Categories of sets with a group action

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1 Introduction

1.1 Abstract

In the 40s, Mac Lane and Eilenberg introduced categories. Although by some referred to as abstract nonsense, the idea of categories allows one to talk about mathematical objects and their relationions in a general setting. Its origins lie in the field of algebraic topology, one of the topics that will be explored in this thesis. First, a concise introduction to categories will be given. Then, a few examples of categories will be presented. After this, two specific categories will be singled out and treated in more detail, namely the category of π-sets and the category of covering spaces for space X (with certain conditions) with π the fundamental group of X. The main theorem that will be proved is that these two categories are "equivalent". This means that we can translate problems from one category, in this case the category of covering spaces, to problems in the category of G-sets. In certain instances this proves to be fruitful as certain problems are more easily solved algebraically than topologically. As an application, a slightly weaker form of the famous **Seifert-van Kampen theorem** will be proved using the equivalence of categories.

1.2 Working method

The books that have been referenced in this thesis can be found in the references. For the section on categories the book written by Saunders Mac Lane (see [1]) was used as a reference. The section on algebraic topology can be found in more detail in the free online textbook written by Allen Hatcher (see [2]). The chapter on the Seifert-van Kampen theorem is based on own ideas and suggestions made by Bas Edixhoven. While there exist many versions of the van Kampen theorem, I have not encountered this proof in the existing literature : either it is proved topologically or in greater generality (for example using the groupoid).

• Notation

In most topology textbooks it is customary to write the composition of paths γ and τ as $\gamma \cdot \tau$ where τ is traversed after γ. Since we are taking a categorical approach we employ the reverse notation $\tau \cdot \gamma$ in order to stay consistent with the right-to-left notation for the composition of arrows. Except for the composition of arrows in categories we shall write fg for the composition of functions f and g.

In chapter 3 and 4 several functors between categories will be defined. We shall not be too pedantic about the notation. For a functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$, an object $c \in \mathcal{C}_1$ and morphism $f : c \to c'$ in $\mathcal{C}_1$ we shall often write $\mathcal{F}(c)$ and $\mathcal{F}(f)$ (or $\mathcal{F}(f : c \to c')$) and understand from context whether the functor is being applied to an object or an arrow.

"I didn't invent categories to study functors; I invented them to study natural transformations."

Saunders Mac Lane
(1909-2005)

2 Categories

In this chapter a concise description of category theory will be given. We will start with a definition and then proceed to give some illuminating examples.

2.1 Basics

First, a **graph** is a set of **objects** O and a set of **arrows**, or **morphisms**, denoted A, together with two functions dom, ran : $A \to O$, specifying the beginning and endpoints of the arrows. The set of all **composable arrows** is defined as

$$A \times_O A \stackrel{\text{def}}{=} \{\ (g, f) \in A \times A \mid \text{dom } g = \text{ran } f\ \}.$$

A **category** is then a graph, as defined above, together with an identity function id : $O \to A$ given by $c \mapsto \text{id}_c$, and a composition function $\circ : A \times_O A \to A$ given by $(g, f) \mapsto g \circ f$ such that the following criteria are met:

- $\text{dom }(\text{id}_a) = \text{ran }(\text{id}_a) = a$ for all $a \in O$;
- $\text{dom }(g \circ f) = \text{dom } f$ and $\text{ran }(g \circ f) = \text{ran } g$ for all $(g, f) \in A \times_O A$;
- $\text{id}_a \circ f = f = f \circ \text{id}_b$ holds for all $f : a \to b$ in A;
- $(f \circ g) \circ h = f \circ (g \circ h)$ for objects and arrows with configuration $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$.

For ease of notation, we will write $x \in \mathcal{C}$ when referring to an object and f in $\mathcal{C}$ when referring to an arrow. Let us consider some examples.

2.1. Examples.

- the category **Set** where the objects are all small[1] sets, and the arrows are functions between them.

[1]In order to avoid contradictions, we restrict oursevles to sets that are contained in some universe U (see for example the Grothendieck universe explained in SGA 4). From now on we will drop the adjective 'small' in our examples and assume that we are working in some large enough universe.

- the category **Grp** where the objects are groups, and the arrows are group homomorphisms.
- the category **Top** with topological spaces as objects, and continuous functions between them as arrows.

If we fix a category $\mathcal{C}$, we say that $f : a \to b$ is an **isomorphism** between a and b if there exists an arrow $g : b \to a$ such that

$$g \circ f = \mathrm{id}_a \quad \text{and} \quad f \circ g = \mathrm{id}_b. \tag{2.2}$$

Applying this to the examples in 2.1 gives us bijections, group isomorphisms, and homeomorphisms respectively.

2.3. Example. Even more abstractly, we have the category of (small) categories **Cat**, where the arrows are **functors**. The next section will investigate these morphisms of categories.

• Functors

2.4. Definition. A **functor** $T : \mathcal{C} \to \mathcal{D}$ assigns to each object $c \in \mathcal{C}$ an object $T(c) \in \mathcal{D}$ and to each arrow $f : c \to c'$ in $\mathcal{C}$ an arrow $Tf : T(c) \to T(c')$ in $\mathcal{D}$ such that

$$\begin{aligned} T(\mathrm{id}_c) &= \mathrm{id}_{T(c)} \\ T(g \circ f) &= T(g) \circ T(f). \end{aligned}$$

Indeed, the category **Cat** from example 2.3 forms a category with functors as morphisms. For example, every category $\mathcal{C}$ has an identity functor $\mathrm{id}_{\mathcal{C}}$ that assigns to each object and each arrow the object and arrow itself. The other criteria for a category are also easily verified.

2.5. Remark. In fact, such a functor is called a **covariant functor**. If a functor reverses all arrows, then we call it a **contravariant functor**. We refer the reader to Mac Lane [1] for details.

A possible functor between $\mathbf{Grp} \in \mathbf{Cat}$ and $\mathbf{Set} \in \mathbf{Cat}$ is $T : \mathbf{Grp} \to \mathbf{Set}$ that assigns to each group the underlying set (the arrows basically stay the same since a group homomorphism is in particular a mapping). This functor is commonly called a **forgetful functor** as it forgets some (or all) of the structure of an object. Another functor that will be treated in more detail later is $T : \mathbf{Top}_* \to \mathbf{Grp}$ sending a topological space X with a given basepoint x_0 to $\pi_1(X, x_0)$, the fundamental group of X at x_0. The arrows of $\mathbf{Top}_*$ are sent to the homomorphisms that are induced by the continuous maps (see section 3.2.3).

Let us again consider the category **Cat**. If $f : \mathcal{C} \to \mathcal{D}$ is a functor between categories $\mathcal{C}$ and $\mathcal{D}$ and there exists a functor $g : \mathcal{D} \to \mathcal{C}$ such that the conditions in equation (2.2) are satisfied, then $\mathcal{C}$ and $\mathcal{D}$ are isomorphic through f. However, it turns out that a weaker condition, namely that of **equivalence** is a more useful concept. Let us illustrate this with the following example.

2.6. Example. Let **Finord** be the category of all finite ordinal numbers and $\mathbf{Set}_f$ the category of all finite sets. We have the obvious functor $S : \mathbf{Finord} \to \mathbf{Set}_f$. To define the reverse functor $\# : \mathbf{Set}_f \to \mathbf{Finord}$, first notice that every finite set of size n can be bijectively mapped to the ordinal number n. For each $X \in \mathbf{Set}_f$ choose $\alpha_X : X \to \#X$ to be such a bijection. Given any arrow $f : X \to Y$ between finite sets, we may consider $\#f : \#X \to \#Y$ given by $\#f = \alpha_Y f \alpha_X^{-1}$. It is clear that $\# \circ S = 1$ is the identity functor on **Finord**. However, we do not have $S \circ \#$ equal to the identity functor on $\mathbf{Set}_f$ since different finite sets X and Y with the same cardinality are mapped onto the same object in **Finord** by $\#$. These sets X and Y differ by an isomorphism in $\mathbf{Set}_f$. We would like to qualify the relation between the identity functor on $\mathbf{Set}_f$ and the functor $S \circ \#$. This brings us to **natural transformations**.

• Natural transformations

Before returning to example 2.6, let us give a definition. For two given categories $\mathcal{C}$ and $\mathcal{D}$ we can look at the category $\mathcal{A}$ of all functors $S : \mathcal{C} \to \mathcal{D}$. The morphisms between functors are called **natural transformations**.

2.7. Definition. A **natural transformation** $\alpha : S \to T$, also called a **morphism of functors**, between functors $S, T : \mathcal{C} \to \mathcal{D}$ associates to each object $c \in \mathcal{C}$ a morphism $\alpha_c : S(c) \to T(c)$ in $\mathcal{D}$ such that for every arrow $g : c \to c'$ in $\mathcal{C}$ the following diagram is commutative:

$$\begin{tikzcd} S(c) \arrow[r, "\alpha_c"] \arrow[d, "S(g)"] & T(c) \arrow[d, "T(g)"] \\ S(c') \arrow[r, "\alpha'_c"] & T(c'). \end{tikzcd}$$

It is again trivial to verify that the class of all functors between $\mathcal{C}$ and $\mathcal{D}$ together with their natural transformations forms a category. Just as in any category, the notion of isomorphism arises. In terms of natural transformations this means that for each object $c \in \mathcal{C}$, the morphism α_c is an isomorphism (as in equation 2.2) in $\mathcal{D}$. In this case we talk of a natural isomorphism $\alpha : S \xrightarrow{\sim} T$, also denoted $S \cong T$.

Returning to example 2.6, we found that $\# \circ S$ is equal to the identity functor but that $S \circ \#$ is not. However, the functor $S \circ \#$ is naturally isomorphic to the identity functor. In this case we associate to a set X the function α_X. One easily checks that this makes the diagram above commute since $\#f\alpha_X = \alpha_Y f$. Also, α_X is clearly an isomorphism in $\mathbf{Set}_f$. Hence, we have $S \circ \# \cong \mathrm{id}_{\mathbf{Set}_f}$. Clearly, we have $\# \circ S \cong \mathrm{id}_{\mathbf{Finord}}$. The functor $\#$

is then called an equivalence of categories and the categories $\mathbf{Set}_f$ and **Finord** are then called **equivalent**.

2.8. Definition. Let $S : \mathcal{C} \to \mathcal{D}$ be a functor between categories. If there exists a functor $T : \mathcal{D} \to \mathcal{C}$ such that there exists natural isomorphisms $\alpha : S \circ T \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ and $\alpha' : T \circ S \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$ then we call S an **equivalence of categories** between $\mathcal{C}$ and $\mathcal{D}$.

The notion of an equivalence of categories is weaker but more useful concept than that of an isomorphism of categories. Often categories are not isomorphic (as in the previous example) and can be of completely different 'sizes', but we can still think of them as essentially the same. Any property that can be formulated in categorical terms in one category, also holds for all its equivalent categories. In what follows, we give an examples of a categorical constructions that will used in chapter 4 where we give an application of the main theorem.

2.2 Categorical constructions

One of the strengths of category theory is the use of categorical constructions. These can be expressed solely in terms of diagrams of morphisms such as for example

$$\begin{tikzcd}
 & a \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "j"] & \\
b & & c \\
 & d, \arrow[ul, "h"] \arrow[ur, "i"'] &
\end{tikzcd}$$

where such a diagram is labelled **commutative** if the composition of the arrows on any path between two fixed objects results in the same arrow. For this particular diagram to be commutative we require that $h \circ j = f$ and $i \circ j = g$.

• Products and coproducts

One of the more basic constructions is the categorical **product**. Let $\mathcal{C}$ be a category and let $a, b \in \mathcal{C}$ be two objects. The product of a and b in $\mathcal{C}$ is a triple (p, f, g) with $p \in \mathcal{C}$ and $f : p \to a$ and $g : p \to b$ two morphisms such that when given two morphisms $\varphi : c \to a$ and $\psi : c \to b$, there exists a unique morphism $h : c \to p$ making the following diagram commute:

$$
\begin{tikzcd}
 & c \arrow[dl, "\varphi"'] \arrow[d, dotted, "\exists! h"] \arrow[dr, "\psi"] & \\
a & p \arrow[l, "f"] \arrow[r, "g"'] & b.
\end{tikzcd}
$$

This last property can be described by saying that the product has a **universal property**. It also tells us that the product is uniquely defined up to a unique isomorphism.

2.9. Remark. This definition naturally generalizes to products over a family of objects, something we will not do here.

2.10. Examples.

- If we take $\mathcal{C}$ to be **Set**, and we let A and B be two sets, then by setting $P = A \times B$ and letting $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ be the respective projections on the first and second coordinates, then (P, π_1, π_2) is a product in the category of sets. To prove this, let $f : C \to A$ and $g : C \to B$ be two morphisms. Our sought-after $h : C \to A \times X$ is given by $c \mapsto (f(c), g(c))$, giving us existence. This morphism is unique since its image in the first coordinate is given by f and its image in the second coordinate is given by g.

- The categorical product in the categories **Top**, **Grp**, and **Cat** give us product topologies, direct products, and product categories respectively.

- The category **Fld** with fiels as objects and inclusions as morphisms has no product.

An object P in a category $\mathcal{C}$ is called **universally attracting** if for each object of $\mathcal{C}$ there exists a unique morphism into P. If for each object C of $\mathcal{C}$ there exists a unique morphism from P into C, such an object is called **universally repelling**.

2.11. Example. The trivial group in the category **Grp** is both universally attracting and repelling. In the category **Set** the object $\emptyset$ is universally repelling but not universally attracting.

The product of a and b in $\mathcal{C}$ as we have just defined it, is universally attracting in the category $\mathcal{D}$ that has pairs of morphisms $f : c \to a$ and $g : b \to a$ as objects and morphisms $h : c \to c'$ in $\mathcal{C}$ making the diagram

$$
\begin{tikzcd}
 & c \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
a & c' \arrow[l, "f'"] \arrow[r, "g'"'] & b
\end{tikzcd}
$$

commute as the morphisms from objects (f, g) to (f', g').

Whenever we have a certain construction, such as the product, and we reverse all arrows in the diagram describing the construction, we speak of the **dual** of this construction. Let us apply this to the product. Let $\mathcal{C}$ be an arbitrary category and let a, $b \in \mathcal{C}$. The **coproduct** of a and b in $\mathcal{C}$ is a triple (c, f, g) with $c \in \mathcal{C}$ and $f : a \to c$ and $g : b \to c$ such that when given two morphism $\varphi : a \to c'$ and $\psi : b \to c'$, there exists a unique morphism $h : c \to c'$ making the following diagram commute:

$$\begin{tikzcd} & c' & \\ a \arrow[ur, "\varphi"] \arrow[r, "f"'] & c \arrow[u, dotted, "\exists! h"] & b. \arrow[ul, "\psi"'] \arrow[l, "g"] \end{tikzcd}$$

If it exists, the coproduct of a and b in $\mathcal{C}$ is often denoted $a \coprod b$.

2.12. Examples.

- Let us again consider the category **Set** and let A, $B \in$ **Set**. If we let $i_1 : A \hookrightarrow A \sqcup B$ and $i_2 : B \hookrightarrow A \sqcup B$ be the obvious inclusions, then $(A \sqcup B, i_1, i_2)$ is a coproduct in **Set**. The required unique morphism $h : A \sqcup B \to C'$ that makes the diagram

$$\begin{tikzcd} & C & \\ A \arrow[ur, "\varphi"] \arrow[r, "i_1"'] & A \sqcup B \arrow[u, dotted, "\exists! h"] & B. \arrow[ul, "\psi"'] \arrow[l, "i_2"] \end{tikzcd}$$

  commute is given by $x \mapsto \varphi(x)$ for $x \in A$ and $x \mapsto \psi(x)$ for $x \in B$. It is trivial to show that this h is unique.

- A less trivial example is the coproduct in the category **Grp**. Let G and $H \in$ **Grp**. Then the coproduct of G and H is the triplet $(G*H, i_1, i_2)$ with $G*H$ the free product of G and H, and $i_1 : G \hookrightarrow G * H$ and $i_2 : G \hookrightarrow G * H$ the obvious inclusions. Let $\varphi : G \to C$ and $\psi : H \to C$ be two group homomorphisms. The map $h : G * H \to C$ is then given by sending a word $\prod_i a_i$ to $\prod_i f_i(a_i)$ with $f_i = \varphi$ if a_i is in the chosen set of generators of G and otherwise $f_i = \psi$. It remains to verify that the diagram

$$\begin{tikzcd} & C & \\ G \arrow[ur, "\varphi"] \arrow[r, "i_1"'] & G * H \arrow[u, dotted, "\exists! h"] & H. \arrow[ul, "\psi"'] \arrow[l, "i_2"] \end{tikzcd}$$

  indeed commutes but this is clear from construction. If h' is another mapping that makes the diagram commute it is easy to show by first restricing h' to G and then to H to see that $h = h'$ showing uniqueness of h.

- In the category **Ab** of abelian groups, the coproduct is isomorphic to the direct product.

• Fibered products and fibered coproducts

As we will only be needing the fibered coproduct, only this construction will be given. By the previous section it should not be a surprise that the fibered product is the dual notion of the fibered coproduct. Let $\mathcal{C}$ be a category and let $z \in \mathcal{C}$. Now consider the category $\mathcal{C}_z$ whose objects are morphisms $f : z \to a$. Given another object $g : z \to b$, a morphism between f and g is a morphism $h : a \to b$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} & z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ a \arrow[rr, "h"'] & & b. \end{tikzcd}$$

A coproduct in $\mathcal{C}_z$ is called the **fibered coproduct** of f and g in $\mathcal{C}$, also denoted $a \coprod_z b$. An equivalent way of describing the fibered coproduct of f and g in $\mathcal{C}$ is saying that the fibered coproduct of f and g is a triplet $(a \coprod_z b, i_1, i_2)$ with $i_1 : a \to c$ and $i_2 : b \to c$ two morphisms such that the following diagram commutes

$$\begin{tikzcd} & z \arrow[dl, "f"] \arrow[dr, "g"'] & \\ a \arrow[dr, "i_1"'] & & b. \arrow[dl, "i_2"] \\ & a \coprod_z b, & \end{tikzcd}$$

and that when another such triplet (c, φ, ψ) exists, there exists a unique morphism h such that the following diagram is commutative:

$$\begin{tikzcd} & z \arrow[dl, "f"] \arrow[dr, "g"'] & \\ a \arrow[dr, "i_1"'] \arrow[ddr, bend right, "\varphi"'] & & b. \arrow[dl, "i_2"] \arrow[ddl, bend left, "\psi"] \\ & a \coprod_z b \arrow[d, dotted, "\exists! h"] & \\ & c. & \end{tikzcd}$$

The map i_1 is called the **pushout** of g by f and i_2 is called the pushout of f by g. Let us consider some examples.

2.13. Examples.

- In the category **Set** the fibered coproduct of $f : X \cap Y \to X$ and $g : X \cap Y \to Y$ is $(X \sqcup Y, i_1, i_2)$ with $i_1 : X \to X \sqcup Y$ and $i_2 : Y \to X \sqcup Y$ the natural inclusions.
- In the category **Top** the fibered coproduct gives us **adjunction spaces**, the glueing together of topological spaces. Let X and Y be topological spaces and let A be a subspace of Y together with a continuous map $f : A \to X$. Let $i : A \to Y$ be the inclusion map. The fibered coproduct of f and i is then the adjunction space $X \cup_f Y := (X \sqcup Y)/\{f(A) \sim A\}$.
- When considering the category **Ab**, the fibered coproduct of group homomorphisms $f : 0 \to A$ and $g : 0 \to B$ is the direct sum $A \bigoplus B$.
- Let us again return to the category **Grp**. Let $f : F \to G$ and $g : F \to H$ be two group homomorphisms. The fibered coproduct of f and g in **Grp** is the free product with amalgamation of G and H with respect to f and g. This is the group $(G*H)/N$ where N is the normal closure of the set $\{f(x)g(x)^{-1} | x \in F\}$ (we identity the elements in this set as elements of $G * H$ through the natural inclusions). Dividing out by this normal subgroup ensures that the upper-left square in diagram 2.14 commutes. To show that $((G * H)/N, i_1, i_2)$, with i_1 and i_2 the obvious inclusion, is indeed the fibered coproduct, we must show if we are given another such triplet (M, ψ_1, ψ_2) with group homomorphisms $\psi_1 : G \to M$ and $\psi_2 : H \to M$, that there exists a unique group homomorphism $\overline{h} : (G * H)/N \to M$ such that the diagram

(2.14)
$$\begin{tikzcd}
F \arrow[r, "g"] \arrow[d, "f"'] & H \arrow[d, "i_2"] \arrow[ddr, bend left, "\psi_2"] & \\
G \arrow[r, "i_1"'] \arrow[drr, bend right, "\psi_1"'] & (G * H)/N \arrow[dr, dotted, "\exists! \overline{h}"] & \\
 & & M.
\end{tikzcd}$$

  commutes. The properties of a fibered coproduct described above already gives a unique group homomorphism $h : G * H \to M$ by considering the lower-right part of the diagram (disregarding the morphism f and g) as a coproduct. The upper-left square commutes by construction of N, and since N is the kernel of h, this gives us a well-defined group homomorphism $\overline{h} : (G * H)/N \to M$ that is unique by uniqueness of h and N.

It is worth mentioning that the product and fibered product are specific instances of a **limit** and coproducts and fibered coproducts are specific instances of **colimit**, the dual of a limit. Since we will not be needing this generalization in this thesis we refer the interested reader to Mac Lane ([1]) for further explanations of these categorical concepts.

The main theorem that we will be working towards, pertains to an **equivalence of categories**, namely between the categories G-**Set** and **Cov**(X) for a certain group G and a certain topological space X. Before we can make this precise we will have to introduce these two categories.

3
An equivalence of categories

3.1 G-sets

A natural thing to consider is the action of a group on a set. For example, the group S_n **acts** in a natural way on the set $X = \{1, 2, \ldots, n\}$. A permutation $\sigma \in S_n$ acts on X by sending the element i to $\sigma(i)$ for $i = 1, 2, \ldots, n$. One can view the group elements as being associated to certain symmetries of a set. In the case of the given example the symmetries are bijections into itself. Let us generalize this notion by giving a formal definition of a group action.

3.1. Definition. Let G be a group and X a set. A (left-) **action** of G on X is a group homomorphism $\phi : G \to \mathrm{Sym}(X)$ where $\mathrm{Sym}(X)$ is the symmetric group of X.

Equivalently, a (left-) action is a map $G \times X \to X$ given by $(g, x) \mapsto g \circ x$ (or gx) satisfying

- $1x = x$;
- $(gh)x = g(hx)$;

for all $g, h \in G$, $x \in X$ and with 1 the identity element of G. If we are given a group homomorphism $\phi : G \to \mathrm{Sym}(X)$ then $(g, x) \to (\phi g)x$ is a map $G \times X \to X$ that satisfies these criteria. Vice versa, if a map satisfies these criteria, it corresponds to a group homomorphism $\phi : G \to \mathrm{Sym}(X)$. The set X is referred to as a G-set.

3.2. Remark. A **right group action** is defined analogously by a map $f_r : X \times G \to X$ sending (x, g) to xg such that $(xh)g = x \cdot (hg)$ and $x1 = x$ for all $g, h \in G$ and $x \in X$. Equivalently, a **right action** is given by an **anti-homomorphism** $\phi : G \to \mathrm{Sym}(X)$ which means that $\phi(gh) = \phi(h)\phi(g)$. It is easy to verify that this right group action f_r can be turned into a left group action $f_l : G \times X \to X$ by sending (g, x) to $f_r(x, g^{-1})$. Thus in essence, the theory of right group actions is the same as the theory of left group actions.

3.3. Remark. Instead of writing $(\phi g)(x)$, the notation gx or xg (in the case of a right action) will be used in the absense of ambiguation.

3.4. Example. Any group G is a left G-set with action

$$\begin{aligned} \phi : \quad G &\longrightarrow \mathrm{Sym}(G) \\ g &\longmapsto (\sigma_g : x \mapsto gx). \end{aligned}$$

This action is sometimes referred to as action by **left translation**.

3.5. Example. Let X be a G-set with corresponding action $\phi : G \to \mathrm{Sym}(X)$. Let $f : G' \to G$ be a group homomorphism. Then f makes X into a G'-set with action $\psi : G' \to \mathrm{Sym}(X)$ given by $g' \mapsto \phi(f(g'))$.

3.6. Definition. Let X and Y be G-sets. A (left) G**-map** is a map $f : X \to Y$ that respects the action of G: $f(gx) = g(f(x))$ for all $x \in X$ and $g \in G$.

For a G-set X, the identity mapping on X is the identity morphism. For G-sets X, Y, and Z with morphisms $f : X \to Y$ and $g : Y \to Z$ one checks that $g \circ f$ is again a G-map:

$$\begin{aligned}(g \circ f)(\sigma x) &= g(\sigma f(x)) \\ &= \sigma(g \circ f)(x).\end{aligned}$$

The other properties also follow trivially. Hence, the set of G-sets is a category with G-maps as the morphisms. Notice that for a bijective G-map $f : X \to Y$, the inverse map f^{-1} is also a G-map making f an isomorphism of G-sets. For an action of G on X one can consider the **orbit** Gx of an element x:

$$Gx \stackrel{\text{def}}{=} \{gx : g \in G\} \subset X.$$

It is easy to see that the orbits of a set X partition X by the equivalence relation $x \sim y$ iff there exists a $g \in G$ such that $gx = y$. If there is at most one orbit, and thus $Gx = X$ (or $X = \emptyset$), then the action is called **transitive**. Equivalently, for every x, $y \in X$ there exists a $g \in G$ such that $gx = y$. One can also consider the **orbit space** of X, denoted X/G, which consists of all equivalence classes Gx. The set

$$G_x \stackrel{\text{def}}{=} \{g \in G : gx = x\}$$

is called the **stabilizer** of x. Note that this is a subgroup of G. For $H \subset G$ a subgroup of G, the set $G/H = \{gH : g \in G\}$ of (left-) cosets is a transitive G-set with (left) translation as action.

There is a relation between stabilizers and orbits. For $x \in X$, define the G-map $f_x : G/G_x \to Gx$ by $gG_x \mapsto gx$. This is a well-defined map since if $gG_x = g'G_x$, then $g = g'h$ with $h \in G_x$ giving us $gx = g'hx = g'x$. Also, if $gx = g'x$, then $g'^{-1}gx = x$ or $g'^{-1}g \in G_x$ which means that $gG_x = g'G_x$. Clearly, f_x respects the action of G giving us an isomorphism $G/G_x \leftrightarrow Gx$ of G-sets. This tells us in particular that the length of the orbit of x is equal to $[G : G_x]$, the index of the stabilizer of x in G.

3.7. Lemma. Let X be a non-empty transitive G-set, fix $x \in X$, let $H = G_x$, and let $NH = \{g \in G | gH = Hg\}$ be the normaliser of H in G. Then, $\varphi : NH/H \longrightarrow \mathrm{Aut}_{G\text{-}\mathbf{Set}}(X)$ given by $nH \mapsto \sigma_n$ with $\sigma_n : gx \mapsto gnx$, is an isomorphism.

Proof. Note that NH/H is indeed a group since NH is the largest group that has H as normal subgroup. It is clear that σ_n is a G-isomorphism for each $n \in G$. Also, φ is clearly a group homomorphism. Let $\sigma \in \mathrm{Aut}_{G\text{-}\mathbf{Set}}(X)$ be given. Since X is transitive, we must have $\sigma(x) = nx$ for some $n \in G$. For a given $h \in H$ we have $hnx = \sigma(hx) = \sigma(x) = nx$, or $n^{-1}hnx = x$, giving us $n^{-1}hn \in H$ since it stabilizes x. Hence, $n \in NH$. Then $\varphi(nH) = \sigma$ since the G-automorphism of a transitive G-set X is completely determined by the image of a single $x \in X$. □

3.8. Definition. Let X be a G-set. If $G_x = \{1\}$ for all $x \in X$, the action of G on X is called a **free action**.

3.9. Corollary. Let X be a non-empty G-set such that the action of G on X is transitive and free. Then $\mathrm{Aut}_{G\text{-}\mathbf{Set}}(X) \cong G$.

Proof. This is a direct consequence of the previous lemma. □

These are all the tools that we will need. Let us continue with covering spaces.

3.2 Covering spaces

Before introducing the category $\mathbf{Cov}(X)$ of covering spaces for X, let us go through some fundamentals of algebraic topology.

• The fundamental group

By matter of convention in topology, a *map* is considered a continuous function. The unit interval $[0, 1]$ is denoted by I.

3.10. Definition. A **path** in X is a map $\gamma : I \to X$ with **endpoints** $\gamma(0)$ and $\gamma(1)$. Furthermore, if $\gamma(0) = \gamma(1)$, the map γ is called a **loop** based at $\gamma(0)$. The path $\gamma : I \to \{x_0\}$ is called the **constant path** at x_0, often denoted by c_{x_0} or just c when x_0 is clear from context.

3.11. Definition. A **homotopy** is a family of maps $f_t : Y \to X$ indexed by I, such that the function $F : Y \times I \to X$ associated to the f_t by $F(y, t) = f_t(y)$ is a map. The maps f_0 and f_1 are then called **homotopic**, also denoted by $f_0 \simeq f_1$. The associated map F is also referred to as a homotopy. A **homotopy of paths** is a homotopy $f_t : I \to X$ such that the endpoints are independent of t. A loop that is homotopic to the constant path is called **null-homotopic**.

3.12. Example. Any two paths f_0 and f_1 lying in a convex subset $X \subset \mathbb{R}^n$ with equal endpoints are homotopic by the **linear homotopy** $f_t = (1-t)f_0 + tf_1$. In fact, they are null-homotopic to any constant path lying in X.

Given two paths f and g such that $f(1) = g(0)$, let the composition $g \cdot f$ be the path defined by

$$(g \cdot f)(s) = \begin{cases} f(2s), & 0 \leq s \leq \frac{1}{2} \\ g(2s-1). & \frac{1}{2} \leq s \leq 1 \end{cases}$$

For a topological space X and a point $x_0 \in X$, consider the set of all loops starting and ending in x_0. This wieldy set becomes more interesting by identifying f and g iff $f \simeq g$. It is not hard to check that $\simeq$ is an equivalence relation on L. The quotient space is denoted $\pi_1(X, x_0)$.

3.13. Theorem. The set $\pi_1(X, x_0)$ is a group under compositions of loops.

Proof. This is a simple verification. The inverse elements are the paths traced backwards and the unit element is the constant path at x_0. See also [2, p. 26, proposition 1.3] □

The group $\pi_1(X, x_0)$ is called the **fundamental group** of X at the basepoint x_0. By example 3.12 the group $\pi_1(X, x_0)$ is the trivial group in the case that X is a convex subset of $\mathbb{R}^n$. A topological space that has a slightly more interesting fundamental group is the circle S^1. Intuitively, one might suspect that $\pi_1(S^1, 1) \cong \mathbb{Z}$. In order to prove this, some preliminary work has to be done.

• Covering spaces and the homotopy lifting property

3.14. Definition. A **covering space** for a topological space X is a space E together with a map $p : E \to X$, called a **covering map**, that satisfies the following condition:

- There exists an open cover $\{U_\alpha\}$ for X such that for all α the preimage $p^{-1}(U_\alpha)$ is a disjoint union of open sets in E, each of which is homeomorphic to U_α through p.

Let us give some examples of covering maps.

3.15. Examples.

- Any homeomorphism between spaces is a covering map;
- The map $p : \bigsqcup_{i \in I} X \to X$ (for any non-empty index set I) acting as the identity on all components X;
- The map $p : S^1 \to S^1$ defined by $z \mapsto z^n$ for $n \in \mathbb{Z} \setminus \{0\}$.

3.16. Definition. If we have two covering spaces $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$, then a **morphism of covering spaces** is a continuous function $f : E_1 \to E_2$ that respects the covering maps. In order words, it is a map that makes the following diagram commutative:

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X. & \end{tikzcd}$$

This makes the class of covering spaces, denoted $\mathbf{Cov}(X)$ into a category. It follows that two covering spaces are considered isomorphic iff there exists a homeomorphism between the two spaces that respects the covering maps.

In the context of a covering space, a map $\tilde{f} : Y \to E$ is called a **lift** of a map $f : Y \to X$ if $f = p\tilde{f}$. This is also expressed by saying that $\tilde{f}$ lifts f. One interesting property of covering spaces is the (unique) **homotopy lifting property**. This is made precise in the next theorem.

3.17. Theorem. Let $p : E \to X$ be a covering space, let $f_t : Y \to X$ be a homotopy, and let $\tilde{f}_0 : Y \to E$ be a map lifting f_0. Then there exists a unique homotopy given by a family $\tilde{f}_t : Y \to E$ with $\tilde{f}_0$ as given.

Proof. First, an outline of the proof.

1. A lift of f_t will be constructed on $N \times I$ with N an open neighborhood of y_0;
2. The construction will be proved unique;
3. Lifts constructed on $N_1 \times I$ and $N_2 \times I$ will be shown to coincide on $(N_1 \cap N_2) \times I$.

Consider the family of maps f_t as the map $F : Y \times I \to X$ given by the equality $f_t(y) = F(y, t)$. Let $\{U_\alpha\}$ be a cover for X with the property as described in definition 3.14. Fix $y_0 \in Y$. For any $t \in [0, 1]$ there is a U_α with $F(y_0, t) \in U_\alpha$. By continuity, there exists an open neighborhood $N_t \times (a, b)$ of (y_0, t) such that $F(N_t \times (a, b))$ is contained in this U_α. Since $\{y_0\} \times I$ is compact there is a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ of I and a single N (namely the intersection of the finite number of $N_{t_i} \times (t_{i-1}, t_i)$) such that $F(N \times [t_{i-1}, t_i])$ is contained in some U_α, denoted U_i. The first point will be proved by induction.

The map $\tilde{F}$ has been constructed on $N \times [0, t_0]$ by the given $\tilde{f}_0$. Assume that $\tilde{F}$ has been constructed on $N \times [0, t_i]$. Let $\tilde{U}_i$ be the disjoint open subset in $p^{-1}(U_i)$ that contains $\tilde{F}(y_0, t)$. By intersecting $N \times \{t_i\}$ with the preimage of $\tilde{U}_i$ under $\tilde{F}$ restricted to $N \times \{t_i\}$ the open set $N \times \{t_i\}$ satisfies $\tilde{F}(N \times \{t_i\}) \subset \tilde{U}_i$. Then, by continuity, it is clear that $\tilde{F}(N \times [t_i, t_{i+1}]) \subset \tilde{U}_i$. To complete the induction step, define $\tilde{F}$ on $N \times [t_i, t_{i+1}]$ by the map $p^{-1}F$.

The second point, uniqueness of $\tilde{F}$, is proved for the case that Y is a point. Suppose that $\tilde{F}$ and $\tilde{F}'$ are two lifts of $F : I \to X$. As before, let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a partition of $[0, 1]$ such that $F([t_i, t_{i+1}]) \subset U_i$. By the given map $\tilde{f}_0$, the homotopies F and $\tilde{F}$ agree on $[0, t_0] = \{0\}$. Assume that $F = \tilde{F}$ on $[0, t_i]$. As $\tilde{F}$ is continuous, and because $[t_i, t_{i+1}]$ is connected, $\tilde{F}([t_i, t_{i+1}])$ is connected. This means that $\tilde{F}([t_i, t_{i+1}])$ lies in a single disjoint open set $\tilde{U}_i$. Since the same argument can be applied to $\tilde{F}'$, and as $\tilde{F}(t_i) = \tilde{F}'(t_i)$ they must both map to the same $\tilde{U}_i$. From $p\tilde{F} = p\tilde{F}'$, it follws by injectivity of p that $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$. This completes the induction.

To prove the last point, use the fact that $\tilde{F}$ restricted to a $\{y_0\} \times I$ is unique. This completes the proof as $\tilde{F}$ can be considered a unique map on open sets $N_y \times I$ for all $y \in Y$ where the last point says that $\tilde{F}$ is indeed well-defined. $\square$

This theorem has two useful corollaries (keeping in mind the covering space for X as above).

3.18. Corollary. For each path $f : I \to X$ starting at a point $x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde{f} : I \to E$ starting at $\tilde{x}_0$.

Proof. This is a special case of the previous theorem with Y consisting of a single point. $\square$

3.19. Corollary. For each homotopy $f_t : I \to X$ of paths starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{f}_t : I \to E$ of paths starting at $\tilde{x}_0$.

Proof. First obtain a unique lift $\tilde{f}_0$ of the map f_0 by applying the previous corollary. By setting $Y = I$, a unique lift $\tilde{f}_t$ of the homotopy f_t is obtained by the previous theorem. Now look at the paths formed by the endpoints of $\tilde{f}_t$. These are lifts of constants paths (since a homotopy of paths was considered) and hence must be constant paths themselves by the uniqueness given in the previous corollary. $\square$

To give an immediate application of the previous set of results, we will show that $\pi_1(S^1, 1) \cong \mathbb{Z}$. Notice that the map $p : \mathbb{R} \to S^1$ given by $x \mapsto \mathrm{e}^{2\pi ix}$ is a covering map. For the cover of S^1, take any two open arcs whose union is S^1.

3.20. Theorem. The function $\phi : \mathbb{Z} \to \pi_1(S^1, 1)$ that sends an integer n to the loop $[p\gamma_n]$ based at 1, with γ_n the path defined by $\gamma_n(s) = ns$, is an isomorphism.

Proof. Consider the map $\tau_v : \mathbb{R} \to \mathbb{R}$ as the translation over v given by $x \mapsto x + v$. Take m, $n \in \mathbb{Z}$. First, the verification that ϕ is a homomorphism:

$$\begin{aligned}
\phi(m+n) &= [p\gamma_{m+n}] \\
&= [p((\tau_n\gamma_m) \cdot \gamma_n))] \\
&= [p\tau_n\gamma_m][p\gamma_n] \\
&= [p\gamma_m][p\gamma_n] \\
&= \phi(m)\phi(n),
\end{aligned}$$

where $[p\tau_n\gamma_m] = [p\gamma_m]$ since $e^{2\pi ix} = e^{2\pi i(x+n)}$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. For surjectivity, take any loop $[f]$ in $\pi_1(S^1, 1)$. By corollary 3.18 there exists a unique lift $\tilde{f}$ starting at 0. This path necessarily ends at some integer n since $p^{-1}(1) = \mathbb{Z}$. By noticing that this path must be homotopic to γ_n it follows that $\phi(n) = [p\gamma_n] = [p\tilde{f}] = [f]$. Finally, to show injectivity, assume that $[p\gamma_n] = [p\gamma_m]$. There exists a homotopy f_t of paths between $p\gamma_m$ and $p\gamma_n$. By corollary 3.19 there exists a lifted homotopy of paths $\tilde{f}_t$ in $\mathbb{R}$ starting at 0. By corollary 3.18 the lifts of $p\gamma_m$ and $p\gamma_n$ are unique, hence $\tilde{f}_0 = \gamma_m$ and $\tilde{f}_1 = \gamma_n$. Since $\tilde{f}_t$ is a homotopy of paths, the endpoints are independent of t. For $t = 0$ the endpoint is m, and for $t = 1$, the endpoint is n, thus $m = n$. $\square$

• Induced homomorphisms

Let $\varphi : X \to Y$ be a map taking the basepoint $x_0 \in X$ to $y_0 \in Y$ denoted by $\varphi : (X, x_0) \to (Y, y_0)$. Then φ induces a homomorphism

$$\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, y_0).$$

by $\varphi_*[f] = [\varphi f]$. Since $[\varphi(g \cdot f)] = [(\varphi g) \cdot (\varphi f)] = [\varphi g][\varphi f]$, it is indeed a homomorphism. Induced homomorphisms have the property that $(\varphi\psi)_* = \varphi_*\psi_*$ for a composition

$$(X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\psi} (Z, z_0).$$

Also, $\mathbf{1}_* = \mathbf{1}$, with $\mathbf{1}$ the identity map, which is not much news. Notice that this gives a functor between the categories $\mathbf{Top}_*$ and $\mathbf{Grp}$. If there exists maps $f : X \to Y$ and $g : Y \to X$ such that $(fg)_* \simeq \mathbf{1}_*$ and $(gf)_* \simeq \mathbf{1}_*$, then we call X and Y homotopically equivalent. In that case we have an isomorphism $\pi_1(X, x) \cong \pi_1(Y, f(x))$ for each $x \in X$.

3.21. Theorem. Let $p : (E, \tilde{x}_0) \to (X, x_0)$ be a covering map. Then $p_* : \pi_1(E, \tilde{x}_0) \to \pi_1(X, x_0)$ is an injective group homomorphism. The image of p_* consists precisely of loops in X based at x_0 that lift to loops in E at $\tilde{x}_0$.

Proof. Take a loop $\gamma \in \pi_1(E, \tilde{x}_0)$ for which $[p\gamma] = [c]$. This means that there exists a homotopy of paths f_t with $f_0 = p\gamma$ and $f_1 = c$. By corollary 3.19 there is a unique homotopy of paths $\tilde{f}_t$ in E with $\tilde{f}_0 = \gamma$ and $\tilde{f}_1$ the constant path at $\tilde{x}_0$ by uniqueness of lifts. A loop in X at x_0 lifting to a loop E at x_0 is certainly in the image of p_*. Conversely, a loop $\gamma \in p_*(\pi_1(E, \tilde{x}_0))$ is homotopic to a loop having such a lift. By corollary 3.18 the loop itself must also have such a lift. □

• Classifying covering spaces through the fundamental group

Before giving an equivalence of categories, we will classify the covering spaces with help of the fundamental group. It turns out that after putting certain conditions on X, that the subgroups of the fundamental group are in correspondence with the path-connected covering spaces by associating a covering space $p : (E, \tilde{x}_0) \to (X, x_0)$ with $p_*(\pi_1(E, \tilde{x}_0)) \subset \pi_1(X, x_0)$. Let us begin by describing the conditions.

3.22. Definition. A topological space X is called **locally path-connected** if for each x and every open neighborhood N of x there exists a neighborhood B with $x \in B \subset N$ such that B is **path-connected**[1].

3.23. Remark. It is easily proved that a space X that is both locally path-connected and connected is **path-connected**, that is, the only sets that are both open and closed are $\emptyset$ and X. Note that a covering space for X inherits the property of being locally path-connected and thus is connected iff it is path-connected.

3.24. Definition. A topological space X is called **semilocally simply-connected** if for each $x \in X$ there exists an open neighborhood N such that the inclusion $i : N \to X$ induces the trivial homomorphism $i_* : \pi_1(N, x) \to \pi_1(X, x)$. This says that any loop at $x \in N$ will be null-homotopic when viewed in the larger space X.

Most spaces that one encounters are semilocally simply-connected. The following shows that not all spaces have this property.

[1] The space B is non-empty and each two points in B are connected by a path in B.

3.25. Example. Let $S_i \subset \mathbb{R}^2$ be the circle with radius $\frac{1}{i}$ and center $(\frac{1}{i}, 0)$. Then $X = \bigcup_{i=1}^{\infty} S_i$ is a space that is not semilocally simply-connected. To see this, notice that any neighborhood of $(0,0)$ will contain a circle. A path traversing this circle will not be null-homotopic in X.

There is a more general form of the homotopy lifting property that we will need for our classification. It is given in the following lemma that is often called the **lifting criterion**.

3.26. Lemma. Let $p : (E, \tilde{x}_0) \to (X, x_0)$ be a covering space, and let $f : (Y, y_0) \to (X, x_0)$ be a map with Y path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (E, \tilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, \tilde{x}_0))$.

Proof. Let us first give a diagram of the different spaces and corresponding fundamental groups.

$$\begin{tikzcd} & (E, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (X, x_0) \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(E, \tilde{x}_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"] & \pi_1(X, x_0) \end{tikzcd}$$

It is clear that if a lift $\tilde{f}$ exists, then $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, \tilde{x}_0))$ since for any loop $[\gamma] \in \pi_1(Y, y_0)$ we have $[f\gamma] = [p\tilde{f}\gamma]$ with $[\tilde{f}\gamma] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Now assume that $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, \tilde{x}_0))$. To define $\tilde{f}$ on $y \in Y$, let γ be a path in Y from y_0 to y. The path $f\gamma$ can be lifted to a path $\widetilde{f\gamma}$ in E starting at $\tilde{x}_0$. Now define $\tilde{f}(y) = \widetilde{f\gamma}(1)$. To show that $\tilde{f}$ is well-defined, let γ' be another loop from y_0 to y in Y. Then the loop $[(\overline{f\gamma}) \cdot (f\gamma')] = [h_0]$ is contained in $p_*(\pi_1(E, \tilde{x}_0))$ by assumption. There exists a homotopy h_t of h_0 to a loop h_1 that lifts to a loop $\tilde{h}_1$ in E based at $\tilde{x}_0$. This homotopy can be lifted to a homotopy $\tilde{h}_t$. Since $\tilde{h}_1$ is a loop, $\tilde{h}_0$ must be also a loop. Since paths are lifted uniquely we have $\tilde{h}_0 = (\widetilde{\overline{f\gamma}}) \cdot (\widetilde{f\gamma'})$ and thus $\widetilde{f\gamma}'(1) = \widetilde{f\gamma}(1)$.

To show that $\tilde{f}$ is continuous take an open neighborhood $N \subset X$ of $f(y)$ that has a lift $\tilde{N} \subset E$ containing $\tilde{f}(y)$ such that $p : \tilde{N} \to N$ is a homeomorphism. Let V be a path-connected neighborhood of V of y such that $f(V) \subset N$, this exists by virtue of continuity of f. All paths from $y_0 \in Y$ to any point $y' \in V$ can be written as $\eta \cdot \gamma$ with γ a fixed path from y_0 to y and η a path from y to y'. The paths $(f\eta) \cdot (f\gamma)$ have lifts $(\widetilde{f\eta}) \cdot (\widetilde{f\gamma})$ with $\widetilde{f\eta} = p^{-1}f\eta$. Hence we see that $\tilde{f}(V) \subset \tilde{N}$ and $\tilde{f}|V = p^{-1}f$. This proves the continuity of $\tilde{f}$. $\square$

Another lemma that we will need tells us that when two lifts agree on a single point, they must be equal.

3.27. Lemma. Let $p : E \to X$ be a covering space, let Y be connected, let $f : Y \to X$ be a map, and let $\tilde{f}_1, \tilde{f}_2 : Y \to E$ be two lifts of f that agree on at least one point of Y, then $\tilde{f}_1 = \tilde{f}_2$ on Y.

Proof. It will be shown that the set of points where $\tilde{f}_1$ and $\tilde{f}_2$ agree is both open and closed. Since they agree on at least one point and because Y is connected, the lifts must agree everywhere on Y.

For a point $y \in Y$, let U be an open neighborhood of $f(y)$ in X with $p^{-1}(U)$ a disjoint union of open sets $\tilde{U}_\alpha$ that are all homeomorphic to U through p. Let $\tilde{U}_1$ and $\tilde{U}_2$ be the sets containing $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$ respectively. Since $\tilde{f}_1$ and $\tilde{f}_2$ are both continuous there exists a neighborhood N of y that is mapped into $\tilde{U}_1$ and $\tilde{U}_2$ by $\tilde{f}_1$ and $\tilde{f}_2$ respectively. If $\tilde{f}_1(y) = \tilde{f}_2(y)$ then $\tilde{U}_1 = \tilde{U}_2$ so that $p\tilde{f}_1 = p\tilde{f}_2$ on N and thus $\tilde{f}_1 = \tilde{f}_2$ on N by injectivity of p (on $\tilde{U}_1 = \tilde{U}_2$). From this we can conclude that the set of points where the lifts agree is open. Otherwise, If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ then $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$ so that $\tilde{f}_1 \neq \tilde{f}_2$ on the whole of N. And thus the set of points where the lift agree is closed. This is what we wanted to prove. □

3.28. Lemma. Let X be a semilocally simply-connected, (path-)connected, and locally path-connected topological space, and let $x_0 \in X$. The covering spaces $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ with respective basepoints $\tilde{x}_1 \in p_1^{-1}(x_0)$ and $\tilde{x}_2 \in p_2^{-1}(x_0)$ are isomorphic (preserving basepoints) iff $p_{1*}(\pi_1(E_1, \tilde{x}_1)) = p_{2*}(\pi_1(E_2, \tilde{x}_2))$.

Proof. Given an isomorphism $f : (E_1, \tilde{x}_1) \to (E_2, \tilde{x}_2)$ we have $p_2 f = p_1$ and $p_1 f^{-1} = p_2$. Hence $p_{1*}(\pi_1(E_1, \tilde{x}_1)) \subset p_{2*}(\pi_1(E_2, \tilde{x}_2))$ and vice versa. Conversely, assume that $p_{1*}(\pi_1(E_1, \tilde{x}_1)) = p_{2*}(\pi_1(E_2, \tilde{x}_2))$. Applying lemma 3.26 twice we get lifts $\tilde{p}_1 : (E_1, \tilde{x}_1) \to (E_2, \tilde{x}_2)$ and $\tilde{p}_2 : (E_2, \tilde{x}_2) \to (E_1, \tilde{x}_1)$ with $p_2\tilde{p}_1 = p_1$ and $p_1\tilde{p}_2 = p_2$. Since $\tilde{p}_1\tilde{p}_2$ and $\tilde{p}_2\tilde{p}_1$ both fix the basepoints $\tilde{x}_1$ and $\tilde{x}_2$ respectively, they must be the identity map on E_1 and E_2 respectively by lemma 3.27. This proves that $\tilde{p}_1$ and $\tilde{p}_2$ are inverse isomorphisms that preserve the basepoints. □

In the next theorem, a special covering space called the **universal covering space** will be constructed for X. In a way, this is the 'biggest' connected covering space that X has. For this covering space to exist, our space X must meet some criteria.

3.29. Theorem. Let X be a semilocally simply-connected, (path-)connected, and locally path-connected topological space and let $x_0 \in X$. Then there exists a covering $p : U \to X$ with U simply-connected. This covering space is called the **universal covering space**.

Proof. Let us first define the set U and the map $p : U \to X$:

$$U \overset{\text{def}}{=} \{\ [\gamma]\ :\ \gamma \text{ a path in } X \text{ starting at } x_0\ \},$$

$$\begin{aligned} p :\ & U && \longrightarrow X \\ & [\gamma] && \longmapsto \gamma(1). \end{aligned}$$

Note that p is a well-defined map since the equivalence relation $\simeq$ for paths ensures that the endpoints are independent of the chosen representative.

Let $\mathcal{B}$ be the collection of all open sets B in X that are path-connected and for which the induced homomorphism $\pi_1(B) \to \pi_1(X)$ is trivial. This is a basis for a topology on X. To see this, note that every point $x \in X$ has a neighborhood N_1 for which the induced inclusion homomorphism $i_* : \pi_1(N_2, x) \to \pi_1(X, x)$ is trivial by semilocally simply-connectedness of

X. By locally path-connectedness this neighborhood must contain an open neighborhood N_2 that is path-connected. The induced inclusion homomorphism $i_* : \pi_1(N_2, x) \to \pi_1(X, x)$ is trivial since we have the inclusions $\pi_1(N_2, x) \to \pi_1(N_1, x) \to \pi_1(X, x)$ of which the second is trivial. Hence $\mathcal{B}$ covers X. If we take two elements N_1 and N_2 in $\mathcal{B}$ then $i_* : \pi_1(N_1 \cap N_2, x) \to \pi_1(X, x)$ is again trivial. Take a path-connected neighborhood N of x contained in this intersection and $N \in \mathcal{B}$ proving that $\mathcal{B}$ is a basis for a topology on X.

Given a set $B \in \mathcal{B}$ and a path γ starting at x_0 and ending in B we define

$$B_{[\gamma]} = \{[\eta \cdot \gamma] \mid \eta \text{ a path in } \mathcal{B}\}.$$

Notice that $p : B_{[\gamma]} \to B$ is surjective because B is path-connected, and injective because $\pi_1(B) \to \pi_1(X)$ is the trivial induced homomorphism. A simple fact that we will use in this proof is

$$[\gamma] \in B_{[\gamma']} \Longrightarrow B_{[\gamma]} = B_{[\gamma']}.$$

The claim is that all sets of the form $B_{[\gamma]}$ form a basis for a topology on U. Let two elements $B_{[\gamma]}$ and $B'_{[\gamma']}$ be given with $[\gamma''] \in B_{[\gamma]} \cap B'_{[\gamma']} = B_{[\gamma'']} \cap B'_{[\gamma'']}$. Choose an open set $B'' \in \mathcal{B}$ with $B'' \subset B \cap B'$ that contains $\gamma''(1)$. Then $[\gamma''] \in B''_{[\gamma'']}$ and $B''_{[\gamma'']} \subset B_{[\gamma]} \cap B'_{[\gamma']}$. Hence it is a basis for a topology on U.

The map $p : B_{[\gamma]} \to B$ is a homeomorphism because it gives a bijection between the basis elements $B'_{[\gamma']} \subset B_{[\gamma]}$ and the sets $B' \in \mathcal{B}$ contained in B. It is clear that $p(B'_{[\gamma']}) = B'$ and $p^{-1}(B') \cap B_{[\gamma]} = B'_{[\gamma']}$ for $[\gamma'] \in B_{[\gamma]}$ with $\gamma'(1) \in B'$.

This all implies that p is a continuous map. To see that is in fact a covering map, we notice that for a fixed $B \in \mathcal{B}$ the sets $B_{[\gamma]}$ partition $p^{-1}(B)$ as $[\gamma]$ varies. This can be seen by observing that if $[\gamma''] \in B_{[\gamma]} \cap B_{[\gamma']}$, then $B_{[\gamma]} = B_{[\gamma']} = B_{[\gamma'']}$.

The claim is that U is simply-connected. First, it will be shown that U is path-connected. Take a point $[\gamma] \in U$. Let γ_t be the path in X that is γ on $[0, t]$ and $\gamma(t)$ on $[t, 1]$. The function $t \mapsto \gamma_t$ is a path in U that lifts γ starting at $[x_0]$ and ending at $[\gamma]$. It remains to show that $\pi_1(U, [x_0]) = 1$. Since p_* is injective we can just as well show that $p_*(\pi_1(U, [x_0])) = 1$. The elements in the image of p_* are loops γ at x_0 that lift to loops at $[x_0]$. We saw that $t \mapsto \gamma_t$ lifts γ. If this is to be a loop we must have that $[\gamma_1] = [x_0] = [\gamma]$. This means that γ is null-homotopic and hence that U is simply-connected. □

3.30. Theorem. Let X be a semilocally simply-connected, (path-)connected, and locally path-connected topological space and let $x_0 \in X$. Then for each subgroup $H \subset \pi_1(X, x_0)$ there exists a covering space $p : E_H \to X$ with a selected basepoint $\tilde{x}_0 \in p^{-1}(x_0)$ such that $p_*(\pi_1(E_H, \tilde{x}_0)) = H$.

Proof. First we prove the statement for $H = 1$. This comes down to finding a covering space E_1 and basepoint $\tilde{x}_0$ such that $p_*(\pi_1(E_1, \tilde{x}_0)) = 1$. Since p_* is injective, an equivalent conditions is given by $\pi_1(E_1, \tilde{x}_0) = 1$. This means we are looking for a simply-connected covering space. By theorem 3.29 this exists. Let $p : U \to X$ be this simply-connected covering space.

Now let H be any subgroup of $\pi_1(X, x_0)$. Define an equivalence relation $\sim$ on U by

$$[\gamma] \sim [\gamma'] \Longleftrightarrow \gamma(1) = \gamma'(1) \text{ and } [\overline{\gamma'} \cdot \gamma] \in H,$$

for all $[\gamma]$, $[\gamma'] \in U$. The three requirements for an equivalence relation (reflexivity, symmetry, and transitivity) follow from the fact that H is a group (H contains the identity element, H is closed under inverses, and H is closed under the operation). Let this quotient space (with the quotient topology) be called E_H. For our covering space we take the natural projection $p : E_H \to X$ given by $[\gamma] \mapsto \gamma(1)$. Take two basic open sets $B_{[\gamma]}$ and $B_{[\gamma']}$ of U in the notation of theorem 3.29. If two points in these open sets are identified then the whole neighborhoods are identified: when $\gamma(1) = \gamma'(1)$, then $[\gamma] \sim [\gamma'] \Leftrightarrow [\eta \cdot \gamma] \sim [\eta \cdot \gamma']$. This means that p is a covering map.

Now let $\tilde{x}_0 \in E_H$ be the equivalence class of the constant path c at x_0. We show that $p_*(\pi_1(E_H, \tilde{x}_0)) = H$. Take a loop γ based at x_0 in X. If we lift the path to U, the path starts at $[c]$ and ends at $[\gamma]$. This path is a loop if and only if $[c] \sim [\gamma]$ or $[\gamma] \in H$. This completes the proof. $\square$

3.31. Theorem. Let X be a semilocally simply-connected, (path-)connected, and locally path-connected topological space and let $x_0 \in X$. Then there is a bijection

$$\phi : \left\{ \begin{array}{l} \text{basepoint preserving} \\ \text{isomorphism classes of} \\ \text{connected covering} \\ \text{spaces} \;\; p : (E, \tilde{x}_0) \to (X, x_0) \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{subgroups} \\ H \subset \pi_1(X, x_0) \end{array} \right\}$$

where ϕ sends a path-connected covering space $p : (E, \tilde{x}_0) \to (X, x_0)$ to $p_*(\pi_1(E, \tilde{x}_0))$.

Proof. The function ϕ is injective by lemma 3.28 and surjective by theorem 3.30 $\square$

3.32. Remark. If we are given a covering space $p : (E, \tilde{x}_0) \to (X, x_0)$, then applying ϕ gives us a subgroup $H = p_*(\pi_1(E, \tilde{x}_0)$ by the previous theorem. If we change the basepoint of the covering space to $\tilde{x}_1$, then this covering space corresponds to the subgroup $H' = p_*(\pi_1(E, \tilde{x}_1)$. If we let $\tilde{\gamma}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$, then $H' = [p\tilde{\gamma}]H[\overline{p\tilde{\gamma}}]$. This means that if we look at isomorphism classes that do not necessarily preserve basepoints in the previous theorem, we get a bijection between these isomorphism classes and the set of subgroups of $\pi_1(X, x_0)$ modulo conjugacy.

Of course there is a reason for the name "universal cover". The universal covering space $p : U \to X$ satisfies a universal property. That is, if we are given any connected covering space $r : E \to X$, there exists a unique basepoint-preseving morphism $f : U \to E$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[d, "f"'] \arrow[dr, "p"] & \\ E \arrow[r, "r"] & X. \end{tikzcd}$$

This covering map is simply the quotient map $q : U \to U/\sim$, with $\sim$ the equivalence relation as defined in theorem 3.30. To prove uniqueness of U, let $p' : U' \to X$ be another universal covering space. Then there exists a unique morphism $q : U' \to U$ such that $p' = pf$. Choose $\tilde{x}_0 \in p'^{-1}(x_0)$, then $p'_*(\pi_1(U', \tilde{x}_0) = p_*(f_*(\pi_1(U', \tilde{x}_0))) \subset 1$. This means that the covering space $p' : U' \to X$ is sent to the trivial group by ϕ. In other words, U' and U are isomorphic by theorem 3.31, proving uniqueness.

Theorem 3.31 gives us an insight into the connected covering spaces for our space X. However, we can give a more meaningful description. This time we will not ask our covering spaces to be connected. Let X be a semilocally simply-connected, (path-)connected, locally path-connected topological space with a selected basepoint x_0, and let $p : E \to X$ be a covering space, not necessarily connected. Then there is an action of the fundamental group $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$. For $\tilde{x}_0 \in p^{-1}(x_0)$ and $[\gamma] \in \pi_1(X, x_0)$ we define $[\gamma]\tilde{x}_0 = \tilde{\gamma}(1)$ where $\tilde{\gamma}$ is the lift of γ starting at $\tilde{x}_0$. It turns out that associating a covering space with its fiber over x_0 (with the action described above) gives us a functor from the category of covering spaces into the category G-**Set** with $G = \pi_1(X, x_0)$. The last category shall be referred to as π-**Set** from now on. All of this will be explored in the next section.

3.3 The equivalence

In all that follows X is a topological space that is semilocally simply-connected, locally path-connected and x_0 is a chosen basepoint. If $p : E \to X$ is a covering space, then we can consider the (discrete) set $F = p^{-1}(x_0)$. We shall refer to the category G-**Set** as π-**Set** since we are considering the group action of $\pi = \pi_1(X, x_0)$. For $[\gamma] \in \pi$ define

$$\begin{aligned} \sigma_\gamma :\ & F \longrightarrow F \\ & \tilde{x}_0 \longmapsto \tilde{\gamma}_{\tilde{x}_0}(1), \text{ with } \tilde{\gamma} \text{ the lift of } \gamma \text{ with } \tilde{\gamma}(0) = \tilde{x}_0 \end{aligned}$$

This map is well defined since lifts of homotopic paths γ and γ' will have the same endpoints and hence give rise to the same bijection.

3.33. Lemma. Let $p : E \to X$ be a covering space. The map σ_γ defined above is a bijection and the map $\phi : \pi \to \mathrm{Sym}(p^{-1}(x_0))$ given by $[\gamma] \to \sigma_\gamma$ is an action.

Proof. The inverse mapping is given by $\sigma_{\overline{\gamma}}$. Take $x \in p^{-1}(x_0)$. If we lift the path $\overline{\gamma}$ starting at $\sigma_\gamma(x)$ then the path ends at x by uniqueness of lifts. This shows that $\sigma_\gamma \sigma_{\overline{\gamma}} = \mathrm{id}$. A similar argument shows that $\sigma_{\overline{\gamma}} \sigma_\gamma = \mathrm{id}$.

Let $[\gamma]$, $[\gamma'] \in \pi$. For a given $\tilde{x}_0 \in p^{-1}(x_0)$, the endpoint of the lift of $\gamma' \cdot \gamma$ to $\widetilde{\gamma' \cdot \gamma}$ starting at $\tilde{x}_0$ is the same as the endpoint of the lift of γ' to $\tilde{\gamma}'$ starting at $\tilde{\gamma}(1)$ with $\tilde{\gamma}$ the lift of γ starting at $\tilde{x}_0$. This follows easily from the uniqueness of lifts. In other words $\phi([\gamma' \cdot \gamma]) = \sigma_{\gamma' \cdot \gamma} = \sigma_{\gamma'} \sigma_\gamma = \phi([\gamma'])\phi([\gamma])$. This proves that ϕ is a (left) action. $\square$

3.34. Lemma. For the universal covering space $p : U \to X$, we have $p^{-1}(x_0) \cong \pi$, with π a π-set with left-translation as action.

Proof. Let $p : U \to X$ be the universal covering space. Consider the function $f : p^{-1}(x_0) \to \pi$ given by $\tilde{x}_0 \mapsto [p\tilde{\gamma}]$ with $\tilde{\gamma}$ the path starting from $[c]$ to $\tilde{x}_0$ in U. This is well-defined because U is simply-connected: different paths between $[c]$ and $\tilde{x}_1$ are lifts of the same path since there exists a homotopy between them in U. Take an element $[\gamma] \in \pi$. Lift γ to $\tilde{\gamma}$ starting at $[c]$ in U. Thus $f(\tilde{\gamma}(1)) = [\gamma]$. For injectivity, notice that when paths $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ from $[c]$ to $\tilde{x}_1$ and $\tilde{x}_2$ respectively are lifts of γ that the endpoints are fixed and hence $\tilde{x}_1 = \tilde{x}_2$. The action on the set π is the action described above. □

3.35. Lemma. Consider the universal cover U as the set of equivalence classes of paths starting at $x_0 \in X$ as in theorem 3.29. The map $\psi : \pi \to \mathrm{Aut}(U)$ given by $[\gamma] \mapsto \tau_\gamma$ with $\tau_\gamma : U \to U$ given by $[\lambda] \mapsto [\lambda \cdot \gamma]$ is a right action.

Proof. First, we must prove that for $[\gamma] \in \pi$, the function τ_γ is an automorphism of covering spaces. Take $[\lambda] \in U$. Then $\tau_\gamma([\lambda \cdot \overline{\gamma}]) = [\lambda]$, proving surjectivity. Take $[\lambda], [\lambda'] \in U$. If $[\lambda \cdot \gamma] \simeq [\lambda' \cdot \gamma]$, then it is clear that $[\lambda] \simeq [\lambda']$, proving injectivity. It is also clear that τ_γ respects the covering map since $p(\tau_\gamma[\lambda]) = \lambda\gamma(1) = p(\tau_\gamma([\lambda]))$. Clearly, τ_γ is a continuous map since basic open sets from $\mathcal{B}$ (as in theorem 3.29) are mapped onto basic open sets. The same arguments hold for the inverse $\tau_{\overline{\gamma}}$. Hence τ_γ is an automorphism of covering spaces. It remains to check that $\psi([\gamma \cdot \gamma']) = \psi([\gamma'])\psi([\gamma])$, or that $\tau_{\gamma \cdot \gamma'} = \tau_{\gamma'}\tau_\gamma$. Let $[\lambda] \in U$ be given. Then $\tau_{\gamma \cdot \gamma'}([\lambda]) = [\lambda \cdot \gamma \cdot \gamma'] = \tau_{\gamma'}(\tau_\gamma([\lambda]))$. □

3.36. Lemma. Let $p : E \to X$ be a covering space. Then the orbits of $F = \mathcal{F}(p : E \to X)$ are in bijective correspondence with the connected components of E. In particular, the covering space $p : E \to X$ is connected iff the action on F is transitive.

Proof. Take an orbit $\pi\tilde{x} \subset F$. It is clear that any $\tilde{x}_0, \tilde{x}_1 \in \pi\tilde{x}$ lie in the same path-connected component $E_{\tilde{x}}$ (and hence connected component) since there exists a loop in X whose lift to E starts at $\tilde{x}_0$ and ends at $\tilde{x}_1$. Conversely, any connected component corresponds to an orbit. Take any point $\tilde{x}_0$ in the connected component. Then this connected component corresponds to $\pi\tilde{x}_0$. It is clear that these two associations are each others inverse. The last statement follows immediately. □

• The functors

First, define the functor $\mathcal{F} : \mathbf{Cov}(X) \to \pi\text{-}\mathbf{Set}$ given by $\mathcal{F}(p : E \to X) = p^{-1}(x_0)$ with left-action

$$\begin{aligned} \phi : \quad \pi \quad &\longrightarrow \mathrm{Sym}(p^{-1}(x_0)) \\ [\gamma] \quad &\longmapsto \sigma_\gamma, \end{aligned}$$

and $\mathcal{F}(f : E_1 \to E_2) = f|_{p_1{}^{-1}(x_0)}$. Let $p_1 : E_1 \to X$, $p_2 : E_2 \to X$ be two covering spaces together with a morphism $f : E_1 \to E_2$. We must check that the function f restricted to $p^{-1}(x_0)$ is indeed a π-map. It is clear that f sends $p_1^{-1}(x_0)$ to $p_2^{-1}(x_0)$ since f respects the covering maps. It remains to check $f([\gamma]\tilde{x}_0) = [\gamma]f(\tilde{x}_0)$ for all $\tilde{x}_0 \in p_1^{-1}(x_0)$ and all $[\gamma] \in \pi$. Let $\tilde{\gamma}_1$ be the lift of γ starting at $\tilde{x}_0 \in E_1$ and $\tilde{\gamma}_2$ the lift of γ starting at $f(\tilde{x}_0) \in E_2$. Since f respects the covering maps we have $p_2 f\tilde{\gamma}_1 = p_1\tilde{\gamma}_1 = p_2\tilde{\gamma}_2$. Because $f\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are both lifts of γ starting at $f(\tilde{x}_0)$ we must have $f\tilde{\gamma}_1 = \tilde{\gamma}_2$ by uniqueness of lifts, and thus $f(\tilde{\gamma}_1(1)) = \tilde{\gamma}_2(1)$. It is clear that the identity morphism of covering spaces is sent to the identity π-map by $\mathcal{F}$. Let

$$E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3$$

be two morphisms of covering spaces $p_1 : E_1 \to X$, $p_2 : E_2 \to X$, and $p_3 : E_3 \to X$. Also, $\mathcal{F}(g \circ f) = (g \circ f)|_{p_1^{-1}(x_0)} = \mathcal{F}(g) \circ \mathcal{F}(f)$ and $\mathcal{F}(\mathrm{id}_{p:E\to X}) = \mathrm{id}_{p:E\to X}|_{p^{-1}(x_0)} = \mathrm{id}_{\mathcal{F}(p:E\to X)}$. Hence, $\mathcal{F}$ is indeed a functor.

Let us proceed by giving the functor from π-**Set** to $\mathbf{Cov}(X)$. The right action of $[\lambda] \in \pi$ on $[\gamma] \in U$ will be written as $[\gamma][\lambda]$ and for the left action of $[\lambda] \in \pi$ on $\tilde{x}_0 \in p^{-1}(x_0)$ we shall use the notation $[\lambda]\tilde{x}_0$. In all that follows, let U be the universal covering space for X. Define the functor $\mathcal{C} : \pi\text{-}\mathbf{Set} \to \mathbf{Cov}(X)$ given by $\mathcal{C}(F) = p$ with

$$\begin{aligned} p : \quad (U \times F)/{\sim} &\longrightarrow X \\ ([\gamma], \tilde{x}_0) &\longmapsto \gamma(1), \text{ with } ([\gamma], \tilde{x}_0) \sim ([\gamma][\overline{\lambda}], [\lambda]\tilde{x}_0)) \end{aligned}$$

and $\mathcal{C}(f : F \to F') = h$ with

$$\begin{aligned} h : \quad (U \times F)/{\sim} &\longrightarrow (U \times F')/{\sim} \\ ([\gamma], \tilde{x}_0) &\longmapsto ([\gamma], f(\tilde{x}_0)). \end{aligned}$$

The map p above is a covering space for X. First of all, note that p is well-defined since $\gamma(1) = \gamma \cdot \overline{\lambda}(1)$ for all $[\lambda] \in \pi$. Furthermore, p is a continuous function since the maps $(U \times F) \to U$ and $U \to X$ by $[\gamma] \mapsto \gamma(1)$ are continuous. It is clear that h respects the map p as the map does not depend on the second coordinate. It is also clearly continuous since it is continuous in both coordinates (identity map on the left coordinate, and the right coordinate has the discrete topology) and the quotient map is continuous. Given morphisms $f : F_1 \to F_2$ and $g : F_2 \to F_3$, we have $\mathcal{G}(g \circ f) = q$ with $q : (U \times F_1)/{\sim} \to (U \times F_3)/{\sim}$ given by $([\gamma], \tilde{x}_0) \mapsto ([\gamma], (g \circ f)(\tilde{x}_0))$ which is clearly the composition of covering maps $\mathcal{G}(g) \circ \mathcal{G}(f)$. Also, $\mathcal{G}(\mathrm{id}_F) = \mathrm{id}_{\mathcal{G}(F)}$ by construction of the map h above (letting $f = \mathrm{id}_F$). We conclude that $\mathcal{C}$ is indeed a functor.

3.37. Lemma. The composite functor $\mathcal{F} \circ \mathcal{C}$ is naturally isomorphic to $\mathrm{id}_{\pi\text{-}\mathbf{Set}}$ in the category of functors from π-**Set** to itself.

Proof. To prove that $\mathcal{F} \circ \mathcal{C} \cong \mathrm{id}_{\pi\text{-}\mathbf{Set}}$ we must give a natural isomorphism $\tau : \mathcal{F} \circ \mathcal{C} \to \mathrm{id}_{\pi\text{-}\mathbf{Set}}$. This means that τ assigns to each object $F \in \pi$-**Set** an invertible arrow $\tau_F : (\mathcal{F} \circ \mathcal{C})(F) \to F$ such that for every morphism $f : F \to F'$ between objects F and F' in π-**Set** the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{F}\circ\mathcal{C})(F) \arrow[r, "\tau_F"] \arrow[d, "(\mathcal{F}\circ\mathcal{C})(f)"'] & F \arrow[d, "f"] \\
(\mathcal{F}\circ\mathcal{C})(F') \arrow[r, "\tau_{F'}"] & F'.
\end{tikzcd}$$

Hence, let $F \in \pi$-**Set** be a general π-set. Let $p : (U \times F)/\!\sim\ \longrightarrow X$ be the covering space $\mathcal{C}(F)$. Then $(\mathcal{F}\circ\mathcal{C})(F) = p^{-1}(x_0)$. Let $f_F : F \to F'$ be a morphism of π-sets and let $p' : (U \times F')/\!\sim\longrightarrow X$ be the covering space $\mathcal{C}(F')$. Applying $\mathcal{F}$ to p' gives us the map

$$\begin{aligned}(\mathcal{F}\circ\mathcal{C})(f_F) :\ \ (\pi \times p^{-1}(x_0))/\!\sim\ &\longrightarrow (\pi \times p'^{-1}(x_0))/\!\sim \\ ([\gamma], \tilde{x}_0)\ &\longmapsto ([\gamma], f_F(\tilde{x}_0)).\end{aligned}$$

By lemma 3.34, the preimage $p^{-1}(x_0)$ is precisely set of elements $\pi \times F$ modulo the equivalence relation $\sim$. Hence $(\mathcal{F}\circ\mathcal{C})(F) = (\pi \times F)/\!\sim$. The action of π on $(\pi \times F)/\!\sim$ is the obvious action: $[\lambda]([\gamma], \tilde{x}_0) = ([\gamma][\overline{\lambda}], [\lambda]\tilde{x}_0)$. Let

$$\begin{aligned}\tau_F :\ \ (\pi \times F)/\!\sim\ &\longrightarrow F \\ ([\gamma], \tilde{x}_0)\ &\longmapsto [\gamma]\tilde{x}_0.\end{aligned}$$

The map τ_F is well-defined because $\tau_F([\gamma], \tilde{x}_0) = [\gamma]\tilde{x}_0 = [\gamma][\overline{\lambda}][\lambda]\tilde{x}_0 = \tau_F([\gamma][\overline{\lambda}], [\lambda]\tilde{x}_0)$ for all $[\lambda] \in \pi$. We go on to show that τ_F is an isomorphism of π-sets. We have $\tau_F(([c], \tilde{x}_0)) = \tilde{x}_0$ for any $\tilde{x}_0 \in F$ proving surjectivity. Let $([\gamma], \tilde{x}_0)$ and $([\gamma'], \tilde{x}_1)$ be given and assume that $[\gamma]\tilde{x}_0 = [\gamma']\tilde{x}_1$. Then $\tilde{x}_1 = [\overline{\gamma'}][\gamma]\tilde{x}_0$. Hence $([\gamma'], \tilde{x}_1) = ([\gamma'], [\overline{\gamma'}][\gamma]\tilde{x}_0) \sim ([\gamma], \tilde{x}_0)$ proving injectivity. It remains to show that τ_F is a π-map. Let $([\gamma], \tilde{x}_0) \in (\pi \times F)$ and $[\lambda] \in \pi$ be given and consider $[\lambda]\tau_F(([\gamma], \tilde{x}_0)) = [\lambda][\gamma]\tilde{x}_0$. Alternatively, if we apply $[\lambda]$ to $([\gamma], \tilde{x}_0)$ we move to the point $([\lambda][\gamma], \tilde{x}_0)$. We get $\tau_F(([\lambda][\gamma], \tilde{x}_0)) = [\lambda][\gamma]\tilde{x}_0$ proving that τ_F is a π-map and thus a morphism of π-sets. Since the inverse is automatically also a π-map we conclude that τ_F is an isomorphism. Similarly, let

$$\begin{aligned}\tau_{F'} :\ \ (\pi \times F')/\!\sim\ &\longrightarrow F' \\ ([\gamma], \tilde{x}_0)\ &\longmapsto [\gamma]\tilde{x}_0.\end{aligned}$$

It remains to show that the diagram above commutes. In other words, we want to verify that $f_F \circ \tau_F = \tau_{F'} \circ (\mathcal{C}\circ\mathcal{F})(f_F)$. Take $([\gamma], \tilde{x}_0) \in p^{-1}(x_0)$. We get $(f_F \circ \tau_F)(([\gamma], \tilde{x}_0)) = f_F([\gamma]\tilde{x}_0)$. On the other hand, $(\tau_{F'} \circ (\mathcal{C}\circ\mathcal{F})(f_F))(([\gamma], \tilde{x}_0)) = \tau_{F'}(([\gamma], f_F(\tilde{x}_0))) = [\gamma]f_F(\tilde{x}_0)$. Since $\tau_{F'}$ is a π-map we have proved that the diagram commutes. This proves the natural isomorphism. $\square$

3.38. Lemma. We have $\mathcal{C}\circ\mathcal{F} \cong \mathrm{id}_{\mathbf{Cov}(X)}$.

Proof. To prove that $\mathcal{C}\circ\mathcal{F} \cong \mathrm{id}_{\mathbf{Cov}(X)}$ we must give a natural isomorphism $\tau : \mathcal{C}\circ\mathcal{F} \to \mathrm{id}_{\mathbf{Cov}(X)}$. This means that τ assigns to each object $E \in \mathbf{Cov}(X)$ an invertible arrow $\tau_E : (\mathcal{C}\circ\mathcal{F})(E) \to E$ such that for every morphism $f : E \to E'$ between objects E and E' in $\mathbf{Cov}(X)$ the following diagram commutes:

$$\begin{tikzcd}
(\mathcal{C} \circ \mathcal{F})(E) \arrow[r, "\tau_E"] \arrow[d, "(\mathcal{C}\circ\mathcal{F})(f)"'] & E \arrow[d, "f"] \\
(\mathcal{C} \circ \mathcal{F})(E') \arrow[r, "\tau_{E'}"] & E'.
\end{tikzcd}$$

Let $p : E \to X$ be a covering space for X. Consider the covering space $(\mathcal{C} \circ \mathcal{F})(E)$ given by $q : (U \times p^{-1}(x_0))/{\sim} \to X$. Let $p' : E' \to X$ be another covering space such that there is a morphism $f : E \to E'$ of covering spaces. If we apply $\mathcal{F}$ to f we get the restriction of f to $p^{-1}(x_0)$. Now apply $\mathcal{C}$ to get

$$\begin{array}{rll} (\mathcal{C} \circ \mathcal{F})(f|_{p^{-1}(x_0)}) : & (U \times p^{-1}(x_0))/{\sim} & \longrightarrow (U \times p'^{-1}(x_0))/{\sim} \\ & ([\gamma], \tilde{x}_0) & \longmapsto ([\gamma], f(\tilde{x}_0)). \end{array}$$

The mappings τ_E and $\tau_{E'}$ will be constructed. Then, after verifying that the diagram above commutes with the given mappings, the theorem is proved.

We begin by defining τ_E:

$$\begin{array}{rll} \tau_E : & (U \times p^{-1}(x_0))/{\sim} & \longrightarrow E \\ & ([\gamma], \tilde{x}_0) & \longmapsto \tilde{\gamma}(1),\ \tilde{\gamma} \text{ the lift of } \gamma \text{ to } E \text{ from } \tilde{x}_0. \end{array}$$

We need to show that τ_E is an isomorphism of covering spaces. The map τ_E is clearly continuous. Let γ be a path starting at x_0 in X. Let B be a basic neighborhood of $\gamma(1)$ breaking up in a disjoint union of homeomorphic sets in U such that $B_[\gamma]$ is the corresponding neighborhood in the universal cover. Let $\tilde{x}_0 \in p^{-1}(x_0)$. Then $B_{[\gamma]} \times \{\tilde{x}_0\}$ is an open neighborhood in $(U \times p^{-1}(x_0))/{\sim}$ that is homeomorphic to B through τ_E. Since X is path-connected τ_E is surjective. To prove injectivity, assume that $\tau_E(([\gamma], \tilde{x}_0)) = \tau_E(([\gamma'], \tilde{x}_1))$. That means that γ' and γ are both paths starting at x_0 and ending at the same point. Hence $\lambda = \overline{\gamma'} \cdot \gamma$ is a loop at x_0. It is easy to see that $\tilde{x}_1 = [\lambda]\tilde{x}_0$. Also $[\gamma\overline{\lambda}] = [\gamma']$. Hence, $([\gamma], \tilde{x}_0) \sim ([\gamma'][\overline{\lambda}], [\lambda]\tilde{x}_0)$. Conversely, if we take $[\lambda] \in \pi$, then $\tau_E(([\gamma], \tilde{x}_0)) = \tau_E(([\gamma][\overline{\lambda}], [\lambda]\tilde{x}_0)$ proving that τ_E is well-defined. We conclude that τ_E is a homeomorphism. For τ_E to be a morphism, the relation $p\tau_E = q$ must hold. But this is clear since $p\tau_E(([\gamma], \tilde{x}_0)) = p(\tilde{\gamma}(1)) = \gamma(1) = q(([\gamma], \tilde{x}_0))$.

Similarly, let

$$\begin{array}{rll} \tau'_E : & (U \times p'^{-1}(x_0))/{\sim} & \longrightarrow E' \\ & ([\gamma], \tilde{x}_0) & \longmapsto \tilde{\gamma}(1),\ \tilde{\gamma} \text{ the lift of } \gamma \text{ to } E' \text{ from } \tilde{x}_0. \end{array}$$

It remains to verify that the diagram above commutes, in other words $f \circ \tau_E = \tau'_E \circ (\mathcal{C} \circ \mathcal{F})(f|_{p^{-1}(x_0)})$. Let $([\gamma], \tilde{x}_0) \in (U \times p^{-1}(x_0))$ be given. Then $(f \circ \tau_E)(([\gamma], \tilde{x}_0)) = f(\tilde{\gamma}(1))$ with $\tilde{\gamma}$ the lift of γ to E from $\tilde{x}_0$. On the other hand, $(\tau'_E \circ (\mathcal{C} \circ \mathcal{F})(f_E|_{p^{-1}(x_0)}))([\gamma], \tilde{x}_0)) = \tau'_E([\gamma], f(\tilde{x}_0)) = \tilde{\gamma}'(1)$ where $\tilde{\gamma}'$ is the lift of γ starting at $f(\tilde{x}_0)$. Certainly we have $f_E(\tilde{\gamma}(1)) = \tilde{\gamma}'(1)$ since both paths $f\tilde{\gamma}$ and $\tilde{\gamma}'$ are lifts of γ in E' starting at $f(\tilde{x}_0)$. This proves the natural isomphism. $\square$

3.39. Theorem. (Main theorem) The functor $\mathcal{F} : \mathbf{Cov}(X) \to \pi\text{-}\mathbf{Set}$ is an equivalence of categories.

Proof. This follows from the previous two lemmas. $\square$

In the next chapter, we shall take advantage of this equivalence by classifying the automorphism group of a covering space and by proving the famous Seifert-van Kampen theorem in pure algebraic and categorical terms using the construction of the fibered coproduct as explained in section 2.2.2

4

Applications and examples

In this section a number of applications of the main theorem 3.39 will be presented. First, we shall give some account of the automorphisms of covering spaces, and then an alternative proof (to the usual topological proofs involving paths) of the Seifert-van Kampen theorem will be given. Finally, a few concrete examples will be worked out. Again, in this chapter, we will assume X to be a topological space that is semilocally simply-connected, locally path-connected, and x_0 is a chosen basepoint.

4.1 Automorphisms and recovering the fundamental group

4.1. Definition. Let $p : E \to X$ be a covering space. The elements in the automorphism group $\mathrm{Aut}(E)$ are called **deck transformations**. This group is sometimes also denoted $G(E)$.

4.2. Lemma. Let $p : E \to X$ be a covering space. The group $G(E)$ is isomorphic to the automorphisms of the π-set $\mathcal{F}(E) = p^{-1}(x_0)$.

Proof. This follows directly from the main theorem (3.39). □

Let $\tilde{x}_0 \in \mathcal{S}(E)$ be given. Assume that $[\gamma]\tilde{x}_0 = \tilde{x}_0$, then γ is a loop at x_0 that lifts to a loop at $\tilde{x}_0 \in E$. This means that $[\gamma] \in p_*(\pi_1(E, \tilde{x}_0))$. Conversely, if we are given an element $[\gamma] \in p_*(\pi_1(E, \tilde{x}_0))$ then $[\gamma]\tilde{x}_0 = \tilde{x}_0$. Hence, the stabilizer $\pi_{\tilde{x}_0}$ of $\tilde{x}_0$ equals $p_*(\pi_1(E, \tilde{x}_0))$.

We are now ready to completely describe the deck transformations of $p : E \to X$.

4.3. Theorem. Let $p : E \to X$ be a connected covering space and let $\tilde{x}_0 \in p^{-1}(x_0)$ and let $H = p_*(\pi_1(E, \tilde{x}_0))$. The group of deck transformations $G(E)$ is isomorphic to the group NH/H.

Proof. This follows from the previous discussion, theorem 3.39, and lemma 3.7. Notice that the basepoint $\tilde{x}_0$ can be chosen arbitrarily as choosing $\tilde{x}_1$ will give rise to $\pi_1(E, \tilde{x}_1)$, a conjugate subgroup of $\pi_1(E, \tilde{x}_0)$. □

4.4. Lemma. Let $p : E \to X$ be a connected covering space and let $\sigma : E \to E$ be an automorphism of covering spaces. If $\sigma(x) = x$ for some $x \in E$, then $\sigma = \mathrm{id}_E$.

Proof. This follows from lemma 3.27 by setting $Y = E$ and $f = p$. □

This means that only the trivial element in π stabilizes any $\tilde{x}_0 \in p^{-1}(x_0)$. In terms of π-sets this tells us that the action of π is free.

4.5. Lemma. The group of deck transformations of the universal covering space $p : U \to X$ is isomorphic to π.

Proof. We have $\mathcal{F}(U) = \pi$ where π acts transitive on itself by left-translation (since U is path-connected) connected by lemma 3.36. It is also a free action by the previous discussion. The corollary then follows from theorem 4.3 by realizing that U is simply-connected, and that the normalizer of the trivial group is the whole group itself. □

This lemma allows to recover the group that is acting upon the fiber over x_0 if we are given the category **Cov**(X). In fact, we have the following useful corollary.

4.6. Corollary. Let **Cov**(X) and **Cov**(Y) be two equivalent categories. Then $\pi_1(X) \cong \pi_1(Y)$. Also, when G-**Set** and H-**Set** are equivalent categories, given that G and H are fundamental groups of a fundamental spaces satisfying the criteria in theorem 3.39, then $G \cong H$.

Proof. We leave the basepoints out, since we are only interested in the fundamental group up to isomorphism. Since **Cov**(X) is equivalent to $\pi_1(X)$-**Set** the automorphism group of the universal cover is isomorphic to $\pi_1(X)$. Likewise, the automorphism group of the universal cover of **Cov**(Y) is isomorphic to $\pi_1(Y)$. Since **Cov**(X) and **Cov**(Y) are equivalent, the universal cover of X is equivalent to the universal cover of Y, and hence their automorphism groups must be isomorphic. The second statement follows directly from the first by applying theorem 3.39. □

4.2 The Seifert-van Kampen theorem

The Seifert-van Kampen theorem is a useful tool to calculate the fundamental group of X in terms of the fundamental groups of smaller spaces.

4.7. Theorem. Let X be a connected, locally path-connected, semilocally simply-connected topological space and let $x_0 \in X$. Let X_1 and X_2 be open subspaces of X both containing x_0 such that $X_{1,2} = X_1 \cap X_2$ is connected and $X = X_1 \cup X_2$. Let $i_1 : X_{1,2} \to X_1$ and $i_2 : X_{1,2} \to X_2$ be the natural inclusions. Let N be the normal closure of the set of all elements of the form $i_{1*}([\lambda])(i_{2*}([\lambda]))^{-1}$ with $[\lambda] \in \pi_1(X_{1,2}, x_0)$. Then $\pi_1(X, x_0) \cong (\pi_1(X_1, x_0) * \pi_1(X_2, x_0))/N$.

In order to prove this we shall employ the equivalence that we have proved in the previous chapter extensively. The functors $\mathcal{F}$ and $\mathcal{C}$ from section 3.3.1 shall be used for different base spaces. It is clear that by replacing X by another space satisfying the same criteria, such as X_1, we get another equivalence between $\mathbf{Cov}(X_1)$ and $\pi_1(X_1)$-$\mathbf{Set}$. To indicate that we are using this equivalence we shall write $\mathcal{F}_{X_1}$ and $\mathcal{G}_{X_1}$. Occasionally, we shall denote the G-set S as S_G to avoid ambiguity. For ease of notation, we write $\pi = \pi_1(X, x_0)$, $\pi_1 = \pi_1(X_1, x_0)$, $\pi_2 = \pi_1(X_2, x_0)$, $\pi_{1,2} = \pi_1(X_{1,2}, x_0)$. Whereas the idea of the proof is quite interesting, the proof itself often comes down to mere verifications. In order not to bore the reader, some obvious details of proofs have been hinted at, or have even been left out.

The idea of the proof, is to view coverings for X, as two coverings: one for X_1, and one for X_2, including an isomorphism of coverings of $X_{1,2}$. By applying the functor $\mathcal{F}$ to these coverings, we end up in the world of π_1-sets, π_2-sets and $\pi_{1,2}$-morphisms. By using a categorical construction, we can easily prove the theorem, without resorting to using anything topological.

• The categories $\mathcal{C}_1$, $\mathcal{C}_2$, and π_P-Set

Let $\mathcal{C}_1$ be the category whose objects are triplets $(p_1 : E_1 \to X_1, p_2 : E_2 \to X_2, \alpha)$, with p_1 and p_2 coverings of X_1 and X_2 respectively, and $\alpha : p_1^{-1}(X_{1,2}) \to p_2^{-1}(X_{1,2})$ an isomorphism in $\mathbf{Cov}(X_{1,2})$. A morphism between $(p_1 : E_1 \to X_1, p_2 : E_2 \to X_2, \alpha)$ and $(p_1' : E_1' \to X_1, p_2' : E_2' \to X_2, \alpha')$ is given by a couple of covering morphisms (f_1, f_2) in $\mathbf{Cov}(X_1)$ and $\mathbf{Cov}(X_2)$ respectively, such that the following diagram commutes:

$$\begin{tikzcd} p_1^{-1}(X_{12}) \arrow[rr, "\alpha"] \arrow[dr, "p_1"] \arrow[dd, "f_1"'] & & p_2^{-1}(X_{12}) \arrow[dl, "p_2"'] \arrow[dd, "f_2"] \\ & X_{12} & \\ {p'}_1^{-1}(X_{12}) \arrow[rr, "\alpha'"'] \arrow[ur, "p_1'"] & & {p_2'}^{-1}(X_{12}) \arrow[ul, "p_2'"'] \end{tikzcd} \tag{4.8}$$

It is a trivial exercise to verify that $\mathcal{C}_1$ is indeed a category. By employing our functor $\mathcal{F}$ from the previous chapter, we get to the category $\mathcal{C}_2$ whose objects are triplets (S_1, S_2, φ) with S_1 a π_1-set, S_2 a π_2-set, and $\varphi : S_1 \to S_2$ an isomorphism of $\pi_{1,2}$-sets. A morphism between objects (S_1, S_2, φ) and (S_1', S_2', φ') is given by a 2-tuple $(f_1 : S_1 \to S_1', f_2 : S_2 \to S_2')$ of morphisms of π_1-sets and π_2-sets respectively such that the following diagram commutes:

$$\begin{tikzcd} S_1 \arrow[r, "\varphi"] \arrow[d, "f_1"'] & S_2 \arrow[d, "f_2"] \\ S_1' \arrow[r, "\varphi'"'] & S_2'. \end{tikzcd} \tag{4.9}$$

The last category that is of our interest, is the category π_P-**Set** where π_P is the fibered coproduct of the inclusion group homomorphisms $i_{1*} : \pi_{1,2} \to \pi_1$ and $i_{2*} : \pi_{1,2} \to \pi_2$ in the category **Grp** (whose existence we have hinted at in examples 2.13):

$$\begin{tikzcd} & \pi_{1,2} \arrow[dl, "i_{1*}"'] \arrow[dr, "i_{2*}"] & \\ \pi_1 \arrow[dr, "j_1"'] & & \pi_2 \arrow[dl, "j_2"] \\ & \pi_P. & \end{tikzcd} \tag{4.10}$$

The approach is, that we will show that $\mathbf{Cov}(X)$ is equivalent, through categories $\mathcal{C}_1$ and $\mathcal{C}_2$, to π_P-**Set**. Then, we will conclude that the fundamental group of X is $\pi_P \cong (\pi_1 * \pi_2)/N$ with N the normal closure of all elements of the form $i_{1*}{i_{2*}}^{-1}$, thus proving theorem 4.7.

• The functors

First, define the functor $\mathcal{S} : \mathbf{Cov}(X) \to \mathcal{C}_1$ given by

$$\begin{aligned} \mathcal{S}(p : E \to X) &= (p_1 : p^{-1}(X_1) \to X_1, p_2 : p^{-1}(X_2) \to X_2, \mathrm{id}_{p:E\to X}) \\ \mathcal{S}(f : E \to E') &= (f|_{p^{-1}(X_1)}, f|_{p^{-1}(X_2)}). \end{aligned}$$

and the functor $\mathcal{T} : \mathcal{C}_1 \to \mathbf{Cov}(X)$ given by

$$\begin{aligned} \mathcal{T}((p_1 : E_1 \to X_1, p_2 : E_2 \to X_2, \alpha)) &= p : E_1 \sqcup_\alpha E_2 \to X \\ \mathcal{T}((f_1 : E_1 \to E_1', f_2 : E_2 \to E_2')) &= f : E_1 \sqcup_\alpha E_2 \to E_1' \sqcup_\alpha E_2' \end{aligned}$$

where p is given by $e \mapsto p_i(e)$ and f is given by $e \mapsto f_i(e_i)$ for $e \in E_i$.

4.11. Lemma. The categories $\mathbf{Cov}(X)$ and $\mathcal{C}_1$ are equivalent.

Proof. The fact that $\mathcal{T}$ and $\mathcal{S}$ are indeed functors is a trivial verification. We will only verify that $(\mathcal{T} \circ \mathcal{S}) \cong \mathrm{id}_{\mathbf{Cov}(X)}$. We must associate to each covering space $p : E \to X$ an isomorphism $\tau_E : (\mathcal{T} \circ \mathcal{S})(E) \to E$ such that for any morphism $f : E \to E'$ the following diagram commutes:

$$\begin{tikzcd} (\mathcal{T} \circ \mathcal{S})(E) \arrow[r, "\tau_E"] \arrow[d, "(\mathcal{T}\circ\mathcal{S})(f)"'] & E \arrow[d, "f"] \\ (\mathcal{T} \circ \mathcal{S})(E') \arrow[r, "\tau_{E'}"] & E'. \end{tikzcd}$$

We have $(\mathcal{T} \circ \mathcal{S})(E) = p^{-1}(X_1) \sqcup_{\mathrm{id}} p^{-1}(X_2)$ which is isomorphic to E where the isomorphism is given by the inclusions $p^{-1}(X_1) \hookrightarrow E$ and $p^{-1}(X_2) \hookrightarrow E$. Also,

$$(\mathcal{T} \circ \mathcal{S})(f : E \to E') = f : p^{-1}(X_1) \sqcup_{\mathrm{id}} p^{-1}(X_2) \to p'^{-1}(X_1) \sqcup_{\mathrm{id}} p'^{-1}(X_2).$$

It is now a trivial matter to verify that the diagram above commutes. $\square$

4.12. Lemma. The categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent.

Proof. This follows quite formally from the functors $\mathcal{F}$ and $\mathcal{C}$ that we constructed in the previous chapter. For a triplet $(p_1 : E_1 \to X_1, p_2 : E_2 \to X_2, \alpha)$ we can apply the functors $\mathcal{F}_{X_1}(p_1)$, $\mathcal{F}_{X_2}(p_2)$, and $\mathcal{F}_{X_{1,2}}(\alpha)$ to get a π_1-set S_1, a π_2-set S_2, and a morphism $f : S_1 \to S_2$ of $\pi_{1,2}$ sets. By going back with the functor $\mathcal{C}$ it is not hard to verify that this gives an equivalence. □

The next step is to show that the categories π_P-**Set** and $\mathcal{C}_2$ are equivalent. Note that we can consider π_P a π_1-set and π_2-set through the homomorphisms j_1 and j_2 respectively. Define the functor $\mathcal{M} : \pi_P\text{-}\mathbf{Set} \to \mathcal{C}_2$ by

$$\begin{aligned} \mathcal{M}(S_{\pi_P}) &= (S_{\pi_1}, S_{\pi_2}, \mathrm{id}_{S_{\pi_{1,2}}}) \\ \mathcal{M}(f : S_{\pi_P} \to S'_{\pi_P}) &= (f_1, f_2). \end{aligned}$$

where f_i is the by j_i induced morphism of π_i-sets for $i = 1, 2$. Since $\pi_{1,2}$ has the same action on S_{π_1} and S_{π_2} by commutativity of diagram 4.10, we can use $\mathrm{id}_{S_{\pi_{1,2}}}$ as the isomorphism of $\pi_{1,2}$-sets between S as π_1-set and S.

Vice versa, we can define the functor $\mathcal{N} : \mathcal{C}_2 \to \pi_P$-**Set**. Let $\mathcal{N}((S_1, S_2, \varphi)) = S \stackrel{\text{def}}{=} S_1 \sqcup_\varphi S_2$. There is an natural action ϕ_1 of π_1 on S: for $g \in \pi_1$ we define $gs = f(g)s$ for $s \in S_1$ and $gs = (\varphi f(g) \varphi^{-1})(s)$ for $s \in S_2$ with f the action of π_1 on S_1. Likewise there is a natural action ϕ_2 of π_2 on S, and these actions coincide on $\pi_{1,2}$. This means that we have a diagram

(4.13)
$$\begin{tikzcd} & \pi_{1,2} \arrow[dl, "i_{1*}"'] \arrow[dr, "i_{2*}"] & \\ \pi_1 \arrow[r, "j_1"] \arrow[dr, "\phi_1"'] & \pi_P \arrow[d, dotted, "\exists! h"] & \pi_2 \arrow[l, "j_2"'] \arrow[dl, "\phi_2"] \\ & \mathrm{Sym}(S_1 \sqcup_\varphi S_2), & \end{tikzcd}$$

where the outer (tilted) square commutes. By the universal property of the fibered coproduct π_P, there exists a unique group homomorphism $h : \pi_P \to \mathrm{Sym}(S_1 \sqcup_\varphi S_2)$ making the diagram commute, so S is indeed a π_P-set. Also, for a morphism we have $\mathcal{N}((g_1 : S_1 \to S_1', g_2 : S_2 \to S_2')) = g : S_1 \sqcup_\varphi S_2 \to S_1' \sqcup_\varphi S_2'$ where g is induced in the most natural way by g_1 and g_2.

4.14. Lemma. The categories π_P-**Set** and $\mathcal{C}_2$ are equivalent.

Proof. Let us begin to prove that $\mathcal{M} \circ \mathcal{N} \cong \mathrm{id}_{\mathcal{C}_2}$. We are then required to give the natural isomorphism $\tau : (\mathcal{M} \circ \mathcal{N}) \to \mathrm{id}_{\mathcal{C}_2}$. Let $S_{1,2} \stackrel{\text{def}}{=} (S_1, S_2, \varphi) \in \mathcal{C}_2$ be given. Applying $(\mathcal{M} \circ \mathcal{N})$ gives us $(S_1 \sqcup_\varphi S_2, S_1 \sqcup_\varphi S_2, \mathrm{id})$. Let the isomorphism $\tau_{S_{1,2}}$ be given by (σ_1, σ_2) with

$$
\begin{aligned}
\sigma_1 : \quad S_1 \sqcup_\varphi S_2 \to \quad & S_1 \\
s \longmapsto \quad & \begin{cases} s & s \in S_1 \\ \varphi^{-1}(s) & s \in S_2 \end{cases} \\
\sigma_2 : \quad S_1 \sqcup_\varphi S_2 \to \quad & S_2 \\
s \longmapsto \quad & \begin{cases} \varphi(s) & s \in S_1 \\ s & s \in S_2 \end{cases}
\end{aligned}
$$

These are easily to be seen π_1-set and π_2-set isomorphisms respectively. Let a morphism $(f_1, f_2) : (S_1, S_2, \varphi) \to (S_1', S_2', \varphi')$ be given. In order for τ to be a natural isomorphism, the following diagram must commute:

$$
\begin{tikzcd}
(S_1 \sqcup_\varphi S_2, S_1 \sqcup_\varphi S_2, \mathrm{id}) \arrow[r, "(\sigma_1,\sigma_2)"] \arrow[d, "(f,f)"'] & (S_1, S_2, \varphi) \arrow[d, "(f_1,f_2)"] \\
(S_1' \sqcup_{\varphi'} S_2', S_1' \sqcup_{\varphi'} S_2', \mathrm{id}) \arrow[r, "(\sigma_1',\sigma_2')"] & (S_1', S_2', \varphi).
\end{tikzcd}
$$

where $\tau_{S_{1,2'}}$ is (σ_1', σ_2') with given by the description above and where (f, f) is the morphism with $f : S_1 \sqcup_\varphi S_2 \to S_1' \sqcup_{\varphi'} S_2'$ induced by j_1 in the first coordinate, and induced by j_2 in the second coordinate. It is now a trivial verifcation that the diagram commutes using the fact that diagram 4.13 commutes.

It remains to show that $\mathcal{N} \circ \mathcal{M} \cong \mathrm{id}_{\pi_P\text{-}\mathbf{Set}}$. We will give the natural isomorphism $\tau : (\mathcal{N} \circ \mathcal{M}) \to \mathrm{id}_{\pi_P\text{-}\mathbf{Set}}$. Let $S \in \pi_P$-**Set** be given. Applying $(\mathcal{N} \circ \mathcal{M})$ gives us $S \sqcup_{\mathrm{id}} S$ which is clearly isomorphic to S in π_P-**Set**. Given a morphism $f : S \to S'$ it remains to verify that the following diagram commutes:

$$
\begin{tikzcd}
S \sqcup_{\mathrm{id}} S \arrow[r, "\tau_S"] \arrow[d, "(\mathcal{N}\circ\mathcal{M})(f)"'] & S \arrow[d, "f"] \\
S' \sqcup_{\mathrm{id}} S' \arrow[r, "\tau_{S'}"] & S'.
\end{tikzcd}
$$

where $(\mathcal{N} \circ \mathcal{M})(f) : S \sqcup_{\mathrm{id}} S \to S' \sqcup_{\mathrm{id}} S'$ is the obvious map. It is now a trivial verifcation that the diagram commutes. □

We are now ready to prove the Seifert-van Kampen theorem.

Proof. (Theorem 4.7) By transitivity of equivalences of categories we conclude by theorem 3.39 and lemma's 4.11, 4.12, and 4.2.2 that π-**Set** and π_P-**Set** are equivalent. Then, by corollary 4.6, $\pi_P \cong \pi = \pi_1(X, x_0)$. The fibered coproduct π_P is isomorphic to the group $(\pi_1(X_1, x_0) * \pi_1(X_2, x_0))/N$ by example 2.13, proving the theorem. □

- ## Example

To illustrate the basic use of the Seifert-van Kampen theorem, we will give an example.

4.15. Example. Let $X_1 = \{z \in \mathbb{C} : 1-\epsilon < |z-1| < 1+\epsilon\}$ and $X_2 = \{z : 1-\epsilon < |z+1| < 1+\epsilon\}$ for ϵ small. We can apply the Seifert-van Kampen theorem to X and find that $\pi_1(X) \cong X_1 * X_2$ since $X_1 \cap X_2$ is simply-connected. Since X deformation retracts onto the wedge of two circles, we conclude that $S^1 \vee S^1$ has the same fundamental group.

The universal cover of the space $S^1 \vee S^1$ from example 4.15 can be easily given by imitating the construction in theorem 3.29. Let a be a generator of the fundamental group of one circle, and b a generator of the other. Now embed the universal cover U of $S^1 \vee S^1$ in $\mathbb{R}^2$ and identify the constant path, at the point $0 \in \mathbb{C}$, with $(0,0)$. Any path in $S^1 \vee S^1$ starting at $0 \in \mathbb{C}$ can be described by a group element $g_n g_{n-1} \ldots g_0$ consisting of the generators a, b, and their inverses, plus the shortest segment (by the linear homotopy) γ of length x (with $0 < x < 2\pi$) that either goes in the direction of a, a^{-1}, b, or b^{-1}, not revolving a complete circle. To find the point in $\mathbb{R}^2$ we associate this path with, we start by considering g_0. If $g_i = a$ then go $\left(\frac{2}{3}\right)^i$ units to the right (or left for a^{-1}) starting from $(0,0)$. If g_i is b then go $\left(\frac{2}{3}\right)^i$ up (or down for b^{-1}). Continue this process up until g_n. Suppose the short line segment γ of length x is going in the direction of the generator a, then go $\frac{x}{2\pi}\left(\frac{2}{3}\right)^{n+1}$ units to the right (left for the inverse). Similarly for the other generator (up and down instead of right and left). To get an idea of this construction, consult figure 4.1.

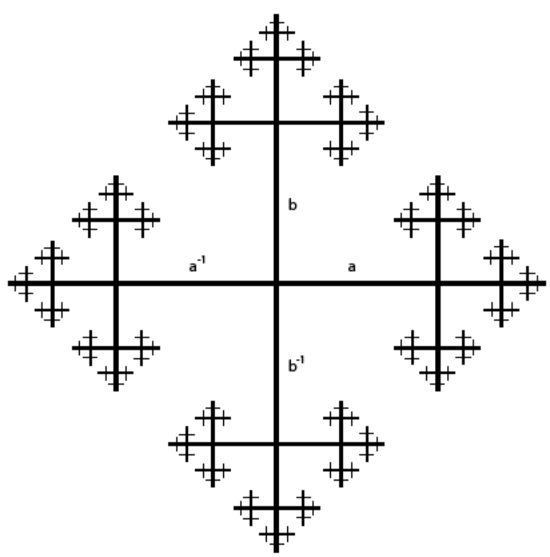

Figure 4.1: The universal cover for $S^1 \vee S^1$.

We can also completely describe the automorphisms of the universal cover U. By lemma 4.5, this group is isomorphic to $\mathbb{Z} * \mathbb{Z}$. The automorphisms act on U by right translation as described in chapter 3. In graphic terms, applying the automorphism induced by multiplying by a, comes down to shifting all line segments to the right. Similarly for multiplying with b, b^{-1}, or a^{-1}.

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