KAM and Melnikov theory describing island chains in plasmas

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Abstract

In this thesis toroidal plasmas are described by a Hamiltonian system parameterising the magnetic field lines. Application of KAM and Melnikov theory explains the occurance of chaotic island chains bounded by invariant manifolds. In plasma simulations and experiments island chains have shown to be rather stable. This thesis provides a theoretical but nonrigorous argument for this stability and proposes the description of magnetohydrodynamics as a Hamiltonian field theory as a method to rigorously study the stability. Hypotheses are given for a generalisation to nontoroidal structures.

Introduction

Since the 1950's toroidal plasmas have been of great importance due to their application in nuclear fusion reactors, for which it is needed create a stable plasma under extreme conditions. An ideal way to achieve this goal is to use a plasma that is stable due to its own structure. One approach to obtain self-stability is to use a plasma with a high magnetic helicity, which corresponds to magnetic field line linking. This creates special interest in a plasma with the magnetic field derived from the Hopf map of which all field lines are circles linked with all other field lines exactly once. In addition to that the field lines fill tori, such that the whole space is foliated by tori.

This bachelor project has been performed in the Dirk Bouwmeester group at Leiden University, which does experimental and theoretical research on self-organising knotted magnetic structures, including structures similar to the one with the Hopf field as magnetic field. Results from their simulations of nonideal plasmas ([1]) show toroidal structures of which the magnetic field is similar to structures described by KAM and Melnikov theory for small perturbations of completely integrable Hamiltonian systems.

A completely integrable Hamiltonian system has toroidal invariant manifolds in phase space, each orbit lies on such a torus. This is similar to the tori filled with field lines in the magnetic field derived from the Hopf map. Perturbations to the Hamiltonian system lead to chaotic island chains described by Melnikov theory. In plasma physics magnetic island chains with a similar structure are observed. In fusion reactors magnetic islands can blow up and explode onto the wall or distort the plasma stability in other ways, which makes their behaviour very important. In Reference [1] the dynamics of magnetic island chains is different: they shrink and move to the center of the plasma where they disappear. Despite the chaotic island chains KAM theory proves the preservation of invariant manifolds in phase space in small perturbations of completely integrable Hamiltonian systems. Toroidal plasmas also contain tori completely filled with field lines that form barriers between chaotic island chains.

Aim of this project

Despite the similarity described above plasma physics and Hamiltonian systems are two completely different concepts, the correspondence between the two is nontrivial. The aim of this thesis is to make this correspondence exact and to show how KAM theory and the Melnikov theory can be used to explain observations in plasma physics, namely the preservation of impenetrable manifolds and the existence of chaotic island chains. Besides that an argument will be given for the relatively stable behaviour of these plasma structures. Structures with similar properties but a nontoroidal shape have been observed ([2]), hypotheses will be given about their behaviour.

Two correspondences between plasma physics and Hamiltonian systems

This thesis is based on two correspondences between plasma physics and Hamiltonian systems. The first one is based on the idea that the magnetic field of a plasma represents the structure of the plasma. The magnetic field can be related to a Hamiltonian system through its divergencelessness, the correspondence between divergenceless vector fields and Hamiltonian systems has been shown in general in Reference [3]. For the second correspondence a set of partial differential equations that describes the plasma is written as an infinite-dimensional Hamiltonian dynamical system to which Hamiltonian field theory can be applied. This shows the Hamiltonian nature of that particular model for plasma physics, ideal magnetohydrodynamics. The infinite-dimensional dynamical system is mathematically far more complex than the finite-dimensional Hamiltonian system of the first approach, therefore the first will be used throughout this thesis while the second is proposed as a possible direction for further research.

Overview of this thesis

This thesis roughly consists of two parts: Chapters 1 - 6 describe theories and models for plasma physics and Hamiltonian systems, which are used to explain plasma physcis observations in Chapters 7 - 8. An overview per chapter is given below.

Chapter 0 gives preliminary remarks about smoothness and the definition of a manifold.

Paragraphs 1.1,1.2 introduce magnetohydrodynamics (MHD), a model for plasma physics. Paragraphs 1.3,1.4 describe an ideal stationary plasma situation with a magnetic field derived from the Hopf map and generalisations of that field. It will be used as the primary examples of ideal toroidal plasmas.

Chapter 2 introduces Hamiltonian systems and related concepts and methods, including complete integrability, invariant manifolds, action-angle variables, the reduction of Hamiltonian systems with an invariant and Poincaré maps.

Chapter 3 makes the first connection between dynamical systems and vector fields. It describes a vector field as a dynamical system using functions that parameterise field lines of the vector field, which are orbits of the dynamical system. This correspondence can be used to study the behaviour of zeroes of the vector field and field lines converging to or diverging from a zero.

Chapter 4 uses a similar method to describe divergenceless vector fields. The divergencelessness is used to give the dynamical system a Hamiltonian structure. Intuitively the volume preservation of flux tubes moving along vector field lines corresponds to the volume preservation in Hamiltonian phase space. This chapter discusses one very explicit method to describe this correspondence, which makes it useful for applications. Other correspondences are mentioned in the introduction to the chapter. In Paragraph 4.4 this method is applied to the Sagdeev fields studied in Paragraph 1.4, giving a Hamiltonian system with a toroidal motion in phase space, which can be generalised for other toroidal fields. Paragraph 4.5 shows how symmetry can be used to describe vector fields with a nontoroidal structure in terms of a toroidal Hamiltonian system.

In Chapter 5 the Hamiltonian system derived in Paragraph 4.4 is generalised to Hamiltonian systems containing island chains. These systems contain both idealised and realistic models for island chains and they are used to describe the basic island chain properties. Paragraph 5.3 describes singularities that come into play if the method of Chapter 4 is used to relate these generalised Hamiltonian systems to divergenceless vector fields, a correction for singularities is given as well.

Chapter 6 covers Hamiltonian system theories that describe the structure of island chains present in the models of Chapter 5. It gives two variants of a KAM-theorem describing the preservation of invariant tori for "small enough" perturbations of completely integrable Hamiltonian systems and describes the emergence of chaotic island chains in systems with such perturbations. The chaotic structure is stated with results from Melnikov theory. For larger perturbations the chaotic structure is preserved and an estimative but nonrigorous method shows that often many invariant tori are preserved.

Paragraphs 7.1,7.2 bring the theory of Chapters 4 - 6 together to explain the rather stable structure of chaotic island chains bounded by impenetrable manifolds observed in time dependent plasmas. Paragraph 7.3 describes the time dependent ideal MHD equations of Paragraph 1.1 as an infinite-dimensional dynamical system with the form of a Hamiltonian field theory, which can be used to study the stability of the predescribed time-dependent structures.

The outlook of Chapter 8 describes the current view on some recent and partly unpublished observations in simulations. This view can be used as hypotheses for further research. It includes the relation between the shape of a nontoroidal slowly decaying structure and the symmetry of the initial condition of the simulation. Besides that the (im)possibility to describe those structures by a Hamiltonian system with a toroidal structure is discussed as well as the motion of island chains due to different profiles of the rotational transform as observed in the simulations of Reference [1].

Contents

Chapter 0

Preliminary remarks

Smoothness

In this thesis all structures are smooth, unless stated otherwise. As an example the term “manifold” is used for a differentiable manifold, defined below. Familiarity with differentiable manifolds is not needed to understand this thesis, it is enough to keep in mind the intuitive idea of a set that is locally diffeomorphic to $\mathbb{R}^d$.

Definition 1. *A set $M \subset \mathbb{R}^n$ is a* smooth manifold in $\mathbb{R}^n$ *if for each $p \in M$ there exists a an open set $U \in \mathbb{R}^d$ and a diffeomorphism $\varphi : U \to M$ such that $0 \mapsto p$. The map φ is called a* parameterisation *and d is the* dimension *of the manifold.*

Manifolds defined as above are sometimes referred to as *smooth manifolds in $\mathbb{R}^n$ without boundary*.

Chapter 1

Magnetohydrodynamics

Plasma is a phase of matter where nearly all atoms are ionised, a plasma consists of ions and separate electrons. As all particles are charged, they interact with electromagnetic fields. A model for the charged particles interacting with electromagnetic fields can be used as the description of a plasma, but for practical purposes simplified models have been constructed, one of which is magnetohydrodynamics (MHD). It treats the plasma as a fluid interacting with electromagnetic fields. In principle it is a combination of Maxwell's equations and fluid dynamics, simplified for a specific regime of plasma parameters. This approximation is a huge simplification, but it turns out to describe most properties of interest very well, even outside the regimes for which it was set up in the first place. This is also the case for the plasma structures that are the topic of this thesis, which justifies the use of MHD.

The following paragraphs give the equations that define ideal and resistive magnetohydrodynamics: IMHD in Paragraph 1.1 and RMHD in Paragraph 1.2, for reference see [4]. The specific plasma conditions used by Kamchatnov and Sagdeev are discussed in Paragraph 1.3 while the fields they studied are described in Paragraph 1.4. Those plasma situations are important because they form the basis for explicit plasma models studied in Chapter 5.

1.1 Ideal magnetohydrodynamics (IMHD)

This subsection gives the equations that define ideal magnetohydrodynamics. IMHD is a form of magnetohydrodynamics in which there is no electromagnetic resistivity or viscosity and in which the conductance is perfect.

Definition of quantities:

τ	Time
ρ	Mass density
$\vec{v}$	Velocity field field
p	Scalar hydrodynamic pressure
$\vec{B}$	Magnetic field
γ	Poisson constant, ratio of specific heats

$$\frac{\partial \rho}{\partial \tau} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1.1}$$

$$\frac{\partial \vec{B}}{\partial \tau} - \nabla \times (\vec{v} \times \vec{B}) = 0 \tag{1.2}$$

$$\rho \left(\frac{\partial \vec{v}}{\partial \tau} + \vec{v} \cdot \nabla \vec{v} \right) + \nabla p - \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} = 0 \tag{1.3}$$

$$\left(\frac{\partial p}{\partial \tau} + \vec{v} \cdot \nabla p \right) + \gamma p \nabla \cdot \vec{v} = 0 \tag{1.4}$$

$$\nabla \cdot \vec{B} = 0 \tag{1.5}$$

Equation 1.1 describes the conservation of mass, and 1.2 is Gauss' law simplified using the absense of electromagnetic resistivity: $\vec{E} + \vec{v} \times \vec{B} = 0$. The force balance is given by 1.3 and 1.4 is the energy equation. Although 1.5 should be fulfilled at any time, it is enough to impose it as an initial condition, because Equation 1.2 implies that it remains fulfilled.

1.2 Resistive magnetohydrodynamics (RMHD)

This subsection describes the equations that define RMHD. The nonidealities which are introduced with respect to IMHD are resistivity and viscosity. Other nonidealities can be introduced as well, they are not included here. The aim of this subsection is to give an example of a nonideal model, not to give a complete overview of nonidealities.

Definition of quantities:

τ	Time
ρ	Mass density
$\vec{v}$	Velocity field
p	Hydrodynamic pressure (scalar field)
$\vec{B}$	Magnetic field
$\vec{E}$	Electric field
γ	Poisson constant, ratio of specific heats
$\vec{j}$	Electrical current density
$\vec{F}_{\text{visc}}$	Viscosity force
η	Electromagnetic resistivity

$$\frac{\partial\rho}{\partial\tau} + \nabla\cdot(\rho\vec{v}) = 0 \tag{1.6}$$
$$\frac{\partial\vec{B}}{\partial\tau} + \nabla\times\vec{E} = 0 \tag{1.7}$$
$$\rho\left(\frac{\partial\vec{v}}{\partial\tau} + \vec{v}\cdot\nabla\vec{v}\right) + \nabla p - \vec{j}\times\vec{B} = 0 \tag{1.8}$$
$$\left(\frac{\partial p}{\partial\tau} + \vec{v}\cdot\nabla p\right) + \gamma p\nabla\cdot\vec{v} = \vec{F}_{\text{visc}} \tag{1.9}$$
$$\nabla\cdot\vec{B} = 0 \tag{1.10}$$
$$\vec{E} + \vec{v}\times\vec{B} = \eta\vec{j} \tag{1.11}$$
$$\vec{j} = \frac{1}{\mu_0}\nabla\times\vec{B} \tag{1.12}$$

Viscosity has been added to Equation 1.9, while electromagnetic resistivity is described by Equation 1.11. IMHD corresponds to $\vec{F}_{\text{visc}} = 0, \eta = 0$.

1.3 Conditions used by Kamchatnov and Sagdeev

In Reference [5] Kamchatnov studied a stationary solution of the IMHD equations, which means that the time derivatives of $\vec{B}, \vec{v}, p, \rho$ vanish. Besides that the fluid was assumed to be incompressible ($\nabla\cdot\vec{v} = 0$) and the mass density to be homogenious ($\nabla\rho = 0$).

The incompressibility implies that the ratio of specific heats, $\gamma = \frac{C_p}{C_v}$, is infinitely large. In the set of MHD equations used by Kamchatnov Equation 1.4 is reduced to $\nabla\cdot\vec{v} = 0$

by neglecting the first term. The resulting equation is equal to the assumption of an incompressible fluid. There has been a discussion about the validity of this reduction, in this thesis it will just be assumed.

After the application of stationarity and incompressibility the IMHD equations are as given below, homogeneity has not been expressed in these equations.

$$\vec{v} \cdot \nabla \rho = 0 \tag{1.13}$$
$$\nabla \times (\vec{v} \times \vec{B}) = 0 \tag{1.14}$$
$$\rho \vec{v} \cdot \nabla \vec{v} + \nabla p - \frac{1}{\mu_0} (\nabla \times \vec{B}) \times \vec{B} = 0 \tag{1.15}$$
$$\nabla \cdot \vec{v} = 0 \tag{1.16}$$
$$\nabla \cdot \vec{B} = 0 \tag{1.17}$$

Homogeneity solves Equation 1.13. Equations 1.14,1.16,1.17 have been solved by taking $\vec{v} \parallel \vec{B}$ for a divergenceless vector field $\vec{B}$. This assumption is supported by the fact that charged particles will gyrate by the magnetic field, which drastically restricts the motion perpendicular to the magnetic field, particles (nearly) move parallel to magnetic field lines.

The only equation that has not been solved is 1.15, which is equivalent to:

$$\vec{v} \cdot \nabla \vec{v} - \frac{1}{\mu_0 \rho} (\vec{B} \cdot \nabla \vec{B}) + \frac{1}{\rho} \nabla \left(p + \frac{B^2}{2\mu_0} \right) = 0 \tag{1.18}$$

This equation has been solved by taking $\vec{v}$ to be the Alfvén speed and by taking a specific pressure profile

$$\vec{v} = \pm \frac{\vec{B}}{\sqrt{\mu_0 \rho}} \tag{1.19}$$
$$p = p_\infty - \frac{B^2}{2\mu_0} \tag{1.20}$$

for a constant p_∞. The constant p_∞ is equal to the pressure at infinity if it is assumed that $\vec{B}$ converges to 0 at infinity. Note that p is determined by $\vec{B}$ up to the constant p_∞.

With this construction Kamchatnov found a specific solution of equations 1.14 - 1.17 based on a divergenceless field $\vec{B}$ - the Hopf field. The plasma configuration is fully described by the magnetic field (apart from the density and p_∞, which are free parameters).

The assumptions of equations 1.19,1.20 can be made for every divergenceless vector field $\vec{B}$, which gives a stationary solution of the IMHD equations for each such vector field $\vec{B}$. This supports the idea that a divergenceless field describes a plasma situation, in the rest of this thesis it will be assumed that a divergenceless vector field represents the structure of the plasma situation.

The Hopf field used by Kamchatnov and its generalisations by Sagdeev described in the next paragraph will be used as an example on which more general models for plasma physics will be based. Besides the solution of IMHD studied by Kamchatnov and Sagdeev there are other similar stationary solutions of IMHD, for example the solutions with a compressive fluid studied in Reference [6]. They can be used as examples instead of those by Kamchatnov and Sagdeev.

This paragraph has shown that under certain assumptions the structure of a stationary solution of IMHD is completely described by the magnetic field - a divergenceless vector field defined on $\mathbb{R}^3$. Examples of such fields will be described in the next paragraph.

1.4 Magnetic field of the IMHD solutions studied by Kamchatnov and Sagdeev

Formulas for Kamchatnov and Sagdeev fields

Kamchatnov studied an IMHD solution of which the magnetic field is derived from the Hopf map ([5]). In terms of cartesian coordinates it is given below.

$$\vec{B} \;=\; \frac{4\sqrt{a}}{\pi(1+x^2+y^2+z^2)^3}\begin{pmatrix} 2y-2xz \\ -2x-2yz \\ (x^2+y^2-z^2-1) \end{pmatrix} \tag{1.21}$$

In this formula a is a parameter that determines the magnetic field strength at the origin. It follows from its construction that all field lines except the field line filling the z-axis are circles which are linked exactly once with each other and with the z-axis (Figure 1.1).

The linked field lines give nonzero magnetic helicity $H_m = \int \vec{A}\cdot\vec{B}d^3x$, which a conserved quantity in IMHD.

In Reference [7] Sagdeev described a generalisation of the field studied by Kamchatnov, with the following magnetic field:

$$\vec{B} = \frac{4\sqrt{a}}{\pi(1+x^2+y^2+z^2)^3}\begin{pmatrix} 2\omega_2 y-2\omega_1 xz \\ -2\omega_2 x-2\omega_1 yz \\ \omega_1(x^2+y^2-z^2-1) \end{pmatrix} \tag{1.22}$$

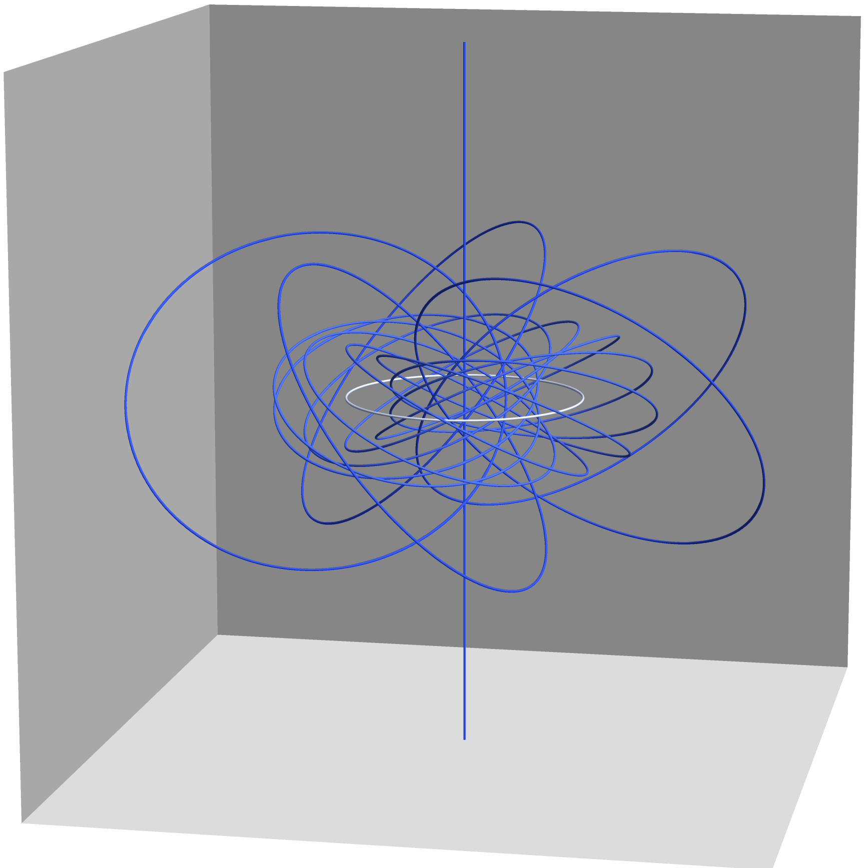


Figure 1.1: *Field lines of the divergenceless vector field derived from the Hopf map.*

The followig equation gives a corresponding vector potential (a vector field $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$):

$$\vec{A} = \frac{\sqrt{a}}{\pi(1+x^2+y^2+z^2)^2} \begin{pmatrix} 2\omega_1 y - 2\omega_2 xz \\ -2\omega_1 x - 2\omega_2 yz \\ \omega_2(x^2+y^2-z^2-1) \end{pmatrix} \tag{1.23}$$

Note that the Sagdeev field with $\omega_1 = 1 = \omega_2$ is equal to the Hopf field used by Kamchatnov.

This thesis will use a slightly modified form of these formulas, which simplifies later expressions. $a = \omega_2^{-2}$ will be used and the gradient [1]

$$\frac{\omega_1}{2\pi\omega_2} \begin{pmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix}$$

[1]This expression is the gradient $\frac{\omega_1 \nabla t^r}{2\pi\omega_2}$ where t^r is the toroidal coordinate c_1 as defined in appendix B.

will be added to the vector potential after filling in $a = \omega_2^{-2}$. These modifications can be used without loss of generality for $\omega_2 \neq 0$. If $\omega_2 = 0$ the unmodified version can be used.

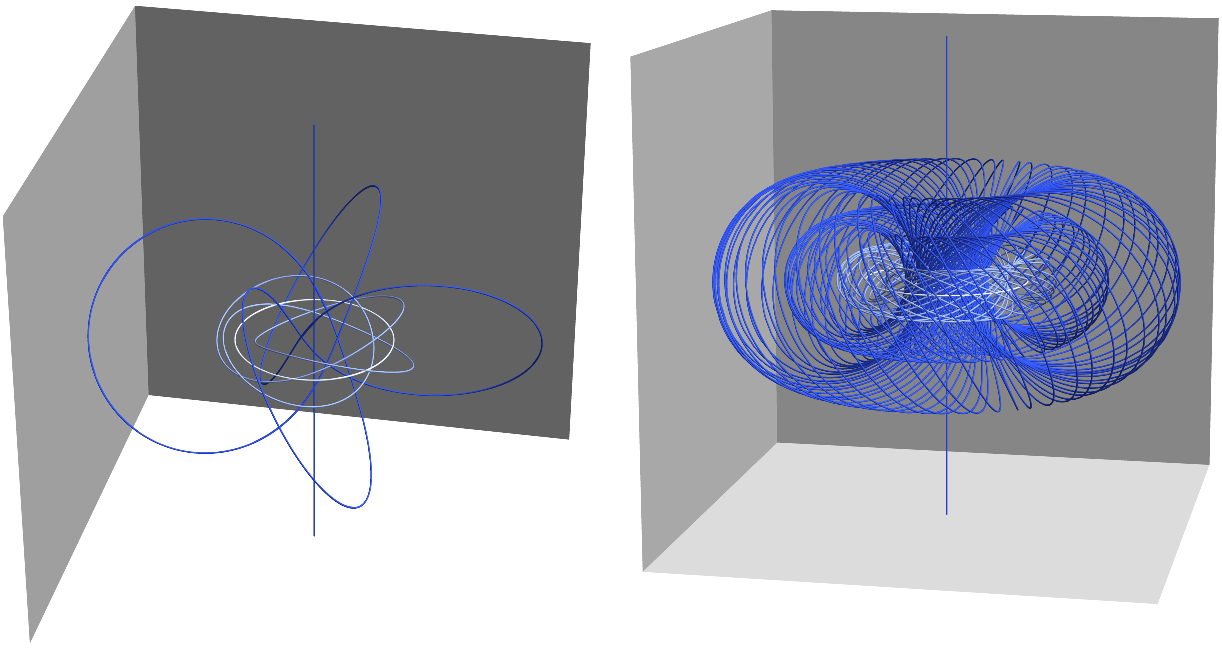
The modifications give the following expressions:

$$\vec{B} = \frac{4}{\pi(1+x^2+y^2+z^2)^3}\begin{pmatrix} 2y - 2\frac{\omega_1}{\omega_2}xz \\ -2x - 2\frac{\omega_1}{\omega_2}yz \\ \frac{\omega_1}{\omega_2}(x^2+y^2-z^2-1) \end{pmatrix} \quad (1.24)$$

$$\vec{A} = \frac{1}{\pi(1+x^2+y^2+z^2)^2}\begin{pmatrix} 2\frac{\omega_1}{\omega_2}y - 2xz \\ -2\frac{\omega_1}{\omega_2}x - 2yz \\ (x^2+y^2-z^2-1) \end{pmatrix} + \frac{\omega_1}{2\pi\omega_2}\begin{pmatrix} -\frac{y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \\ 0 \end{pmatrix} \quad (1.25)$$

Field line structures for the Sagdeev fields

For $\frac{\omega_1}{\omega_2} \in \mathbb{Q}$ the field lines of the Sagdeev fields are closed, they are torus knots, while for $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$ field lines densely fill a torus (Figure 1.2).



(a) *(b)*

Figure 1.2: *Field lines of the Sagdeev fields (a)* $\iota = \frac{4}{3}$ *(b)* $\iota \approx 1.618$

For the Sagdeev fields all field lines lie on tori, except for a field line on the unit circle and a field line on the z-axis. Therefore the field can best be described in a toroidal coordinate system, as discussed in Paragraph 4.4. Appendix B describes a suitable coordinate system (q_1, ρ, q_2) in which the vector potential of the Sagdeev fields is given by Equation 4.26, which is copied below. The notation $\nabla q_1, \nabla q_2$ is used for basis vector fields, it can be interpreted as a gradient with respect to another coordinate system.

$$\vec{A} = \frac{\operatorname{sech}^2\rho}{2\pi}\nabla q_2 - \frac{\omega_1 \operatorname{sech}^2\rho}{2\pi\omega_2}\nabla q_1$$

Chapter 2

Concepts and methods for Hamiltonian systems

This section gives some definitions and concepts for Hamiltonian dynamical systems. A completely integrable Hamiltonian system is the aim of the construction in Chapter 4 and the starting point for the theory of Chapter 6.

Many concepts discussed in this chapter are widely used, those concepts will not be discussed thoroughly.

2.1 Definitions of Hamiltonian systems, separability and integrability

Definition 2. *A* Hamiltonian system with n degrees of freedom *is a dynamical system of* $2n$ *functions depending on t:* $p_1^t(t), \ldots, p_n^t(t), q_1^t(t), \ldots, q_n^t(t)$ *satisfying* Hamilton's equations*: the following system of* $2n$ *ordinary differential equations.*

$$\frac{dp_i^t(t)}{dt} = -\frac{\partial H}{\partial q_i}(p^t(t), q^t(t), t)\ ;\ \frac{dq_i^t(t)}{dt} = \frac{\partial H}{\partial p_i}(p^t(t), q^t(t), t) \tag{2.1}$$

The differential equations depend on the function $H(p_1, \ldots, p_n, q_1, \ldots, q_n, t)$*, which is called the* Hamiltonian*. The variable t is called* Hamiltonian time variable*. The functions* $p_i^t(t), q_i^t(t)$: $1 \leq i \leq n$ *solving 2.1 together form an* orbit *of the Hamiltonian system. Orbits will be denoted as follows:*

$$\begin{aligned} p^t(t) &:= (p_1^t(t), \ldots, p_n^t(t)) \\ q^t(t) &:= (q_1^t(t), \ldots, q_n^t(t)) \end{aligned}$$

q_i are called position variables*, while p_i are* generalised momenta*. For a fixed i the variables p_i, q_i are a pair of* conjugate variables*.*

On orbits the Hamiltonian can be seen as a function of t:

$$H^t(t) := H(p^t(t), q^t(t), t) \tag{2.2}$$

Hamilton's equations imply that H^t only depends on t through its partial derivative with respect to time:

$$\begin{aligned}\frac{dH^t(t)}{dt} &= \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}(p^t(t), q^t(t), t)\frac{dp_i^t(t)}{dt} + \frac{\partial H}{\partial q_i}(p^t(t), q^t(t), t)\frac{dq_i^t(t)}{dt} + \frac{\partial H}{\partial t}(p^t(t), q^t(t), t) \\ &= \frac{\partial H}{\partial t}(p^t(t), q^t(t), t) \end{aligned} \tag{2.3}$$

Definition 3. *If the Hamiltonian does not directly depend on time, i.e. $\frac{\partial H}{\partial t} = 0$, the system is called* autonomous*.*

Every nonautonomous Hamiltonian system with $n-1$ degrees of freedom (having $n-1$ p_i's and q_i's) can be transformed into an autonomous n degree of freedom Hamiltonian system (having n p_i's and q_i's). An example of this procedure is explained on page 258 of Reference [8] and in essence it is the inverse transformation of the reduction described in Paragraph 2.4.

Such systems are said to have $n - \frac{1}{2}$ degrees of freedom, independent of the number of p_i, q_i's in the description. In this thesis a different convention will be used: the different notations will be denoted as the $n-1$ degree of freedom system and the n degree of freedom system.

Separability

Hamiltonian systems for which p_i, q_i and p_j, q_j do not interact for $i \neq j$ are called *separable*. The Hamiltonian can be written in the form

$$H(p, q, t) = F_1(p_1, q_1, t) + \ldots + F_n(p_n, q_n, t) \tag{2.4}$$

for n functions F_i. Separable Hamiltonian systems can be seen as n independent one degree of freedom Hamiltonian systems with Hamiltonians F_i. This means that separable systems have a relatively simple structure.

Invariants and integrability

Another important property of Hamiltonian systems is the existence of invariants:

Definition 4 ([9] page 333)**.** *A function $f(p, q)$ is an* invariant *if*

$$\frac{df(p^t(t), q^t(t))}{dt} = 0 \tag{2.5}$$

for all orbits of the Hamiltonian system. Invariants are assumed to have nonvanishing derivatives.

Invariants are important because they constrain the orbits of the Hamiltonian system to manifolds on which the invariants are constant. Therefore it "reduces the dimension of the system", just as the dynamics of separable dynamical systems is reduced to one-dimensional Hamiltonian systems.

The existence of invariants is often called *integrability*. A notion of integrability that is important for the KAM theorems in Paragraph 6.1 is defined below.

Definition 5 ([9] page 368)**.** *A Hamiltonian system is* completely integrable *in the sense of Liouville if there exist n invariants which are linearly independent almost everywhere and for which the following equation holds.*

$$0 = \{f_j, f_k\} := \sum_{i=1}^{n} \frac{\partial f_j}{\partial p_i}\frac{\partial f_k}{\partial q_i} - \frac{\partial f_k}{\partial p_i}\frac{\partial f_j}{\partial q_i} \tag{2.6}$$

Two invariants satisfying 2.6 are said to be in involution.

For autonomous Hamiltonian systems the Hamiltonian is an invariant. Therefore all autonomous systems with one degree of freedom are completely integrable.

2.2 Geometric representation of a Hamiltonian system

Just as all other dynamical systems Hamiltonian systems can be represented geometrically by vector fields, which is one of the main ideas of the mathematical field of dynamical systems. It allows the use of geometric arguments in the study of differential equations. This paragraph is a short overview of this correspondence. Besides that the important concept of invariant manifolds will be introduced.

In this paragraph the Hamiltonian system will be assumed to be autonomous.

A vector field on phase space

The domain of the vector field representing the Hamiltonian system is a $2n$-dimensional manifold $\mathcal{M}$ that contains all values (p, q) that orbits of the dynamical system can assume. It is called *phase space* and has coordinate fields $p_1, \ldots, p_n, q_1, \ldots, q_n$.

An autonomous Hamiltonian system can be seen as a family of maps $\{\varphi_t\}_{t\in\mathbb{R}}, \varphi_t : \mathcal{M} \to \mathcal{M}$ such that $\forall s, t \in \mathbb{R} : \varphi_t \circ \varphi_s = \varphi_{t+s}$. This family of maps is called the *flow*.

The orbits p^t, q^t of such a system can also be defined as the orbit of a point in phase space $(p^t(t_0), q^t(t_0))$ under the flow:

$$(p^t(t), q^t(t)) = \varphi_{t-t_0}(p^t(t_0), q^t(t_0)) \tag{2.7}$$

Hamilton's equations define a vector field on phase space, namely the t-derivative of the flow ([9] page 108):

$$\frac{d\varphi_t}{dt}(p,q) = \left(\frac{dp_1^t}{dt}(p,q), \ldots, \frac{dp_n^t}{dt}(p,q), \frac{q_1^t}{dt}(p,q), \ldots, \frac{dq_n^t}{dt}(p,q)\right) \tag{2.8}$$

Orbits of the dynamical system correspond to field lines of this vector field. The field lines can be parameterised by t, just as the orbits of the dynamical system. Expressions like $\frac{dp_1^t}{dt}$ correspond to partial derivatives of the Hamiltonian, which do not depend on t, which means that the vector field described above does not depend on t.

Invariant manifolds

Invariants can also be described geometrically, in terms of invariant manifolds and foliations.

Definition 6. *A submanifold of phase space is an* invariant manifold *if for all for all $t \in \mathbb{R}$ and all points (p_0, q_0) in the manifold the image $\varphi_t(p_0, q_0)$ under the flow lies in that manifold.*

The interpretation of invariant manifolds in terms of the vector field on phase space is that for all points in an invariant manifold the field line through that point completely lies in the manifold. This implies that field lines of the vector field cannot cross invariant manifolds. That is what makes the existence of invariant manifolds important for the description of plasma physics: invariant manifolds act as boundaries for the dynamics.

For an invariant $f(p, q)$ its level sets $\{f = c\}$ are $2n - 1$-dimensional invariant manifolds. The collection $\{f = c\}_{c\in\mathbb{R}}$ foliates phase space:

Definition 7. *A family of pairwise disjoint submanifolds of phase space* foliates *phase space if the union of these submanifolds equals phase space.*

Completely integrable systems have n independent invariants, corresponding to n families of $2n-1$-dimensional invariant manifolds such that each family foliates phase space. Intersections of invariant manifolds are invariant manifolds themselves. This leads to the following proposition:

Proposition 1 ([9] page 368 Theorem 9.19)**.** *The phase space of a completely integrable Hamiltonian system is foliated by n-dimensional invariant manifolds.*
If those manifolds are compact and connected, they are diffeomorphic to n-tori, $\underbrace{S^1 \times \ldots \times S^1}_{n \text{ times}}$.

This structure is similar to the structure of the Sagdeev fields discussed in paragraph 1.4, where field lines lie on two-dimensional tori. The intuitive similarity between the two is made exact in chapter 4.

2.3 Action-angle variables

Completely integrable autonomous Hamiltonian systems with compact and connected invariant manifolds have a very rich structure. In order to stress this structure and to simplify its use such systems are often transformed to "action-angle variables" defined below. Transformations should preserve the Hamiltonian structure, which is why *canonical* coordinate transformations are used. In order to ensure that the coordinate transformation is canonical it can be constructed using a *generating function*, a definition and an explanation can be found in on page 126 of Reference [10]. If the generating function solves the stationary Hamilton-Jacobi equation, a partial differential equation, then the resulting coordinate system is given in action-angle variables ([10] page 127). That shows the difficulty of constructing action-angle variables.

Definition 8. *A pair of conjugate variables p_i, q_i are* action-angle variables *if the Hamiltonian does not depend directly on q_i:*

$$\frac{\partial H}{\partial q_i} = 0 \Rightarrow \frac{dp_i^t}{dt} = 0\,; \; \frac{\partial H}{\partial p_i} = \frac{dq_i^t}{dt}$$

In this case p_i are called action-variables *and q_i are called* angle-variables*.*

This definition implies that action-variables p_i are independent invariants which are in involution. If all n pairs of conjugate variables are given in action-angle form and if the manifolds of constant p are compact and connected, then manifolds of constant p are invariant tori by Proposition 1 and the variables q are coordinates on the tori of constant p.

The dynamics on invariant tori is completely determined by a frequency which describes the motion in different angular directions.

Definition 9. *The* frequency vector Ω *is defined as*

$$\Omega(p) = (\Omega_1, \Omega_2, \ldots, \Omega_n)(p, q) := \left(\frac{\partial H_0}{\partial p_1}, \frac{\partial H_0}{\partial p_2}, \ldots, \frac{\partial H_0}{\partial p_n} \right)(p, q). \tag{2.9}$$

For systems given in action-angle variables the frequency only depends on p, it is constant on invariant tori of constant p. In the definition it depends on p, q, as the notation Ω will also be used for nearly integrable systems.

Definition 10. *For two degree of freedom systems with $\Omega_1 \neq 0$ the ratio of the frequences is defined as the* rotational transform*:*

$$\iota := \frac{\Omega_2}{\Omega_1} \tag{2.10}$$

2.4 Reduction of Hamiltonian systems

If p_n, q_n is a pair of action-angle variables for an n degree of freedom autonomous Hamiltonian system, then p_n is an invariant and the flow is restricted to manifolds of constant p_n. This allows a local reduction of the n degree of freedom system to a family of $n-1$ degree of freedom systems. In these reduced systems p_n acts as a Hamiltonian, while q_n acts as a Hamiltonian time variable. This paragraph shows this reduction, it follows the discussion on page 214 of Reference [11].

The original n degree of freedom system will be denoted by H, p, q, t, the reduced $n-1$ degree of freedom system by H^r, p^r, q^r, t^r.

On each manifold $\{H = c'\}$ a reduced system will be defined. An expression for $H^r := p_n$ in terms of $(p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}, q_n)$ and c' can be found by inverting

$$H(p_1, \ldots, p_n, q_1, \ldots, q_n) = c' \tag{2.11}$$

to the following expression:

$$p_n = H^r(p_1, \ldots, p_{n-1}, q_1, \ldots, q_{n-1}, q_n; c') \tag{2.12}$$

Under the assumption that $\frac{\partial H}{\partial p_n} \neq 0$ the inverse used in Equation 2.12 exists locally, in general a global inverse does not have to exist.

By the assumption that $\frac{\partial H}{\partial p_n} \neq 0$ it also follows that $q_n^t(t)$ is strictly monotonic. This implies that $q_n^t(t)$ can be inverted to $t(q_n)$. The following definitions give a Hamiltonian system.

$$\begin{aligned}
t^r &:= q_n && (2.13)\\
q^r &:= (q_1, \ldots, q_{n-1}) && (2.14)\\
p^r &:= (p_1, \ldots, p_{n-1}) && (2.15)\\
q^{r,t}(t^r) &:= (q_1^t(t(t^r)), \ldots, q_{n-1}^t(t(t^r))) && (2.16)\\
p^{r,t}(t^r) &:= (p_1^t(t(t^r)), \ldots, p_{n-1}^t(t(t^r))) && (2.17)
\end{aligned}$$

$$H^r(p_1^r, \ldots, p_{n-1}^r, q_1^r, \ldots, q_{n-1}^r, t^r) \tag{2.18}$$

Now it will be proven that H^r, p^r, q^r, t^r as defined above satisfy Hamilton's equations. Implicit differentiation of Equation 2.11 after filling in 2.12 for p_n leads to the following:

$$\frac{\partial H}{\partial q} + \frac{\partial H}{\partial p_n}\frac{\partial H^r}{\partial q} = 0\ ;\ \frac{\partial H}{\partial p} + \frac{\partial H}{\partial p_n}\frac{\partial H^r}{\partial p} = 0 \tag{2.19}$$

This gives the following equations:

$$\begin{aligned}
\frac{\partial H^r}{\partial p_i^r}(p^r, q^r, t^r; c') &= \frac{\frac{\partial H}{\partial p_i}}{\frac{\partial H}{\partial p_n}} = \frac{\frac{dq_i}{dt}}{\frac{dq_n}{dt}} = \frac{dq_i^r}{dt^r} && (2.20)\\
-\frac{\partial H^r}{\partial q_i^r}(p^r, q^r, t^r; c') &= -\frac{\frac{\partial H}{\partial q_i}}{\frac{\partial H}{\partial p_n}} = \frac{\frac{dp_i}{dt}}{\frac{dq_n}{dt}} = \frac{dp_i^r}{dt^r} && (2.21)
\end{aligned}$$

These are exactly Hamiltons equations for the reduced system. [1] The phase space of the reduced system is a $(2n-2)$-dimensional manifold $\mathcal{M}^r$.

Note that the periodicity of H with respect to q_n gives the same periodicity of H^r with respect to t^r.

Remark 2. *The rotational transform ι has been defined for the two degree of freedom system (Definition 10), but it is also visible in the reduced Hamiltonian system. Therefore the term will also be used for the frequency $\frac{\partial H^r}{\partial p^r}$ of the reduced Hamiltonian system.*

2.5 Commutativity of subflows

For a completely integrable Hamiltonian system given in action-angle variables, the system can be reduced as described in the previous paragraph. For systems with more

[1] Note that there is a minus sign mistake in the derivation, which is also there in Reference [11]. It can be corrected by redefining H^r or t^r.

than one degree of freedom the system can be reduced with respect to any pair p_i, q_i of action-angle variables giving corresponding reduced Hamiltonian time variables t_i^r. Reduction with respect to each pair of variables induces a subflow $\varphi_{t_i^r} =: \varphi_i$ on the manifold of constant p_i.

It follows from the fact that the invariants p_i, p_j are in involution that on the intersection of manifolds of constant p_i, p_j the subflows φ_i, φ_j commute ([9] page 369). This can be seen as the reason why for completely integrable Hamiltonian systems compact and connected invariant manifolds of constant p are diffeomorphic to tori: a torus is the only compact and connected n-manifold that allows n independent commuting flows ([9] pages 368 - 369).

2.6 Poincaré map

Each periodic Hamiltonian system has a corresponding discrete dynamical system based on the Poincaré map. This paragraph introduces the concept of a Poincaré map using the assumption that the Hamiltonian system is periodic in t with period T.

It needs the notion of a cross section ([11] page 214 Equation 4.8.11): for $t_0 \in \mathbb{R}$ a *cross section* is defined as

$$\Sigma^{t_0} := \{(p, q, t) \in \mathcal{M}^r \times \mathbb{R} : t = t_0\}. \tag{2.22}$$

For a set $U \subset \Sigma^{t_0}$ the orbits of the Hamiltonian system induce the following map.

$$P^{t_0} : U \to \Sigma^{t_0} \tag{2.23}$$

This map is called the *Poincaré map* ([11] page 214). In this thesis Poincaré maps will be used for reduced Hamiltonian systems defined in Paragraph 2.4. The family of reduced systems corresponding to one original Hamiltonian system is parameterised by c', which can be displayed in notation as follows: $\Sigma_{c'}^{t_0}, P_{c'}^{t_0}$.

The Poincaré map shows the time progress of the system during one period. Due to the periodicity of the Hamiltonian system the Poincaré map can be composed with itself to study the time evolution of the system. This defines a discrete dynamical system with the cross section as phase space.

It follows from Liouville's theorem that the discrete dynamical system defined by the Poincaré map of a Hamiltonian system is volume preserving, which is explained in Reference [11] on pages 216 - 217. Liouville's theorem itself follows from Hamilton's equations.

The discrete dynamical system will be used in Chapter 6 because the system is much simpler than the continuous dynamical system, but it can still be used to study the most important dynamics. A lot of information is lost in the step from the continuous to the discrete system, but the results derived for the discrete system can be interpreted in terms of the Hamiltonian system using the knowledge about how the discrete system follows from the continuous one.

Chapter 3

Dynamical system description of a vector field

In this chapter the behaviour of zeroes of the magnetic field and (semi-)finite field lines is shortly discussed in terms of a dynamical system. It introduces the way in which a vector field can be described by a dynamical system, also used in the next chapter.

A vector field $\vec{B}$ defined on $\mathbb{R}^3$ with coordinates x, y, z can be identified with a dynamical system by using functions x^t, y^t, z^t of t which solve the following autonomous system of ordinary differential equations:

$$\frac{dx^t}{dt} = [B \cdot \nabla x](x^t, y^t, z^t) \tag{3.1}$$

$$\frac{dy^t}{dt} = [B \cdot \nabla y](x^t, y^t, z^t) \tag{3.2}$$

$$\frac{dz^t}{dt} = [B \cdot \nabla z](x^t, y^t, z^t) \tag{3.3}$$

If x^t, y^t, z^t satisfy these equations they parameterise field lines: the derivative of $(x^t(t), y^t(t), z^t(t))$ is a rescaled form of $\vec{B}(x, y, z)$. The rescaling is due to the fact that the coordinate system is not assumed to be normalised: the basis vector fields denoted by $\nabla x, \nabla y, \nabla z$ can have lengths different from 1, which allows $\frac{d}{dt}(x^t(t), y^t(t), z^t(t))$ to be different from $\vec{B}(x^t(t), y^t(t), z^t(t))$.

In terms of this dynamical system the zeroes of $\vec{B}$ correspond to fixed points. (Semi-) finite field lines correspond to orbits of the dynamical system converging to a zero for $t \to +\infty$ or $-\infty$. In dynamical systems theory all points that converge to a fixed point for $t \to \infty$ are the *stable manifold* of the fixed point. All points that converge to the fixed point for $t \to -\infty$ are the *unstable manifold*. It has been shown that for smooth dynamical systems these manifolds are differential manifolds ([11] pages 14 and 18).

All zeroes and (semi-)finite field lines of a divergenceless vector field $\vec{B}$ can be seen as fixed points and (un)stable manifolds of the dynamical system defined by Equations 3.1 - 3.3. This gives them a smooth structure which is enough to study the relevant features of their dynamics.

Chapter 4

A correspondence between divergenceless vector fields and Hamiltonian systems

The aim of this section is to show a method by which a divergenceless vector field can be described by a Hamiltonian system. The existence of a correspondence between a divergenceless vector field and a Hamiltonian system is a general property of such fields. This fact has been proven for general divergenceless vector fields in general curvilinear coordinate systems, for example in Reference [12]. That approach uses a coordinate system defined by the vector field $\vec{B}$. It is a useful approach to see how the divergencelessness leads to the Hamiltonian structure. The disadvantage is that it is not immediately clear whether or how replacing the Hamiltonian fuction by another function of p, q again describes a divergenceless vector field. That is required to show how results of Chapter 6 about the influence of perturbations transfer to physical vector fields. Another correspondence is described in Reference [3], where the correspondence between the vector field and the Hamiltonian system does not depend on the explicit form of the vector field, but on the the symmetry of the system, which means that it does not have the disadvantage of the method in Reference [12].

This thesis will study a more explicit construction of a Hamiltonian system than the two referred to above. Once an appropriate coordinate system has been constructed a parameterisation of field lines takes the form of Hamilton's equations. A property of this construction is that the relation between the divergenceless vector field and the Hamiltonian system is very direct, which can be used to give conditions for the coordinate system that lead to a completely integrable Hamiltonian system with invariant tori in phase space. Small perturbations of such Hamiltonian systems are the subject of Chapter 6.

Under mild conditions small perturbations of the Hamiltonian function to another function of p, q give a Hamiltonian system that describes a divergenceless field as well. Therefore this method is useful to study the influence of small perturbations to an ideal system describing a vector field, it is widely used in plasma physics, for example in Reference [13].

Throughout this chapter the divergenceless vector field will be denoted by $\vec{B}$. In the context of plasma physics $\vec{B}$ is often taken to be the magnetic field, but it can also be the velocity field of an incompressible fluid.

Outline of this chapter

Paragraph 4.1 discusses the prerequisites for the coordinate system, 4.2 describes how $\vec{B}$ field lines can be parameterised and 4.3 shows that there is a Hamiltonian function such that the functions parameterising field lines satisfy Hamilton's equations. This construction is applied to the Sagdeev fields in Paragraph 4.4, that procedure applies to other toroidal fields as well. Paragraph 4.5 explains how the construction can be applied to nontoroidal fields with rotational symmetries, of which field lines lie on surfaces of genus unequal to 1.

4.1 Coordinate system prerequisites

This paragraph describes properties of the coordinate system needed for the construction of a Hamiltonian system in Paragraphs 4.2 - 4.3. Besides that it is described what properties of the coordinate system lead to a completely integrable Hamiltonian system with invariant tori in phase space given in action-angle variables.

Coordinate fields

In this thesis coordinate function are defined as follows.

Definition 11. *A function $c : U_c \to R_c$ with $U_c \subset \mathbb{R}^3$ be an open subset and R_c a one-dimensional manifold is a* coordinate field *if its derivative is nonzero on all points of U_c. The* range *of c is its image, with notation:* range $c :=$ im $c \subset R_c$.

This thesis often uses *angular* coordinate functions, with the following range:

$$\text{range}\, c = R_c = S^1 \cong \mathbb{R}/2\pi\mathbb{Z} \tag{4.1}$$

Let c_1, c_2, c_3 be three coordinate functions defining a *coordinate system* for $\mathbb{R}^3$, which means that related basis vector fields $\nabla c_1, \nabla c_2, \nabla c_3$ are independent almost everywhere on $U := U_{c_1} \cap U_{c_2} \cap U_{c_3}$. The previous definitions imply that the following map ψ is a smooth diffeomorphism when restricted to neighbourhoods of points where $\nabla c_1, \nabla c_2, \nabla c_3$ are linearly independent.

$$\psi : U \to \text{range } c_1 \times \text{range } c_2 \times \text{range } c_3 : x \mapsto (c_1, c_2, c_3) \tag{4.2}$$

Remark 3. *In this thesis an exception is made with respect to usual definitions of coordinate systems: the map ψ does not need to be injective.*

Singularities related to angular coordinate fields

The use of angular coordinates introduces singularities. A property of such coordinates is that they are not defined at the center around which the angle is defined. These singularities are allowed on curves and points of U, the coordinate transformation from new coordinates to known coordinates (x, y, z) may be noninjective on curves if the curve is a field line of $\vec{B}$ and if its direction can be described by a basis vector field ∇c.

Note that the predescribed singularities are features of the basis vector fields, they are not part of the dynamical system and therefore these singularities have no influence on the application of mathematical theories to the dynamical system. However, it does change the implications of those theories for divergenceless vector fields on U.

Domain of interest

Not all field lines can be parameterised by the procedure of Paragraph 4.2. Therefore the construction of this chapter is restricted to a subset $D \subset U \subset \mathbb{R}^3$, the *domain of interest*. D must have the following properties:

- D does not contain any zeroes of $\vec{B}$, any (semi-)finite field lines or any singularities of the coordinate fields
- For all points in D the whole $\vec{B}$ field line going through that point lies in D
- ∇c_1 is defined for all points in D and is nonzero

These are the sufficient requirements for D. There is a freedom in D that can be used to exclude difficulties from calculations.

$\vec{B}$-dependent prerequisites for the coordinate system

As Paragraph 4.2 aims to parameterise field lines of $\vec{B}$, one coordinate field, c_1, will be chosen such that at all points in D the field $\vec{B}$ has a nonzero component in the c_1 direction:

$$\vec{B} \cdot \nabla c_1 \neq 0 \tag{4.3}$$

Note that this is well defined as ∇c_1 exists and is nonzero on D. Moreover, Appendix A requires

$$\frac{\vec{B} \cdot \nabla c_1}{\nabla c_2 \times \nabla c_3 \cdot \nabla c_1} \tag{4.4}$$

to be a function of c_2 not depending on c_1, c_3. The numerator is nonzero by assumption 4.3 and the denominator is proportional to the determinant of the inverse Jacobian, which is nonzero. Therefore expression 4.4 is a nonzero function of c_2.

This paragraph has shown that given a vector field $\vec{B}$ on $\mathbb{R}^3$ the field lines in a domain of interest D will be studied, where D is a union of field lines such that it does not contain any zeroes of $\vec{B}$ or any (semi)finite field lines. On an open set $U : \mathbb{R}^3 \supset U \supset D$ a coordinate system can be constructed. The sufficient requirements for this coordinate system needed in the rest of this chapter are summarised by the following definition.

Definition 12. *A* pre-Hamiltonian coordinate system *is a coordinate system of coordinate fields c_1, c_2, c_3 for which expression 4.4 is a nonzero function of c_2. The coordinate function c_1 is called the* Hamiltonian time coordinate field. *Any singularities due to angular coordinate fields should be compatible with the field $\vec{B}$: on D they have to be described in terms of $\nabla c_1, \nabla c_2, \nabla c_3$.*

4.2 Field line parameterisation

Let (c_1, c_2, c_3) be a pre-Hamiltonian coordinate system on $U; D \subset U \subset \mathbb{R}^3$. The following diffeomorphism φ can be defined for any field line in D and any point $\hat{x} := (\hat{c}_1, \hat{c}_2, \hat{c}_3) \in D$ on that field line:

$$\varphi : \mathbb{R} \to D : t \mapsto (c_1^t(t), c_2^t(t), c_3^t(t)) \; ; \; \hat{c}_1 \mapsto \hat{x} \tag{4.5}$$

The point $\hat{x}$ has only been used to fix a point for which $t = 0$. The fact that it is locally diffeomorphic to $\mathbb{R}$ follows from 4.4, while the fact that the field line is not (semi-)finite implies that it is globally diffeomorphic. The assumption $\vec{B} \cdot \nabla c_1 \neq 0$ implies that $\frac{dc_1^t}{dt} \neq 0$, which means that it is a strictly monotonic function. Together with the fact that φ is a diffeomorphism this means that the map φ can be rescaled such that

$$c_1^t(t) = t + \hat{c}_1 \tag{4.6}$$

All field lines in D can be parameterised with base points $\hat{x} = (0, \hat{c}_2, \hat{c}_3)$, which gives $t \equiv q_1$.

Using the assumption that $\frac{dc_1^t}{dt} = 1$ it follows that $\frac{dc_2^t}{dt}$ is the ratio of the $\vec{B}$ component in the ∇c_2 direction and the component in the ∇c_1 direction. That ratio is proportional to the ratio of $\vec{B} \cdot \nabla c_2$ and $\vec{B} \cdot \nabla c_1$. Proportionality follows from the fact that $\nabla c_2, \nabla c_1$ may have lengths different from 1. The same is true for $\frac{dc_3^t}{dt}$.

This leads to the following expressions:

$$\frac{dc_2^t}{dt} \propto \frac{\vec{B} \cdot \nabla c_2}{\vec{B} \cdot \nabla c_1}$$
$$\frac{dc_3^t}{dt} \propto \frac{\vec{B} \cdot \nabla c_3}{\vec{B} \cdot \nabla c_1}$$

Rescaling c_2^t, c_3^t gives equalities similar to those in Chapter 3.

$$\frac{dc_2^t}{dt} = \frac{\vec{B} \cdot \nabla c_2}{\vec{B} \cdot \nabla c_1} \tag{4.7}$$
$$\frac{dc_3^t}{dt} = \frac{\vec{B} \cdot \nabla c_3}{\vec{B} \cdot \nabla c_1} \tag{4.8}$$

This paragraph has shown that under the assumptions made in Paragraph 4.1 there exists a parameter t which parameterises field lines in D such that they are diffeomorphic to $\mathbb{R}$. For each field line this leads to three functions $c_1^t(t), c_2^t(t), c_3^t(t)$ which parameterise that field line. Rescaling of these functions gives $c_1^t(t) = t$.

4.3 Hamilton's equations for divergenceless vector fields

This paragraph shows that there exists a Hamiltonian function such that the field line parameterisations derived in paragraph 4.2 satisfy Hamilton's equations. It is based on the results derived in Appendix A: with $\vec{B}$ and a pre-Hamiltonian coordinate system (q_1, ρ, q_2) a new pre-Hamiltonian coordinate system (q_1, p_2, q_2) has been constructed such that in these coordinates a vector potential for $\vec{B}$ can be written in the following form:

$$\vec{A}(q_1, \rho, q_2) = A_{q_1}(q_1, p_2, q_2)\nabla q_1 + p_2 \nabla q_2 \tag{4.9}$$

This equation follows from A.15,A.16, leaving the tildes. For every field written in this form the function h can be defined:

$$h(q_1, p_2, q_2) := -A_{q_1}(q_1, p_2, q_2) \tag{4.10}$$

This definition gives the following expression for $\vec{B}$:

$$\vec{B} = \nabla \times \vec{A} = \nabla p_2 \times \nabla q_2 - \nabla h(q_1, p_2, q_2) \times \nabla q_1 \tag{4.11}$$

This has effectively put all structural information about the vector field $\vec{B}$ into the function h, because all other functions are determined by the coordinate system.

In field lines of $\vec{B}$ can be parameterised by functions $q_1^t(t) = t, p_2^t(t), q_2^t(t)$. In the rest of this paragraph it will be shown that with a function $p_1^t(t)$ that will be defined later the functions $p_1^t, q_1^t, p_2^t, q_2^t$ satisfy Hamilton's equations with Hamiltonian

$$H(p_1, q_1, p_2, q_2) := p_1 + h(q_1, p_2, q_2) \tag{4.12}$$

In order to prove Hamilton's equations a function p_1^t has to be defined such that

$$\frac{dp_1^t(t)}{dt} = -\frac{\partial H}{\partial q_1}(p_1^t(t), q_1^t(t), p_2^t(t), q_2^t(t))$$

The following function satisfies that condition:

$$p_1^t(t) := \int_0^t -\frac{\partial h}{\partial q_1}(q_1^t(t), p_2^t(t), q_2^t(t))dt \tag{4.13}$$

Before Hamilton's equations will be proved for the parameterisations some intermediate results will be given. Note that the left hand side of Equation 4.14 is undefined at coordinate system singularities described in Paragraph 4.1, which is why they were excluded from D.

$$\begin{aligned}
\nabla q_1 \cdot \nabla p_2 \times \nabla q_2 &\neq 0 && (4.14)\\
\nabla h &= \partial_{q_1} h \nabla q_1 + \partial_{p2} h \nabla p_2 + \partial_{q_2} h \nabla q_2 && (4.15)\\
\vec{B} \cdot \nabla p_2 &= -\nabla p_2 \cdot \nabla h \times \nabla q_1 \\
&= -\partial_{q_2} h \nabla q_1 \cdot \nabla p_2 \times \nabla q_2 \\
&= -\partial_{q_2} H \nabla q_1 \cdot \nabla p_2 \times \nabla q_2 && (4.16)\\
\vec{B} \cdot \nabla q_2 &= \partial_{p2} h \nabla q_1 \cdot \nabla p_2 \times \nabla q_2 \\
&= \partial_{p2} H \nabla q_1 \cdot \nabla p_2 \times \nabla q_2 && (4.17)\\
\vec{B} \cdot \nabla q_1 &= \nabla q_1 \cdot \nabla p_2 \times \nabla q_2 && (4.18)
\end{aligned}$$

The following calculations show that the functions $p_1^t, q_1^t, p_2^t, q_2^t$ satisfy Hamilton's equations:

$$\frac{dp_1^t}{dt} = -\frac{\partial H}{\partial q_1} \tag{4.19}$$

$$\frac{dq_1^t}{dt} = 1 = \frac{\partial H}{\partial p_1} \tag{4.20}$$

$$\frac{dp_2^t}{dt} = \frac{\vec{B}\cdot\nabla p_2}{\vec{B}\cdot\nabla q_1} = -\partial_{q_2} H \tag{4.21}$$

$$\frac{dq_2^t}{dt} = \frac{\vec{B}\cdot\nabla q_2}{\vec{B}\cdot\nabla q_1} = \partial_{p2} H \tag{4.22}$$

Equations 4.19,4.20 directly follow from 4.13,4.6, while 4.21,4.22 follow from 4.7,4.8 for $(c_1, c_2, c_3) = (q_1, p_2, q_2)$.

Equations 4.19 - 4.22 represent a two degree of freedom autonomous Hamiltonian system which has a four-dimensional manifold $\mathcal{M}$ as phase space. $\mathcal{M}$ will be written as a product

$$\mathcal{M} = \mathbb{R} \times \mathcal{M}_3 \tag{4.23}$$

where $\mathbb{R}$ is the range of p_1 and

$$\mathcal{M}_3 \subset R_{q_1} \times R_{p_2} \times R_{q_2} \tag{4.24}$$

which can be related to U by coordinate functions $(c_1, c_2, c_3) = (q_1, p_2, q_2)$ and the map ψ defined in 4.2.

Conditions under which the Hamiltonian system will be completely integrable and given in action-angle variables

The theory of Chapter 6 requires the Hamiltonian system to be completely integrable and in action-angle variables. It is useful to define the coordinate fields such that the Hamiltonian system will be in action-angle variables if that is possible, as the construction of action-angle coordinates is difficult (Paragraph 2.3). In order to describe $\vec{B}$ by a Hamiltonian system in action-angle variables two of the coordinate fields q_1, q_2 should be angular: range $c_i = R_{c_i} = S^1$. Besides that the coordinates q_1, q_2 should commute as follows from the discussion in Paragraph 2.5 taking in mind that the Hamiltonian time variables t_i^r correspond to the variables q_i of the original system, which are coordinates on invariant tori in phase space. The third coordinate field p_2 has to correspond to an action variable which is an invariant, which implies that $\vec{B}$ field lines should lie on manifolds of constant p_2.

So far this paragraph has shown that in the (q_1, p_2, q_2) coordinate system (Appendix A) there exists a Hamiltonian function such that the functions q_1^t, p_2^t, q_2^t parameterising $\vec{B}$ field lines (Paragraph 4.2) satisfy Hamilton's equations. The Hamiltonian system will be in action-angle variables if q_1, q_2 are angular coordinates that commute and if field lines lie on manifolds of constant p_2.

Properties of this construction

A strength of the construction of Paragraphs 4.1 - 4.3 is that adding small perturbations to h under mild conditions directly defines another divergenceless field: h is one of the components of the vector potential, replacing h gives another vector potential that corresponds to another divergenceless vector field. The mild condition is that the corresponding vector fields have to behave correctly at the boundary of the range of coordinate functions. The problem that can occur is that the new Hamiltonian system has orbits that start inside the range of the coordinate fields q_1, p_2, q_2 for $q_1^t(0) = \hat{q}_1$, but has points $(q_1^t(t), p_2^t(t), q_2(t))$ outside the range for some values of t.

The results of this paragraph can be seen as opposite to the geometrical interpretation of dynamical systems described in Paragraph 2.2. There a Hamiltonian system is represented by a vector field in phase space, while this paragraph describes a vector field in $\mathbb{R}^3$ as a Hamiltonian system. The difference is that the vector fields in Paragraph 2.2 are defined on phase space, while this paragraph starts with a vector field on $\mathbb{R}^3$ and gives a Hamiltonian system, which has a corresponding vector field on phase space $\mathcal{M}$. That vector field restricted to $\mathcal{M}_3$ for a specific value of p_1 is locally diffeomorphic to the vector field on $\mathbb{R}^3$ on the interior of the range of the coordinate fields. It is not diffeomorphic on the interior as the coordinate functions do not have to be injective. This fact is used in Paragraph 4.5 to describe nontoroidal vector fields by the same dynamical system as constructed for toroidal vector fields in Paragraph 4.4, which has invariant tori in phase space. The differences between the fields are incorporated in the coordinate fields, while the similarities are described by the Hamiltonian system.

4.4 A Hamiltonian system for the Sagdeev fields

In this paragraph the procedure of paragraphs 4.1 - 4.3 will be applied to the Sagdeev fields (paragraph 1.4). First the results of the procedure will be worked out in detail, while a more abstract overview of the construction of coordinate fields will be given at the end. The abstract ideas of the construction can be used for general toroidal vector fields as well as for non-toroidal vector fields by the construction in paragraph 4.5.

Coordinate fields and the domain of interest

The first angular coordinate function q_1 is defined such that for most field lines the field $\vec{B}$ has a nonzero component in the ∇q_1 direction. For the Sagdeev fields q_1 can be taken to be the angle around the z-axis. All field lines have a nonzero component in the ∇q_1 direction, except the field line on the z-axis, which will be excluded from D. ρ can be taken to be a specifically scaled distance to the unit circle such that surfaces of constant ρ define the tori on which field lines of the Sagdeev fields lie.

For simplicity $\nabla \rho$ is chosen to be perpendicular to ∇q_1. q_2 is chosen to be the angle around the unit circle, which is a field line of $\vec{B}$ and will be excluded from D. For simplicity ∇q_2 is perpendicular to $\nabla \rho$ and ∇q_1.

A specific coordinate system that fulfills these requirements is the toroidal coordinate system described in Appendix B. Another coordinate system with differently scaled ρ, q_2 could have been used as well, but this coordinate system was chosen because the Sagdeev fields have a very simple expression in these coordinates, as is shown below.

For the sagdeev fields D is taken to be

$$D := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 > 0\} - \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, z = 0\} \tag{4.25}$$

The Sagdeev fields have no zeroes, which implies that all field lines are bi-infinite or closed. Together with the exclusion of coordinate field singularities this implies that D as in Equation 4.25 satisfies the requirements stated in Paragraph 4.1 and that the coordinate system is a pre-Hamiltonian coordinate system as needed for the construction of Appendix A.

Expressions for the vector potential and magnetic field

Computations with Wolfram Mathematica 10.3 have led to an explicit description of a vector potential for the Sagdeev fields in the toroidal coordinates of appendix B:

$$\vec{A} \;=\; \frac{\operatorname{sech}^2\rho}{2\pi}\nabla q_2 - \frac{\omega_1 \operatorname{sech}^2\rho}{2\pi\omega_2}\nabla q_1 \tag{4.26}$$

It follows from assumptions made between Equation 1.23 and 1.25 that $\omega_2 \neq 0$, which means that the expressions in 4.26 are well defined.

Equation 4.26 shows that for the Sagdeev fields $A_\rho = 0$, and $A_{q_2}(\rho) = p_2(\rho)$ does not depend on q_1, q_2, which means that the vector potential is in the form of Equation 4.9. Comparing 4.26 and 4.10 gives an expression for p_2, h:

$$p_2 \;=\; \frac{\operatorname{sech}^2\rho}{2\pi} \tag{4.27}$$

$$h \;=\; \frac{\omega_1}{\omega_2}\frac{\operatorname{sech}^2\rho}{2\pi} = \frac{\omega_1}{\omega_2}p_2 = \iota p_2 \tag{4.28}$$

Here ι is the rotational transform, which is constant. It follows that:

$$\frac{dq_2^t}{dt} = \frac{\partial H}{\partial p_2} = \frac{\partial h}{\partial p_2} = \iota$$

As both $q_2; q_1 \equiv t$ are 2π-periodic this implies that field lines are closed if $\iota = \frac{\omega_1}{\omega_2} \in \mathbb{Q}$ and that field lines are not closed for $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$, in that case they densely fill the invariant tori.

As described in Appendix A p_2 will be used as a coordinate field instead of ρ (Figure 4.1).

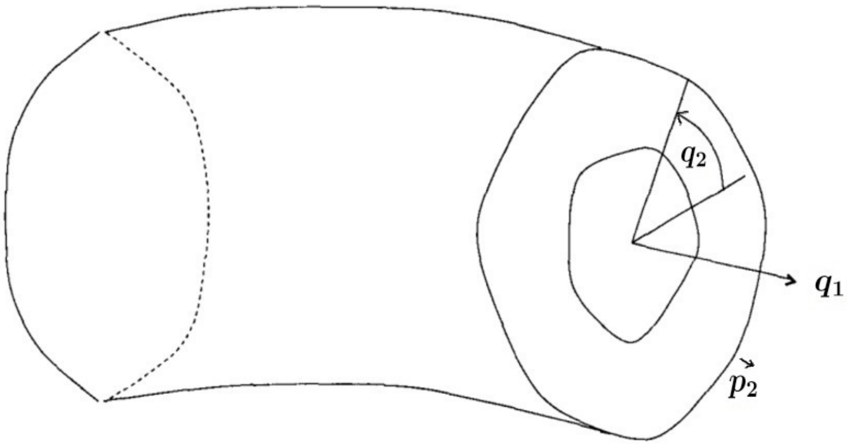


Figure 4.1: *Toroidal coordinate system q_1, p_2, q_2. Figure from [14].*

The Hamiltonian system

The previous results give the following autonomous, separable two degree of freedom Hamiltonian system:

$$H(p_1, q_1, p_2, q_2) = p_1 + h(p_2) = p_1 + \frac{\omega_1}{\omega_2} p_2 = p_1 + \iota p_2 \quad (4.29)$$

$$\frac{dp_1^t}{dt} = 0 \quad (4.30)$$

$$\frac{dq_1^t}{dt} = 1 \quad (4.31)$$

$$\frac{dp_2^t}{dt} = 0 \quad (4.32)$$

$$\frac{dq_2^t}{dt} = \iota \quad (4.33)$$

These equations imply that p_1, p_2, h, H are invariants defining toroidal invariant manifolds, where p_1, h are two independent invariants as in Definition 5, just as p_1, p_2, both pairs define the same invariant tori. The system is completely integrable and given in action-angle variables.

The results of this paragraph can also be stated in the reduced, one degree of freedom Hamiltonian system, which is done below. That system is simpler, which is why it will be used in Chapters 5,6.

Toroidal symmetry

The Sagdeev fields have a toroidal symmetry, the angles q_1, q_2 have an equivalent role. That implies that h has a role similar to p_2. If q_2 satisfies the prerequisites for a Hamiltonian time coordinate field, then the pairs h, p_2 and q_1, q_2 can be mutually interchanged, leading to a similar Hamiltonian system.

The reduced system

The Hamiltonian system of Equations 4.29 - 4.33 can be reduced as described in Paragraph 2.4. The Hamiltonian and the variables of the reduced system are related to that of the two degree of freedom system by:

$$t^r \equiv q_1, q^r \equiv q_2, p^r \equiv p_2, H^r \equiv h$$

The reduced, autonomous one degree of freedom Hamiltonian system is given by the following equations. The value c' used in the reduction is taken to be 0, as that value does not have any implications for the physical vector field it represents.

$$\begin{aligned} H^r(p^r, q^r, t^r) &= \frac{\omega_1}{\omega_2} p^r = \iota p^r && (4.34) \\ \frac{dH^{r,t}}{dt^r} &= 0 && (4.35) \\ \frac{dp^{r,t}}{dt^r} &= 0 && (4.36) \\ \frac{dq^{r,t}}{dt^r} &= \iota && (4.37) \end{aligned}$$

In this system both H^r, p^r are invariants (defining the same invariant manifolds), the system is completely integrable.

The construction of a coordinate system described in this paragraph can also be described in a more abstract way. In fact, it only used the existence of a core field line, the unit circle, around which other field lines spiral. q_1 was defined as the coordinate along the core that parameterises field lines. ρ was defined as a specifically scaled distance from the core, defining the surfaces on which field lines lie. q_2 was defined as the angle around the core. The corresponding domain of interest is $\mathbb{R}^3$ except the unit circle and the z-axis. This procedure can be applied to any divergenceless toroidal vector field. The resulting two degree of freedom Hamiltonian system is autonomous, separable, completely integrable and given in action-angle variables.

4.5 Generalisation for rotationally symmetric nontoroidal structures

As will be explained in Chapter 8 results of simulations described in Reference [2] show nontoroidal vector field structures with a behaviour similar to perturbed toroidal Hamiltonian systems described in Chapter 6. Therefore we will now look at a way to describe nontoroidal divergenceless vector fields by a toroidal Hamiltonian system. A construction describing a vector field by a Hamiltonian system and also depending on symmetry can be found in Reference [3].

Every toroidal Hamiltonian constructed using the method in Paragraphs 4.1 - 4.3 defines a vector field on $\mathcal{M}_3$. The three-dimensional manifold $\mathcal{M}_3$ is introduced in Equations 4.23,4.24, while the vector field on $\mathcal{M}_3$ is introduced at the end of Paragraph 4.3 as the restriction of the vector field on $\mathcal{M}$ (Paragraph 2.2) to $\mathcal{M}_3$ by fixing one value of p_1. This vector field represents the toroidal structure of the Hamiltonian system. Using this vector field every correspondence of a toroidal Hamiltonian system with a nontoroidal vector field can be seen as the correspondence between a toroidal vector field on $\mathcal{M}_3$ and a nontoroidal vector field on $D \subset \mathbb{R}^3$. This means that the correspondence basically comes down to identifying a nontoroidal vector field by a toroidal one, which is a purely geometric challenge, independent of dynamical systems.

As defining properties for the toroidal field it can be used that the field lines lie on two-dimensional manifolds with commuting angular coordinates. This description is based on the discussion of Paragraph 2.5, taking in mind that the Hamiltonian time variables of the reduced flows, t_i^r, correspond to the variables q_i of the original system, which are coordinates on invariant tori in phase space, hence also coordinates on tori in $\mathcal{M}_3$.

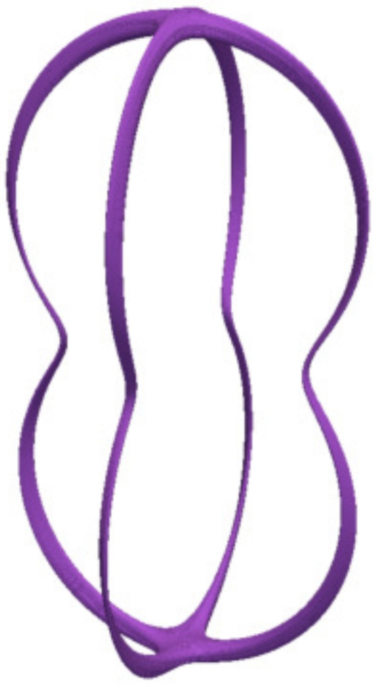
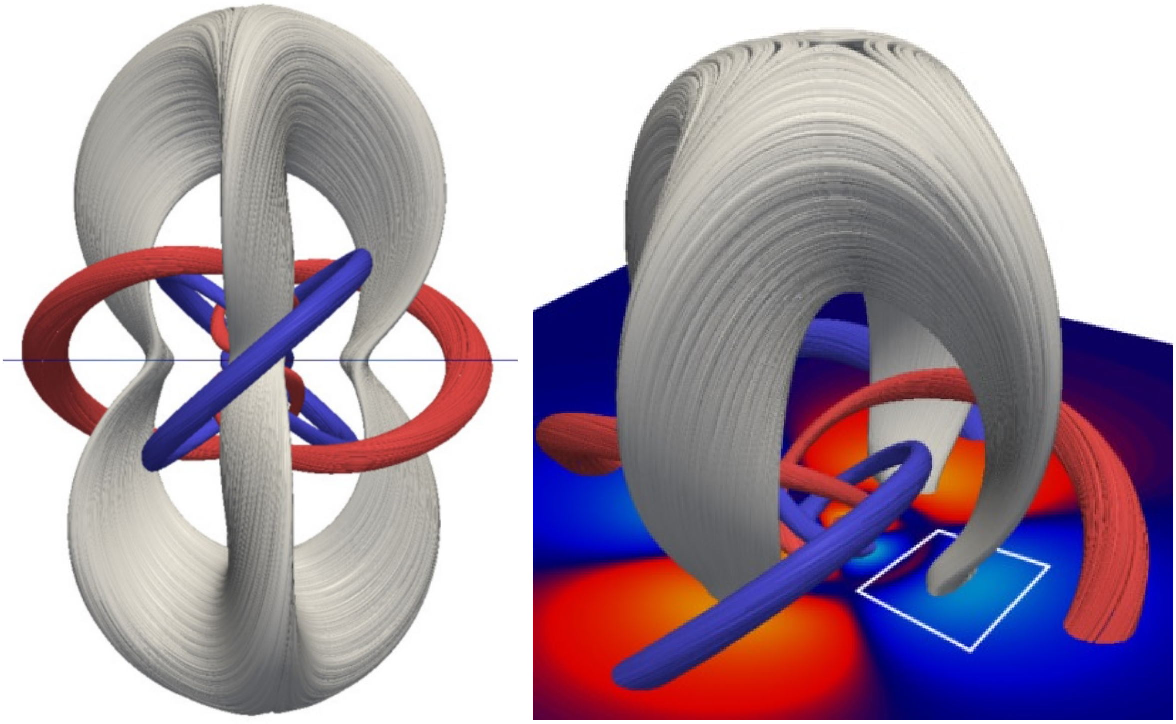


Figure 4.2: *Shape of a surface of genus* 3 *as observed in RMHD simulations ([2]). Figure from [2].*

Thus the goal is to foliate the domain of the nontoroidal vector field by two-dimensional manifolds on which there are two commuting angular coordinates. This is possible if the nontoroidal vector field has a symmetry. As an example a structure will be used where the field lines lie on surfaces of genus 3, which has a null line on the z-axis and a 180° rotational symmetry around the z-axis. This structure has been found in simulations described in Reference [2]. Figure 4.2 shows the shape of the surfaces of genus 3, while Figure 4.3 shows field lines on such a surface.



(a) *(b)*

Figure 4.3: *Field lines on surfaces of genus* 3*, shown in grey. Other colours show the surroundings of the genus* 3 *surface as they arise in RMHD simulations ([2]) and are not the topic of the current discussion. Figure from [2]. (a) Side view of the complete structure (b) Close-up of the upper half*

If the core is defined to be the shape in Figure 4.2 in the limit of zero thickness, then q_1, ρ, q_2 can be defined as in Paragraph 4.4: q_1 the direction along the core, ρ a scaled distance from the core and q_2 the angle around the core. The resulting coordinate fields are shown in Figure 4.4.

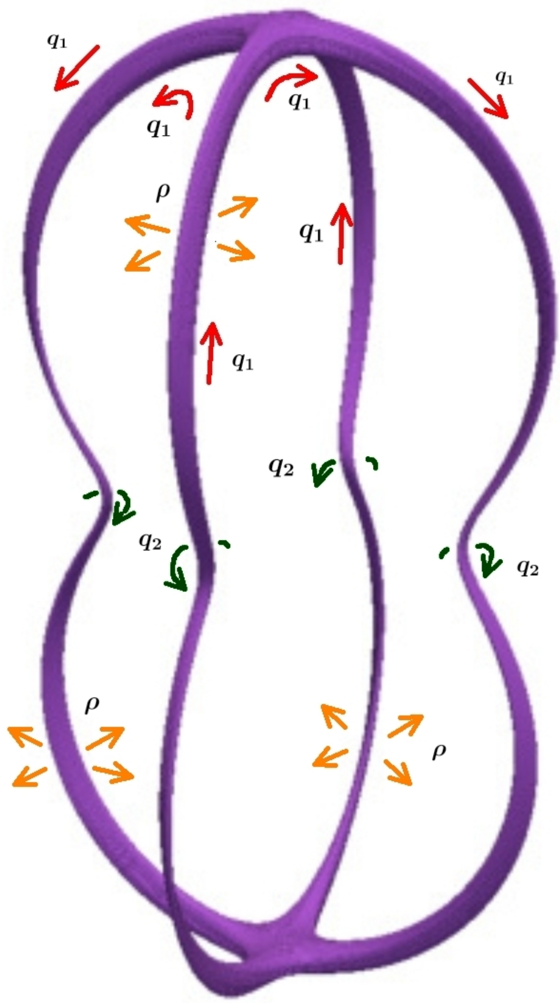


Figure 4.4: *Coordinate fields for the description of a vector field with a genus* 3 *shape. Figure based on [2].*

If the 180° rotational symmetry around the z-axis is used to identify points, then it follows that q_1, q_2 commute. This means that the construction of Paragraphs 4.1 - 4.3 gives a toroidal completely integrable Hamiltonian system in action-angle variables describing the nontoroidal vector field. If $\iota = \frac{dq_2^t}{dt} = \frac{dq^r}{dt^r}$ is constant then the Hamiltonian system is equal to the one derived for a Sagdeev field. Otherwise the system has a Hamiltonian with shear as defined in Paragraph 5.0.1.

The introduction of new coordinate fields gives a smooth map

$$U \to R_{q_1} \times R_\rho \times R_{q_2} = S^1 \times [\rho_{\min}, \rho_{\max}] \times S^1$$

from the domain of the nontoroidal vector field to the toroidal space. If the domain of this map is restricted such that it does not contain two points that are identified by the symmetry, and if the codomain is restricted to the image, then this is a local diffeomorphism which identifies the nontoroidal vector field with the toroidal field. There is no global diffeomorphism as the identification by symmetry makes this map noninjective.

Similar constructions are possible for other nontoroidal fields with symmetries, for example all structures covered by Paragraph 8.1.

Chapter 5

Hamiltonian models for magnetic islands

This chapter studies *island chains* which are regions in Hamiltonian phase space bounded by an invariant manifold where orbits spiral around a periodic orbit. This definition is derived from the definition of a *magnetic island chain* known in plasma physics, where magnetic field lines inside a flux tube spiral around a closed field line. This chapter gives Hamiltonian systems modeling island chains and derives basic properties, theoretical overview of their behaviour is given in the next chapter.

Outline of this chapter

First of all the term shear will be introduced in 5.0.1 as island chains are only possible in Hamiltonian systems with shear. Paragraph 5.1 studies the generalisations needed to go get a simplified, completely integrable model for island chains. Paragraph 5.2 shortly describes more general perturbations forming a realistic model for island chains. Paragraph 5.3 covers nonphysicalities caused at singularities of angular coordinate fields that arise when a perturbed Hamiltonian system is identified with a vector field via the identification of Chapter 4. A solution is given in the form of an expansion of the Hamiltonian system.

The results of this chapter will be derived for the reduced one degree of freedom Hamiltonian system. If results for the two degree of freedom system give additional insight, they will be stated at the end of a (sub)paragraph.

5.0.1 A Hamiltonian model with shear

For the Sagdeev fields the rotational transform is constant, namely $\iota = \frac{\omega_1}{\omega_2}$. This can be generalised to the situation where ι depends on p^r such that it is constant on each manifold of constant p^r, H^r, but that it differs between different manifolds. This p-dependence of rotatitonal transform is called shear:

Definition 13. Shear *is defined as*

$$\iota' := \frac{\partial^2 H_0^r}{\partial p^{r2}} = \frac{\partial \iota}{\partial p^r}. \tag{5.1}$$

Hamiltonian systems with nonzero shear are called nondegenerate*, while shearless Hamiltonian systems are called* degenerate*.*

The general one degree of freedom Hamiltonian system in action-angle variables with shear is as follows:

$$H_0^r(p^r) = \int \iota(p^r) dp^r \tag{5.2}$$

The subscript 0 is used to distinguish this Hamiltonian from other generalisations which will be derived in this chapter. Integration constants in the Hamiltonian will be neglected, as they do not change the resulting dynamics. Equation 5.2 implies

$$\frac{\partial H_0^r}{\partial p^r} = \iota(p^r). \tag{5.3}$$

The simplest example of a nonconstant ι is a linear dependence of ι on p^r.

$$\frac{\partial H_0^r}{\partial p^r} = \iota(p^r) = c_1 p^r \tag{5.4}$$

A corresponding Hamiltonian is

$$H_0^r = \int \iota(p^r) dp^r = \frac{1}{2} c_1 (p^r)^2. \tag{5.5}$$

The effect of nonzero shear is shown in figure 5.1, which shows field lines of the divergenceless vector field corresponding to this Hamiltonian.

Nondegenerate Hamiltonian systems describing toroidal vector fields can be used as an ideal model for toroidal plasmas, the generalisations that follow in the rest of this chapter will be seen as perturbations of the system with Hamiltonian 5.2.

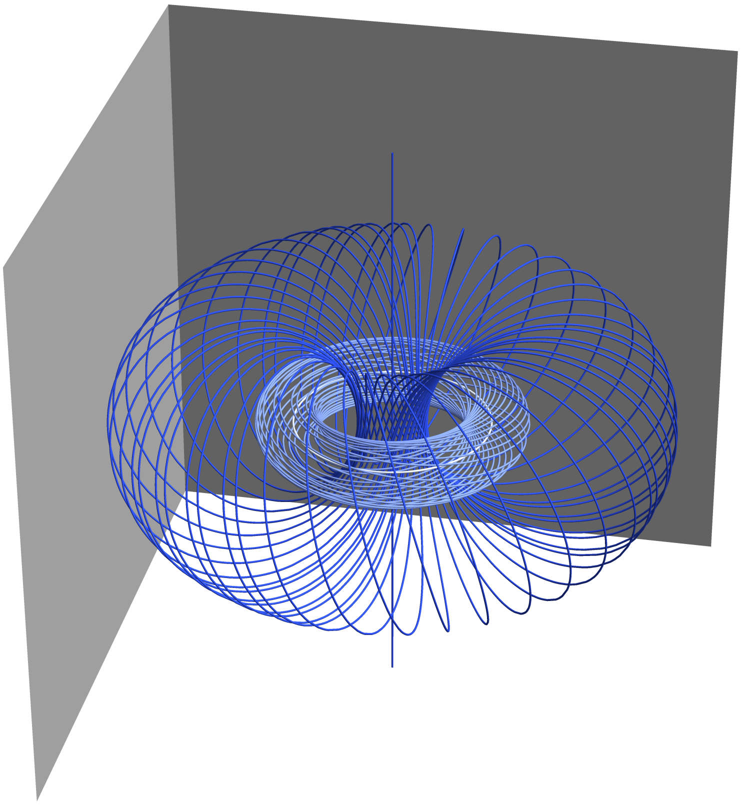


Figure 5.1: *Shear shown by the divergenceless vector field corresponding to Hamiltonian 5.5, visible through the different ratio of rotation around the unit circle and around the z-axis. Two field lines are partly shown, apart from the field lines on the z-axis and the unit circle.*

The two degree of freedom analogue

The following equations describe the corresponding two degree of freedom system:

$$H_0(p_1, p_2) = p_1 + \int \iota(p_2) dp_2 \tag{5.6}$$

$$\iota(p_2) = \frac{\partial H_0}{\partial p_2} \tag{5.7}$$

$$\iota'(p_2) = \frac{\partial^2 H_0}{\partial {p_2}^2} = \frac{\partial \iota}{\partial p_2} \tag{5.8}$$

5.1 Completely integrable models

Overview of this paragraph

Subparagraph 5.1.1 describes one specific perturbation of the Hamiltonian system in 5.0.1 for which the perturbed system is still completely integrable. Subparagraph 5.1.2 studies the orbit types of that perturbation, showing that the system contains an island chain, it can be used as a simplified model for them. It is also discussed to what field lines the orbits of the Hamiltonian system correspond via the results of Chapter 4. Subparagraph 5.1.3 studies generalisations of the perturbation described in 5.1.1 that arise from period doubling bifurcations.

5.1.1 A model with one Fourier term in q^r, t^r

The Hamiltonian system of a divergenceless vector field with shear (Equation 5.2) can be perturbed by a term $\delta H_1^r(p^r, q^r, t^r)$ where δ is a small parameter determining the perturbation strength. If this model is used to describe physical situations δ can depend on physical quantities and other parameters, but for the moment δ is studied as an independent parameter. Once the behaviour for given δ is known the dependence of δ on physical quantities gives a complete description of the physical situation.

The perturbed Hamiltonian can be written as

$$H^r(p^r, q^r, t^r) = H_0^r(p^r) + \delta H_1^r(p^r, q^r, t^r). \tag{5.9}$$

It is assumed that H_1^r does not depend on p^r, but that it does depend on q^r, t^r. Because q^r, t^r are periodic with period 2π the function

$$H_1^r(q^r, t^r) = \cos(nq^r - mt^r) \tag{5.10}$$

is an elementary choice: all other 2π-periodic functions of q^r, t^r can be Fourier decomposed in terms of this form. In such decompositions each cosine term also has a phase: $\cos(nq^r - mt^r - \zeta_{m,n})$. In this project the phases will be neglected because that does not change any qualitative behaviour.

The resulting Hamiltonian is

$$H^r(p^r, q^r, t^r) = \int \iota(p^r) dp^r + \delta \cos(nq^r - mt^r). \tag{5.11}$$

In Paragraph 9.15 of Reference [9] this model is studied under the name *one wave resonance model*. The fact that H^r can be seen as a function with two arguments, namely $p^r, q^r - \frac{m}{n} t^r$ can be interpreted as the fact that in this reduced system the islands behave like waves.

The orbits $p^{r,t}(t^r), q^{r,t}(t^r)$ of the Hamiltonian system satisfy the following equations.

$$\frac{dp^{r,t}}{dt^r} = -\frac{\partial H^r}{\partial q^r} = -\frac{\partial H_1^r}{\partial q^r} \neq 0 \tag{5.12}$$

$$\frac{dH^{r,t}}{dt^r} = -\frac{\partial H^r}{\partial t^r} = -\frac{\partial H_1^r}{\partial t^r} \neq 0 \tag{5.13}$$

Equations 5.12,5.13 show that manifolds of constant p and manifolds of constant H are no longer invariant manifolds. However, the system is completely integrable as can be seen by the introduction of a variable replacing q^r.

$$Q^r := q^r - \frac{m}{n}t^r \tag{5.14}$$

In terms of this new variable the the system has another Hamiltonian $\tilde{H}$:

$$\tilde{H}^r(p^r, Q^r) := \int i(p^r)dp^r + \delta\cos(nQ^r) \tag{5.15}$$

$$i(p^r) := \iota(p^r) - \frac{m}{n} \tag{5.16}$$

$$\frac{d\tilde{H}^{r,t}}{dt^r} = \frac{\partial \tilde{H}^r}{\partial t^r} = 0 \tag{5.17}$$

Remark 4. *The difference between $\iota(p^r), i(p^r)$ is a constant, which implies that their derivatives describing the shear are equal.*

$$i'(p^r) = \iota'(p^r) \tag{5.18}$$

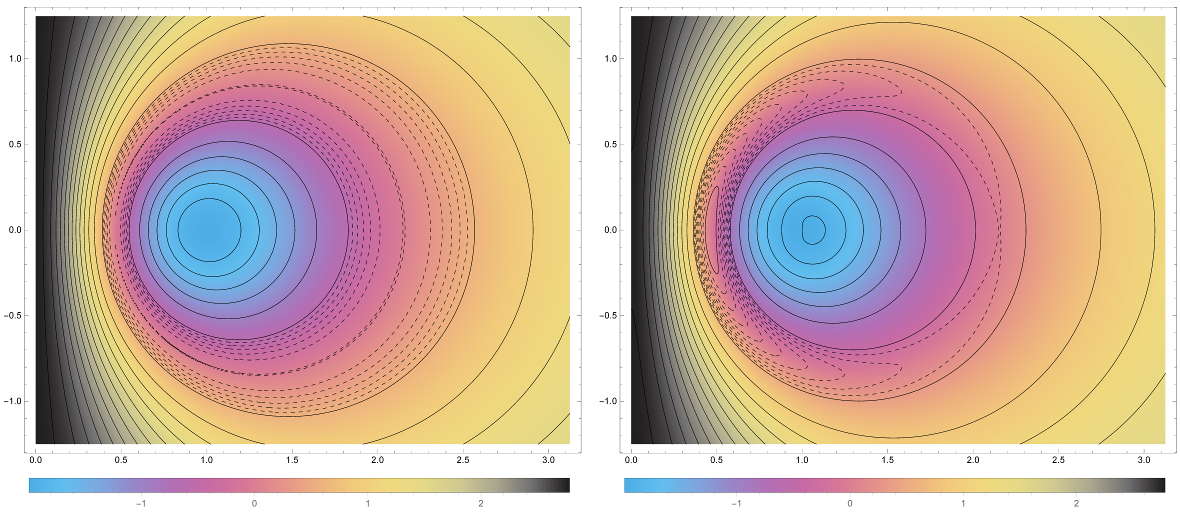
Both ι, i will be referred to as the rotational transform.

The resulting Hamiltonian is independent of t^r, so the system is completely integrable. This also proves the claim that the Hamiltonian of Equation 5.11 is completely integrable. The new variables are preferable as they better shows the integrable structure of the system. The behaviour is shown in plots of the phase plane (Figure 5.2).

It will be shown for the two degree of freedom analogue how $\tilde{H}$ follows from H, which can be used to derive equation 5.15 for $\tilde{H}^r$.

Averaged rotational transform

Because the Hamiltonian 5.15 depends on Q^r it follows that $\frac{dp^r}{dt^r} = -\frac{\partial \tilde{H}^r}{\partial Q^r} \neq 0$ and that $\frac{dQ^r}{dt^r}(p^r)$ is not constant along orbits of the system. Therefore $i(p^r)$ cannot be used to label invariant manifolds, which is needed in Paragraph 6.1. However, the average of i can be used to label the manifolds. The same is true for ι. The averages can be defined in a very direct sense using the periodicity of the Hamiltonian in t^r.



(a) Without perturbation, Hamiltonian 5.2 *(b) With perturbation, Hamiltonian 5.11*

Figure 5.2: *The effect of a one $n = 1, m = 2$ Fourier term perturbation to the Hamiltonian system shown in a transformed phase plane. Lines show curves where the Hamiltonian is constant, orbits lie on such lines. The colour scale shows the rotational transform i for $n = 1, m = 2$. The Hamiltonian variables p^r, Q^r have been transformed to cylindrical coordinates R, z using the results of Paragraph 4.4. The horizontal scale is the distance R from the z-axis, which is the left border of the figure. The vertical scale is the z-coordinate.*

Definition 14. *If the Hamiltonian system is periodic in t^r with period T then the* average rotational transform *is defined as follows.*

$$\iota_{avg} := T^{-1}\int_0^T \frac{dq^r(t^r)}{dt^r}dt^r \tag{5.19}$$

$$i_{avg} := T^{-1}\int_0^T \frac{dQ^r(t^r)}{dt^r}dt^r = \iota_{avg} - \frac{m}{n} \tag{5.20}$$

A less direct but more constructive approach is to define a Hamiltonian system closely related to that of $\tilde{H}^r$ in which $\frac{dQ^r}{dt^r}$ is constant on invariant manifolds. This system has a rational transform, that can be seen as an "average" as well. The technique used to construct such systems is called *averaging* and it is described in Paragraphs 4.1-4.4 of Reference [11].

The two degree of freedom analogue

In the two degree of freedom system the perturbation corresponds to a $\delta H_1(q_1, p_2, q_2)$ term. The general form of the Hamiltonian is

$$H(p_1, q_1, p_2, q_2) = H_0(p_1, p_2) + \delta H_1(q_1, p_2, q_2). \tag{5.21}$$

Periodic perturbations in q_1, q_2 not depending on p_1, p_2 can again be Fourier decomposed in cosines of the form $\cos(nq_2 - mq_1 - \zeta_{m,n})$ of which the phase $\zeta_{m,n}$ will be neglected. If H_1 is taken to be one such term and if H_0 is as in Equation 5.6, then the resulting Hamiltonian system is

$$H(p_1, q_1, p_2, q_2) = p_1 + \int \iota(p_2)dp_2 + \delta\cos(nq_2 - mq_1). \tag{5.22}$$

An orbit $p_1^t(t), q_1^t(t), p_2^t(t), q_2^t(t)$ of the Hamiltonian system satisfies the following equations.

$$\begin{aligned} \frac{dp_2^t}{dt} &= -\frac{\partial H}{\partial q_2} = -\frac{\partial H_1}{\partial q_2} \neq 0 && (5.23)\\ \frac{dp_1^t}{dt} &= -\frac{\partial H}{\partial q_1} = -\frac{\partial H_1}{\partial q_1} \neq 0 && (5.24)\\ \frac{dH^t}{dt} &= \frac{\partial H}{\partial t} = 0 && (5.25) \end{aligned}$$

This shows that p_1, p_2 are not invariants for the system, but that H is an invariant. However, this does not show that the system is completely integrable, as that requires the existence of two independent invariants. Just as for the one degree of freedom system there exists a canonical transformation to variables that directly show the integrable structure. Two new coordinates Q, P can be introduced, which replace q_2, p_1 respectively.

$$\begin{aligned} Q &:= q_2 - \frac{m}{n}q_1 && (5.26)\\ P &:= p_1 + \frac{m}{n}p_2 && (5.27) \end{aligned}$$

This variable transformation has a corresponding generating function (Paragraph 2.3). That proves that this transformation is canonical - it correctly transfers the Hamiltonian structure to the new variables. The expression for $\tilde{H}$ follows by substitution of the following two expressions into 5.11.

$$\begin{aligned} q_2 &= Q + \frac{m}{n}q_1\\ p_1 &= P - \frac{m}{n}p_2\\ \tilde{H}(P, p_2, Q) &= P + \int \iota(p_2)dp_2 - \frac{m}{n}p_2 + \delta\cos(nQ)\\ &= P + \int i(p_2)dp_2 - \delta\cos(nQ) && (5.28) \end{aligned}$$

As a function $i(p_2) := \iota(p_2) - \frac{m}{n}$ is equal to i as defined in Equation 5.16. The Hamiltonian 5.15 is derived from this equation by reduction of the Hamiltonian system.

For the new Hamiltonian P is an invariant, as $\tilde{H}$ does not depend on q_1. Besides that $\tilde{H}$ is an invariant as well, as it does not explicitly depend on t. Complete integrability follows from the fact that $P, \tilde{H} - P$ are independent invariants as in definition 5.

5.1.2 Orbit types in completely integrable models for island chains - and the corresponding field line types

Orbit types

First of all the fixed points of the Hamiltonian system in the new variables will be calculated. The orbit types will be derived using the structure of the fixed points, which satisfy the following equations:

$$0 = \frac{dQ^{r,t}}{dt^r} = i(p^r) = \iota(p^r) - \frac{m}{n} \quad ; \quad 0 = \frac{dp^{r,t}}{dt^r} = n\delta \sin(nQ^r) \tag{5.29}$$

This implies that for fixed points $p^r = p^r_*$ which is defined by $\iota(p^r_*) = \frac{m}{n}$. Besides that $Q^r = Q^r_*$ with $nQ^r_* \in \pi\mathbb{Z}$, which means that nQ^r_* is an integer multiple of π.

The system can be linearised around the fixed points (p^r_*, Q^r_*):

$$\dot{Q}^r = \ddot{Q}^r(p^r_*, Q^r_*)(p^r - p^r_*) = \iota'(p^r_*)(p^r - p^r_*) \tag{5.30}$$
$$\dot{p}^r = \ddot{p}^r(p^r_*, Q^r_*)(Q^r - Q^r_*) = n\delta(-1)^{nQ^r_*/\pi}(Q^r - Q^r_*) \tag{5.31}$$

Note that $\iota'(p^r_*) \neq 0$ by Equation 5.29. Fixed points are hyperbolic for:

- $\iota' > 0$ and $nQ^r_* \in 2\pi\mathbb{Z}$, which means that nQ^r_* is a multiple of 2π
- $\iota' < 0$ and $nQ^r_* \in \pi + 2\pi\mathbb{Z}$, which means that nQ^r_* is π plus a multiple of 2π

The following fixed points are elliptic:

- $\iota' < 0$ and $nQ^r_* \in 2\pi\mathbb{Z}$, which means that nQ^r_* is a multiple of 2π
- $\iota' > 0$ and $nQ^r_* \in \pi + 2\pi\mathbb{Z}$, which means that nQ^r_* is π plus a multiple of 2π

When looking at the fixed points for a fixed p^r_* it follows that the hyperbolic and elliptic fixed points alternate. That gives the phase plane of Figure 5.3. The elliptic fixed points are denoted by "O" and are often called *O-points* in physics, while the hyperbolic fixed points are denoted by "X" and are often called *X-points*.

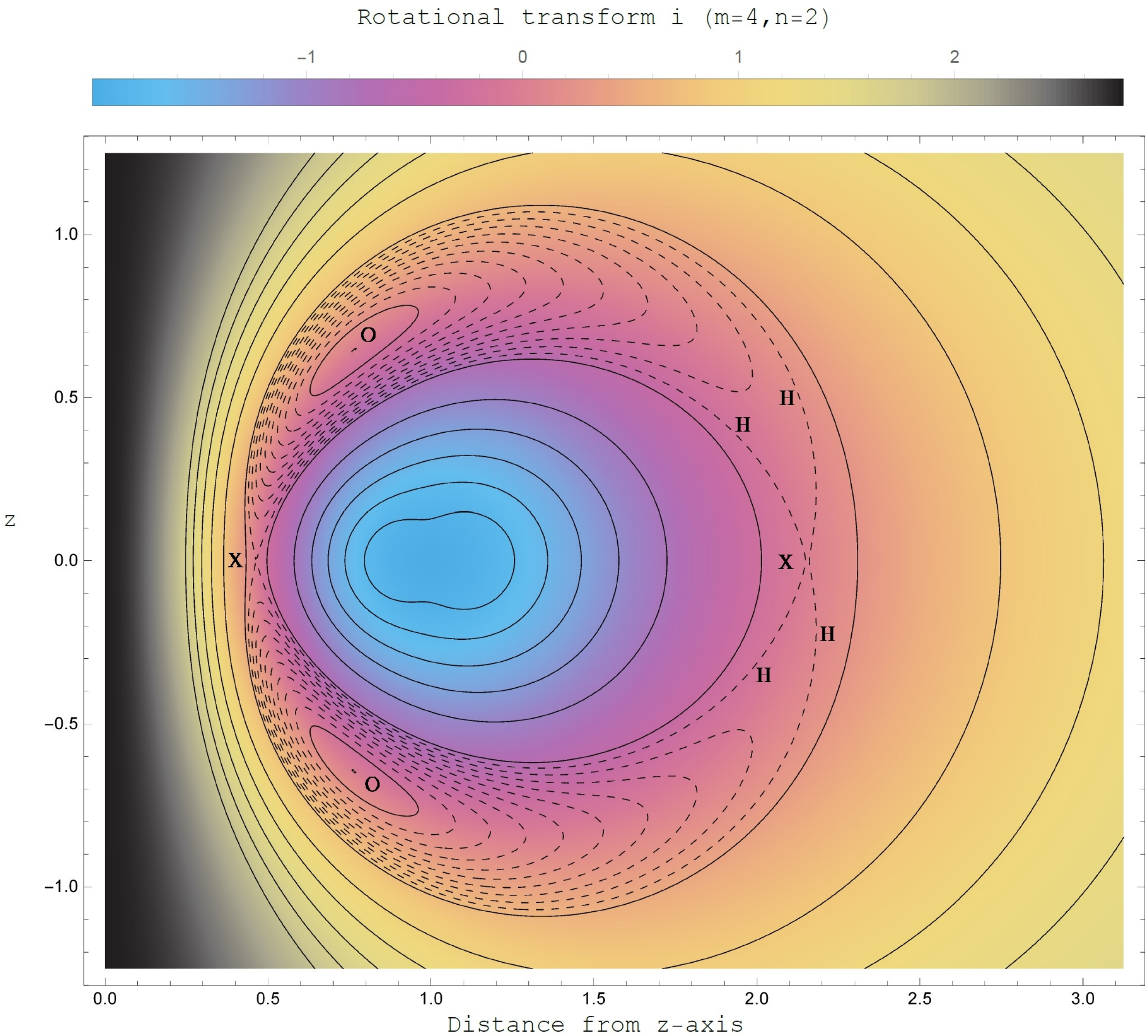


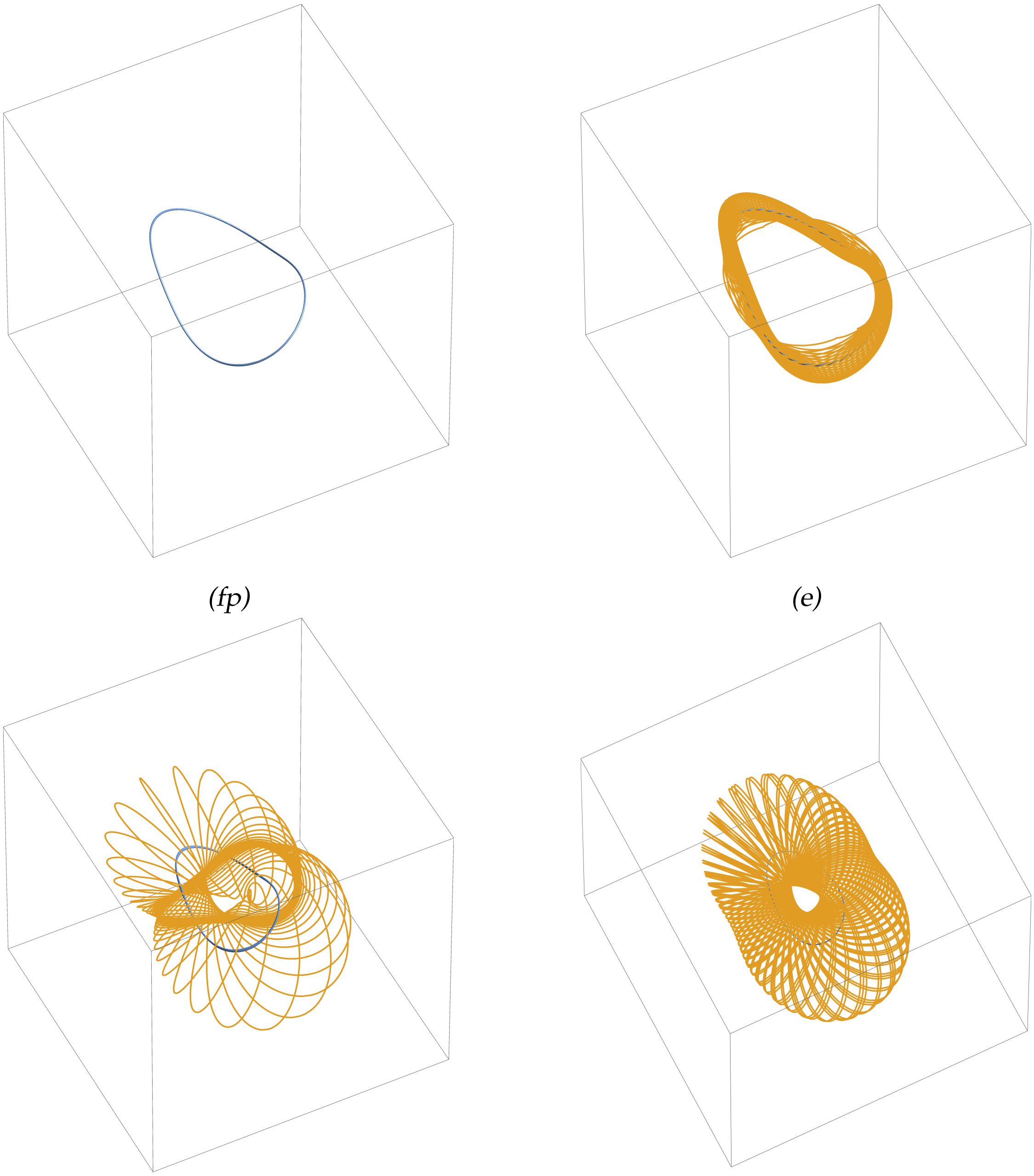
Figure 5.3: *Fixed points in an $m = 4, n = 2$ island chain, shown in the phase plane. Orbits lie on curves are of constant $\tilde{H}^r$, of which some are shown in black.*
"O": elliptic fixed points, "X": hyperbolic fixed points, "H": heteroclinic orbits

The system contains so-called *heteroclinic orbits* connecting two different hyperbolic fixed points: heteroclinic orbits converge to fixed points for $t^r \to +\infty$ and $t^r \to -\infty$, but in both cases to different fixed points. Heteroclinic orbits are denoted by "H". These heteroclinic orbits fill the stable and unstable manifolds of neighbouring hyperbolic fixed points defined in Chapter 3, which coincide in this system. If an island chain contains only one hyperbolic fixed point (for example the system in Figures 5.2,5.4,5.5), then the description is analogous. The only difference is that the heteroclinic orbits are replaced by *homoclinic orbits*, converging to the same fixed point for $t \to \pm\infty$.

The union of the heteroclinic orbits and hyperbolic fixed points is called *separatrix*, as it separates all orbits spiralling around the elliptic fixed points inside the island chain from the orbits that have the same topology as in the unperturbed system of Equation 5.2. The orbits inside the separatrix circling around one of the elliptic fixed point is an *island*. The union of the separatrix and all orbits circling around the elliptic fixed points is an *island chain*.

Field line types

Using the coordinate fields constructed in Chapter 4 the orbits of the reduced Hamiltonian system correspond to field lines of a divergenceless vector field. The figures below show the four types of field lines present in the vector field corresponding to the system with Hamiltonian 5.15.



(fp) (e)

(h) (t)

Figure 5.4: *Field lines corresponding to different orbit types of the Hamiltonian system with a* $m = 2, n = 1$ *perturbation, which are: (fp) (Elliptic) fixed point (blue), corresponding to a closed field line. (e) Orbit spiralling around the elliptic fixed point (yellow), the elliptic fixed point is shown as a reference (blue). (h) Homoclinic orbit (yellow), the elliptic fixed point is shown as a reference (blue). The thick yellow curve corresponds to the hyperbolic fixed point, which is the limit of the homoclinic orbit. (t) Toroidal orbit equivalent to the unperturbed system.*

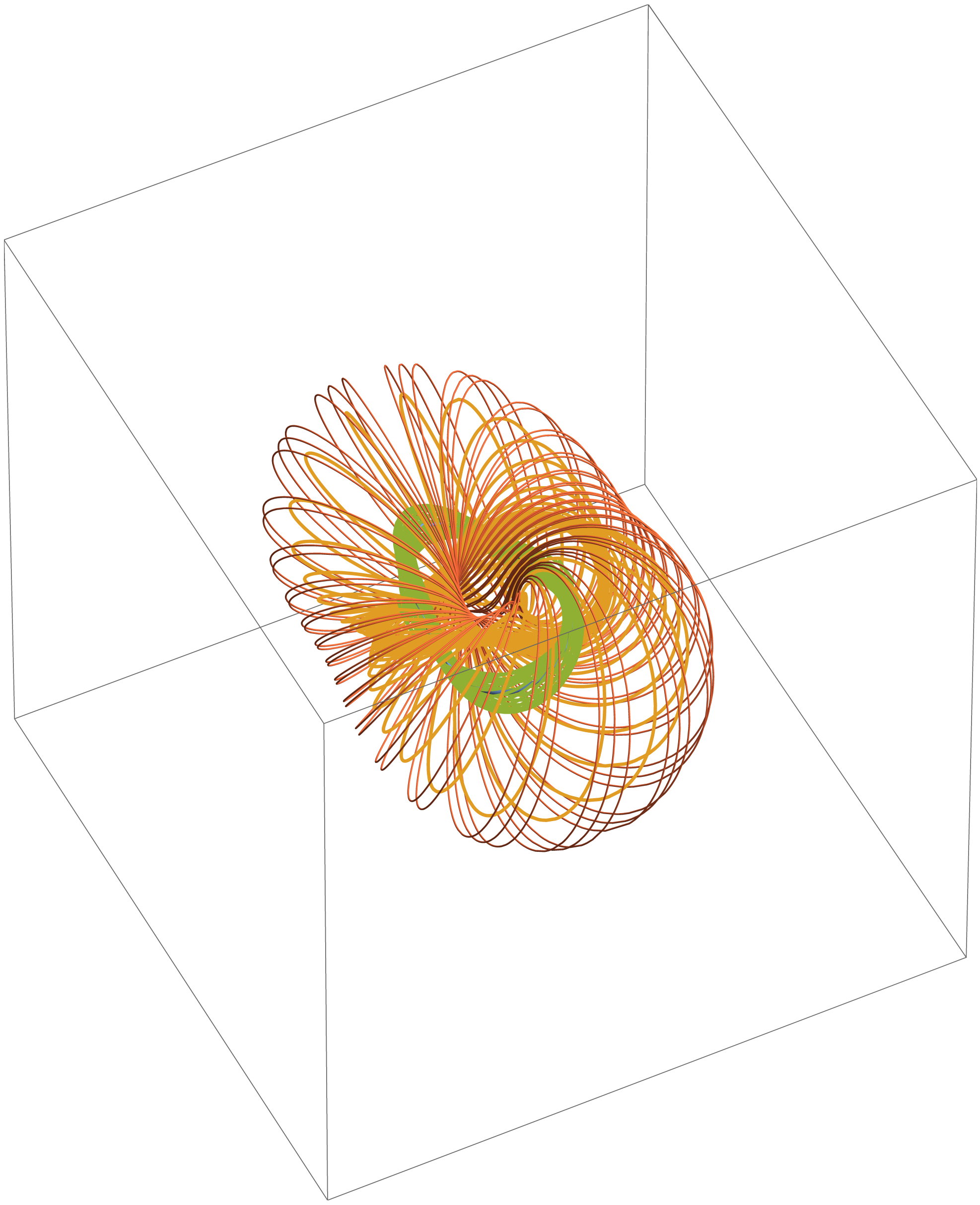


Figure 5.5: *All predescribed field line types of the system with a $m = 2, n = 1$ perturbation shown together (elliptic fixed point in blue, orbit spiralling around elliptic fixed point in green, homoclinic orbit in yellow, toroidal orbit in red).*

The closed field lines corresponding to the elliptic fixed points of the reduced Hamiltonian system will be called *island core field lines*.

The two degree of freedom analogue

The fixed points of the one degree of freedom system correspond to periodic orbits of the two degree of freedom system. A description in terms of the two degree of freedom system gives results equivalent to those above, where the counterparts of elliptic and hyperbolic fixed points are elliptic and hyperbolic periodic orbits.

5.1.3 Integrable models with multiple Fourier terms in q^r, t^r

Subparagraph 5.1.1 has studied perturbations of Hamiltonian 5.2 with one Fourier term $\cos(nq^r - mt^r)$, which gives a completely integrable system. For perturbations with several (or infinitely many) Fourier terms the same canonical transformation of variables as in Subparagraph 5.1.1 can be used. If the ratio $\frac{m}{n}$ is equal for all terms that implies that t^r can be eliminated by the substitution of Q^r as defined in 5.14, which again gives a completely integrable Hamiltonian system.

Adding such higher order terms leads to new fixed points in phase space: elliptic fixed points will be replaced by a hyperbolic fixed point between two new elliptic ones, as can be seen in the phase plane of Figure 5.6. The emergence of higher order terms corresponds to a period doubling bifurcation of the system. Figure 5.6 should be compared to Figure 5.2.

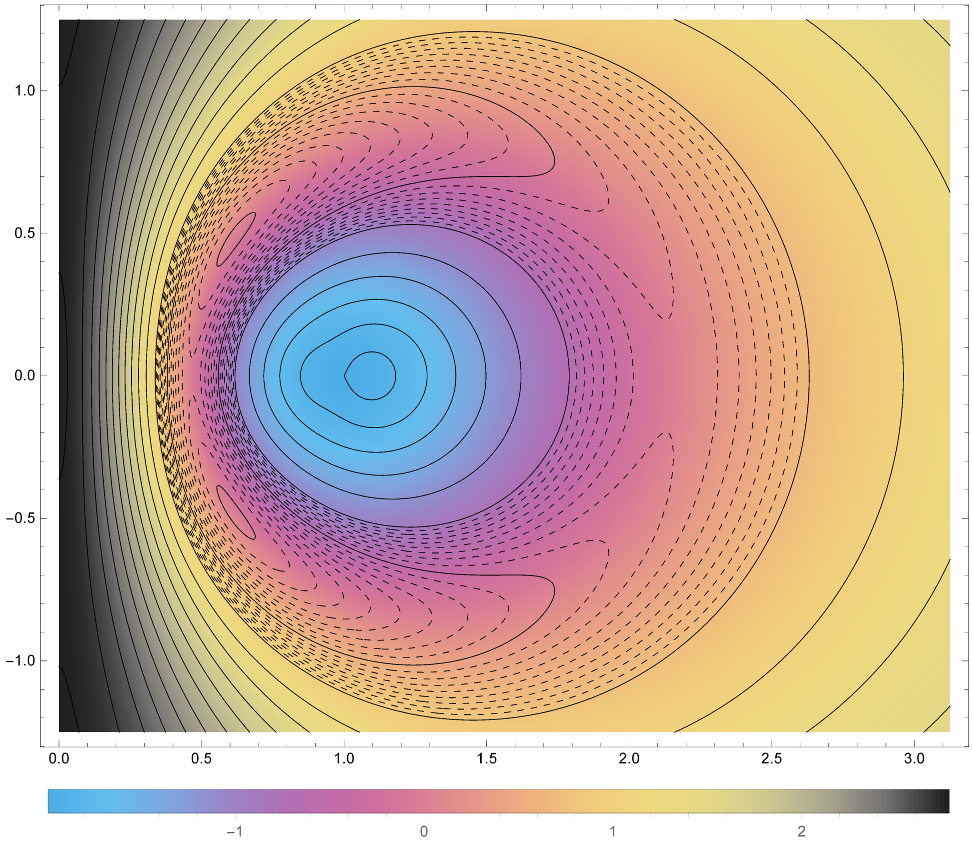


Figure 5.6: *Two elliptic orbits inside an $m = 2, n = 1$ island chain, arising from a second Fourier term with $m = 4, n = 2$ added to the Hamiltonian. The difference with Figure 5.2 is that the left hyperbolic fixed point is not connected to the right hyperbolic fixed point, but that it is contained in field lines with the topology of the island in 5.2.*

5.2 Nonintegrable models

In Paragraph 5.1 only perturbations consisting of Fourier terms independent of p^r and with fixed ratios $\frac{m}{n}$ have been studied. The simplest perturbation that is still independent of p^r consists of two Fourier terms with different ratios $\frac{m}{n}$. Adding a term $H_2^r(q^r, t^r) := \varepsilon \cos(n_2 q^r - m_2 t^r)$ to the Hamiltonian in 5.11 gives

$$H^r(p^r, q^r, t^r) = \int \iota(p^r) dp^r + \cos(nq^r - mt^r) + \varepsilon \cos(n_2 q^r - m_2 t^r) \tag{5.32}$$

where $\frac{m_2}{n_2} \neq \frac{m}{n}$. This is a special case of the *multiple wave resonance model* in Paragraph 9.16 of Reference [9]. It is also the topic of Reference [8], their results are described in Paragraph 6.4.

New variables for this system

Equations 5.12,5.13 are true for this system as well. A new variable defined by 5.14 can be used just as before.

$$\tilde{H}^r(p^r, Q^r, t^r) = \int i(p^r)dp^r + \delta\cos(nQ^r) + \varepsilon\cos(n_2Q^r - \tilde{m}t^r) \tag{5.33}$$

$$\tilde{m} := n_2\left(\frac{m_2}{n_2} - \frac{m}{n}\right) = m_2 - n_2\frac{m}{n} = \frac{nm_2 - n_2m}{n} \tag{5.34}$$

$$i(p^r) = \iota(p^r) - \frac{m}{n} \tag{5.35}$$

$$\frac{d\tilde{H}^{r,t}}{dt^r} = \frac{\partial\tilde{H}^r}{\partial t^r} \neq 0 \tag{5.36}$$

With the new variable Q^r the Hamiltonian depends on t^r through the second Fourier term. This dependence on t^r makes the system dependent on t^r with period period $\frac{2\pi}{\tilde{m}} = \frac{2\pi n}{nm_2 - n_2m}$. Therefore this form of the Hamiltonian cannot be used to prove that the system is completely integrable. The opposite can be proven, as discussed in Paragraph 6.2, namely that this system exhibits chaotic behaviour and that it is not completely integrable.

Regimes of perturbation sizes

As before δ, ε are used as independent parameters determining the size of the perturbation. It is discussed below how regimes of parameter choices can be used: $\delta \approx \varepsilon$ represents general nonintegrable perturbations of the Hamiltonian 5.2 and $\delta \gg \varepsilon$ can be used to study perturbations to ideal island chains.

For $\delta \approx \varepsilon$ the system represents general small perturbations of 5.2. The fact that this specific Hamiltonian represents general perturbations is supported by Reference [15], page 318. Chirikov has studied the perturbed Hamiltonian of equation 5.32, but also the system where the Hamiltonian 5.2 has been perturbed by a cosine times a Dirac delta function:

$$H_1^r(q^r, t^r) = \varepsilon\sum_{m=-\infty}^{\infty}\cos(nq^r - mt^r) = \varepsilon\cos(nq^r)\delta_{\mathrm{Dirac}}(t^r) \tag{5.37}$$

That gives the following Hamiltonian.

$$H^r(p^r, q^r, t^r) = \int \iota(p^r)dp^r + \varepsilon\sum_{m=-\infty}^{\infty}\cos(nq^r - mt^r) \tag{5.38}$$

Chirikov calculated that the sizes of the largest few island chains of the system with Hamiltonian 5.33 are similar to those of the Hamiltonian 5.38, which supports the idea that the qualitative behaviour of those systems is equivalent.

Another regime is $\delta \gg \varepsilon$, where ε is small but δ can be large. That models small perturbations of the integrable model for island chains. This system with $\varepsilon = 0$ can be seen as an ideal system to which the theory of chapter 6 can be applied, giving the preservation of invariant tori in island chains and giving chaotic island chains inside island chains. This interesting repetition of structure on different scales is also described in paragraph 6.3.

5.3 Singularities at core field lines

Problems occuring at angle-related singularities described in Paragraph 4.1 do not allow a physical interpretation of some orbits in the Hamiltonian systems of Paragraphs 5.1,5.2. In those perturbations the curves on which the coordinate system has singularities are not field lines of the system (required in Paragraph 4.1). The problems with singularities are only important in the regions close to the core field lines (the z-axis and the unit circle), therefore they can be neglected in most practical cases. If the behaviour of island chains near the core field lines is of interest then it is important to understand how the real behaviour of divergenceless fields, in which coordinate system singularities play no role, corresponds to extra terms in the Hamiltonian model.

In this paragraph the Hamiltonian 5.15 is used to discuss the singularities. Similar results can be obtained for all Hamiltonian models discussed in Paragraphs 5.1,5.2.

The following equatio holds for orbits $p^{r,t}(t^r), Q^{r,t}(t^r)$ of the Hamiltonian system:

$$p^{r,t}(t^r) = \int_0^{t^r} \frac{\partial \tilde{H}^r}{\partial Q^r}(p^{r,t}(t^r), Q^{r,t}(t^r))dt^r + p^{r,t}(0) = -\int_0^{t^r} n\delta \sin(nQ^{r,t}(t^r))dt^r + p^{r,t}(0)$$

As $\sin(nQ^{r,t}(t^r))$ reaches negative values this can give values of $p^{r,t}(t^r)$ outside its domain $[p_{\text{min}}, p_{\text{max}}]$. This does not represent any physical situation: field lines seem to disappear and to reappear at the singularity, as can be seen in Figure 5.7.

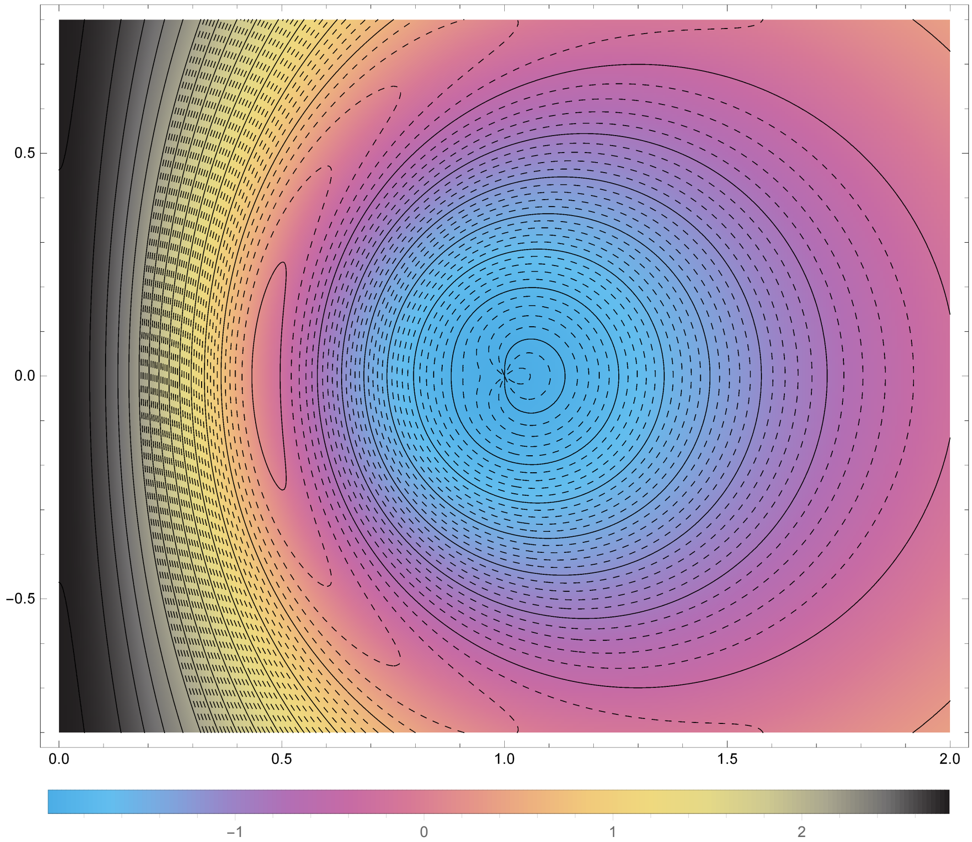


Figure 5.7: *Problems with the singularity at the centers shown in a close-up of the* $m = 2, n = 1$ *island chain in figure 5.2. In this figure the singularities are the point* $(1, 0)$ *(unit circle) and where the horizontal scale is* 0 *(the* z*-axis).*

The Hamiltonian can be changed to solve the issues at singularities. To prevent variables from attaining values outside their domain $\frac{\partial \tilde{H}^r}{\partial Q^r}$ should be small for values of p^r close to the core field lines. This means that the $\cos(nQ^r)$ term should be multiplied by a correction function in p^r, giving the following Hamiltonian.

$$\tilde{H}^r(p^r, Q^r, t^r) = \int i(p^r)dp^r + \delta C(p^r)\cos(nQ^r) \tag{5.39}$$

An example of such a correction function is

$$C_{k,l}(p^r) = \frac{(p^r_{\text{max}} - p^r)^l (p^r - p^r_{\text{min}})^k}{\left(p^r_{\text{max}} - \frac{kp^r_{\text{max}} + lp^r_{\text{min}}}{k+l}\right)^l \left(\frac{kp^r_{\text{max}} + lp^r_{\text{min}}}{k+l} - p^r_{\text{min}}\right)^k} \tag{5.40}$$

where k, l are nonzero and positive. Figure 5.8 shows the effect of the correction function. This correction is smooth and gives a k-th order decay of the perturbation strength near p_{min} and an l-th order decay near p_{max} (different orders can occur in physical systems). Moreover, it is normalised such that its maximum value is 1.

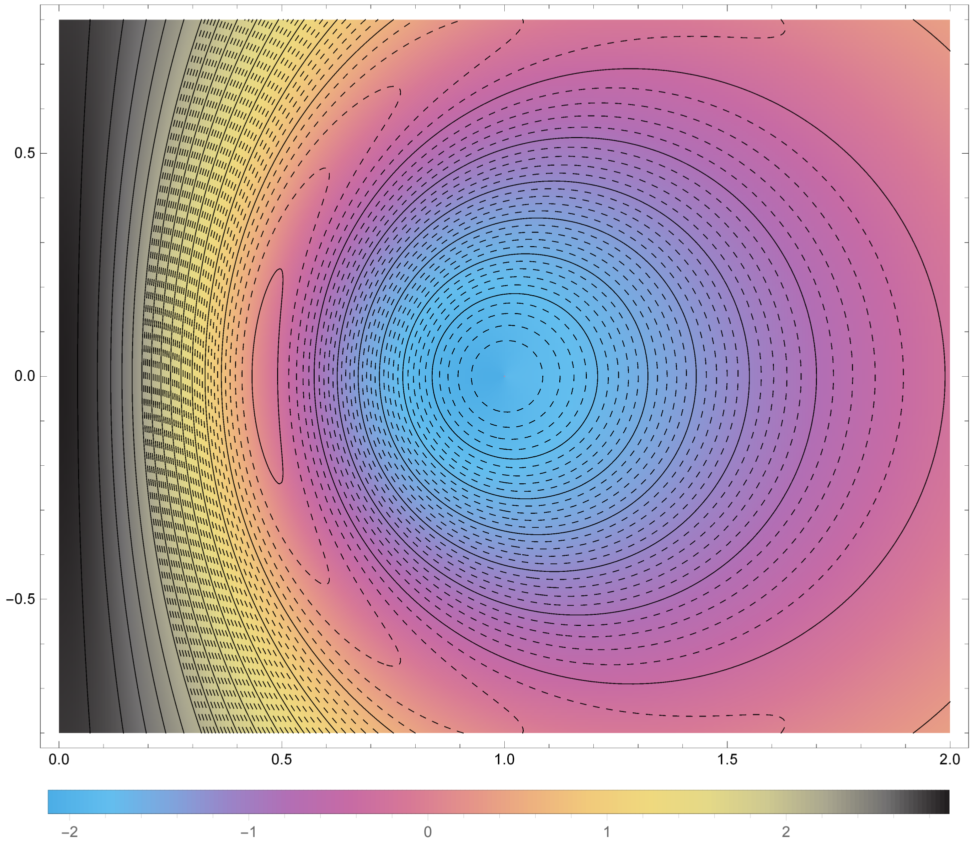


Figure 5.8: *Effect of the correction function with $k = 1 = l$ removing the issues at singularities.*

Implications of singularity corrections

The use of such a correction function has more implications than the absense of singularities: $\frac{\partial \tilde{H}^r}{\partial Q^r}$ has become dependent on p^r (as required), but $\frac{\partial \tilde{H}^r}{\partial p^r}$ has also become dependent on Q^r. The variation in $\frac{dQ^r}{dt^r} = \frac{\partial \tilde{H}^r}{\partial p^r}$ depends on the size of the derivative of the correction function. This can be used to determine the p^r-dependence of the perturbation strength for specific empirical divergenceless fields, or vice versa.

More abstractly the discussion of this paragraph has shown that the assumption that the perturbation does not depend on p^r is not realistic if the Hamiltonian system has to correspond to a divergenceless vector field via the correspondence of Chapter 4. Despite that the rest of this thesis uses the uncorrected model, as the problems are only important close to core field lines.

Chapter 6

Theories about perturbations of completely integrable Hamiltonian systems with invariant tori

This chapter describes the results of some theories about perturbations of completely integrable Hamiltonian systems with connected and compact invariant manifolds in phase space, namely invariant tori. The first results of this type were derived by Kolmogorov, Arnold and Moser, therefore their results are called KAM-theorem. The theory discussed in this chapter expands the ideas of that theorem, therefore it is often referred to as *KAM-theory*. The range of the term "KAM-theory" is not very strict, sometimes it is restricted to the results of Paragraph 6.1 and techniques to prove Theorems 5,6, but it is also used for a field of research that is much larger than the scope of this thesis.

Overview of this chapter

In Paragraph 6.1 two versions of a KAM-theorem will be discussed, which state the preservation of invariant tori for small perturbations of the Hamiltonian system. Paragraph 6.2 describes the structure of island chains for small nonintegrable perturbations. Subparagraph 6.2.1 uses the area preservation of Poincaré maps derived from a reduced Hamiltonian system to show how island chains appear for general perturbations of completely integrable Hamiltonian systems. In Subparagraph 6.2.2 it will be explained that for general perturbations island chains are chaotic. Melnikov theory can be used to prove the existence of heteroclinic tangles, which have a chaotic structure. The theories showing the chaotic structures are not discussed in this paragraph, the results of those theories are given instead. Paragraph 6.3 gives an overview of completely integrable Hamiltonian systems and perturbations of those systems, globally describing the properties for small perturbations and the effect of increasing the perturbation size. Paragraph 6.4 describes a method that estimates for what perturbation size an invariant torus will be destroyed, which can be used to show that the structure described by paragraph 6.3 is also rather stable for realistic perturbation sizes.

6.1 The KAM theorems

The unperturbed system

The following discussion is based on Reference [11] pages 218 - 220.

Let an autonomous completely integrable two degree of freedom Hamiltonian system be given, for which invariant manifolds are compact and connected, diffeomorphic to tori. It is assumed that the system is given in action-angle variables. The corresponding Hamiltonian will be denoted by $H_0(p_1, p_2)$. It is assumed that H_0 is separable and that $\frac{\partial H}{\partial p_1} \neq 0$. The Hamiltonian can be written as

$$H_0(p_1, p_2) = F_1(p_1) + F_2(p_2) \tag{6.1}$$

Due to the assumption that $\frac{\partial H_0}{\partial p_1} \neq 0$ the system can be reduced with respect to p_1, q_1. The reduced system is denoted by

$$H_0^r(p^r) = F_1{}^{-1}(c' - F_2(p^r)) \tag{6.2}$$

The resulting reduced systems is given below. Note that it is completely integrable and given in action-angle variables.

$$\frac{dp^{r,t}}{dt} = 0 \; ; \; p^t(t^r) = p^{r,t}(0)$$

$$\frac{dq^{r,t}}{dt} = \frac{\Omega_2}{\Omega_1}(p^r) = \iota(p^r) \; ; \; q^{r,t}(t^r) = \iota(p^r)t^r + q^{r,t}(0)$$

The cross section and the Poincaré map corresponding to the reduced system will be denoted by

$$\Sigma = \Sigma_{c'}^{t_0} \; ; \; P_0 = P_{c'}^{t_0}$$

where the old sub- and superscripts have been left out. The new subscript in P_0 displays the correspondence of this map to H_0.

Note that in the reduced system the invariant tori show up as invariant closed curves: invariant manifolds are manifolds of constant p_1, p_2. In the cross section, having coordinates $p_2, q_2 \equiv p^r, q^r$, they show up as curves of constant p_r.

The perturbed system

The KAM theorem studies systems that can be seen as small perturbations of the "ideal" Hamiltonian H_0. The new Hamiltonian is given by the following formula, where $\varepsilon > 0$ is a small parameter:

$$H(p,q) = H_0(p) + \varepsilon H_1(p,q) \tag{6.3}$$

The system is autonomous and for small perturbations $\frac{\partial H}{\partial p_1} \neq 0$, which means that this system can be reduced as well, giving a Hamiltonian H_0^r. The reduced system also has a Poincaré map, denoted by P_ε.

The perturbed Hamiltonian may depend on q_1 and q_2, which means that p_1, p_2 do not have to be invariants of the perturbed system, the perturbed system does not have to be completely integrable, it can be chaotic as discussed in Paragraph 6.2.

Despite the chaos that can be present in these systems many of the invariant tori of the ideal situation with H_0 are preserved for small perturbations. This idea is made explicit by the following theorems, which can both be referred to as KAM-theorem.

Theorem 5 ([11] page 219 Theorem 4.8.1)**.** *Let the Hamiltonian system be as defined above and let J be the set of invariant closed curves of the Poincaré map P_0. If*

$$\frac{d\iota(p^r)}{dp^r} \neq 0 \tag{6.4}$$

and if ε is sufficiently small, then the Poincaré map P_ε has a set J_ε of positive Lebesgue measure $\mu(J_\varepsilon)$ of invariant closed curves close to those in J. Moreover, $\lim_{\varepsilon\to 0} \mu(\varepsilon) = \mu(J)$. All surviving closed orbits in J_ε have irrational averaged rotational transform (Definition 14).

Equation 6.4 means that the reduced system is nondegenerate, as in Definition 13.

The result of this theorem is important as it gives global information about the perturbed system, even if it is locally chaotic. Another important aspect is the description of invariant manifolds by the frequency of the flow on these manifolds, the rotational transform ι. In plasma physics the q-factor is often used to describe the frequency. q is related to ι by $q = 1/\iota$.

In terms of invariant tori this theorem states that nearly all invariant tori survive, although all invariant tori with rational ι are destroyed. This shows the fractal structure of the set J_ε. The invariant tori that do survive are often called *KAM-tori*.

To make this more precise, a second, more technical theorem will be stated, explaining the same idea. The theorem uses the C^s norm for s times differentiable functions, which is defined as:

$$\|f(p^r,q^r)\|_s := \sum_{k=0}^{s} |D^k f(p^r,q^r)| \tag{6.5}$$

Theorem 6 ([11] pages 219 - 220 Theorem 4.8.2)**.** *Let an area preserving map P_0 be given and let P_ε be another area preserving map which is an ε-small perturbation of P_0. Assume $\iota(p^r) \in C^s; s \geq 5$ and $\iota'(p^r) \geq \nu > 0$ on an annulus $\mathcal{R} = \{(p^r, q^r) | a \leq p^r \leq b\}$.*

Then there exists a δ depending on $\varepsilon, \iota(p^r)$ such that if P_ε satisfies

$$\sup_{(p^r,q^r)\in\mathcal{R}} \{\|\varepsilon F\|_r + \|\varepsilon G\|_r\} < \nu\delta \tag{6.6}$$

then P_ε possesses an invariant curve $\Gamma_\varepsilon \subset \mathcal{R}$ of the form

$$p^r = p_0^r + U(\zeta) \; ; \; a < p_0^r < b \; ; \; q^r = q_0^r + V(\zeta) \tag{6.7}$$

where U, V are periodic with period 2π and

$$U, V \in C^1 \; ; \; \|U\|_1 + \|V\|_1 < \varepsilon \tag{6.8}$$

P_ε induces a map on Γ_ε:

$$P_\varepsilon|_{\Gamma_\varepsilon} : q^r \to q^r + 2\pi\lambda \tag{6.9}$$

where λ satisfies the following requirements for some $\alpha, \gamma > 0$.

$$\forall m, n \in \mathbb{N}_{>0} : \left|\lambda - \frac{m}{n}\right| \geq \gamma m^{-\alpha} \tag{6.10}$$

Each λ in the range of $\iota(p^r)$ satisfying (6.10) corresponds to an invariant curve.

The invariant curves of P_0 satisfying condition 6.10 are called *nonresonant*, while other invariant curves are *resonant*. The same nomenclature is used for the corresponding invariant tori.

Note that the area preservation by P_ε is essential, otherwise invariant curves do not have to exist.

Remark 7. *Theorem 6 explicitly describes sufficient criteria for an invariant torus to be preserved: the invariant tori with relatively irrational rotational transform are preserved, while those with (nearly) rational rotational transform may be destroyed.*

Remark 8. *For both theorems the criteria for the preservation of invariant manifolds are valid for "ε small enough", which can be far smaller than ε found in real systems: these theorems cannot be applied directly to systems found in physics.*

6.2 Island chains in small perturbations

Theorems 5,6 state that many invariant tori with an irrational rotational transform survive, the set of frequencies for which tori with that frequency survive is of positive Lebesgue measure. However, those theorems do not explain what happens to invariant surfaces with (nearly) rational rotational transform. That will be shown in this paragraph.

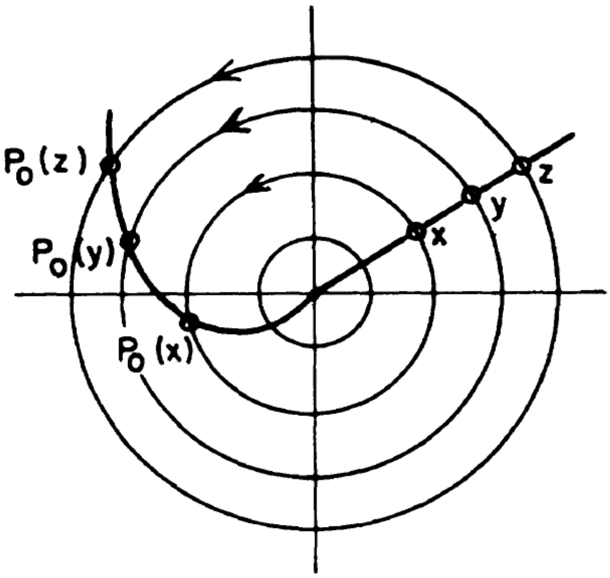


Figure 6.1: Shear shown in a trasformed phase plane. p^r is the distance from the origin, while q^r is the angle around it. Figure from [11].

6.2.1 Presence of island chains described with the Poincaré map

The following derivation is based on pages 220 - 222 of Reference [11].

For a fixed $\iota_0 = \frac{m}{n} \in \mathbb{Q}$ let two frequencies ι_l, ι_h be given which satisfy 6.10 such that $\iota_l < \iota_0 < \iota_h$. Both ι_l, ι_h correspond to invariant curves of the perturbed system forming a boundary for the behaviour described below.

For the ideal system with $\varepsilon = 0$ the values $\iota_l, \iota_0, \iota_h$ correspond to tori where p^r is equal to p^r_l, p^r_0, p^r_h. It follows that points with p^r_l rotate less than $2\pi\iota_0$ in q^r during one application of the Poincaré map (which is a 2π step in t^r). Points with $p^r = p^r_0$ rotate exactly $2\pi\iota_0$, while points with p^r_h rotate more than $2\pi\iota_0$. This shear is shown in Figure 6.1.

Under the Poincaré map P_0 the points with $p^r = p^r_0$ have periodic orbits with period n, they rotate $2\pi m$ in q^r under the map P^n_0. Because q^r is defined modulo 2π this means that they are fixed points of P^n_0. Points with p^r_l, p^r_l rotate less respectively more than $2\pi m$ in q^r under P^n_0. This is shown in Figure 6.2.

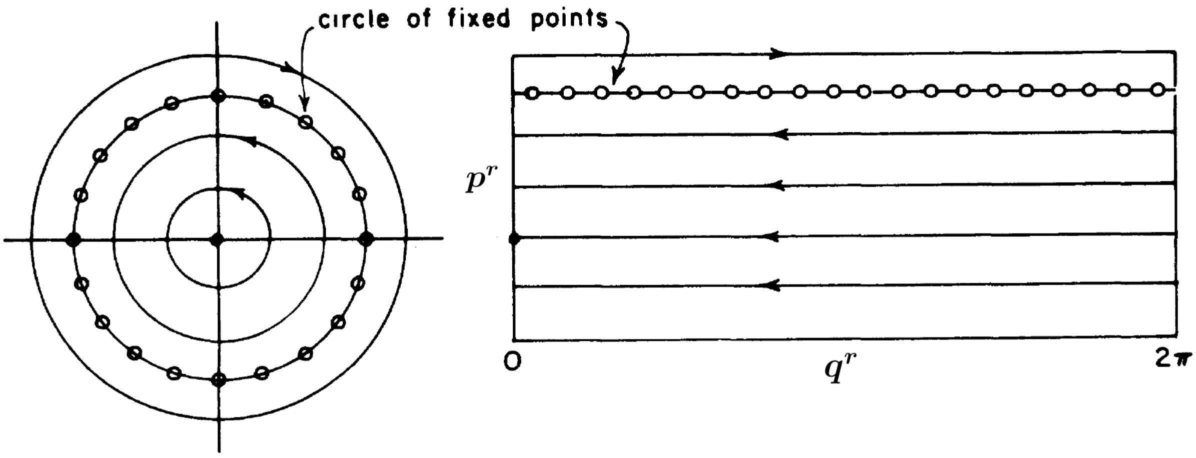


Figure 6.2: *Shear shown in P_0^n, both in phase space (right) and in a transformed phase space (left) where p^r is the distance from the origin and q^r is the angle around it. Figure from [11].*

Under the perturbed Poincaré map P_ε there are closed curves close to $\{p^r = p_l^r\}, \{p^r = p_h^r\}$ for which the averaged rotational transform is ι_l, ι_h, respectively. This implies that orbits on those invariant manifolds rotate $2\pi\iota_l, 2\pi\iota_h$ under the map P_ε and $2\pi n\iota_l, 2\pi n\iota_h$ under P_ε^n. Because $\frac{dq^r}{dt^r}$ is a smooth function of p^r, q^r of which the derivative is assumed to be nonzero, there is a unique p^r for each q^r such that P_ε^n rotates the point (p^r, q^r) with $2\pi n\iota_0 = 2\pi m \equiv 0$ in q^r. The smoothness implies that the points that rotate $2\pi m$ under P_ε^n form a smooth closed curve Γ_ε (Figure 6.3). The smoothness of P_ε with respect to ε implies that $\lim_{\varepsilon\to 0} \Gamma_\varepsilon = \Gamma_0 = \{p^r = p_0^r\}$.

The image $P_\varepsilon(\Gamma_\varepsilon)$ is again a closed curve, which encloses the same area as Γ_ε because P_ε is area preserving. The same is true for P_ε^n. In the general case $\Gamma_\varepsilon, P_\varepsilon^n(\Gamma_\varepsilon)$ will intersect with transverse intersections, which implies that the amount of intersection points is even.

Each intersection point is a fixed point under P_ε^n, which is shown by the following argument. Intersection points lie on Γ_ε which means that the image of any intersection point x lies on $P_\varepsilon^n(\Gamma_\varepsilon)$. Besides that q^r of $P_\varepsilon^n(x)$ differs from that of x by $2\pi m \equiv 0$. The only point for which q^r is equal to the value q^r of x and $P_\varepsilon^n(x)$ lying on $P_\varepsilon^n(\Gamma_\varepsilon)$ is x itself, which means that any intersection point x is a fixed point of P_ε^n. That implies that the intersection points have periodic orbits under P_ε.

Poincaré has proven a more precise description of the number of intersection points in Reference [16], which states that there are $2mk$ intersection points for some $k \in \mathbb{N}_{\geq 1}$. The same result can be derived more constructively using Melnikov's method, as described in Reference [11] on page 223 Theorem 4.8.3.

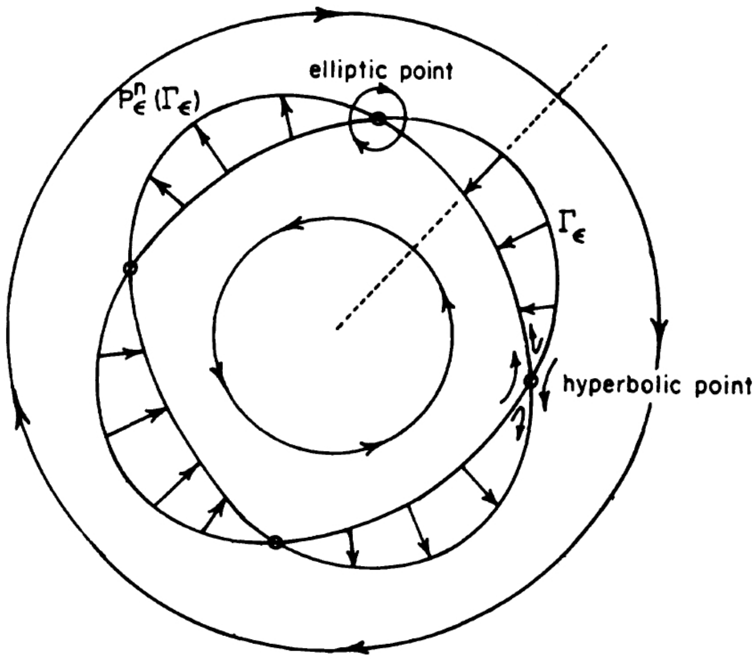


Figure 6.3: *Visualisation of the argument of this subparagraph proving the existence of fixed points in small perturbations of a completely integrable system at manifolds of rational rotational transform. p^r is the distance from the center, while q^r is the angle around it. The dashed line shows a line of constant q^r, under P_ε^n points on Γ_ε move along this line. Figure from [11].*

The behaviour of the fixed points of P_ε^n can be classified by the eigenvalues λ_1, λ_2 of the linearised system. The area preservation of P_ε^n implies that $|\det P_\varepsilon^n| = 1$, which implies that $\lambda_1\lambda_2 = 1$. This gives two possible types of fixed points: hyperbolic fixed points with $\lambda_1, \lambda_2 \in \mathbb{R} : 0 < \lambda_1 < 1 < \lambda_2$ or elliptic fixed points with $\lambda_2 = \bar{\lambda_1}; |\lambda_1| = |\lambda_2| = 1$. It follows from a closer inspection of Figure 6.3 that half of the fixed points is hyperbolic and half is elliptic, such that each elliptic fixed point lies between hyperbolic fixed points, and vice versa. This is shows that the system contains island chains covered in Chapter 5.

Corollary 9. *If the Poincaré map is derived from a continuous Hamiltonian dynamical system, then the hyperbolic and elliptic nature of the periodic points of P_ε which are fixed points of P_ε^n transfers to the Hamiltonian dynamical system: the continuous periodic orbits represented by those points of the Poincaré map also show a hyperbolic respectively elliptic behaviour.*

6.2.2 The chaotic structure of island chains

Orbit types resulting from transverse intersections

For general perturbed Hamiltonian systems different from the completely integrable model described in Paragraph 5.1 the stable and unstable manifolds do not coincide, they can intersect. If there is a transverse intersection of a stable and an unstable manifold, then there are some heteroclinic orbits, which converge to two different fixed points for $t^r \to \pm\infty$. The other points lying on the stable manifold do not converge for $t^r \to -\infty$. Similarly the other points on the unstable manifold do not converge for $t^r \to \infty$. The structures arising from transverse intersections of stable and unstable orbits are called *homoclinic* or *heteroclinic tangles*, which are well described in Chapter 5 of Reference [11]. It describes the theory for homoclinic tangles, but the theory applies equivalently to heteroclinic tangles as well. Some results will be stated by Theorem 10 and will be used to show that the system is chaotic conform Definition 15.

Theorem 10 ([11] page 110 Proposition 2.4.1, page 235 Theorem 5.1.2 and page 252 Theorem 5.3.5)**.** *Let $P_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with fixed point p and a point $q \neq p$ which is a transversal intersection of the stable and unstable manifold of p. Then P_ε has a* hyperbolic invariant set Λ *containing a dense orbit, a countable set of periodic orbits with arbitrarily large periods and an uncountable set of nonperiodic motions. The periodic orbits are all of hyperbolic type and they are dense in Λ.*

More technically $P_\varepsilon|_\Lambda$ is topologically equivalent to a subshift of finite type. This implies that there is an $N \in \mathbb{N}_{>0}$ such that for $n \geq N$ the map P_ε^n is topologically equivalent to a shift on two symbols which gives it a horseshoe. This implies that P_ε^n has a countable set of periodic orbits containing orbits of all periods.

This theorem states the existence of a hyperbolic invariant set, a formal definition is given in Reference [11] page 238 Definition 5.2.6. Chapter 5 of [11] studies the properties of hyperbolic structures.

Corollary 11. *The results of Theorem 10 for the Poincaré map can be translated to the Hamiltonian system, giving it a hyperbolic invariant set Λ', having a dense orbit, a countable set of periodic orbits that is dense in Λ' and an uncountable set of nonperiodic orbits.*

Corollary 12 ([11] page 224 Corollary 4.8.5)**.** *Hamiltonian systems for which the Poincaré map P_ε satisfies the conditions of Theorem 10 are not completely integrable.*

The idea behind the proof of this corollary is as follows: if the system is completely integrable there exists an invariant of the Hamiltonian sytem and the orbits lie on the manifolds in phase space on which the invariant is constant, which are of lower dimension than phase space itself. There is no submanifold of phase space of dimension lower than phase space itself such that the dense orbit lies on that manifold.

Chaos

The orbit types present in the hyperbolic invariant set show that the motion around the transversely intersecting (un)stable manifolds is "chaotic", that concept will now be defined for the flow of a Hamiltonian system (paragraph 2.2).

Definition 15 ([9] pages 243 - 246)**.** *A flow φ is* chaotic *on a compact invariant set Λ if it has the following two properties:*

- topological transitivity*: for every pair of open sets $X \supset U, V \neq \emptyset$ there is a $t > 0$ such that $\varphi_t(U) \cap V \neq 0$.*
- a sensitive dependence on initial conditions*: there is a fixed d such that $\forall x \in \Lambda, \forall \zeta > 0$ there is a nearby $y \in B_\zeta(x) \cap \Lambda$ such that $|\varphi_t(x) - \varphi_t(y)| > d$ for some $t \geq 0$.*

Remark 13. *Note that this definition of chaos does not contradict the fact that φ can be deterministic. The chaos is a feature emerging from the properties of φ, no stochasticity is involved.*

Corollary 14. *The results of Corollary 11 imply that the structure around the (un)stable manifolds is chaotic.*

Structural stability

It is important to know whether the behaviour described in Theorem 10 persists under small perturbations of the system. A form of such persistence is structural stability.

Definition 16 ([11] page 39 Definition 1.7.4)**.** *A map $f : \mathbb{R}^n \to \mathbb{R}^n; f \in C^r$ is* structurally stable *if there is an $\varepsilon > 0$ such that all C^1, ε perturbations of f are topologically equivalent to f.*

Proposition 15 ([11] page 110 Proposition 2.4.1)**.** *The map $f|_\Lambda$ is structurally stable.*

The existence of transverse intersections

The existence of transverse intersections can be proven using Melnikov theory. A good description is given in Paragraphs 4.5 - 4.7 of Reference [11]. Numerical calculations using Wolfram Mathematica 10.3 have been used to show that for the system with Hamiltonian 5.33 the Melnikov function of the separatrix has a simple zero if the the $\delta, \varepsilon > 0$ case is seen as a perturbation of $\delta > 0, \varepsilon = 0$. This proves the existence of transverse intersections for this model.

The specific structure of the systems corresponding to divergenceless vector fields described in Chapter 6 (for example volume preservation in the Hamiltonian system) seems to imply that there has to be a simple zero in the Melnikov function of the separatrix. In the current variables the orbits on the separatrix do not have a simple analytic description without elliptic integrals, which makes it difficult to analytically prove the existence of a simple zero of the Melnikov function. In this project it is left as an open question.

Independent of the existence of a formal proof the presence of homo- and heteroclinic tangles is an important topic in plasma physics. An example of an article showing how this theory applies to plasma physcis is Reference [17], which describes software that detects the structure of interest. Besides that the book [13] gives an extensive overview of chaotic structures with examples from many plasma reactors.

6.3 The global structure of perturbations of completely integrable Hamiltonian systems

This paragraph brings the results of Paragraphs 6.1,6.2 together to describe the global structure of such systems. A similar description is given in Reference [8].

The global structure for small perturbations

The structure of the unperturbed system has been described in Chapter 2, while examples of small perturbations to that model have been described in Paragraph 5.2. For such perturbations with ε of the order needed for Theorems 5,6 described by a Hamiltonian H^r and Poincaré map P_ε the global structure is as follows.

At tori where the averaged rotational transform is rational chains of elliptic and hyperbolic periodic orbits emerge. The behaviour on the boundary between orbits with an elliptic motion and orbits which behave like in the unperturbed system is determined by the stable and unstable manifolds and how they intersect. In general these intersections are transverse, which implies that the boundary is a chaotic region. The observed chaos does not contradict the deterministicity of the system.

Such a chaotic region exists near all tori of rational averaged rotational transform, which means that the system has a countable collection of chaotic regions. The result of Theorems 5,6 is that there is a set of positive measure of frequencies for which the invariant closed orbits survive under the perturbation. These invariant manifolds form a boundary for the chaos which gives the system countably many perfectly separated chaotic regions that are small. As there is a countable family of chaotic island chains between any two island chains, they are not of equal width. The island width depends on the rotational transform $\iota = \frac{m}{n}$: small m, n give "very rational" ι corresponding to wide island chains, while larger m, n correspond to very thin chains.

Islands in islands

The orbits circling around the elliptic periodic orbits (fixed points of P_ε^n) essentially have the same structure as the total system: all field lines circle around an island core field line with a toroidal motion. The ideal motion inside island chains as described by Paragraph 5.1 is completely integrable and describes a toroidal motion. That means that the behaviour inside an insland chain can be modelled by a similar Hamiltonian system as the motion of the whole system. This gives the same structure as for the whole system, but at a smaller scale: there are preserved invariant tori and chaotic island chains inside island chains, et cetera. Thus such a chaotic structure exists at any scale. However, only the largest scales have an influence on the global structure, as the small scale chaos is restricted to small regions bounded by invariant manifolds.

The effect of increasing perturbations

So far ε has been a fixed parameter, chosen to be "small enough" for the desired results to be true (Remark 8). When ε is increased the size of the chaotic region following from homoclinic tangles is increased. As a result invariant manifolds are destroyed. The remainder of an invariant torus is an invariant set that is a strict subset of a torus. It has the structure of a cantor set, containing infinitely many small holes. These sets are often called *cantori*, referring to the strong relation with KAM-tori. Although most of the structure of KAM-tori and cantori coincides, cantori do have holes by which neighbouring chaotic regions are connected.

For perturbations just large enough for the invariant manifold to be destructed the area of the holes in the cantorus is very small with respect to the area of the original KAM-torus, which means that the cantorus acts as a relatively strong but imperfect separation between chaotic regions. That explains why after the KAM-torus has been destroyed a KAM-torus-like structure can be observed, despite the fact that there are orbits crossing the cantorus. For larger perturbations the area of holes increases and the separating property decreases. All invariant manifolds will be destroyed in this way, generally the tori with the "most irrational rotational transform" will survive the longest.

If the destruction of invariant tori has lead to large scale chaos, its dynamics is best described in terms of statistical quantities. Two important quantities are the *diffusion constant D* which describes the diffusion through cantori and the *Kolmogorov entropy* which describes the rate at which neighbouring orbits diverge ([14] pages 18 - 26).

6.4 Estimating the destruction or preservation of a specific invariant manifold with a renormalisation method

Although there is no rigorous mathematical theory that can prove the preservation of specific invariant manifolds for large perturbations, nonrigorous estimates can be used to show the preservation in specific practical situations. One way to give such estimates is by a renormalisation method, based on the idea that a certain structure is shows up on different scales. An early example of such an approach is described in Reference [8]. Their method is explained for the Hamiltonian system of Paragraph 5.2, but it can also be used for general perturbations of two degree of freedom systems by using local approximations (Section 6 of [8]).

The method uses the fact that for each invariant torus there are chaotic island chains on both sides of the invariant torus, regardless of the length scale which is used perpendicular to the torus. Starting with perturbations by two main resonances causing wide, nonoverlapping island chains two smaller resonances are determined which give island chains closer to a possibly invariant manifold. Those two resonances are close to the manifold which supports the approximation to neglect all other resonances. The strength of those two resonances can be used to estimate whether they give one combined chaotic region where an invariant manifold cannot exist, or whether the manifold of interest can exist between two small chaotic island chains. In the last case the renormalisation can be repeated as the structure is similar to the one started with. This is shown in Figure 6.4.

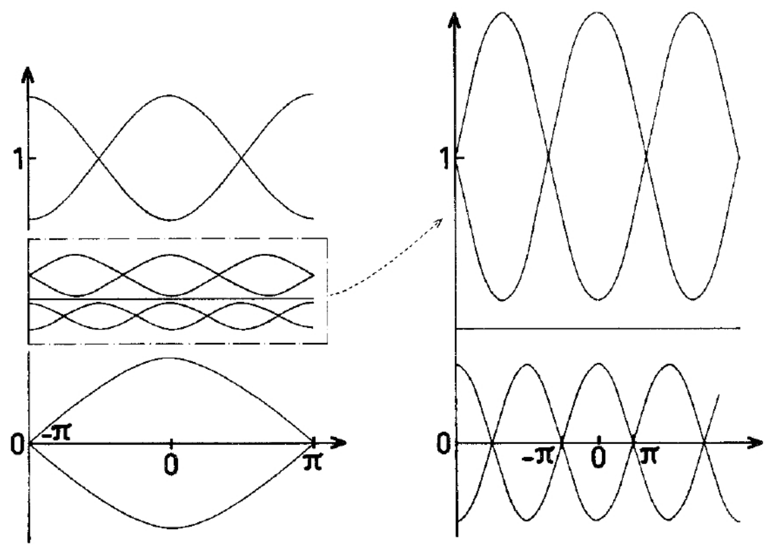


Figure 6.4: *The same structure occurs at different length scales, which is the basis for the renormalisation method. Figure from [8].*

This method has been used to show that many manifolds with "very irrational" rotational transform persist for some perturbations of realistic size. In that sense the structure of chaotic island chains bounded by invariant manifolds is also rather stable for realistic perturbation sizes.

Chapter 7

Hamiltonian description of nonstationary plasmas

So far the translation of plasma physics to Hamiltonian systems was given by the application of the construction of paragraphs 4.1 - 4.3 to the Sagdeev fields, which are the magnetic field of stationary solutions of ideal MHD. It has been shown that after the introduction of shear perturbations to the resulting Hamiltonian system give chaotic island chains bounded by invariant manifolds. This chapter shows how these mathematical results can be used to explain the behaviour of realistic time-dependent plasmas.

7.1 The description of a time dependent magnetic field by a Hamiltonian system

The basic idea is that a construction like that of Paragraphs 4.1 - 4.3 can be applied to "snapshots" of a nonstationary nonideal plasma situation on a specific moment in time. If τ is the real time in which the plasma evolves, then for every $\tau \in \mathbb{R}$ the magnetic field $\vec{B}(\tau)$ is a divergenceless vector field that can be described by a Hamiltonian system. Other divergenceless vector fields such as the velocity field of an incompressible fluid can be used instead.

The Hamiltonian system has a Hamiltonian time variable t, which should not be confused with the physical quantity τ. Via the construction of Chapter 4 the Hamiltonian variable t corresponds to an angular coordinate field on $\mathbb{R}^3$. In a similar way many elements of the Hamiltonian system have counterparts in terms of the vector field $\vec{B}$, the translation between the two has been summarised in Tables 7.1,7.2. In the rest of this paragraph the one degree of freedom system of Table 7.2 will be used.

Two degree of freedom Hamiltonian system	**Vector field $\vec{B}(\tau_0)$ on $\mathbb{R}^3$ at the instant τ_0**
q_1, q_2 Hamiltonian variables: generalised coordinates	q_1, q_2 Angular coordinate fields
p_1 Hamiltonian variable: conjugate momentum	p_1 No geometric or physical interpretation
p_2 Hamiltonian variable: conjugate momentum	p_2 Component of the vector potential, coordinate field describing distance from core
t Hamiltonian variable: time	Via $q_1 \equiv t$ Angular coordinate field
h An invariant	h Component of the vector potential
H The Hamiltonian	H No geometric or physical interpretation
$p_1^t(t), q_1^t(t), p_2^t(t), q_2^t(t)$ Orbit of the Hamiltonian system	$q_1^t(t) \equiv t, p_2^t(t), q_2^t(t)$ Parameterisation of a $B(\tau_0)$ field line
Invariant manifold in phase space (torus)	Inpenetrable manifold in $\mathbb{R}^3$, giving confinement
Completely integrable Hamiltonian system: phase space is foliated by invariant manifolds	Space is foliated with manifolds, field lines lie on such manifolds
Completely integrable Hamiltonian systems with Hamiltonian 4.29	The Sagdeev fields

Table 7.1: *Summary of a correspondence between a two degree of freedom Hamiltonian system and a divergenceless vector field $\vec{B}$ at the moment τ_0.*

One degree of freedom Hamiltonian system	**Vector field $\vec{B}(\tau_0)$ on $\mathbb{R}^3$ at the instant τ_0**
q^r Hamiltonian variable: generalised coordinate	q^r Angular coordinate field
p^r Hamiltonian variable: conjugate momentum	p^r Component of the vector potential, coordinate field describing distance from core
t^r Hamiltonian variable: time	t^r Angular coordinate field
H^r The Hamiltonian	H^r Component of the vector potential
$p^{r,t}(t^r), q^{r,t}(t^r)$ Orbit of the Hamiltonian system	$p^{r,t}(t^r), q^{r,t}(t^r)$ Parameterisation of a $B(\tau_0)$ field line
Invariant manifold in phase space (closed curve)	Intersection of an inpenetrable manifold in $\mathbb{R}^3$ with a plane
Completely integrable Hamiltonian systems with Hamiltonian 4.35	The Sagdeev fields

Table 7.2: *Summary of a correspondence between a reduced one degree of freedom Hamiltonian system and a divergenceless vector field $\vec{B}$ at the moment τ_0.*

A correspondence as described in Table 7.2 exists for every τ. If it is assumed that for all $\tau \in [\tau_{\text{min}}, \tau_{\text{max}}]$ two coordinate functions q^r, t^r independent of τ satisfy the prerequisites stated in Paragraph 4.1 then both p^r, H^r can be described in terms of the same q^r, t^r for all $\tau \in [\tau_{\text{min}}, \tau_{\text{max}}]$. In general p^r, H^r depend on τ, by using fixed q^r, t^r they can be compared for different $\tau \in [\tau_{\text{min}}, \tau_{\text{max}}]$.

As we are interested in the structure of $\vec{B}(\tau)$, rather than its exact value, it is enough to take into account the direction of $\vec{B}(\tau)$. It follows from the assumption that $\vec{B} \cdot \nabla t^r \neq 0$ made in Paragraph 4.1 that the direction of $\vec{B} = \nabla p^r \times \nabla q^r - \nabla H^r \times \nabla t^r$ can be described by H^r alone. That means that it is possible to use a fixed p^r and a Hamiltonian $H^r(\tau)$ that depends on τ, which can be seen as a parameter of the system.

A similar correspondence between a time dependent divergenceless vector field $\vec{B}(\tau)$ and a Hamiltonian system is given in Reference [3]. The variables constructed in that article are also independent of the exact expression of the vector field, they only depend on the symmetry of the Hamiltonian system.

7.2 Small perturbations of ideal time-dependent vector fields studied using Hamiltonian systems

For many nonideal time-dependent plasmas the difference with a solution of ideal MHD is small. That means that the Hamiltonian $H^r(p^r, q^r, t^r; \tau)$ corresponding to the nonideal vector field $\vec{B}(\tau)$ is a small perturbation of $H_0^r(p^r, q^r, t^r)$, the Hamiltonian corresponding to the ideal vector field $\vec{B}_0$. As an example H_0^r can be taken to be completely integrable with invariant tori in phase space, and $H^r(\cdot; \tau)$ can be taken as in Equation 5.32 with $\delta(\tau), \varepsilon(\tau)$.

If the perturbation is small enough the theory of Chapter 6 describes the structure of the system with Hamiltonian $H^r(\cdot; \tau)$. The theory predicts chaotic island chains close to surfaces where H_0^r has rational rotational transform and many invariant tori surviving the perturbation, the chaotic regions are bounded. As discussed in Paragraph 6.3 the island chains are thin for large m, n ($\frac{m}{n} = \iota$). Therefore mainly the island chains with small m, n are visible in real plasmas and simulations: many of the other island chains are too thin to be distinguished from orbits lying on invariant manifolds.

That description is true for all $\tau \in [\tau_{\min}, \tau_{\max}]$ for which $H^r(\cdot; \tau) - H_0^r(\cdot)$ is small enough. For physical plasmas this difference can be much larger than allowed by the theory in paragraphs 6.1,6.2 (Remark 8). However, as described by Paragraph 6.4 it is expected that many invariant manifolds are preserved for larger perturbations as well, giving a similar global behaviour. Besides that it has been shown that the structure of the chaotic heteroclinic tangles is structurally stable (Proposition 15), while the results Paragraph 6.4 estimate that many invariant tori will be preserved. In that sense the structure described in Chapter 6 is rather stable.

So far the theories of chapter 6 have only been used to explain the behaviour of a plasma for specific τ. Besides that it can be assumed that the physical forces acting on $\vec{B}(\tau)$ described by RMHD are small, which implies that $H^r(\cdot; \tau) - H_0^r(\cdot)$ will change relatively slowly as a function of τ. In combination with the relatively stable behaviour of the structure described above this can be seen as an explanation for the fact that the structure of chaotic island chains bounded by invariant manifolds is observed often and is rather stable in nonideal plasmas. Note that this explanation depends on the results of Paragraph 6.4, therefore this explanation can be seen as a good estimate, but it is not mathematically rigorous. In order to make this stability estimate rigorous those time dependent fields should be studied as solutions of the time dependent system (MHD), as discussed in Paragraph 7.3.

Remark 16. *In the previous description $\vec{B}_0$ does not have to be independent of τ, it can also be taken to correspond to a time dependent solution of IMHD. That implies that $H_0^r(\tau)$ depends on the parameter τ as well.*

Bifurcations

As $H(\cdot;\tau)$ evolves as a function of τ the Hamiltonian can undergo structural changes, for example the period doubling of an island chain (Paragraph 5.1.3) or the destruction of an invariant manifold. In terms of dynamical systems theory these structural changes are called *bifurcations* with τ acting as a *bifurcation parameter*.

Thus the real time-dependence of $\vec{B}$ can be included in the Hamiltonian description by adding τ as a bifurcation parameter. The correspondence that has been described in this paragraph is summarised in Table 7.3.

Hamiltonian system	**Vector field $\vec{B}(\tau)$ on $\mathbb{R}^3$**
t or t^r Hamiltonian variable: time	t or t^r Angular coordinate field
Orbits (functions of t or t^r)	Field lines parameterised by t or t^r
τ Bifurcation parameter	τ Time (physical quantity)
Bifurcations as a function of the parameter τ	Structural changes in the time evolution

Table 7.3: *Summary of a correspondence between a Hamiltonian system with a bifurcation parameter and a time dependent divergenceless vector field.*

7.3 Magnetohydrodynamics as an infinite-dimensional dynamical system

The system of partial differential equations defining ideal magnetohydrodynamics (Equations 1.1 - 1.5) can be seen as an infinite-dimensional dynamical system. Stationary solutions of that system have the role that fixed points have for finite-dimensional dynamical systems: stationary solutions can be stable or unstable and if the time dependent system converges, the limit is a stationary solution. That makes stationary solutions of IMHD important for the time-dependent system, on the other hand the stability in the time dependent system determines the importance of stationary solutions. Therefore studying the PDE-system has a large potential to improve the description and explanation of the plasma behaviour, at the same time it is mathematically far more difficult than its finite-dimensional counterpart. As an example of the complexity there are many types of waves coming into play. The nonlinearity of the IMHD equations implies that the interaction of waves with other structures is non-trivial, it is not enough to study waves separately.

7.3.1 Ideal magnetohydrodynamics as a Hamiltonian field theory

A possible direction for further research is to use Hamiltonian field theory to study the PDE's that define IMHD. Just as the field without time dependence can be described by a Hamiltonian system of ordinary differential equations, the partial differential equations that describe time dependent IMHD (1.1 - 1.5) can be written as a Hamiltonian system of PDE's that is the basis for field theory. Equations 7.6,7.7 look similar to their finite-dimensional counterparts, but where $H, p = (p_1, p_2), q = (q_1, q_2), t$ used to be scalars and vectors, the infinite-dimensional variables $\mathcal{H}, \vec{\mathcal{P}}, \vec{\mathcal{Q}}, t$ are scalar fields and vector fields on $\mathbb{R}^3$. Therefore $\mathcal{H}$ is called a Hamiltonian density, as it is a function of $\vec{\mathcal{P}}, \vec{\mathcal{Q}}, t$ that also depends on real space $\mathbb{R}^3$. In other words, 7.6,7.7 represent equations at all points of $\mathbb{R}^3$.

In this project the translation of the IMHD equations to an infinite-dimensional Hamiltonian system has been studied parallel to the approach of Chapter 4 to describe plasma physics by a Hamiltonian system. As the field theory is much more complex than the ODE system, the field theoretical approach has not been pursued any further after after the correspondence between the IMHD equations and Hamiltonian field theory had been established. Table 7.4 will give a summary of this correspondence as the result of this explorative research, but the correspondence will first be explained.

One way to write IMHD as a Hamiltonian field theory is described in Reference [18]. It uses a slightly different description of IMHD than that of Chapter 1. In that chapter the plasma was described by $\vec{B}, \vec{v}, p, \rho$, whereas Reference [18] uses $\vec{B}, \vec{v}, \rho, s$ where s is the specific entropy, the entropy per unit mass. Both descriptions of IMHD are assumed to be equivalent, the latter description will be used in this paragraph. In terms of $s, \rho, \vec{B}$ IMHD is defined by the following equations:

$$\partial_t s + \vec{v} \cdot \vec{\nabla} s = 0 \tag{7.1}$$

$$\partial_t \rho + \vec{\nabla} \cdot (\vec{v}\rho) = 0 \tag{7.2}$$

$$\partial_t \vec{B} + \vec{\nabla} \cdot (\vec{v}\vec{B}) - \vec{\nabla} \cdot (\vec{B}\vec{v}) = 0 \tag{7.3}$$

The set of generalised coordinates and conjugated momenta are defined as

$$\vec{\mathcal{Q}} := (s, \rho, \vec{B}) \tag{7.4}$$

$$\vec{\mathcal{P}} := (\alpha, \beta, \vec{\gamma}) \tag{7.5}$$

where in the construction of Reference [18] $\alpha, \beta, \vec{\gamma}$ are defined as Lagrange multipliers of $s, \rho, \vec{B}$, they do not correspond to physical quantities.

In terms of $\vec{\mathcal{Q}}, \vec{\mathcal{P}}$ Equations 7.1 - 7.3 can be written as

$$\partial_t \vec{\mathcal{Q}} = \frac{\partial \mathcal{H}}{\partial \vec{\mathcal{P}}} \tag{7.6}$$

$$\partial_t \vec{\mathcal{P}} = -\frac{\partial \mathcal{H}}{\partial \vec{\mathcal{Q}}} \tag{7.7}$$

$$\mathcal{H} = \frac{1}{2\rho}(\alpha \nabla s - \rho \nabla \beta - \nabla \vec{\gamma} \cdot \vec{B} + \vec{B} \cdot \nabla \vec{\gamma})^2 + \rho U(\rho, s) + \frac{1}{2} B^2 \tag{7.8}$$

$$-\rho \vec{v} = (\alpha \nabla s - \rho \nabla \beta - \nabla \vec{\gamma} \cdot \vec{B} + \vec{B} \cdot \nabla \vec{\gamma}) \tag{7.9}$$

where $U(\rho, s)$ is some potential energy. More details of the construction and the explicit derivation of Equations 7.8,7.9 can be found in Reference [18]. It also describes the role of $\alpha, \beta, \vec{\gamma}$ and how they define the gauge group of the system: the only restriction on $\alpha, \beta, \vec{\gamma}$ is Equation 7.9. $\alpha, \beta, \vec{\gamma}$ together have five degrees of freedom, while 7.9 poses three restrictions. That can be used to determine trasformations of $\alpha, \beta, \vec{\gamma}$ that leave the physical system invariant - that is the gauge group of this system.

A summary of the previous description is given by the following table.

Hamiltonian field theory	**Magnetohydrodynamics** **Scalar and vector fields on $\mathbb{R}^3$**
$\vec{\mathcal{Q}} = (s, \rho, \vec{B})$ Generalised coordinates	$s, \rho, \vec{B}$ Specific entropy, mass density and magnetic field
$\vec{\mathcal{P}} = (\alpha, \beta, \vec{\gamma})$ Conjugate momenta, give rise to a gauge group	No physical interpretation
$\mathcal{H}$ Hamiltonian density	$\mathcal{H}$ Energy density
$U(s, \rho)$ Term in the Hamiltonian density	$U(s, \rho)$ Potential energy density
$\vec{v}$ Defined by Equation 7.9	$\vec{v}$ Fluid field

Table 7.4: *Summary of a correspondence between a Hamiltonian field theory and ideal magnetohydrodynamics.*

7.3.2 Generalisation to resistive magnetohydrodynamics

The system defined in Subparagarph 7.3.1 corresponds to ideal MHD. If nonidealities are small then resistive MHD can be seen as a small perturbation of IMHD, the nonidealities can be added to Equations 7.1 - 7.3 and should be incorporated as a small perturbation to the Hamiltonian density.

The behaviour of RMHD is different from that of IMHD. In RMHD there are energy losses due to nonidealities like electromagetic resistance and viscosity, therefore nontrivial stationary situations are not expected. However, the simulations described in Reference [2] show structures that seem to be meta-stable. These states are formed quickly from out-of-equilibrium initial conditions of the simulation, once constructed their decay is relatively slow. The fact that similar states are formed out of different initial conditions can be seen as a hint that the structures are attracting. Further research is needed to give a mathematically rigorous proof. An outlook based on current knowledge is given in the next chapter.

Chapter 8

Outlook

This chapter describes the current view on some aspects of simulation results. Many ideas have not been proven yet, it is an outlook and a possible starting point for further research. It is based on the work by and discussions with members of the Dirk Bouwmeester group (Leiden University) and Hugo de Blank (DIFFER and Technische Universiteit Eindhoven). It does not describe the main work of the author.

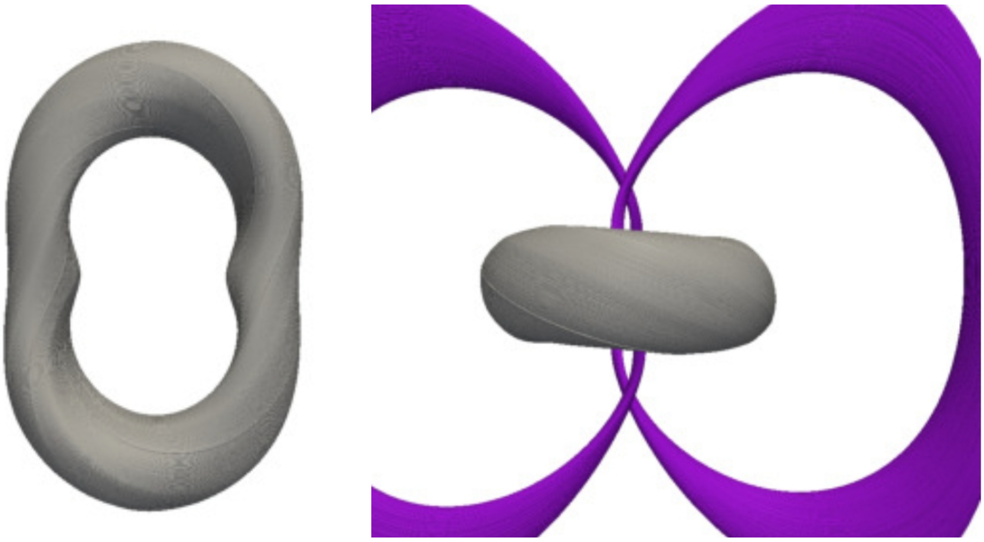
Overview of this chapter

Paragraph 8.1 hypothesises that symmetry determines the slowly decaying structures emerging in plasma simulations. Paragraph 8.2 discusses whether and how the theory of Chapters 6,7 can be applied to nontoroidal structures. Paragraph 8.3 poses the question whether there are solutions of MHD corresponding to the divergenceless vector fields studied in this project and discusses how that is relevant for the results of this project. One way to use such solutions is to study the motion of island chains, for which a hypothesis is given in Paragraph 8.4.

8.1 Symmetry determines shape of slowly decaying structures observed in simulations

In simulations the far-out-of-equilibrium initial conditions are destroyed very quickly, while the resulting structures decay slowly. A possible explanation is that the nonlinear resistances “kill” the higher order, small scale dynamics fast, while the large scale structure persists. The hypothesis is that this large scale slowly decaying structure is determined by the most basic (a)symmetries of the system, in simulations determined by the initial conditions.

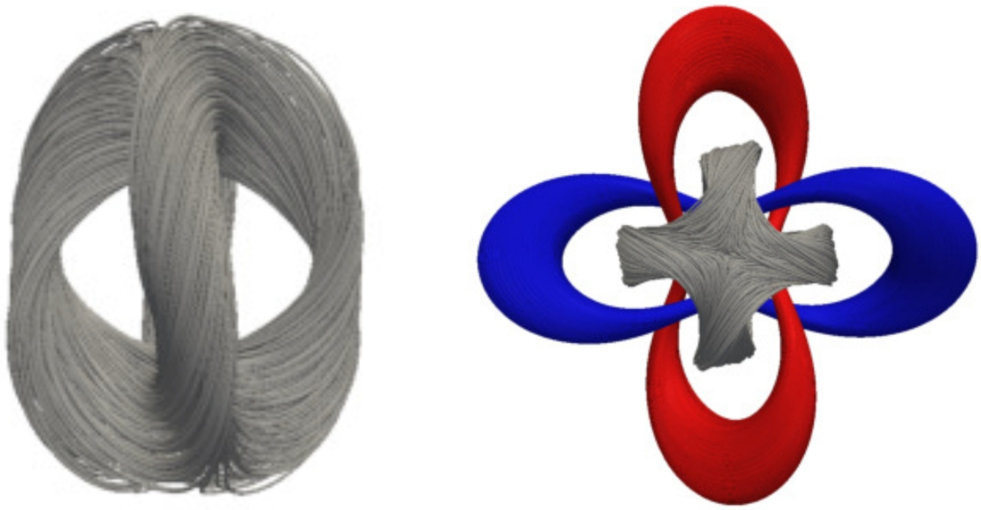
The RMHD simulations starting with linked rings ([1]) and with the Kedia fields with $n_p = 1$ ([2]) all have an overall vertical field strength near the z-axis. Simulations with these initial conditions show a toroidally shaped structure where field lines approximately lie on tori (genus 1).



(a) *(b)*

Figure 8.1: *Structure with field lines lying on surfaces of genus* 1 *(grey). This slowly decaying structure emerged in RMHD simulations with Kedia fields ($n_p = 1$) as initial condition ([2]). Purple field lines show the surroundings of the genus* 1 *surfaces as they arise in simulations and are not the topic of the current discussion. Figure from [2]. (a) Side view (b) Top view*

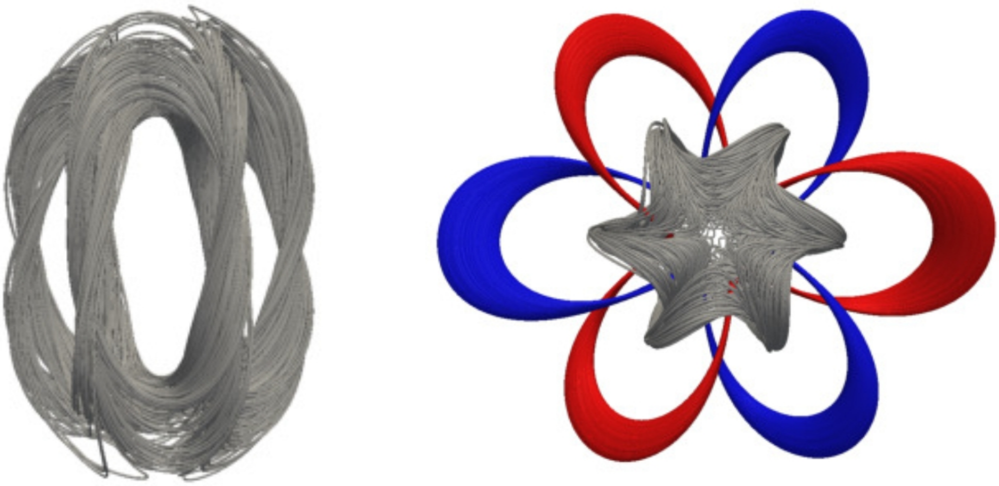
The Kedia fields with $n_p = 2$ have a null line on the z-axis and besides that there is a 180° rotational symmetry around the z-axis. In simulations with those initial conditions a structure appears which also has a null-line on the z-axis and a rotational symmetry around the z-axis, where field lines lie on surfaces of genus 3 (the null-line prehibits the formation of a toroidal structure).



(a) (b)

Figure 8.2: *Structure with field lines lying on surfaces of genus* 3 *(grey). This slowly decaying structure emerged in RMHD simulations with Kedia fields ($n_p = 2$) as initial condition ([2]). Red and blue field lines show the surroundings of the genus* 3 *surfaces as they arise in simulations and are not the topic of the current discussion. Figure from [2]. (a) Side view (b) Top view*

This structure is general for simulations with the Kedia fields as initial condition: for $n_p \geq 2$ there is a null line on the z-axis and $360°/n_p$ rotational symmetry around the z-axis. In the simulations this leads to a structure where field lines lie on surfaces of genus $2n_p - 1$ (a structure with $2n_p$ legs, each leg where the field points upward is neighboured by legs where the field points downward).



(a) (b)

Figure 8.3: *Structure with field lines lying on surfaces of genus* 5 *(grey). This slowly decaying structure emerged in RMHD simulations with Kedia fields (*$n_p = 3$*) as initial condition ([2]). Red and blue field lines show the surroundings of the genus* 5 *surfaces as they arise in simulations and are not the topic of the current discussion. Figure from [2]. (a) Side view (b) Top view*

It would be interesting to see whether a similar on symmetry based structure emerges when an initial condition has null lines on two axes perpendicular to each other and 180° rotational symmetry around both axes (for example the y- and the z-axis). Based on the symmetry a structure with core field lines topologically equivalent to those in Figure 8.4 are be expected.

If this hypothesis of the initial condition symmetry determining the slowly decaying structure is true, then this can be used to classify the possible slowly decaying structures based on the possible symmetries.

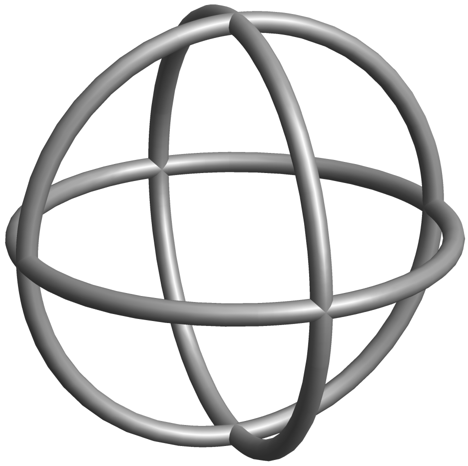


Figure 8.4: *Core with* 180° *rotational symmetry around the y-axis and the z-axis.*

8.2 Applicability of KAM and Melnikov theory to nontoroidal structures

In simulations these structures where field lines lie on surfaces of genus unequal to 1 show the behaviour descriped in Chapter 6: simulations of RMHD ([2]) show chaotic island chains bounded by invariant surfaces. The only difference with the toroidal setting is that the invariant surfaces have genus unequal to 1, they are not toroidally shaped.

That observation was the reason to extend the construction in Paragraphs 4.1 - 4.3 to cover nontoroidal fields and describe them by a toroidal Hamiltonian system, which is done in Paragraph 4.5. The aim was to apply the theory of Chapter 6 to that Hamiltonian system in order prove the similarity with the genus 1 case. The correspondence derived in Paragraph 4.5 depends critically on the symmetry of the field: perturbations of the Hamiltonian system can only correspond to symmetric vector fields, nonsymmetric perturbations of the vector field cannot be studied by direct application of the theory of Chapter 6.

A possible solution is to find another correspondence between the vector field and a Hamiltonian system. The construction of Reference [3] might be useful, and besides that a Hamiltonian system without a toroidal structure can be constructed. The results of Subparagraph 6.2.1 could locally be applied to Poincaré maps of nontoroidal Hamiltonian systems, explaining the occurance of chaotic island chains. However, the theory of Paragraph 6.1 about the preservation of invariant manifolds cannot be transferred that easily. In this project that generalisation to nontoroidal situations is left as an open question.

Despite the predescribed issues the correspondence of Paragraph 4.5 assuming symmetry can still be used for the study of symmetric solutions of MHD, for example for the study of idealisations of symmetric time dependent solutions described in Paragraph 8.1.

8.3 Extension of divergenceless fields to MHD solutions

In Chapter 5 Hamiltonian systems have been used as models for island chains in toroidal divergenceless vector fields. After small corrections the Hamiltonian models also correspond to divergenceless vector fields, as described in Paragraph 5.3. It is an interesting direction for further research to investigate how these vector fields can be extended to solutions of MHD by a suitable choice of a velocity field $\vec{v}$, a scalar pressure p and a density ρ.

The existence of such extensions could show that there exist multiple nonequivalent plasma situations with the same magnetic field. That sheds light on the assumption that the magnetic field describes the structure of the plasma, assumed throughout this thesis. Extensions can also be used to calculate the acting forces, which are assumed to be small in Paragraph 7.2. Besides that they ca be used to study the main motion of island chains, which is described in the next paragraph.

8.4 Motion of island chains

In RMHD simulations of Reference [1] the motion of island chains is largely determined by the profile of the rotational transform, which generally decays over time. A possible explanation for this effect is that the poloidal winding of field lines is an effect on a smaller length scale, which implies that it decays faster under nonlinear resistances and viscosity than the poloidal winding. As a result the ratio of the two decays.

In the RMHD simulations described in Reference [1] the rotational transform is highest at the core (the unit circle) and lowest infinitely far away. The result is that the location of the torus with a fixed rotational transform $\iota_0 = \frac{m}{n}$ moves inward as the rotational transform globaly decays over time. As described in Paragraph 5.3 the island size has to decrease near the core field line, which explains why island chains shrink and disappear at the core field line.

This last statement is based on the assumption that the core field line stays in place. The opposite is also possible, namely that the toroidal shape is deformed, for example if the core field line moves to the border of the toroidal plasma. This is observed, among others, in preliminary results from RMHD simulations by the Dirk Bouwmeester group. In the simulations an $m = 1, n = 1$ island blows up and replaces the main toroidal structure, which shrinks until it is an island chain in the new toroidal structure. This shows that the prediction of shrinking island chains depends on the assumption that the toroidal structure is maintained, as the correspondence between the vector field and the Hamiltonian system is based on that assumption. The stability of the toroidal structure itself is an assumption rather than a result of the given analysis. The study of such structural changes is important as the changes allow violations to the toroidal symmetry and therefore the stability of the predescribed structure. They can be studied as bifurcations of the dynamical system that is described in Chapter 3 if the physical time τ is added as bifurcation parameter. Examples of $m = 1, n = 1$ island chain bifurcations similar to the one mentioned above are described in Chapter 6 of Reference [14].

Chapter 9

Conclusion

The structure of a toroidal plasma has been described by a Hamiltonian system. The Hamiltonian system is based on the magnetic field and uses its divergencelessness to parameterise magnetic field lines, which represent the plasma structure. KAM and Melnikov theory apply to the system explaining the occurance and rather stable behaviour of chaotic island chains bounded by impenetrable manifolds. Hypotheses have been given about the generalisation of this description to nontoroidal plasma structures with the same properties. In addition it has been proposed to use a Hamiltonian field theoretical description of the partial differential equations defining ideal magnetohydrodynamics to study the (meta-)stability of observed plasma structures.

Acknowledgements

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Appendix A

Writing $\vec{B}$ in the required form

The aim of this appendix is to write the divergenceless vector field $\vec{B}$ in the form of Equation 4.11, as needed in Paragraph 4.3. The divergencelessness of $\vec{B}$ implies that there exists a vector potential $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$. $\vec{B}$ will be written in the required form by using a different vector potential $\vec{\tilde{\tilde{A}}}$ for $\vec{B}$ based on $\vec{A}$, which has the form of Equation 4.9.

It is assumed that $\vec{B}$ has been given in a pre-Hamiltonian coordinate system $(c_1, c_2, c_3) = (q_1, \rho, q_2)$ defined in Paragraph 4.1 of which q_1 is the Hamiltonian time coordinate field and q_2 is the other angular coordinate field, field lines lie on manifolds of constant ρ.

Construction of a vector potential for which $A_\rho = 0$

The vector potential $\vec{A}$ can be written in the following form:

$$\vec{A} = A_{q_1} \nabla q_1 + A_\rho \nabla \rho + A_{q_2} \nabla q_2 \tag{A.1}$$

In general A_{q_1}, A_ρ, A_{q_2} are functions of q_1, ρ, q_2. The first step is to find a vector potential $\vec{\tilde{A}}$ for which $\tilde{A}_\rho = 0$ and follows a procedure described on page 10 of Reference [14], it is also used in [13]. The construction uses an auxiliary function G:

$$\begin{aligned} G &:= \int A_\rho d\rho && \text{(A.2)} \\ \nabla G &= \frac{\partial G}{\partial q_1} \nabla q_1 + \frac{\partial G}{\partial \rho} \nabla \rho + \frac{\partial G}{\partial q_2} \nabla q_2 && \text{(A.3)} \end{aligned}$$

Using G equation A.1 can be rewritten as

$$\begin{aligned}\vec{\tilde{A}} &= \tilde{A}_{q_1}\nabla q_1 + \nabla G + \tilde{A}_{q_2}\nabla q_2 &\text{(A.4)}\\ \tilde{A}_{q_1} &:= A_{q_1} - \frac{\partial G}{\partial q_1} &\text{(A.5)}\\ \tilde{A}_{q_2} &:= A_{q_2} - \frac{\partial G}{\partial q_2} &\text{(A.6)}\end{aligned}$$

This leads to the definition of a vector potential for which $\tilde{A}_\rho = 0$:

$$\vec{\tilde{A}} := \vec{A} - \nabla G = \tilde{A}_{q_1}\nabla q_1 + \tilde{A}_{q_2}\nabla q_2 \tag{A.7}$$

$\vec{\tilde{A}}$ and $\vec{A}$ both correspond to the same vector field $\vec{B}$:

$$\nabla \times \vec{\tilde{A}} = \nabla \times \vec{A} - \nabla \times \nabla G = \nabla \times \vec{A} = \vec{B} \tag{A.8}$$

Equation A.7 gives an expression for $\vec{B}$ (from now on the tilde in $\tilde{A}$ will be suppressed in notation).

$$B = \nabla \times \vec{A} = \nabla A_{q_1} \times \nabla q_1 + \nabla A_{q_2} \times \nabla q_2 \tag{A.9}$$

Note that this process starting with a vector potential with the form of Equation A.1 and finding one with the form of equation A.9 works for general coordinate systems and for any vector potential $\vec{A}$.

Implications of the pre-Hamiltonian coordinate system

Now the assumptions about the coordinate system will be discussed. The following calculation is based on Equation A.9.

$$\begin{aligned}\vec{B} \cdot \nabla q_1 &= \nabla A_{q_2} \times \nabla q_2 \cdot \nabla q_1\\ &= \left(\frac{\partial A_{q_2}}{\partial q_1}\nabla q_1 + \frac{\partial A_{q_2}}{\partial \rho}\nabla \rho + \frac{\partial A_{q_2}}{\partial q_2}\nabla q_2\right) \times \nabla q_2 \cdot \nabla q_1\\ &= \frac{\partial A_{q_2}}{\partial \rho}\nabla \rho \cdot \nabla q_2 \times \nabla q_1 &\text{(A.10)}\end{aligned}$$

This implies that the assumptions about the expression in 4.4 are true for $\frac{\partial A_{q_2}}{\partial \rho}$, this is be stated as a lemma.

Lemma 17. *Equation A.10 implies that the expression in 4.4 is equal to $\frac{\partial A_{q_2}}{\partial \rho}$. By the properties of the pre-Hamiltonian coordinate system $\frac{\partial A_{q_2}}{\partial \rho}$ is a nonzero function of ρ that does not depend on q_1, q_2.*

In fact, the assumptions about 4.4 were made to give $\frac{\partial A_{q_2}}{\partial \rho}$ those properties, according to an assumption made in Reference [14]. The rest of this appendix makes the results of this assumption explicit.

The result of Lemma 17 allows the definition of the following function:

$$p_2(\rho) := \int_{\hat{\rho}}^{\rho} \frac{\partial A_{q_2}}{\partial \rho} d\rho = \int_{\hat{\rho}}^{\rho} \frac{\vec{B} \cdot \nabla q_1}{\nabla \rho \cdot \nabla q_2 \times \nabla q_1} d\rho \tag{A.11}$$

The integration constant caused by the choice of $\hat{\rho}$ can be neglected. The following lemma gives properties of this function.

Lemma 18. *Lemma 17 implies that the function $p_2(\rho)$ is strictly monotonic:*

$$\forall \rho : \frac{dp_2(\rho)}{d\rho} = \frac{\partial A_{q_2}}{\partial \rho} \neq 0 \tag{A.12}$$

This gives that range ρ *and* $p_2(\text{range } \rho)$ *are diffeomorphic under the map* $\rho \mapsto p_2(\rho)$.

Construction of a vector potential of the required form

It follows from Lemma 17 that A_{q_2} can be written as

$$A_{q_2}(q_1, \rho, q_2) = p_2(\rho) + f(q_1, q_2) \tag{A.13}$$

for some function f of q_1, q_2. This gives the following expression for $\vec{B}$:

$$\begin{aligned}
\vec{B} &= \nabla \times \vec{A} \\
&= \nabla A_{q_1} \times \nabla q_1 + \nabla A_{q_2} \times \nabla q_2 \\
&= \frac{\partial A_{q_1}}{\partial \rho} \nabla \rho \times \nabla q_1 + \frac{\partial A_{q_1}}{\partial q_2} \nabla q_2 \times \nabla q_1 \\
&+ \frac{\partial A_{q_2}}{\partial q_1} \nabla q_1 \times \nabla q_2 + \frac{\partial A_{q_2}}{\partial \rho} \nabla \rho \times \nabla q_2 \\
&= \frac{\partial A_{q_1}}{\partial \rho} \nabla \rho \times \nabla q_1 + \left(\frac{\partial A_{q_1}}{\partial q_2} - \frac{\partial f}{\partial q_1} \right) \nabla q_2 \times \nabla q_1 + \frac{dp_2}{d\rho} \nabla \rho \times \nabla q_2
\end{aligned}$$

Another use of the gauge freedom based on the derived expression for $\vec{B}$ leads to the vector potential $\tilde{\tilde{\vec{A}}}$ which is of the required form:

$$\tilde{\tilde{A}}_{q_1}(q_1, \rho, q_2) := A_{q_1}(q_1, \rho, q_2) - \int \frac{\partial f}{\partial q_1}(q_1, q_2) dq_2 \tag{A.14}$$

$$\tilde{\tilde{A}}_{q_2}(\rho) := p_2(\rho) = A_{q_2} - f(q_1, q_2) \tag{A.15}$$

$$\tilde{\tilde{\vec{A}}}(q_1, \rho, q_2) := \tilde{\tilde{A}}_{q_1}(q_1, \rho, q_2) \nabla q_1 + \tilde{\tilde{A}}_{q_2}(\rho) \nabla q_2 \tag{A.16}$$

The integration constant can be neglected, as adding a constant to a vector potential does not change the corresponding B. This argument is also the reason why the integration constant was neglected in the Definition of p_2 in Equation A.11, as p_2 is also a component of a vector potential.

It will now be shown that $\vec{\tilde{\tilde{A}}}(q_1, \rho, q_2)$ is also a vector potential for $\vec{B}$:

$$
\begin{aligned}
\nabla \times \vec{\tilde{\tilde{A}}} &= \nabla \tilde{\tilde{A}}_{q_1}(q_1, \rho, q_2) \times \nabla q_1 + \nabla \tilde{\tilde{A}}_{q_2}(\rho) \times \nabla q_2 \\
&= \frac{\partial A_{q_1}}{\partial \rho} \nabla \rho \times \nabla q_1 + \frac{\partial A_{q_1}}{\partial q_2} \nabla q_2 \times \nabla q_1 \\
&\quad - \left(\frac{\partial}{\partial q_2} \int \frac{\partial f}{\partial q_1}(q_1, q_2) dq_2 \right) \nabla q_2 \times \nabla q_1 - \left(\frac{\partial}{\partial \rho} \int \frac{\partial f}{\partial q_1}(q_1, q_2) dq_2 \right) \nabla \rho \times \nabla q_1 \\
&\quad + \frac{dp_2}{d\rho} \nabla \rho \times \nabla q_2 \\
&= \frac{\partial A_{q_1}}{\partial \rho} \nabla \rho \times \nabla q_1 + \left(\frac{\partial A_{q_1}}{\partial q_2} - \frac{\partial f}{\partial q_1} \right) \nabla q_2 \times \nabla q_1 + \frac{dp_2}{d\rho} \nabla \rho \times \nabla q_2 \\
&= \nabla \times \vec{A} \\
&= \vec{B}
\end{aligned}
$$

A change of coordinate system

The result of Lemma 18 implies that p_2 is a coordinate function satisfying the same requirements as ρ and that (q_1, p_2, q_2) is a pre-Hamiltonian coordinate system that can and will replace (q_1, ρ, q_2).

Appendix B

Toroidal coordinates

Toroidal coordinates are used in Paragraph 4.4 and are useful to simplify the calculations for toroidal fields. The following coordinates correspond to the "Toroidal" coordinate system incorporated in Mathematica 10.3, which has the radius of the circle around which the tori are defined as parameter. In this thesis the core is taken to be the unit circle, therefore the parameter is equal to 1. In terms of cartesian coordinates the toroidal coordinates are defined by

$$c_1 := \mathrm{ArcTan}(x, y) \tag{B.1}$$

$$c_2 := \frac{1}{2}\log\left(\frac{(1+\sqrt{x^2+y^2})^2+z^2}{(-1+\sqrt{x^2+y^2})^2+z^2}\right) \tag{B.2}$$

$$c_3 := \mathrm{ArcTan}(-1+x^2+y^2+z^2, 2z) \tag{B.3}$$

where $\mathrm{ArcTan}(a, b)$ is defined as used by Mathematica software, which means that it gives the arc tangent of $\frac{b}{a}$, taking into account the quadrant of the point $(a, b) \in \mathbb{R}^2$.

The metric tensor of the toroidal coordinate system is given by

$$g = \begin{pmatrix} \frac{1}{(\cos c_3 - \cosh c_2)^2} & 0 & 0 \\ 0 & \frac{1}{(\cos c_3 - \cosh c_2)^2} & 0 \\ 0 & 0 & \frac{\sinh^2 c_2}{(\cos c_3 - \cosh c_2)^2} \end{pmatrix} \tag{B.4}$$

which is diagonal, showing that the coordinates are orthogonal. The metric is not equal to the identity matrix, which means that basis vector fields in general do not have norm 1.

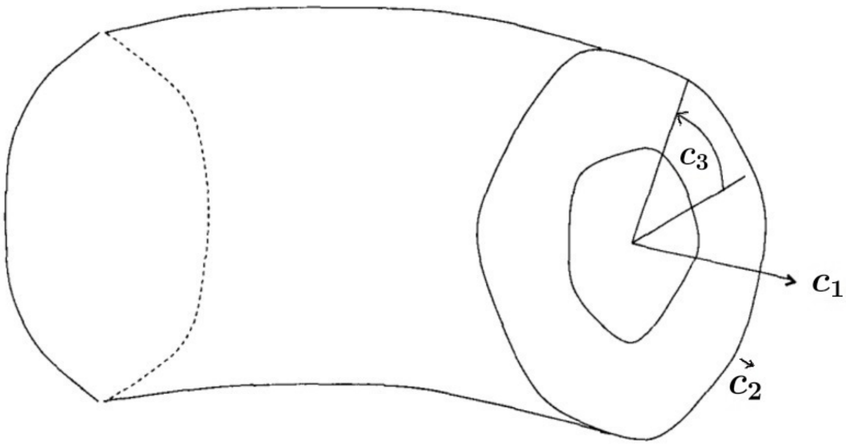

Figure B.1: *Torus coordinates c_1, c_2, c_3. Figure from [14].*

For these coordinate fields the ranges (Definition 11) are as follows:

$$\text{range } c_1 = S^1 \tag{B.5}$$
$$\text{range } c_2 = [0, \infty] \tag{B.6}$$
$$\text{range } c_3 = S^1 \tag{B.7}$$

The limit $c_2 = 0$ is attained for the z-axis $x = y = 0$ and at infinity, while $c_2 = \infty$ for $z = 0, x^2 + y^2 = 1$, which is the unit circle.

Notation of the coordinates used in Paragraph 4.4

The coordinate system is denoted as $(c_1, c_2, c_3) = (q_1, \rho, q_2)$ for the correspondence with a two degree of freedom system, while $(c_1, c_2, c_3) = (t^r, \rho, q^r)$ can be used with the reduced one degree of freedom system.

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