Luuk W. van Rijn

Fractals and their dimensions

Bachelor Thesis, March 10, 2011

Supervisors:

dr. Sander C. Hille
Daniël T. H. Worm

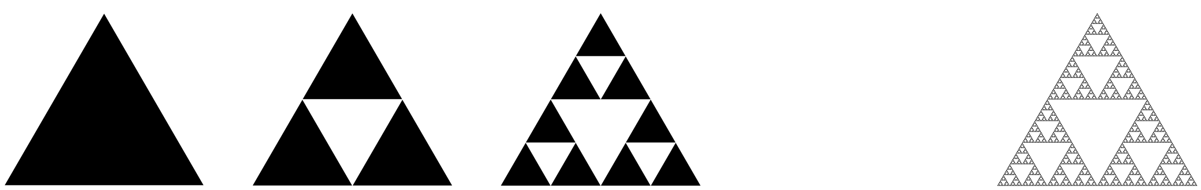
Mathematisch Instituut, Universiteit Leiden

Contents

Introduction

This thesis is about a class of geometric shapes that can be constructed through iterative processes. We shall use the Sierpinski triangle as an example of that kind of geometric shapes. During the development of the theory on the whole, we will return to that example as a guideline. Before we derive some properties of the Sierpinski triangle, we will show how one can construct it.

Example 0.0.1 (Construction of the Sierpinski triangle)**.** *In this example we are going to construct the Sierpinski triangle. We take a solid equilateral triangle T_1 as initial object. Then we obtain a new shape T_2 by taking the triangle, reducing it by a factor of two and placing copies of that reduction in all of the three corners of the triangle. We repeat this iteration again and again using the last shape obtained by the iteration instead of the triangle. In the limit the Sierpinski triangle, a shape that looks like T, appears. That is, $T = \bigcap_{n=1}^{\infty} T_n$.*



T_1 T_2 T_3 T

In this thesis we will use the term *attractor* for the limit set of such an iteration. In the example above the attractor is the Sierpinski triangle T.

In Chapter 1 we will introduce some necessary definitions with regard to metric spaces and contractive functions.

Chapter 2 is about *hyperspaces*. Here we will derive some properties of attractors.

In Chapter 3 we will look at some measure theory. We will construct a unique measure that is related to the attractor obtained in the previous chapter.

In Chapter 4 we are going to introduce a more general notion of *dimension*, the so called *Hausdorff dimension*. Then we will determine the Hausdorff dimension of the attractor. It will turn out that the dimension of the attractor is not an integer in most of the cases. Thus we speak about fractal dimensions.

Chapter 1

Metric spaces

1.1 Definitions

Definition 1.1.1 (Metric space)**.** *Let S be a non-empty set. A metric d is a function $d : S \times S \to \mathbb{R}$ such that for all $x, y, z \in S$:*

(i) $d(x, y) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

The ordered pair (S, d) is called a metric space.

Definition 1.1.2. *A metric space (S, d) is complete if each Cauchy sequence has a limit in S.*

From this point on let (S, d) be a complete metric space.

Definition 1.1.3. *Let (S_1, d_1) and (S_2, d_2) be metric spaces. A function $f : S_1 \to S_2$ is a Lipschitz function if there exists a $c \in \mathbb{R}_{\geq 0}$ such that*

$$d_2(f(x), f(y)) \leq c d_1(x, y) \text{ for all } x, y \in S_1.$$

The Lipschitz constant of f is

$$|f|_{Lip} := \inf\{c \geq 0 : d_2(f(x), f(y)) \leq c d_1(x, y) \text{ for all } x, y \in S_1\}.$$

The set of Lipschitz functions $f : S_1 \to S_2$ is notated by $Lip(S_1, S_2)$. The notation $Lip_{<c}(S_1, S_2)$ is used for the set Lipschitz functions with $|f|_{Lip} < c$. If $|f|_{Lip}$, then f is called contractive or a contraction mapping.

Contraction mappings $f : S \to S$ play an important part in the constuction of geometric objects such as the Sierpinski triangle.

Example 1.1.1. *Let us take a closer look at the Sierpinski triangle, introduced in Example 0.0.1. Since its construction is done in $\mathbb{R}^2$, the Sierpinski triangle is a subset of $\mathbb{R}^2$, which is a metric space. The associated metric d_E is given by*

$$d_E : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

$$d_E(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We refer to d_E as the Euclidean metric.

We can describe the iteration in Example 0.0.1 in terms of functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. For each corner p_i of the triangle we take a function $f_i(x)$ that maps x to the nearest point such that $d_E(f(x), p_i) = \frac{1}{2} d_E(x, p_i)$. We obtain the next subset in the iteration by applying those three functions on the points of the previous subset and taking the union of their outputs. By the way we defined the functions f_i, we see that they are contraction mappings with $|f_i|_{Lip} = \frac{1}{2}$.

Lemma 1.1.1. *Let (S_i, d_i) be a metric space and $f_i \in Lip(S_i, S_{i+1})$ for all $i \in \mathbb{N}$. Then $F_i := f_i \circ \cdots \circ f_1 : S_1 \to S_{i+1}$ is a Lipschitz function with $|F_i|_{Lip} \leq \prod_{i=1}^{n} |f_i|_{Lip}$.*

Proof. For all $x, y \in S_1$:

$$\begin{aligned} d_3(f_2 \circ f_1(x), f_2 \circ f_1(y)) \leq \quad & d_3(f_2(f_1(x)), f_2(f_1(y))) \\ \leq \quad & |f_2|_{Lip} \cdot d_2(f_1(x), f_1(y)) \leq \quad |f_2|_{Lip} \cdot |f_1|_{Lip} \cdot d_1(x, y). \end{aligned}$$

Proceeding inductively we get for all $x, y \in S_1$:

$$d_{i+1}(F_i(x), F_i(y)) \leq d_1(x, y) \cdot \prod_{i=1}^{n} |f_i|_{Lip}.$$

We conclude that $F_i \in \mathrm{Lip}(S_1, S_{i+1})$ with $|F_i|_{Lip} \leq \prod_{i=1}^{n} |f_i|_{Lip}$. $\square$

1.2 Banach Fixed Point Theorem

The well-known Banach Fixed Point Theorem states that a contraction mapping on a complete metric has a unique fixed point. The importance of this theorem will find expression when we are going to prove the uniqueness and existence of attractors.

Theorem 1.2.1 (Banach Fixed Point theorem)**.** *Let (S, d) be a complete metric space and $f : S \to S$ a contraction mapping. Then the following statements hold:*

(i) There exists a unique point $x_f \in S$ such that $f(x_f) = x_f$.

(ii) $\lim_{n\to\infty} f^n(x_0) = x_f$ *for all* $x_0 \in S$.

Proof. Let $c = |f|_{Lip}$. By Lemma 1.1.1 we get that for all $n \in \mathbb{N}$:

$$d(f^n(x), f^n(y)) \le c^n d(x, y).$$

Let $x_0 \in S$ and define the sequence $(x_n)_{n=0}^{\infty}$ by $x_n := f^n(x_0)$. The distance $d(x_0, x_n)$ is bounded:

$$\begin{aligned} d(x_0, x_n) &= d(x_0, f^n(x_0)) \le \sum_{i=0}^{n-1} d(f^i(x_0), f^{i+1}(x_0)) && (1.1)\\ &\le \sum_{i=0}^{n-1} c^i d(x_0, f(x_0)) = d(x_0, f(x_0)) \sum_{i=0}^{n-1} c^i \\ &\le d(x_0, f(x_0)) \sum_{i=0}^{\infty} c^i = \frac{1}{1-c} d(x_0, f(x_0)). && (1.2) \end{aligned}$$

The inequality in (1.1) is justified by the triangle inequality. In justification of (1.2) observe that $\sum_{i=0}^{\infty} c^i$ is a geometric series. Now it is easy to see that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence: let $m, n \in \mathbb{N}$, $m \ge n$, then

$$\begin{aligned} d(x_m, x_n) &= d(f^m(x_0), f^n(x_0)) = d(f^n(f^{m-n}(x_0)), f^n(x_0)) \\ &\le c^n d(f^{m-n}(x_0), x_0) \le \frac{c^n}{1-c} d(x_0, f(x_0)). \end{aligned}$$

Since $\lim_{n\to\infty} \frac{c^n}{1-c} = 0$, it follows that $(x_n)_{n=0}^{\infty}$ is a Cauchy sequence. Because S is complete, $(x_n)_{n=0}^{\infty}$ does have a limit in S, say x_f. Using the continuity of f we see

$$f(x_f) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x_f.$$

So x_f is a fixed point of f.

To prove the uniqueness of x_f we suppose that z and y are fixed points of f. Then

$$d(y, z) = d(f(y), f(z)) \le c d(y, z).$$

Now we find that $d(y, z) = 0$, hence $y = z$. Therefore the fixed point is unique.

We derive that $\lim_{n\to\infty} f^n(x_0) = x_f$ for all $x_0 \in S$ as a direct consequence of the way the sequence $(x_n)_{n=0}^{\infty}$ is defined. $\square$

Chapter 2

Hyperspaces

Let (S, d) be a complete metric space. It is possible to turn the set $\mathcal{H}(S)$ of closed and bounded subsets of S into a metric space with distance function d_H, the *Hausdorff distance*. There is no universal name for $\mathcal{H}(S)$, we will call it the hyperspace of S, following [5]. It will turn out that attractors and in particular the Sierpinski triangle are elements of this space.

2.1 The Hausdorff distance

Let (S, d) be a complete metric space.

We define the distance between a point $x \in S$ and a subset $A \subset S$ by

$$d(x, A) = \inf\{d(x, a) : a \in A\},$$

and we define the semi-distance of two subset $A, B \subset S$ by

$$\delta(A, B) = \sup\{d(b, A) : b \in B\}.$$

Observe that there exist subsets $A, B \subset S$ such that $\delta(A, B) \neq \delta(B, A)$. For instance, we can take $(S, d) = (\mathbb{R}, d_E)$, $A = [1, 2]$, and $B = [0, 3]$. Then $0 = \delta(A, B) \neq \delta(B, A) = 1$. For that reason we refer to $\delta(A, B)$ as the semi-distance from A to B. A proper notion of distance is symmetrical. Therefore we introduce the Hausdorff distance d_H:

Definition 2.1.1 (Hausdorff distance)**.** *Let (S, d) be a metric space and let $A, B \subset S$. The Hausdorff distance between A and B is given by*

$$d_H(A, B) = max\{\delta(A, B), \delta(B, A)\}.$$

Lemma 2.1.1. *Let $A, B, C \subset S$ be bounded subsets of S. Then $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$.*

Proof. Let $a \in A, b \in B, c \in C$. Then

$$d(a, B) \leq d(a, b) \leq d(a, c) + d(c, b).$$

This holds for all $b \in B$, so

$$d(a, B) \leq d(a, c) + d(c, B) \leq d(a, c) + \delta(C, B).$$

The latter inequality holds for all $c \in C$, thus

$$d(a, B) \leq d(a, C) + \delta(C, B).$$

We take the supremum over all $a \in A$ and obtain that

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B).$$

□

Theorem 2.1.1. *$(\mathcal{H}(S), d_H)$ is a metric space.*

Proof. Let $A, B \in \mathcal{H}(S)$. Since A is bounded,

$$\delta(A, B) = \sup\{d(a, B) : a \in A\} < \infty.$$

Since B is bounded, the same is true for $d(B, A)$. So $d_H : \mathcal{H}(S) \times \mathcal{H}(S) \to \mathbb{R}$. We have to verify that d_H is a metric on $\mathcal{H}(S)$:

(i) Let $A \in \mathcal{H}(S)$, then

$$\begin{aligned} d_H(A, A) &= \delta(A, A) = \sup\{\inf\{d(a, b) : b \in A\} : a \in A\} \\ &= \sup\{d(a, a) : a \in A\} = 0. \end{aligned}$$

Let $A, B \in \mathcal{H}(S)$ such that $d_H(A, B) = 0$, then

$$0 = \delta(A, B) = \sup\{\inf\{d(a, b) : b \in B\} : a \in A\}.$$

So for every $a \in A$ we obtain that $\inf\{d(a, b) : b \in B\} = 0$. Consequently $A \subset \overline{B} = B$. Interchanging the role of A and B yields $B \subset \overline{A} = A$, hence $A = B$.

(ii) For all $A, B \in \mathcal{H}(S) : d_H(A, B) = max(\delta(A, B), \delta(B, A)) = d_H(B, A)$.

(iii) From Lemma 2.1.1 it follows that for all $A, B, C \in \mathcal{H}(S)$ we have $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ and $\delta(B, A) \leq \delta(B, C) + \delta(C, A)$. So

$$\begin{aligned} d_H(A, B) &= \max(\delta(A, B), \delta(B, A)) \\ &\leq \max(\delta(A, C) + \delta(C, B), \delta(B, C) + \delta(C, A)) \\ &\leq \max(\delta(A, C), \delta(C, A)) + \max(\delta(C, B), \delta(B, C)) \\ &= d_H(A, C) + d_H(C, B). \end{aligned}$$

Thus d_H is a metric on $\mathcal{H}(S)$. □

2.2 Completeness of hyperspaces

Earlier we mentioned the importance of the Banach Fixed Point Theorem. Before we can apply this theorem to contraction mappings on the hyperspace $\mathcal{H}(S)$, we need $\mathcal{H}(S)$ to be a *complete* metric space.

Theorem 2.2.1. *Let (S, d) be a complete metric space. Then $(\mathcal{H}(S), d_H)$ is a complete metric space.*

Proof. Let $\{A_n\}_{n=1}^{\infty}$ a Cauchy sequence in $\mathcal{H}(S)$. Define

$$A := \bigcap_{k=1}^{\infty} \mathrm{cl}\left(\bigcup_{n \geq k}^{\infty} A_n\right).$$

We observe that A is closed by definition.

Let $\epsilon > 0$. Since $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that $d_H(A_n, A_m) < \epsilon$ for all $n, m \geq N$. We are going to show that A is bounded. Because $d_H(A_n, A_m) < \epsilon$ we see that $\delta(A_n, A_N) < \epsilon$ for all $n \geq N$. And we get

$$A \subset \mathrm{cl}\left(\bigcup_{n \geq N}^{\infty} A_n\right) \subset \{x \in S : d(x, A_N) \leq \epsilon\}.$$

We easily see that $\delta(A, A_N) \leq \epsilon$. A_N is bounded, so we see that A is bounded.

First we will give an upper bound for $\delta(A_N, A)$. Let $y_0 \in A_{N_0} := A_N$. For all $i \in \mathbb{N}$ let $\epsilon_i = \epsilon \cdot 2^{-i}$ and define inductively N_i such that $N_i \geq N_{i-1}$ and $d_H(A_n, A_m) < \epsilon_i$ for all $n, m \geq N_i$. Since $d_H(A_{N_i}, A_{N_{i+1}}) < \epsilon_i$ for all $i \in \mathbb{N}_0$, we can choose an $y_{i+1} \in A_{N_{i+1}}$ such that $d(y_i, y_{i+1}) < \epsilon_i$.

Consider the sequence $\{y_i\}_{i=0}^{\infty}$. For all $n, m \in \mathbb{N}_0$, $n < m$ we see

$$d(y_n, y_m) \leq \sum_{j=n}^{m-1} d(y_j, y_{j+1}) < \sum_{j=n}^{m-1} \epsilon \cdot 2^{-j} < \sum_{j=n}^{\infty} \epsilon \cdot 2^{-j} = \epsilon \cdot 2^{1-n}.$$

So $\{y_i\}_{i=0}^{\infty}$ is a Cauchy sequence. S is complete, therefore $\{y_i\}_{i=0}^{\infty}$ has a limit in S, say y. Now we see that $d(y_0, y) = \lim_{m\to\infty} d(y_0, y_m) < 2\epsilon$.

Observe that for all $m \in \mathbb{N}$:

$$y \in \mathrm{cl}\left(\{y_i : i \geq m\}\right) \subset \mathrm{cl}\left(\bigcup_{i\geq m} A_{N_i}\right) \subset \mathrm{cl}\left(\bigcup_{n\geq m} A_n\right).$$

In the first inclusion we use that $y_i \in A_{N_i}$ for all $i \in \mathbb{N}_0$, in the second that $N_i \geq m$ for all $i \geq m$. Consequently $y \in A$. Since $y_0 \in A_N$ is arbitrary, $\delta(A_N, A) < 2\epsilon$. Recall that $\delta(A, A_N) \leq \epsilon$, so $d_H(A, A_N) < 2\epsilon$.

Now we can show that A is the limit of $\{A_n\}_{n=1}^{\infty}$. For all $n \geq N$:

$$d_H(A, A_n) \leq d_H(A, A_N) + d_H(A_N, A_n) \leq 2\epsilon + \epsilon = 3\epsilon.$$

□

2.3 Hyperspaces and contraction mappings

In the previous section we saw that the condition on (S, d) to be complete ensures the completeness of the hyperspace $\mathcal{H}(S)$. Now we are going to look at contraction mappings that map from $\mathcal{H}(S)$ to $\mathcal{H}(S)$. We know that this kind of functions have a fixed point and in this case it is a closed and bounded subset of S. For an iterative process we will define the Hutchinson function. We will prove that it is a contraction mapping with the property that its fixed point equals the limit set of the iteration.

Let A be any set and f be a function on the elements of A. We define $f(A) := \{f(a) : a \in A\}$.

Lemma 2.3.1. *Let (S_1, d_1) and (S_2, d_2) be metric spaces. Let $f \in Lip(S_1, S_2)$. Then for all $A \subset S_1$:*

$$\mathrm{diam}(f(A)) \leq |f|_{Lip}\mathrm{diam}(A).$$

Proof.

$$\begin{aligned}\mathrm{diam}(f(A)) &= \sup\{d_2(f(a), f(b)) : a, b \in A\} \\ &\leq \sup\{|f|_{Lip} \cdot d_1(a, b) : a, b \in A\} \\ &= |f|_{Lip} \cdot \mathrm{diam}(A).\end{aligned}$$

□

Lemma 2.3.2. *Let (S_1, d_1) and (S_2, d_2) be metric spaces. Let $f \in Lip_{<1}(S_1, S_2)$. Then $\overline{f} : \mathcal{H}(S_1) \to \mathcal{H}(S_2)$ given by $\overline{f}(A) = \mathrm{cl}(f(A))$ is a contraction mapping with $|\overline{f}|_{Lip} \le |f|_{Lip}$.*

Proof. Let $A \in \mathcal{H}(S_1)$.

Observe $\overline{f}(A)$ is closed by definition. Lemma 2.3.1 yields

$$\mathrm{diam}(\overline{f}(A)) = \mathrm{diam}(f(A)) \le |f|_{Lip}\mathrm{diam}(A) < \infty.$$

So $\overline{f}(A)$ is bounded. Therefore $\overline{f}$ maps from $\mathcal{H}(S_1)$ to $\mathcal{H}(S_2)$. Further we need to show that $\overline{f}$ is a contraction mapping with $|\overline{f}|_{Lip} \le |f|_{Lip}$. Let $c = |f|_{Lip}$.

$$\begin{aligned}
\delta(\overline{f}(A), \overline{f}(B)) &= \sup_{a \in \overline{f}(A)} \inf_{b \in \overline{f}(B)} d(a, b) = \sup_{a \in A} \inf_{b \in B} d(f(a), f(b)) \\
&\le \sup_{a \in A} \inf_{b \in B} c \cdot d(a, b) = c \cdot \sup_{a \in A} \inf_{b \in B} d(a, b) \\
&= c \cdot \delta(A, B).
\end{aligned}$$

In a similar way we obtain $\delta(\overline{f}(B), \overline{f}(A)) \le c \cdot \delta(B, A)$. Consequently

$$\begin{aligned}
d_H(\overline{f}(A), \overline{f}(B)) &= \max(\delta(\overline{f}(A), \overline{f}(B)), \delta(\overline{f}(B), \overline{f}(A)) \\
&\le \max(c \cdot \delta(A, B), c \cdot \delta(B, A)) \\
&= c \cdot \max(\delta(A, B), \delta(B, A)) = c \cdot d_H(A, B).
\end{aligned}$$

Thus F is a contraction mapping with $|\overline{f}|_{Lip} \le |f|_{Lip}$.

□

Definition 2.3.1 (Iterated Function System)**.** *An iterated function system (IFS) is a finite set of contraction mappings on a complete metric space. We use the notation*

$$\mathcal{F} := \{f_i : S \to S | i = 1, 2, \cdots, n\}.$$

We define its Lipschitz constant by $|\mathcal{F}|_{Lip} := \max_{1<i<n} |f_i|_{Lip}$.

The (closed) Hutchinson function $\overline{F} : \mathcal{H}(S) \to \mathcal{H}(S)$ associated to this IFS is defined by

$$\overline{F}(A) = \bigcup_{i=1}^{n} \overline{f}_i(A).$$

Note that when we define an IFS $\mathcal{F}$, we implicitly define n, the number of elements of $\mathcal{F}$, and n-number contractive functions $f_i : S \to S$, $i = 1, 2, \cdots, n$. To prevent unnecessary repetition, We mension only $\mathcal{F}$ when no unclearities will arise.

In the next lemma "$\vee$" denotes maximum and "$\wedge$" denotes the minimum of two reals.

Lemma 2.3.3. *Let $A, B, C, D \in \mathcal{H}(S)$, then*

$$d_H(A \cup B, C \cup D) \le [d_H(A,C) \vee d_H(B,D)] \wedge [d_H(A,D) \vee d_H(B,C)]\,.$$

In particular the following inequality is true:

$$d_H(A \cup B, C \cup D) \quad \le \quad d_H(A,C) \vee d_H(B,D). \tag{2.1}$$

Proof. Let $A, B, C, D \in \mathcal{H}(S)$. We observe that for all $E \subset B$:

$$\delta(A,B) = \sup_{a\in A} \inf_{b \in B} d(a,b) \le \sup_{a \in A} \inf_{e\in E} d(a,e) = \delta(A,E).$$

Since $B \subset (B \cup C)$ and $C \subset (B \cup C)$, it is easily been seen that

$$\delta(A, B \cup C) \quad \le \quad \delta(A,B) \wedge \delta(A,C). \tag{2.2}$$

We will need the following equality.

$$\begin{aligned} \delta(A \cup B, C) \quad &= \quad \sup_{a \in A \cup B} d(a,C) = \sup_{a\in A} d(a,C) \vee \sup_{a \in B} d(a,C) \\ &= \quad \delta(A,C) \vee \delta(B,C). \end{aligned} \tag{2.3}$$

Using (2.2) and (2.3) we obain that

$$\begin{aligned} \delta(A \cup B, C \cup D) \quad &= \quad \delta(A, C\cup D) \vee \delta(B, C \cup D) \\ &\le \quad [\delta(A,C) \wedge \delta(A,D)] \vee [\delta(B,C) \wedge \delta(B,D)] \\ &\le \quad [\delta(A,C) \vee \delta(B,D)] \wedge [\delta(B,C) \vee \delta(A,D)]\,. \end{aligned}$$

By definition $\delta(A,B) \le d_H(A,B)$ and $d_H(A,B) = d_H(B,A)$. Now we can derive the requested inequalities.

$$\delta(A \cup B, C \cup D) \le [d_H(A,C) \vee d_H(B,D)] \wedge [d_H(B,C) \vee d_H(A,D)]$$

and

$$\delta(C \cup D, A \cup B) \le [d_H(A,C) \vee d_H(B,D)] \wedge [d_H(B,C) \vee d_H(A,D)]\,.$$

Thus

$$d_H(A \cup B, C \cup D) \le \min(d_H(A,C) \vee d_H(B,D), d_H(B,C) \vee d_H(A,D)).$$

In particular we see that

$$d_H(A \cup B, C \cup D) \le d_H(A,C) \vee d_H(B,D).$$

$\square$

The next lemma states that the Hutchinson function is a contraction mapping. We need Lemma 2.3.2 and (2.1) from Lemma 2.3.3 to prove it.

Lemma 2.3.4. *Let $\mathcal{F}$ be an IFS. Then the associated Hutchinson function $\overline{F}$ is a contraction mapping on $\mathcal{H}(S)$ with Lipschitz constant $|\overline{F}|_{Lip} \leq |\mathcal{F}|_{Lip}$.*

Proof. Let $c_{\mathcal{F}} = |\mathcal{F}|_{Lip}$. For all $A, B \in \mathcal{H}(S)$:

$$\begin{aligned} d_H(\overline{F}(A), \overline{F}(B)) &= d_H(\bigcup_{i=1}^{n} \overline{f}_i(A), \bigcup_{i=1}^{n} \overline{f}_i(B)) \\ &\leq \max_{1 \leq i \leq n} (d_H(\overline{f}_i(A), \overline{f}_i(B)) \\ &\leq \max_{1 \leq i \leq n} (|f_i|_{Lip} \cdot d_H(A, B)) \leq c_{\mathcal{F}} \cdot d_H(A, B). \end{aligned}$$

The first inequality is the result of applying Lemma 2.3.3 (2.1) $(n-1)$ times. The second inequality is justified by Lemma 2.3.2. So $d_H(\overline{F}(A), \overline{F}(B)) \leq c_{\mathcal{F}} \cdot d_H(A, B)$. Thus $\overline{F}$ is a contraction mapping with $|\overline{F}|_{Lip} \leq |\mathcal{F}|_{Lip}$. □

Theorem 2.3.1. *Let $\mathcal{F}$ be an IFS on a complete metric space S and let $\overline{F}$ be the associated Hutchinson funcion. Then there exists a unique $X \in \mathcal{H}(S)$ such that $\overline{F}(X) = X$. Moreover*

$$\lim_{n \to \infty} \overline{F}^n(A) = X \text{ for all } A \in \mathcal{H}(S).$$

Proof. $\mathcal{H}(S)$ is complete by Theorem 2.2.1. $\overline{F}$ is a contraction mapping on $\mathcal{H}(S)$ by Lemma 2.3.4. With the help of the Banach Fixed Point Theorem we conclude that there exists a unique point $X \in \mathcal{H}(S)$ such that $\overline{F}(X) = (X)$ and

$$\lim_{n \to \infty} \overline{F}^n(A) = X \text{ for all } A \in \mathcal{H}(S).$$

□

We call X the *attractor* of the IFS $\mathcal{F}$.

Example 2.3.1. *The IFS of the Sierpinski triangle is given by $\{f_1, f_2, f_3\}$, where f_1, f_2, f_3 are the three contraction mappings from Example 1.1.1. Therefore the Hutchinson function is given by $\overline{F}(A) = \cup_{i=1}^{3} f_i(A)$. By Theorem 2.3.1 we see that T is the unique attractor of this IFS. Observe that we can take any closed bounded subset of S as the initial object. For example we could begin with the solid cube K_1 drawn in figure 2.1.*

Lemma 2.3.5. *Let $\mathcal{F}$ be an IFS. Let $\overline{F}$ be its Hutchinson function. Let $A \in \mathcal{H}(S)$. Then for all $k \in \mathbb{N}$:*

$$\overline{F}^k(A) = \bigcup_{i_k=1}^{n} \bigcup_{i_{k-1}=1}^{n} \cdots \bigcup_{i_1=1}^{n} \overline{f}_{i_k} \circ \overline{f}_{i_{k-1}} \circ \cdots \circ \overline{f}_{i_1}(A).$$

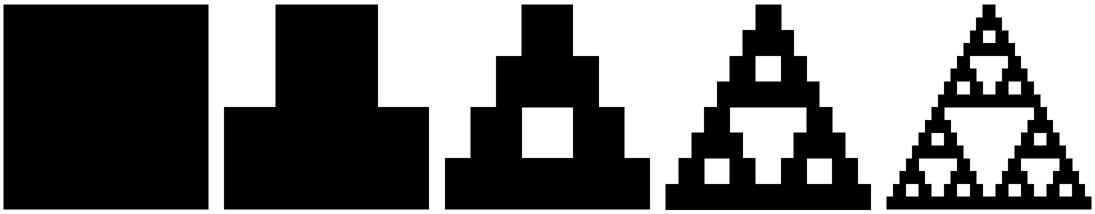


Figure 2.1: from cube to Sierpinski triangle

Proof. Let $A \in \mathcal{H}(S)$. Then

$$\overline{F}^2(A) = \bigcup_{i_2=1}^{n} \overline{f}_{i_2}\left(\bigcup_{i_1=1}^{n} \overline{f}_{i_1}(A)\right).$$

When we use $f(C \cup B) = f(C) \cup f(B)$, we get

$$\bigcup_{i_2=1}^{n} \overline{f}_{i_2}\left(\bigcup_{i_1=1}^{n} \overline{f}_{i_1}(A)\right) = \bigcup_{i_2=1}^{n}\bigcup_{i_1=1}^{n} \overline{f}_{i_2} \circ \overline{f}_{i_1}(A).$$

By induction we easily see that for all $k \in \mathbb{N}$:

$$\overline{F}^k(A) = \bigcup_{i_k=1}^{n}\bigcup_{i_{k-1}=1}^{n} \cdots \bigcup_{i_1=1}^{n} \overline{f}_{i_k} \circ \overline{f}_{i_{k-1}} \circ \cdots \circ \overline{f}_{i_1}(A).$$

□

Lemma 2.3.6. *Let $\mathcal{F}$ be an IFS and let X be its attractor. Then X is compact in S.*

Proof. Let $c_{\mathcal{F}} = |\mathcal{F}|_{Lip}$. Since X is closed by definition, we only need to show that X is totally bounded. By Lemma 2.3.5 we see that for all $k \in \mathbb{N}$:

$$X = \overline{F}^k(X) = \bigcup_{i_k=1}^{n}\bigcup_{i_{k-1}=1}^{n} \cdots \bigcup_{i_1=1}^{n} \overline{f}_{i_k} \circ \overline{f}_{i_{k-1}} \circ \cdots \circ \overline{f}_{i_1}(X).$$

Observe that

$$\begin{aligned} \operatorname{diam}\left(\overline{f}_{i_k} \circ \overline{f}_{i_{k-1}} \circ \cdots \circ \overline{f}_{i_1}(X)\right) &\leq |\overline{f}_{i_k} \circ \overline{f}_{i_{k-1}} \circ \cdots \circ \overline{f}_{i_1}|_{Lip} \cdot \operatorname{diam}(X) \\ &\leq {c_{\mathcal{F}}}^k \cdot \operatorname{diam}(X). \end{aligned}$$

Let $x = \overline{f}_{i_k} \circ \overline{f}_{i_{k-1}} \circ \cdots \circ \overline{f}_{i_1}(x_0)$ for a $x_0 \in X$, then

$$\overline{f}_{i_k} \circ \overline{f}_{i_{k-1}} \circ \cdots \circ \overline{f}_{i_1}(X) \subset B_x({c_{\mathcal{F}}}^k \cdot \operatorname{diam}(X)).$$

It follows that

$$X = \overline{F}^k(X) \subset \bigcup_{x \in \overline{F}^k(\{x_0\})} B_x({c_{\mathcal{F}}}^k \cdot \mathrm{diam}(X)).$$

It can easily been seen that $\overline{F}^k(\{x_0\})$ exists of finitely many point, so for all $\epsilon > 0$ we can choose a $k \in \mathbb{N}$ such that X is covered by finitely many open balls of diameter less or equal to ϵ. So X is totally bounded, hence compact. $\square$

Chapter 3

Measures on attractors

3.1 Measure theory

Let X be a non-empty set. By $\mathfrak{P}(X)$ we denote the power set of X, i.e. the collection of all subsets of X.

Definition 3.1.1. *A non-empty set $\mathcal{M} \subset \mathfrak{P}(X)$ is called a σ-algebra if*

(i) $E \in \mathcal{M}$ implies $E^c \in \mathcal{M}$,

(ii) $E_j \in \mathcal{M}, j \in \mathbb{N}$ implies $\cup_{j=1}^{\infty} E_j \in \mathcal{M}$.

The ordered pair $(X, \mathcal{M})$ is called a measurable space. The elements of $\mathcal{M}$ are called measurable subsets.

Definition 3.1.2. *Let $(X, \mathcal{M})$ be a measurable space.*

A (signed) measure on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to \overline{\mathbb{R}}$ such that

(i) $\mu(\emptyset) = 0$,

(ii) μ attains only one of the values $-\infty$ or $+\infty$,

(iii) for all $E_j \in \mathcal{M}$ such that $E_i \cap E_j = \emptyset$ if $i \neq j$, the following holds

$$\mu\left(\cup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j).$$

The ordered triplet $(X, \mathcal{M}, \mu)$ is called a measure space.

Let μ be a measure. μ is called a positive measure if $\mu(E) \geq 0$ for all $E \in \mathcal{M}$. μ is called a finite measure if $\mu(X) < \infty$. It is a probability measure if $\mu(X) = 1$.

Definition 3.1.3. *Let $(X, \mathcal{T})$ be a topological space. The Borel σ-algebra $\mathcal{B}_X$ is the smallest σ-algebra that contains the open subsets of X.*

A metric space (S, d) is a topological space defined by the open subsets induced by the metric d. Therefore the Borel σ-algebra $\mathcal{B}_S$ is a natural σ-algebra to associate with the metric space (S, d). So we see that $(S, \mathcal{B}_S)$ is a measurable space. A measure on the $\mathcal{B}_S$ is called a Borel measure.

We will use the following notations for certain sets of Borel measures. The set of finite Borel measures is denoted by $\mathcal{M}(S)$. We will use $\mathcal{M}^+(S)$ for the set of positive finite Borel measures and $\mathcal{P}(S)$ is the set of positive probability Borel measures.

Let $E \in \mathcal{B}_S$ be a Borel measurable subset of S. We define addition of Borel measures μ and ν by $(\mu + \nu)(E) := \mu(E) + \nu(E)$. And we define scalar multiplication of a Borel measure μ by $r \in \mathbb{R}$ by $(r\mu)(E) := r\mu(E)$. It is easy to verify that $\mu + \nu$ and $r\mu$ are also Borel measures, so the addition and scalar multiplication are well defined. Moreover one can prove that $\mathcal{M}(S)$ is a vector space over $\mathbb{R}$ with respect to these operators. The necessary properties follow easily from the properties of $\mathbb{R}$.

Definition 3.1.4 (support of a measure)**.** *The support of a measure $\mu \in \mathcal{M}^+(S)$ is defined by*

$$\operatorname{supp}(\mu) := \{x \in S : \mu(U) > 0 \textit{ for all } U \subset S \textit{ such that } U \textit{ is open and } x \in U\}.$$

Lemma 3.1.1. *Let $\mu \in \mathcal{M}^+(S)$. Then* $\operatorname{supp}(\mu)$ *is a closed subset of S.*

Proof. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $\operatorname{supp}(\mu)$ such that $x_n \to x$ in S. Let $U \subset S$ be open such that $x \in U$. Then there is an $N \in \mathbb{N}$ such that $x_n \in U$ for $n \geq N$. Since $x_N \in \operatorname{supp}(\mu)$, we see that $\mu(U) > 0$. So $x \in \operatorname{supp}(\mu)$. We conclude that $\operatorname{supp}(\mu)$ contains all of its limit points. As a consequence $\operatorname{supp}(\mu)$ is closed. $\square$

Definition 3.1.5 (measurable function)**.** *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable if*

$$f^{-1}(E) \in \mathcal{M} \textit{ for all } E \in \mathcal{N}.$$

If X, Y are topological spaces and f is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable, then we say that f is Borel measurable.

Lemma 3.1.2. *Let S_1, S_2 be metric spaces and let $f : S_1 \to S_2$ be a continuous function. Then f is Borel measurable.*

Proof. Observe that if A is an open subset of S_2, then $f^{-1}(A)$ is an open subset of S_1. So we see that $\{A : A \subset S_2, \text{ A is open }\} \subset \{E \in \mathfrak{P}(S_2) : f^{-1}(E) \in \mathcal{B}_{S_1}\}$. Since $f^{-1}(E) \in \mathcal{B}_{S_1}$ implies $f^{-1}(E^C) = f^{-1}(E)^C \in \mathcal{B}_{S_1}$ and $f^{-1}(E_1), f^{-1}(E_2) \in \mathcal{B}_{S_1}$ implies $f^{-1}(E_1 \cup E_2) = f^{-1}(E_1) \cup f^{-1}(E_2) \in \mathcal{B}_{S_1}$, we see that $\{E : f^{-1}(E) \in \mathcal{B}_{S_1}\}$ is a σ-algebra. By definition $\mathcal{B}_{S_2}$

is the smallest σ-algebra containing $\{A : A \subset S_2,\ \text{A is open}\}$, so $\mathcal{B}_{S_2} \subset \{E : f^{-1}(E) \in \mathcal{B}_{S_1}\}$. Thus f is Borel measurable. $\square$

Lemma 3.1.3. *Let $f : S \to S$ be a Borel measurable function. Define $\mu \circ f^{-1}(E) := \mu(f^{-1}(E)), E \in \mathcal{B}_S$. Then $\mu \circ f^{-1}$ is a Borel measure. Moreover if μ is a probability measure, then $\mu \circ f^{-1}$ is a probability measure.*

Proof. Since f is Borel measurable, we see that $f^{-1}(E) \in \mathcal{B}_S$ if $E \in \mathcal{B}_S$. So $\mu \circ f^{-1} : \mathcal{B}_S \to \overline{\mathbb{R}}$. We see that $\mu \circ f^{-1}(\emptyset) = \mu(\emptyset) = 0$. Let $\{E_i\}_{i=1}^{\infty}$, $E_i \in \mathcal{B}_S$ and $E_i \cap E_j = \emptyset$ for all $i \neq j$. We use the fact that $f^{-1}(E_i) \cap f^{-1}(E_j) = \emptyset$ for all $i \neq j$ and obtain that

$$\mu \circ f^{-1}(\cup_{i=1}^{\infty} E_i)) \quad = \quad \mu(\cup_{i=1}^{\infty} f^{-1}(E_i)) = \sum_{i=1}^{\infty} \mu(f^{-1}(E_i)) = \sum_{i=1}^{\infty} \mu \circ f^{-1}(E_i).$$

So $\mu \circ f^{-1}$ is a Borel measure. Moreover, if $\mu(S) = 1$, then $\mu \circ f^{-1}(S) = \mu(S) = 1$. $\square$

3.2 Hutchinson space

In this section we are going to define the *Hutchinson space*, that we denote by $\mathcal{P}_1(S)$. The elements are positive probability measures that have a finite first moment.

$$\mathcal{P}_1(S) := \{\mu \in \mathcal{P}(S) : \text{for some } x_0 \in S, \int_S d(x, x_0) d\mu(x) < \infty\}.$$

Soon we will see that it is a metric space for the *Hutchinson metric*, that we define by

$$d(\mu, \nu) := \sup \left\{ \left| \int_S f d\mu - \int_S f d\nu \right| \text{ such that } f \in \mathrm{Lip}_{\leq 1}(S, \mathbb{R}) \right\}.$$

Lemma 3.2.1. *Let $\mu \in \mathcal{M}^+$ be such that $\int_S d(x, x_0) d\mu(x) < \infty$ for some $x_0 \in S$. Then*

$$\int_S d(x, y) d\mu(x) < \infty \text{ for all } y \in S.$$

Proof. Let $y \in S$. Then

$$\begin{aligned} \int_S d(x, y) d\mu(x) &\leq \int_S (d(x, x_0) + d(x_0, y)) d\mu(x) \leq \int_S d(x, x_0) d\mu(x) + \int_S d(x_0, y) d\mu(x) \\ &= \int_S d(x, x_0) d\mu(x) + d(x_0, y) \cdot \mu(S) < \infty. \end{aligned}$$

$\square$

Lemma 3.2.2. *$(\mathcal{P}_1(S), d)$ is a metric space.*

Proof. We define $D := \mathrm{Lip}_{\leq 1}(S, \mathbb{R})$. Let $x_0 \in S$. For all μ and $\nu \in \mathcal{P}_1(S)$, $f \in D$:

$$\begin{aligned}
\left| \int_S f(x) d\mu(x) - \int_S f(x) d\nu(x) \right| &\leq \left| \int_S f(x) - f(x_0) d\mu(x) - \int_S f(x) - f(x_0) d\nu(x) \right| \\
&\leq \left| \int_S f(x) - f(x_0) d\mu(x) \right| + \left| \int_S f(x) - f(x_0) d\nu(x) \right| \\
&\leq \int_S |f(x) - f(x_0)| d\mu(x) + \int_S |f(x) - f(x_0)| d\nu(x) \\
&\leq \int_S d(x, x_0) d\mu(x) + \int_S d(x, x_0) d\nu(x).
\end{aligned}$$

The latter expression is finite according to Lemma 3.2.1. So $d(\mu, \nu) \in \mathbb{R}$. We still have to verify that d is a metric. Let $\mu, \nu, \rho \in \mathcal{P}_1(S)$. Then

(i) $d(\mu, \nu) = 0$, if $\mu = \nu$. If $d(\mu, \nu) = 0$, then

$$\sup_{f \in D} \left| \int_S f d\mu - \int_S f d\nu \right| = 0.$$

So $\int_S f d\mu = \int_S f d\nu$ for all $f \in D$. By [2, Lemma 6] we get $\mu = \nu$.

(ii) $d(\mu, \nu) = \sup_{f \in D} \left| \int_S f d\mu - \int_S f d\nu \right| = \sup_{f \in D} \left| \int_S f d\nu - \int_S f d\mu \right| = d(\nu, \mu)$.

(iii)

$$\begin{aligned}
d(\mu, \nu) &= \sup_{f \in D} \left| \int_S f d\mu - \int_S f d\nu \right| = \sup_{f \in D} \left| \int_S f d\mu - \int_S f d\rho + \int_S f d\rho - \int_S f d\nu \right| \\
&\leq \sup_{f \in D} \left| \int_S f d\mu - \int_S f d\rho \right| + \left| \int_S f d\rho - \int_S f d\nu \right| \\
&\leq \sup_{f \in D} \left| \int_S f d\mu - \int_S f d\rho \right| + \sup_{f \in D} \left| \int_S f d\rho - \int_S f d\nu \right| = d(\mu, \rho) + d(\rho, \nu).
\end{aligned}$$

$\square$

Theorem 3.2.1. *Let S be a complete metric space. Then $(\mathcal{P}_1(S), d)$ is a complete metric space.*

For the proof of this theorem we refer to [3, Theorem 4.2].

3.3 Contraction mappings and measures

Lemma 3.3.1. *Let $g : S \to \mathbb{R}$ be Borel measurable and let μ be a Borel measure. If $f : S \to S$ is Borel measurable, then*

$$\int_S g \circ f d\mu = \int_S g d[\mu \circ f^{-1}].$$

Proof. Since we can split any measurable function into a positive and a negative part, it suffices to prove the statement for positive g.

Let $\phi(x) = \sum_{i=1}^n a_i \chi_{E_i}(x)$ be a measurable step function, as defined in [4]. Observe that $\chi_E(f(x)) = \chi_{f^{-1}(E_i)}(x)$ for all $E \in \mathcal{B}_S$. So $(\phi \circ f)(x) = \sum_{i=1}^n a_i \chi_{f^{-1}(E_i)}(x)$ is also a measurable step function.

$$\int_S \phi \circ f d\mu = \sum_{i=1}^n a_i \mu(f^{-1}(E_i)) = \sum_{i=1}^n a_i [\mu \circ f^{-1}(E_i)] = \int_S \phi d[\mu \circ f^{-1}]. \qquad (3.1)$$

By [4, Theorem 3.2.1] there exist step functions $\{\phi_n\}_{n=1}^\infty$ such that $0 \leq \phi_n \leq \phi_{n+1} \leq g$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \phi_n(x) = g(x)$ for all $x \in S$. Observe that $0 \leq \phi_n \circ f \leq \phi_{n+1} \circ f \leq g \circ f$ for all $n \in \mathbb{N}$ and that $\lim_{n\to\infty} \phi_n \circ f(x) = g \circ f(x)$ for all $x \in S$. We use (3.1) and the Monotone Convergence Theorem [4, Theorem 3.3.1] twice to obtain that

$$\int_S g d[\mu \circ f^{-1}] = \lim_{n\to\infty} \int_S \phi_n d[\mu \circ f^{-1}] = \lim_{n\to\infty} \int_S \phi_n \circ f d\mu = \int_S g \circ f d\mu.$$

□

Definition 3.3.1. *Let $\mathcal{F} := \{f_i : S \to S | i = 1, 2, \cdots, n\}$ be an IFS. The associated Markov operator $M : \mathcal{P}_1(S) \to \mathcal{P}_1(S)$ is given by*

$$M(\mu) := \frac{1}{n} \sum_{i=1}^n \mu \circ f_i^{-1}.$$

Lemma 3.3.2. *The Markov operator M maps $\mathcal{P}_1(S)$ to $\mathcal{P}_1(S)$.*

Proof. Let $\mu \in \mathcal{P}_1(S)$. Observe that if $\nu, \rho \in \mathcal{M}^+$, then $(\nu + \rho) \in \mathcal{M}^+$, and if $r \in \mathbb{R}, \nu \in \mathcal{M}^+$, then $(r\nu) \in \mathcal{M}^+$. Using this and Lemma 3.1.3 we see that $M(\mu) \in \mathcal{M}^+(S)$.

$$M(\mu)(S) = \frac{1}{n} \sum_{i=1}^n \mu \circ f_i^{-1}(S) = \frac{1}{n} \sum_{i=1}^n 1 = 1.$$

So $M(\mu) \in \mathcal{P}(S)$. We still need to show that $M(\mu)$ has a finite first moment. Fix $x_0 \in S$ and let $d : S \to \mathbb{R} : x \mapsto d(x, x_0)$. Observe that d is continuous, hence it is Borel measurable by Lemma 3.1.2.

$$\begin{aligned}
\int_S d(x,x_0)dM(\mu)(x) &= \int_S d(x,x_0)d(\frac{1}{n}\sum_{i=1}^n \mu \circ f_i^{-1})(x) \\
&= \frac{1}{n}\sum_{i=1}^n \int_S d(x,x_0)d(\mu \circ f_i^{-1})(x) = \frac{1}{n}\sum_{i=1}^n \int_S d(f_i(x),x_0)d\mu(x) \\
&\leq \frac{1}{n}\sum_{i=1}^n \int_S \left[d(f_i(x),f_i(x_0)) + d(f_i(x_0),(x_0))\right] d\mu(x) \\
&\leq \frac{1}{n}\sum_{i=1}^n \int_S |f_i|_{Lip} d(x,x_0)d\mu(x) + \frac{1}{n}\sum_{i=1}^n d(f_i(x_0),x_0) \\
&\leq \frac{1}{n}c_{\mathcal{F}}\int_S d(x,x_0)d\mu(x) + \frac{1}{n}\sum_{i=1}^n d(f_i(x_0),x_0).
\end{aligned}$$

Since $\mu \in \mathcal{P}_1$, the latter expression is finite by Lemma 3.3.1. Thus $M(\mu) \in \mathcal{P}_1(S)$. □

Lemma 3.3.3. *Let $\mathcal{F}$ be an IFS and $M : \mathcal{P}_1(S) \to \mathcal{P}_1(S)$ the associated Markov operator. Then M is a contraction mapping with Lipschitz constant $c_{\mathcal{F}} = |\mathcal{F}|_{Lip}$.*

Proof. Let $\mu, \nu \in \mathcal{P}_1$. We have to show that $d(M(\mu), M(\nu)) \leq c_{\mathcal{F}} d(\mu, \nu)$.

If $c_{\mathcal{F}} = 0$, then each $f_i \in \mathcal{F}(S)$ map onto a single point $x_i \in S$. Therefore

$$\begin{aligned}
d(M(\mu),M(\nu)) &= \sup_{|g|_{Lip}\leq 1} \left| \int_S g dM(\mu) - \int_S g dM(\nu) \right| \\
&= \sup_{|g|_{Lip}\leq 1} \left| \frac{1}{n}\sum_{i=1}^n \int_S g \circ f_i d\mu - \frac{1}{n}\sum_{i=1}^n \int_S g \circ f_i d\nu \right| \\
&= \sup_{|g|_{Lip}\leq 1} \left| \frac{1}{n}\sum_{i=1}^n (g(x_i) - g(x_i)) \right| = 0.
\end{aligned}$$

If $c_{\mathcal{F}} \neq 0$, let $g : S \to \mathbb{R}$ be a Lipschitz function with $|g|_{Lip} \leq 1$. Let $f_i \in \mathcal{F}$. Lemma 1.1.1 yields

$$|g \circ f_i|_{Lip} \leq |g|_{Lip} \cdot |f_i|_{Lip} \leq |g|_{Lip} \cdot c_{\mathcal{F}} \leq c_{\mathcal{F}}.$$

Therefore

$$\{\frac{g \circ f_i}{c_{\mathcal{F}}} : g \in \mathrm{Lip}_{\leq 1}(S,\mathbb{R})\} \subset \mathrm{Lip}_{\leq 1}(S,\mathbb{R}).$$

Thus

$$\begin{aligned}
d(\mu \circ f_i^{-1}, \nu \circ f_i^{-1}) &= \sup_{g:|g|_{Lip}\le 1} \left| \int_S g \circ f_i d\mu - \int_S g \circ f_i d\nu \right| \\
&= c_{\mathcal{F}} \sup_{g:|g|_{Lip}\le 1} \left| \int_S \frac{g \circ f_i}{c_{\mathcal{F}}} d\mu - \int_S \frac{g \circ f_i}{c_{\mathcal{F}}} d\nu \right| \\
&\le c_{\mathcal{F}} \sup_{f:|f|_{Lip}\le 1} \left| \int_S f d\mu - \int_S f d\nu \right| = c_{\mathcal{F}} d(\mu, \nu).
\end{aligned}$$

And we conclude that the Markov operator is a contraction mapping:

$$\begin{aligned}
d(M(\mu), M(\nu)) &= \sup_{g:|g|_{Lip}\le 1} \left| \int_S g dM(\mu) - \int_S g dM(\nu) \right| \\
&= \sup_{g:|g|_{Lip}\le 1} \left| \frac{1}{n} \sum_{i=1}^{n} \int g \circ f_i d\mu - \frac{1}{n} \sum_{i=1}^{n} \int g \circ f_i d\nu \right| \\
&\le \frac{1}{n} \sum_{i=1}^{n} \sup_{g:|g|_{Lip}\le 1} \left| \int_S g \circ f_i d\mu - \int_S g \circ f_i d\nu \right| \\
&\le \frac{1}{n} \sum_{i=1}^{n} c_{\mathcal{F}} d(\mu, \nu) = c_{\mathcal{F}} d(\mu, \nu).
\end{aligned}$$

□

Corollary 3.3.1. *There exists a unique measure $\mu^* \in \mathcal{P}_1$ such that $M(\mu^*) = \mu^*$. Moreover, for all $k \in \mathbb{N}$:*

$$\mu^* = \frac{1}{n^k} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \left(\mu^* \circ f_{i_1}^{-1} \circ \cdots \circ f_{i_k}^{-1} \right). \tag{3.2}$$

Proof. The existence and the uniqueness of μ^* are direct consequences of the Banach Fixed Point Theorem. Equation (3.2) can be obtained by writing out $\mu^* = M^k(\mu^*)$. □

3.4 The support of μ^*

Lemma 3.4.1. *Let $f : S \to S$ be a continuous map and μ any Borel measure on S, then the following statements hold:*

(i) $f(\mathrm{supp}(\mu)) \subset \mathrm{supp}(\mu \circ f^{-1})$,

(ii) if f is a homeomorphism, then equality holds in (i).

Proof. *(i)*: Let $x \in f(\text{supp}(\mu))$ and $U \subset S$ be an open subset, such that $x \in U$. There exists a $y \in \text{supp}(\mu)$ such that $f(y) = x$. Observe that $f^{-1}(U)$ is open and $y \in f^{-1}(U)$. So $\mu \circ f^{-1}(U) = \mu(f^{-1}(U)) > 0$ and $x \in \text{supp}(\mu \circ f^{-1})$.

(ii): We use the result of *(i)* and the continuity of f^{-1}.

$$\text{supp}(\mu \circ f^{-1}) \;=\; f \circ f^{-1}(\text{supp}(\mu \circ f^{-1})) \subseteq f(\text{supp}\mu \circ f^{-1} \circ f)) = f(\text{supp}(\mu)).$$

□

Lemma 3.4.2. *Let μ_i be a Borel measure for $i = 1, 2, \cdots, n$. Then*

$$\text{supp}(\sum_{i=1}^{n} \mu_i) \subset \bigcup_{i=1}^{n} \text{supp}(\mu_i).$$

Proof. Let $x \in \bigcup_{i=1}^{n} \text{supp}(\mu_i)$. Then $x \in \text{supp}(\mu_k)$ for some $k \in \{1, \ldots, n\}$. So for all $U \subset S$ such that U is open and $x \in U$ we get $\sum_{i=1}^{n} \mu_i(U) > 0$. Therefore $x \in \text{supp}(\sum_{i=1}^{n} \mu_i)$. □

Lemma 3.4.3. *Let $\mathcal{F}$ be an IFS and X its attractor. Let μ^* be the fixed point of the associated Markov operator function M. Then* $\text{supp}(\mu^*)$ *is a bounded set. Moreover* $\text{supp}(\mu^*) \subset X$.

Proof. Let $x \in X$ and let δ_x be the dirac measure of x. It is easy to see that $\text{supp}(\delta_x) = \{x\} \in X$.

$$\text{supp}(M(\delta_x)) \;=\; \text{supp}(\frac{1}{n} \sum_{i=1}^{n} \delta_x \circ f_i^{-1}) = \text{supp}(\sum_{i=1}^{n} \delta_{f_i(x)}) = \bigcup_{i=1}^{n} \text{supp}(\delta_{f_i(x)}).$$

Since $f_i(x) \in X$, we get that $\text{supp}(M(\delta_x)) \subset X$. Inductively we can show that $\text{supp}(M^k(\delta_x)) \subset X$ for all $k \in \mathbb{N}$.

Let $f(z) := d(z, X)$. Then $f \in \text{Lip}_{\leq 1}(S, \mathbb{R})$. Since μ^* is the fixed point of the contraction mapping M, we see that

$$\left| \int_S f d[M^k(\delta_x)] - \int_S f d\mu^* \right| \;\leq\; d(M^k(\delta_x), \mu^*) \to 0$$

as we take the limit $k \to \infty$. Using that $f(z) = 0$ for all $z \in X$ and $\text{supp}(M^k(\delta_x)) \subset X$, we obtain $\int_S f d[M^k(\delta_x)] = 0$. Therefore $\int_S f d\mu^* = 0$ and

$$\int_S f d\mu^* \;=\; \int_X f d\mu^* + \int_{X^C} f d\mu^* = 0 + \int_{X^C} f d\mu^*.$$

Since $f(z) > 0$ for $z \notin X$, we get $\mu^*(X^C) = 0$. Since X^C is open, $\text{supp}(\mu^*) \subset X$. Moreover, X is bounded, thus $\text{supp}(\mu^*)$ is bounded. □

Theorem 3.4.1. *Let $\mathcal{F} := \{f_i : S \to S | i = 1, 2, \cdots, n\}$ be an IFS and X its attractor. Let μ^* be the fixed point of the associated Markov operator function M, then* $\mathrm{supp}(\mu^*) = X$*.*

Proof. If we show that $X \subset \mathrm{supp}(\mu^*)$, then Lemma 3.4.3 will complete the proof.

$\mathrm{supp}(\mu^*)$ is closed by Lemma 3.1.1 and bounded by Lemma 3.4.3. So $\mathrm{supp}(\mu^*) \in \mathcal{H}(S)$. By Theorem 2.3.1

$$\lim_{n\to\infty} \overline{F}^n(\mathrm{supp}(\mu^*)) = X. \tag{3.3}$$

We show that $\overline{F}(\mathrm{supp}(\mu^*)) \subset \mathrm{supp}(\mu^*)$ by using Lemma 3.4.2 and Lemma 3.4.1. The removal of closure operator is justified by Lemma 3.1.1.

$$\begin{aligned}\overline{F}(\mathrm{supp}(\mu^*)) &= \bigcup_{i=1}^{n} \overline{f}_i(\mathrm{supp}(\mu^*)) = \bigcup_{i=1}^{n} \mathrm{cl}(f_i(\mathrm{supp}(\mu^*))) \subset \bigcup_{i=1}^{n} \mathrm{cl}(\mathrm{supp}(\mu^* \circ f_i^{-1})) \\ &\subset \mathrm{supp}(\sum_{i=1}^{n} \mu^* \circ f_i^{-1}) = \mathrm{supp}(\frac{1}{n}\sum_{i=1}^{n} \mu^* \circ f_i^{-1}) = \mathrm{supp}(M(\mu^*)) = \mathrm{supp}(\mu^*).\end{aligned}$$

Now it follows easily that $\overline{F}^k(\mathrm{supp}(\mu^*)) \subset \mathrm{supp}(\mu^*)$ for all $k \in \mathbb{N}$. Combining this with (3.3) we obtain that $X \subset \mathrm{supp}(\mu^*)$. $\square$

Chapter 4

Dimension of fractals

4.1 Hausdorff measure and dimension

Let (S, d) be a metric space. For $p \geq 0$, $\delta > 0$ and $A \subset S$ we define

$$H_{p,\delta}(A) := \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(B_i)^p : A \subset \bigcup_{i=1}^{\infty} B_i \text{ and } \operatorname{diam}(B_i) \leq \delta \right\}.$$

Observe that

$$H_{p,\delta}(A) \geq H_{p,\epsilon}(A) \text{ for all } \delta \leq \epsilon,$$

because the infinum is taken over less.

Definition 4.1.1. *The p-dimensional Hausdorff measure of $A \subset S$ is*

$$H_p(A) := \lim_{\delta \downarrow 0} H_{p,\delta}(A).$$

Lemma 4.1.1. *H_p restricted to the Borel sets is a measure.*

For the proof we refer to [1, Lemma 11.16] and [1, Lemma 11.17].

Lemma 4.1.2. *Let $A \in \mathcal{B}_S$ and let $H_p(A)$ be the Hausdorff measure of A. Then there is a unique $p \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that*

$$H_q(A) = 0 \textit{ for all } q > p,$$

$$H_q(A) = \infty \textit{ for all } 0 \leq q < p.$$

Proof. First we show that $H_p(A) < \infty$ implies $H_q(A) = 0$ for all $q > p$.

Suppose $H_p(A) = M < \infty$. Since $H_p(A) = \lim_{\delta \downarrow 0} H_{p,\delta}(A)$, we see that for all $\delta > 0$ holds $H_{p,\delta}(A) \leq M$. Therefore for all $\delta > 0$ there exists a collection of subsets $\{B_i^\delta\}_{i=1}^\infty$ such that *(i)* $A \subset \bigcup_{i=1}^{\infty} B_i^{\delta}$,

(ii) $\operatorname{diam}(B_i^{\delta}) \leq \delta$,

(iii) $\sum_{i=1}^{\infty} \operatorname{diam}(B_i^{\delta})^p \leq M + 1$.

For all $q > p$:

$$\begin{aligned} H_q(A) &= \liminf_{\delta \downarrow 0} H_{q,\delta}(A) \leq \liminf_{\delta \downarrow 0} \sum_{i=1}^{\infty} \operatorname{diam}(B_i^{\delta})^q \\ &= \liminf_{\delta \downarrow 0} \sum_{i=1}^{\infty} \operatorname{diam}(B_i^{\delta})^{q-p} \cdot \operatorname{diam}(B_i^{\delta})^p \leq \liminf_{\delta \downarrow 0} \delta^{q-p} \cdot \sum_{i=1}^{\infty} \operatorname{diam}(B_i^{\delta})^p \\ &\leq \liminf_{\delta \downarrow 0} \delta^{q-p} \cdot (M+1) = 0. \end{aligned}$$

From this it follows that $H_p(A) > 0$ implies $H_q(A) = \infty$ for all $q < p$. Thus there exists a unique $p \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

$$H_q(A) = \infty \text{ for all } 0 \leq q < p,$$

$$H_q(A) = 0 \text{ for all } q > p.$$

□

Definition 4.1.2 (Hausdorff Dimension)**.** *Let $A \in \mathcal{B}_S$. The Hausdorff dimension of A, $\dim_H(A)$, is the unique value of p given by Lemma 4.1.2.*

The next example, Example 4.1.1, will show the calculation of the Hausdorff dimension of a p-dimensional cube. The outcome of the example is in accordance with our intuitive idea of dimension. The method we will use to calculate the Hausdorff dimension is instructive, since it resembles in the main features the determination of the Hausdorff dimension of attractors.

Example 4.1.1. *Let $p, n \in \mathbb{N}, p \leq n$ and let $K^p \subset (\mathbb{R}^n, d_E)$ be the p-dimensional closed unit cube. Then $\dim_H(K^p) = p$.*

Proof. We divide this proof into two parts. First we show that $H_p(K^p) < \infty$ and after that we show that $H_p(K^p) > 0$. Then it follows from the definition that $\dim_H(K^p) = p$.

For all $\delta > 0$ there exists an $N \in \mathbb{N}$ such that $\frac{\operatorname{diam}(K^p)}{N} \leq \delta$. Then K^p can be covered by N^p p-dimensional cubes with diameter $\frac{\operatorname{diam}(K^p)}{N}$. Let $\{K_i\}_{i=1}^{N^p}$ be such a covering. Then for all $\delta > 0$:

$$H_{p,\delta}(K^p) \leq \sum_{i=1}^{N^p} \operatorname{diam}(K_i)^p = N^p \cdot \left(\frac{\operatorname{diam}(K^p)}{N} \right)^p = \operatorname{diam}(K^p)^p < \infty.$$

Therefore $H_p(K^p) = \lim_{d \downarrow 0} H_{p,\delta}(K^p) < \infty$.

To prove $H_p(K^p) > 0$ we need to show that $\lim_{\delta \downarrow 0} H_{p,\delta}(K^p) > 0$. For each covering $\{B_i\}_{i=1}^{\infty}$ of K^p, we construct a new covering $\{R_i\}_{i=1}^{\infty}$ by taking closed p-dimensional cubes R_i with length of the sides $\text{diam}(B_i)$, such that $B_i \subset R_i$. Observe that $\text{diam}(R_i) = \text{diam}(B_i)\sqrt{p}$ and $K^p \subset \cup_{i=1}^{\infty} R_i$.

Let $I(R_i)$ the volume of R_i. It is easy to see that $I(R_i) \leq \text{diam}(B_i)^p$. Now we derive that for each covering $\{B_i\}_{i=1}^{\infty}$ of K^p :

$$\sum_{i=1}^{\infty} \text{diam}(B_i)^p \geq \sum_{i=1}^{\infty} I(R_i) \geq I(K^p) = 1.$$

So for all $\delta > 0$:

$$\begin{aligned} H_{p,\delta}(K^p) &= \inf \left\{ \sum_{i=1}^{\infty} \text{diam}(B_i)^p : K^p \subset \bigcup_{i=1}^{\infty} B_i \text{ and } \text{diam}(B_i) \leq \delta \right\} \\ &\geq \inf \left\{ \sum_{i=1}^{\infty} I(R_i) : K^p \subset \bigcup_{i=1}^{\infty} R_i \text{ and } \text{diam}(R_i) \leq \delta \cdot \sqrt{p} \right\} \\ &\geq I(K^p) = 1. \end{aligned}$$

So $H_p(K^p) = \lim_{\delta \downarrow 0} H_{p,\delta}(K^p) \geq 1 > 0$. Thus $\dim_H(K_p) = p$. □

4.2 The Hausdorff dimension of attractors

In this section we will give expression to the Hausdorff dimension of attractors of iterated function systems. The approach is similar to the calculation of the Hausdorff dimension of the p-dimensional cube in Example 4.1.1: we will show that for the attractor X there exists a p such that $0 < H_p(X)$ and $H_p(X) < \infty$. We will see that the proof of $H_p(X) < \infty$ is easy and can be given for a general IFS. On the other hand $0 < H_p(X)$ is much harder to prove and requires some more conditions on the IFS. On the condition that the IFS has a *separating set* (Definition (4.2.1)) and the elements of the IFS are *similitudes* (Definition (4.2.2)) we will achieve the following result:

Theorem 4.2.1. *Let $\mathcal{F} := \{f_i : \mathbb{R}^m \to \mathbb{R}^m | i = 1, 2, \cdots, n\}$ be an IFS that has a separating set and let X be its attractor. Suppose that each $f_i \in \mathcal{F}$ is a contractive similitude with Lipschitz constant c_i. Then $\dim_H(X) = p$ where p is the unique solution to the equation*

$$\sum_{i=1}^{n} {c_i}^p = 1. \tag{4.1}$$

As we already claimed before, p is not necessarily an integer. The Intermediate value Theorem yields the uniqueness of the solution p to (4.1) since $f(x) := \sum_{i=1}^{n} {c_i}^x$ is a strictly decreasing function.

Lemma 4.2.1. *Let $\mathcal{F} := \{f_i : S \to S | i = 1, 2, \cdots, n\}$ be an IFS on a complete space S. Let c_i be the Lipschitz constant of f_i and let X be the attractor of $\mathcal{F}$. If p is such that*

$$\sum_{i=1}^{n} {c_i}^p = 1,$$

then $H_p(X) < \infty$.

Proof. Let $\delta > 0$ and take $k \in \mathbb{N}$ such that ${c_{\mathcal{F}}}^k \cdot \mathrm{diam}(X) \leq \delta$, where $c_{\mathcal{F}} := \max_{1<i<n} c_i$. By lemma 2.3.5 we see that

$$\overline{F}^k(X) = \bigcup_{i_k=1}^{n} \bigcup_{i_{k-1}=1}^{n} \cdots \bigcup_{i_1=1}^{n} \overline{f}_{i_k} \circ \overline{f}_{i_{k-1}} \circ \cdots \circ \overline{f}_{i_1}(X).$$

From Lemma 1.1.1 it follows that

$$\mathrm{diam}(\overline{f}_{i_k} \circ \overline{f}_{i_{k-1}} \circ \cdots \circ \overline{f}_{i_1}(X)) \leq (c_{i_k} \cdot c_{i_{k-1}} \cdot \ldots \cdot c_{i_2} \cdot c_{i_1})\mathrm{diam}(X) \leq \delta.$$

So X can be covered with n^k subsets with diameter less or equal δ. Observe that

$$\sum_{i_k=1}^{n} \sum_{i_{k-1}=1}^{n} \cdots \sum_{i_1=1}^{n} (c_{i_k} \cdot c_{i_{k-1}} \cdot \ldots \cdot c_{i_2} \cdot c_{i_1})^p \;=\; \sum_{i_k=1}^{n} \sum_{i_{k-1}=1}^{n} \cdots \sum_{i_2=1}^{n} (c_{i_k} \cdot c_{i_{k-1}} \cdot \ldots \cdot c_{i_2})^p \cdot \sum_{i_1=1}^{n} c_{i_1}^p.$$

By repeating this we obtain that

$$\begin{aligned}\sum_{i_k=1}^{n} \sum_{i_{k-1}=1}^{n} \cdots \sum_{i_1=1}^{n} (c_{i_k} \cdot c_{i_{k-1}} \cdot \ldots \cdot c_{i_2} \cdot c_{i_1})^p &= \sum_{i_k=1}^{n} c_{i_k}^p \cdot \sum_{i_{k-1}=1}^{n} c_{i_{k-1}}^p \cdot \ldots \cdot \sum_{i_2=1}^{n} c_{i_2}^p \cdot \sum_{i_1=1}^{n} c_{i_1}^p \\ &= 1,\end{aligned}$$

since $\sum_{i=1}^{n} {c_i}^p = 1$. So

$$\begin{aligned}H_{p,\delta}(X) &\leq \sum_{i_k=1}^{n} \sum_{i_{k-1}=1}^{n} \cdots \sum_{i_1=1}^{n} \left((c_{i_k} \cdot c_{i_{k-1}} \cdot \ldots \cdot c_{i_2} \cdot c_{i_1})\mathrm{diam}(X)\right)^p \\ &= \mathrm{diam}(X)^p \cdot \sum_{i_k=1}^{n} \sum_{i_{k-1}=1}^{n} \cdots \sum_{i_1=1}^{n} (c_{i_k} \cdot c_{i_{k-1}} \cdot \ldots \cdot c_{i_2} \cdot c_{i_1})^p \\ &= \mathrm{diam}(X)^p < \infty.\end{aligned}$$

$\delta > 0$ was arbitrary. Therefore $H_p(X) = \lim_{d \downarrow 0} H_{p,\delta}(X) < \infty$. $\square$

Let $\mathcal{F}$ be an IFS. For $A \subset S$ we define the open Hutchinson funcion by

$$F(A) := \bigcup_{i=1}^{n} f_i(A).$$

Definition 4.2.1. *[separating set] A separating set of an IFS $\mathcal{F}$ is a non-empty bounded open subset U such that*

(i) $F(U) \subset U$,

(ii) $f_i(U) \cap f_j(U) = \emptyset$ *if* $i \neq j$,

where F is the associated open Hutchinson function of $\mathcal{F}$.

The first condition of the separating set ensures that U is not too small. In Lemma (4.2.2) we will prove that this condition implies that $X \subset \overline{U}$. As a result we can obtain that $\overline{f}_i(X) \subset \overline{f}_i(\overline{U})$. The second condition ensures that U is not too big. Consequently if $i \neq j$, then overlap of $\overline{f}_i(X)$ and $\overline{f}_j(X)$ can only occur on their bounderies, so that the counting argument of Lemma (4.2.3) can be used.

Lemma 4.2.2. *Let $\mathcal{F}$ be an IFS and X its attractor. Suppose that U is a separating set of $\mathcal{F}$, then $X \subset \overline{U}$.*

Proof. Let $\overline{F}$ be the Hutchinson function of $\mathcal{F}$ and let F be the open Hutchinson function of $\mathcal{F}$. First we show that $F^k(\overline{U}) \subseteq \overline{U}$ for all $k \in \mathbb{N}$.

For continuous functions f we see that $f(\overline{U}) \subset \overline{f}(U)$. Therefore

$$F(\overline{U}) = \bigcup_{i=1}^{n} f_i(\overline{U}) \subset \bigcup_{i=1}^{n} \overline{f_i}(U) = \overline{F}(U) \subset \overline{U}. \tag{4.2}$$

The last inclusion is a consequence of the separating set property. By repeating (4.2) we obtain that

$$F^k(\overline{U}) \subset F^{k-1}(\overline{U}) \subset \ldots \subset F(\overline{U}) \subset \overline{U}.$$

Due to the definition of the semi-distance δ, if $\delta(X, \overline{U}) = 0$, then $X \subset \overline{U}$. We complete the proof by showing that $\delta(X, \overline{U}) = 0$. Observe that $\lim_{n\to\infty} d_H(X, F^n(\overline{U})) = \lim_{n\to\infty} d_H(X, \overline{F}^n(U)) = 0$ by Theorem 2.3.1.

$$\delta(X, \overline{U}) \;\; \leq \;\; \delta(X, F^n(\overline{U}) + \delta(F^n(\overline{U}), \overline{U}) = \delta(X, F^n(\overline{U})),$$

using $\delta(F^n(\overline{U}), \overline{U}) = 0$, since $F^n(\overline{U}) \subset \overline{U}$. By taking the limit $n \to \infty$ we obtain $\delta(X, \overline{U}) = 0$. Thus $X \subset \overline{U}$. □

Lemma 4.2.3. *Let $\{U_n\}_{n=1}^{\infty}$ be a set of disjoint open subsets in $\mathbb{R}^n$. Let $a, b, \delta \in \mathbb{R}$ be such that for all $n \in \mathbb{N}$:*

(i) U_n contains a ball of radius $a\delta$.

(ii) U_n is contained in a ball of radius $b\delta$.

Let N be the number of non-empty intersections of a ball with radius δ with the sets $\overline{U}_n$. Then $N \leq (2b+1)^n a^{-n}$.

Proof. Let $B = \overline{B_x}(\delta)$ i.e. B is a closed ball with radius δ around x. Suppose that $\overline{U}_m \cap B \neq \emptyset$ for some $m \in \mathbb{N}$. Since $\overline{U}_m$ is contained in a ball of radius $b\delta$ we see that $d(y, B) \leq 2b\delta$ for all $y \in \overline{U}_m$. So $d(y,x) \leq 2b\delta + \delta$ and $\overline{U}_m \subset \overline{B_x}(2b\delta + \delta)$.

Let $\{\overline{U}_i\}_{i=1}^N$ be the elements $\{\overline{U}_n\}_{n=1}^\infty$ that intersect with B. We know that the volume of a ball with radius r in $\mathbb{R}^n$ is given by $c_n r^n$ where c_n is a constant depending on n. Since $U_i \cap U_j = \emptyset$ if $i \neq j$, we see that the volume of $\bigcup_{i=1}^N U_i \geq N c_n (a\delta)^n$. On the other hand we see that de volume of $\overline{B_x}(2b\delta+\delta) \leq c_n((2b+1)\delta)^n$. Then $\bigcup_{i=1}^N \overline{U}_i \subset \overline{B_x}(2b\delta + \delta)$. As a result of that we get

$$N c_n (a\delta)^n \leq c_n((2b+1)\delta)^n$$

or equivalently $N \leq (2b+1)^n a^{-n}$. □

Definition 4.2.2. *Let $f : S_1 \to S_2$ be a Lipschitz function. If for $x, y \in S_1$:*

$$d_2(f(x), f(y)) = |f|_{Lip} d_1(x,y),$$

then f is called a similitude.

We will prove a special case of Theorem 4.2.1. We make the assumption that all contractive similitudes of $\mathcal{F}$ have the same Lipschitz constant $c_{\mathcal{F}} = |\mathcal{F}|_{Lip}$. We will follow the prove of [1]. A full proof of Theorem 4.2.1 can be found in [6].

proof of Theorem 4.2.1 (Special case). The proof of $H_p(X) < \infty$ has already been given in Lemma 4.2.1. We still need to show that $0 < H_p(X)$. Let U be the separating set that is assumed to exist for the IFS.

Since $c_i = c_{\mathcal{F}}$ for all $i \in \{1, \ldots, n\}$, we see $\sum_{i=1}^n {c_i}^p = n{c_{\mathcal{F}}}^p = 1$. So $n = \frac{1}{{c_{\mathcal{F}}}^p}$.

Let $\delta > 0$. Since X is compact (Lemma 2.3.6), there exists a covering $\{E_i\}_{i=1}^\infty$ of X with $\mathrm{diam}(E_i) \leq \delta$ for all $i \in \mathbb{N}$. We are going to prove that $0 < H_p(X)$ by showing that

$$H_{p,\delta}(X) \quad = \quad \inf\left\{\sum_{i=1}^\infty \mathrm{diam}(F_i)^p : X \subset \bigcup_{i=1}^\infty F_i \text{ and } \mathrm{diam}(F_i) \leq \delta\right\} \tag{4.3}$$

is greater than zero for some $\delta > 0$. Let $E \in \{E_i\}_{i=1}^\infty$ and let $k \in \mathbb{N}$ be such that ${c_{\mathcal{F}}}^k \mathrm{diam}(U) < 2\mathrm{diam}(E) \leq {c_{\mathcal{F}}}^{k-1}\mathrm{diam}(U)$. Let $\mathbb{U}_k := \{A \subset \mathbb{R}^m : A = f_{i_k} \circ \cdots \circ f_{i_1}(U), i_1, \ldots, i_k \in \{1, \ldots, n\}\}$ be the collection of sets that can be obtained by applying k-number of functions of the IFS to the separating set U. Moreover, let $\overline{\mathbb{U}}_k$ be the collection of the closure of the same sets. Observe that the sets of $\mathbb{U}_k$ are open. By using Definition 4.2.1 *(ii)* for separating sets inductively, we see that the sets of $\mathbb{U}_k$ are disjoint.

Let $A \in \mathbb{U}_k$. Lemma 2.3.1 yields

$$\mathrm{diam}(A) \quad = \quad {c_{\mathcal{F}}}^k \mathrm{diam}(U) < 2 \cdot \mathrm{diam}(E), \tag{4.4}$$

and

$$\mathrm{diam}(A) \quad = \quad c_{\mathcal{F}} {c_{\mathcal{F}}}^{k-1} \mathrm{diam}(U) \geq 2 \cdot c_{\mathcal{F}} \mathrm{diam}(E). \tag{4.5}$$

The equalities in (4.4) and (4.5) are justified, since the elements of $\mathcal{F}$ are similitudes. From (4.4) it follows that A is contained in a ball of radius $\mathrm{diam}(E)$. Since U is an open subset, it contains a ball with radius $R > 0$. Because each f_i is a similitudes, A contains a ball with radius $R \cdot {c_{\mathcal{F}}}^k$. Using (4.5) we obtain that

$$R \cdot {c_{\mathcal{F}}}^k = \frac{R \cdot \mathrm{diam}(A)}{\mathrm{diam}(U)} \geq \frac{2R \cdot c_{\mathcal{F}}}{\mathrm{diam}(U)} \cdot \mathrm{diam}(E).$$

So A contains a ball of $\frac{R \cdot c_{\mathcal{F}}}{\mathrm{diam}(U)} \cdot \mathrm{diam}(E)$. By Lemma 4.2.3 we see that a ball of radius $\mathrm{diam}(E)$ intersects at most N, the greatest integer less or equal to $\left(\frac{3 \cdot \mathrm{diam}(U)}{R \cdot c_{\mathcal{F}}}\right)^m$, of the elements of $\overline{\mathbb{U}}_k$. Since E can be contained in a closed ball of radius $\mathrm{diam}(E)$, it intersect no more than N of the sets in $\overline{\mathbb{U}}_k$. Observe that N does not depend on E or δ.

Let $i_1, \ldots, i_k \in \{1, \ldots, n\}$. Then

$$\begin{aligned} \mathrm{supp}\left(\mu^* \circ f_{i_1}^{-1} \circ \cdots \circ f_{i_k}^{-1}\right) &= \overline{f}_{i_k} \circ \cdots \circ \overline{f}_{i_1} (\mathrm{supp}(\mu^*)) & (4.6)\\ &= \overline{f}_{i_k} \circ \cdots \circ \overline{f}_{i_1}(X) \subset \overline{f}_{i_k} \circ \cdots \circ \overline{f}_{i_1}(\overline{U}) \in \overline{\mathbb{U}}_k. & (4.7) \end{aligned}$$

In (4.6) we used Lemma 3.4.1 k times. The equality in (4.7) is justified by Theorem 3.4.1. We justify the inclusion in (4.7) by Lemma 4.2.2. Now we can give an upper bound for $\mu^*(E)$. Recall that $n = \frac{1}{{c_{\mathcal{F}}}^p}$ and ${c_{\mathcal{F}}}^k \mathrm{diam}(U) < 2\mathrm{diam}(E)$.

$$\mu^*(E) = \frac{1}{n^k} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \left(\mu^* \circ f_{i_1}^{-1} \circ \cdots \circ f_{i_k}^{-1}(E_i)\right) \leq N \frac{1}{n^k} = N {c_{\mathcal{F}}}^{(kp)} \leq 2^p N \cdot \frac{\mathrm{diam}(E)^p}{\mathrm{diam}(U)^p}.$$

For the first inequality we used the fact that each $\mu^* \circ f_{i_1}^{-1} \circ \cdots \circ f_{i_k}^{-1} \in \mathcal{P}$ and that E intersects no more than N of their supports.

We return to (4.3) and continue:

$$\begin{aligned} \cdots \quad &\geq \quad \inf \left\{ \frac{\mathrm{diam}(U)^p}{2^p N} \cdot \sum_{i=1}^{\infty} \mu^*(F_i) : X \subset \bigcup_{i=1}^{\infty} F_i \text{ and } \mathrm{diam}(F_i) \leq \delta \right\} \\ &\geq \quad \inf \left\{ \frac{\mathrm{diam}(U)^p}{2^p N} \cdot \mu^*(X) \right\} = \frac{\mathrm{diam}(U)^p}{2^p N}. \end{aligned}$$

Taking the limit $\delta \downarrow 0$, we get that $H_p(X) \geq \frac{\mathrm{diam}(U)^p}{2^p N} > 0$. Thus $\dim_H(X) = p$. $\square$

Example 4.2.1. *The IFS of the Sierpinski triangle has a separating set, namely the interior of the initial triangle T_1. By Theorem 4.2.1 the Hausdorff dimension of the Sierpinski triangle T is the unqiue p such that*

$$1 = \sum_{i=1}^{3} \left(\frac{1}{2}\right)^p = \frac{3}{2^p}.$$

So $\dim_H(T) = \log_2 3$.

Bibliography

[1] G.B. Folland, (1984), *Real analysis, modern techniques and their applications*, John Wiley & Sons Inc.,New York.

[2] R.M. Dudley, (1966), Convergence of Baire measures, *Stud. Math.* ***27***, 251–268.

[3] A.S. Kravchenko, (2006), Completeness of the space of separable measures in the Kantorovich–Rubinstein metric, *Sibirsk. Mat. Zh.* ***47***, 85–96.

[4] S.M. Verduyn Lunel and S.C. Hille, (2006), *Syllabus Maat- en Integratietheorie*, Leiden University.

[5] M.F. Barnsley, (1988), *Fractals Everywhere*, Academic Press Inc., Londen.

[6] J.E. Hutchinson, (1981), Fractals and self-similarity, *Indiana Univ. Math. J.* ***30**, no. 5*, 713–747.

[7] K. R. Pathasorathy (1967), *Probability Measures on Metric Spaces*, AMS, Providence.