks and leptons, which gives us hope that the $\mathfrak{su}(5)$ grand unification could be a good model.

But there is something that this unification implicates that seems to make it impossible for $\mathfrak{su}(5)$ to be a good physical theory, namely it predicts *proton decay* in such a rate that it contradicts experimental data. To be precise, there is no experiment that indicates instability of the proton.

The Standard Model forbids us to change quarks into leptons and vice versa, since there is no particle multiplet in figure 8.3 that combines these two kinds of particles. It does allows us to switch the u and the d quark, which in a proton effectively does not change anything (since the proton consists of two u quarks and one d quark), indicating that the proton is stable.

Let us see why the proton should decay as a result of our unification. We have seen that the 5 representation of $\mathfrak{su}(5)$ works on vectors of the form $((d_R)_1, (d_R)_2, (d_R)_3, \bar{e}_R, \bar{\nu}_R)^T$. And proposition 9.2 indicates that the $\bar{10}$ representation works on antisymmetric matrices in $M_5(\mathbf{C})$ corresponding to a mixture of the particles $u_R, e_R, \bar{d}_R$ and $\bar{u}_R$. Now since both representations are irreducible, we see that the $\mathfrak{su}(5)$ force can transform quarks into leptons and vice versa. With the $\mathfrak{su}(5)$ interaction there is nothing that can stop us from doing these transformations, which would result in the decay of the proton.

The lifetime of a proton that the $\mathfrak{su}(5)$ unification predicts is at most $4.5 \times 10^{29\pm1.7}$ years, while experiments have shown that the lifetime is at least 6×10^{31} years (see [4, chapter 18.5]). We see that these numbers are far off, which indicates that the $\mathfrak{su}(5)$ model is not physical. Actually there is an even bigger problem with GUT's known as the *hierarchy problem* (see [4, chapter 18.6]), where the $\mathfrak{su}(5)$-GUT gives even worse predictions. These are all reasons why in physics the $\mathfrak{su}(5)$ unification is not anymore under consideration as a grand unified theory.

10 What is the current condition of GUT's in physics?

As we already noted, the Georgi-Glashow $\mathfrak{su}(5)$ unification model has been ruled out as a candidate for a GUT. But there are still other, more complex unification theories that have not yet been contradicted by experiments.

One thing that would certainly point in the direction of grand unification is the detection of proton decay. In the mean time, even the correctness of the electroweak unification is uncertain, since we have not yet detected the Higgs boson, a particle that is essential for this this theory.

So for now GUT's are pending, waiting for experimental data that will make or break them.

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