On the geography of symplectic manifolds

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Introduction.

The subject of this thesis is symplectic topology. More specifically, we are interested in construction and invariants of symplectic manifolds.

Differential topology and algebraic geometry provide us with some standard operations which can be performed within the differentiable and complex category, respectively. For instance: connected sum along a submanifold, blow-up, construction of fibrations and branched coverings. These operations make sense, under certain conditions, in the symplectic category and thus enable us to produce new examples of manifolds admitting symplectic structures.

Given a symplectic manifold (M,ω) of dimension $2n$, one can define its Chern classes as the Chern classes of a tame almost complex structure J. In general, the Chern classes of an almost complex manifold, that is, a manifold with a complex structure on the tangent bundle, only depend on a connected choice of such complex structure. Since the space of tame almost complex structures for a given symplectic form is connected, the Chern classes of a symplectic manifold are invariants of the symplectic form.

By evaluating top-dimensional products of Chern classes on the fundamental homology class of M, one obtains a system of integer numbers, in fact as many as the partitions of n, which are called the Chern numbers of (M,ω). The problem of determining which combinations of integer numbers may appear as Chern numbers of a closed, connected, symplectic manifold is known in the literature under the name of symplectic geography.

Symplectic geography is a suitable subject of study when considering symplectic and related structures, in particular almost complex and Kähler structures. Symplectic manifolds, in fact, occupy the central position in the sequence of inclusions

$$\text{Kähler} \subsetneq \text{symplectic} \subsetneq \text{almost complex}.$$

These inclusions have long been known to be proper.

One is interested in finding out which properties distinguish symplectic manifolds from the manifolds in the other two classes and which ones do not. Chern numbers can be defined for every (closed) almost complex manifold, so we can compare the geography of manifolds belonging to all classes. The main theorem in this thesis states that in dimension 8 the geography of symplectic manifolds does not differ from that of almost complex ones.

Theorem. *Any ordered quintuple of integers which arises as the system of Chern numbers of an almost complex* 8*-dimensional manifold can also be realised by a closed, connected, symplectic* 8*-manifold.*

The analogous result in dimension 6 is due to Halic ([13]). His proof makes use of two important operations: blow-up and connected symplectic sum. Both of them may be performed inside the symplectic category. We also consider symplectic fibrations, obtained by projectifying a complex vector bundle over a symplectic manifold, and symplectic branched coverings with a given symplectic basis and branching set.

This thesis consists of five parts or, more precisely, four chapters and one appendix.

The first chapter is a collection of some basic definitions and results of symplectic geometry. We define here manifolds, submanifolds and isomorphisms in the symplectic category. Then we recall Darboux's theorem, which states that any two symplectic forms are locally isomorphic, so that symplectic invariants must necessarily be of a global nature. After this, we focus on the relationship between symplectic and almost complex manifolds on one side and symplectic and Kähler manifolds on the other. We hope to have included all the notions which are necessary to comprehend the following material.

The second chapter deals with several ways of constructing symplectic manifolds. We recall some standard operations which may be performed on one or more given symplectic manifolds, to produce a new manifold which again admits a symplectic structure. Some of these constructions come from complex geometry (for instance, blow-up), others are of purely topological nature (for instance, connected sum along a symplectic submanifold). We also show that the branched covering of a symplectic manifold, branched along a symplectic submanifold, must admit a symplectic structure and apply this result to the construction of cyclic branched coverings with a given branching set.

The third chapter starts dealing with invariants of symplectic manifolds, namely with their Chern classes. In particular, we try to describe how to compute them efficiently for the manifolds obtained by performing the constructions of Chapter 2. This is possible because for the symplectic forms arising from these operations there is a well defined, unique homotopy class of almost complex structures. The main section of this chapter is the one about Chern classes of blow-ups. The "blow-up formula" which we obtain was already known for algebraic varieties and we show how the proof may be modified in order to apply to symplectic manifolds. Due to their length, the computations of the invariants in dimension 8 are postponed to the appendix. In this chapter we also apply Donaldson's result about the existence of symplectic submanifolds to the total space of some given symplectic fibrations and compute the Chern classes and numbers of such submanifolds.

The fourth chapter finally studies Chern numbers and the geography of closed, symplectic, 8-dimensional manifolds. The first part of it, though, is completely devoted to a review of the main facts about the complex cobordism ring of Milnor and its relationship with Chern numbers of almost complex manifolds. More precisely, we describe how knowledge of the structure of this ring (it is a polynomial ring and Milnor was able to point out explicit generators for it) eventually enables one to write down some congruence relations which must be satisfied by the Chern numbers of any almost complex manifold, hence any symplectic manifold as well. In dimension 8, for instance, these relations are

$$\begin{aligned} -c_4+c_1c_3+3c_2^2+4c_2c_1^2-c_1^4 &\equiv 0 \pmod{720} \\ 2c_1^4+c_1^2c_2 &\equiv 0 \pmod{12} \\ -2c_4+c_1c_3 &\equiv 0 \pmod{4}. \end{aligned} \tag{1}$$

We thought this review worth the effort, since there seems to be an inclination in the literature towards referring for these matters to a “mythical” part 2 of the paper [21] of Milnor, which was in fact never published. After this overview, we start constructing the examples that will fill our 8-dimensional symplectic geography picture, compute their Chern numbers (the explicit computations are in fact once more relegated to the appendix) and show that, precisely as in the case of almost complex manifolds, we are able to obtain all combinations of numbers which are allowed by system (1) as Chern numbers of a closed, connected, symplectic 8-dimensional manifold.

In the fifth part, the appendix, we carry out the computations: the blow-up formula is applied to get expressions for the Chern classes of blow-ups in dimension 8. From this the top Chern classes and subsequently the Chern numbers are calculated, by using information on the structure of the cohomology ring of the blow-up. We have also collected here the computations of the Chern numbers of the sumbanifolds obtained in Chapter 3 by applying Donaldson’s theorem.

To conclude this introduction, we would like to point out an interesting potential application of symplectic geography. We have seen that Chern numbers of a symplectic manifold are invariants of the symplectic form. When considering invariants of some nature, it is always interesting and natural to ask to what extent these invariants classify. The Chern numbers certainly fail to classify symplectic structures. The homotopy class of tame almost complex structures is an invariant of deformation equivalence, so deformation equivalent forms have the same Chern numbers. Isomorphic symplectic forms also have the same numbers. There are even examples of symplectic forms which are not related by any sequence of isomorphisms and deformation equivalences and which are distinguished by finer invariants (Chern classes, Gromov-Witten invariants), but not by the Chern numbers. In other words, the extent to which Chern numbers fail to classify symplectic structures is considerable. Therefore one may wonder whether they might not be topological invariants. In dimension 4, the Chern numbers c_1^2 and c_2 are indeed topological invariants. In dimension 6, LeBrun has shown that Chern numbers are not topological invariants of complex manifolds, but what happens if we introduce a symplectic form is not known. A better understanding of the geography of symplectic manifolds may be useful in order to answer this question by comparing the Chern numbers with the topology. The symplectic constructions on which Halic’s results and our main theorem rely allow a good control of the cohomological data. In dimension 6 and 8 there are smooth classification theorems (Wall [30] and Müller [23], respectively) based on those data. So one might hope to be able to detect a smooth manifold realising two different combinations of Chern numbers, that is, admitting two distinct symplectic structures, distinguished by the Chern numbers.

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Chapter 1

Basic notions.

1.1 Symplectic manifolds.

Definition 1.1. A **symplectic manifold** is a smooth manifold M with a closed nondegenerate two-form ω. Nondegeneracy means that the top exterior power of ω is nowhere zero, i.e., it is a volume form. Thus, M is always even dimensional and canonically oriented. A map $f : M \to N$ between symplectic manifolds (M, ω_M) and (N, ω_N) is called symplectic if $f^*\omega_N = \omega_M$; a symplectic diffeomorphism is called a **symplectomorphism**.

Example 1.2. Euclidean space $\mathbb{R}^{2n}$ with coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ and the form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is a symplectic manifold. The form ω_0 is called standard or canonical symplectic form.

In fact, Darboux's theorem states that the above example is universal, in the following sense (see [20, p. 95]):

Theorem 1.3. *Every symplectic manifold (M, ω) is locally isomorphic to euclidean space with the standard symplectic form.*

So there are no symplectic local invariants; globally, of course, the situation is different: volume, for example, is preserved by symplectic isomorphisms.

Definition 1.4. Two symplectic forms ω_0 and ω_1 on M are said to be **isotopic** if they can be joined by a smooth family of cohomologous symplectic forms ω_t on M, **strongly isotopic** if there is an isotopy F_t of M such that $F_1^*\omega_1 = \omega_0$.

In general, of course, strong isotopy implies isotopy (by setting $\omega_t = F_t^*\omega_1$), but for a closed manifold the opposite implication is also true as a corollary of the following theorem.

Theorem 1.5 (Moser stability theorem). *If ω_t is a smooth family of cohomologous symplectic forms on a closed manifold M, there exists an isotopy F of the identity of M such that $F_t^*\omega_t = \omega_0$ for all t.*

1.2 Submanifolds and tubular neighbourhoods.

Definition 1.6. A smooth submanifold N of a symplectic manifold (M,ω) is called a **symplectic submanifold** if ω restricts to a symplectic form on TN.

In this case the normal bundle of N in M may be identified with the symplectic complement of the tangent bundle, namely the bundle TN^{ω} defined pointwise at $p \in N$ by

$$(TN^{\omega})_p = \{v \in T_pM \mid \omega(v,w) = 0 \text{ for all } w \in T_pN\}.$$

Then a neighbourhood of N is completely determined by the isomorphism class of the normal bundle. This result is referred to as the Symplectic Neighbourhood Theorem and can be found in [20, p. 101], where it is stated more precisely in the form below.

Proposition 1.7. *Let (M_i,ω_i), $i = 1,2$, be symplectic manifolds. Suppose we are given symplectic embeddings j_1 and j_2 of N in M_1 and M_2, respectively, such that the normal bundles of the two embeddings are isomorphic. Then there exists tubular neighbourhoods U_i of $j_i(N)$ and a symplectomorphism $\phi : U_1 \to U_2$ such that the differential of ϕ induces between the normal bundles the given isomorphism.*

In particular, if N is a symplectic submanifold of M, a tubular neighbourhood of N is always symplectomorphic to a tubular neighbourhood of the zero section of the normal bundle of N in M.

1.3 Symplectic and almost complex structures.

Definition 1.8. An **almost complex structure** on a smooth oriented manifold M is an isomorphism of the tangent bundle $J : TM \to TM$ such that $J^2 = -\mathrm{id}_{TM}$. In other words, an almost complex structure on M is a complex structure on its tangent bundle. Thus M admits an almost complex structure if and only if the structure group of TM may be reduced from $\mathrm{SO}(2n)$ to the unitary group $\mathrm{U}(n)$. A nondegenerate 2-form $\omega \in \Omega^2(M)$ and an almost complex structure J on M are called **compatible** if the bilinear form

$$\langle v,w \rangle := \omega(v,Jw)$$

defines a Riemannian metric on M.

Notice that ω and J are compatible if and only if the following two conditions are satisfied:

(i) $\omega(Jv,Jw) = \omega(v,w)$ for all $v,w \in T_pM$ and $p \in M$;

(ii) $\omega(v,Jv) > 0$ for all $v \in T_pM$, $v \neq 0$, and $p \in M$.

If (ii) alone holds, one says that ω tames J or that J is a **tame** almost complex structure. The taming conditon alone already implies that ω is nondegenerate, hence a closed taming 2-form is automatically symplectic.

If we fix a nondegenerate 2-form, for example a symplectic form, there isn't a unique tame almost complex structure. However, we have the following result (for a proof, see [20, p. 118]).

Proposition 1.9. *Given a nondegenerate 2-form ω, the corresponding spaces of compatible and tame almost complex structures are nonempty and contractible.*

Given a complex vector bundle (ξ, J) with base M, one can define its Chern classes $c_i(\xi) \in H^i(M;\mathbb{Z})$. If the complex rank of ξ is r, one denotes by $c(\xi)$ the total Chern class $\sum_{i=0}^{r} c_i(\xi)$. These classes depend on a connected choice of complex structure on ξ. Since for a symplectic form ω on a manifold M the space of tame almost complex structures is in particular connected, we can define the Chern classes of (M,ω).

Definition 1.10. If J is a tame almost complex structure for the symplectic form ω on M, the **Chern classes** of (M,ω) are by definition the Chern classes of the complex vector bundle (TM, J).

1.4 Kähler manifolds.

If M is a complex manifold of real dimension $2n$, it is possible to define an almost complex structure on M as follows. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a trivialising cover for the tangent bundle $\pi : TM \to M$, that is, φ_α is a fibre-preserving and fibrewise complex linear diffeomorphism $\pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^n$, and write an element of TM in the form (q, v), with $q \in M$ and $v \in T_qM$. If $q \in U_\alpha$, denote by $\varphi_\alpha(q)$ the restriction of the diffeomorphism φ_α to the fibre over q: then $\varphi_\alpha(q)$ is a complex isomorphism $\pi^{-1}(q) \overset{\cong}{\to} \mathbb{C}^n$ and we can define

$$J(q,v) = (q, \varphi_\alpha(q)^{-1} J_0 \varphi_\alpha(q)(v)),$$

where J_0 is the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and 1 denotes the $n \times n$ identity matrix. This definition does not depend on the choice of trivialising neighbourhood. In fact, if $q \in U_\alpha \cap U_\beta$ and $g_{\alpha\beta}$ are the transition matrices of TM, we have

$$\begin{aligned}
\varphi_\beta(q)^{-1} J_0\, \varphi_\beta(q)(v) &= \varphi_\alpha(q)^{-1}\, \varphi_\alpha(q)\, \varphi_\beta(q)^{-1}\, J_0\, \varphi_\beta(q)(v) \\
&= \varphi_\alpha(q)^{-1} g_{\alpha\beta}(q)\, J_0\, \varphi_\beta(q)(v) \\
&= \varphi_\alpha(q)^{-1} J_0\, g_{\alpha\beta}(q)\, \varphi_\beta(q)(v) \\
&= \varphi_\alpha(q)^{-1} J_0\, \varphi_\alpha(q)(v),
\end{aligned}$$

where $g_{\alpha\beta}(q)$ commutes with J_0 because it is an element of $\mathrm{GL}(n,\mathbb{C})$. So J is a well defined almost complex structure on M.

Definition 1.11. Whenever an almost complex structure J on an arbitrary manifold can be represented by the matrix J_0, with respect to some local coordinates, it is called **integrable**.

Every complex manifold can also be endowed with a Hermitian metric, denoted by $\langle \ , \ \rangle$: this is by definition bilinear, $\mathbb{C}$-linear in the first slot, $\mathbb{C}$-antilinear in the second one and satisfies the additional two properties

(i) $\langle v, w \rangle = \overline{\langle w, v \rangle}$;

(ii) $\langle v,v\rangle \geq 0$, with equality if and only if $v=0$.

The imaginary part of such a metric, which can be written as

$$\operatorname{Im}\langle v,w\rangle = -\frac{i}{2}(\langle v,w\rangle - \overline{\langle v,w\rangle}),$$

is skew-symmetric and nondegenerate. Moreover, it is compatible with the standard almost complex structure J defined above. To see this, choose local coordinates with respect to which J is represented by J_0 and $\langle v,w\rangle = \overline{v}^T H w$. Then

$$\begin{aligned}\langle Jv,Jw\rangle &= (\overline{J_0 v})^T H J_0 w = \overline{v}^T \overline{J}_0^T H J_0 w\\ &= \overline{v}^T H \overline{J}_0^T J_0 w = \overline{v}^T H w\\ &= \langle v,w\rangle,\end{aligned}$$

where we have used the fact that H is hermitian, hence it commutes with J_0, and $\overline{J}_0^T J_0 = 1$.

Definition 1.12. If M is a complex manifold, endowed with a Hermitian metric whose imaginary part is closed, then it is called a **Kähler manifold**; in particular, it is symplectic with an integrable almost complex structure.

On the other hand, one can define a Kähler manifold as a symplectic manifold (M,ω) with a compatible integrable almost complex structure J. This definition is equivalent to the one above: one can show that M is then complex, using the fact if some local transition functions commute with the matrix J_0, then they must be elements of the complex general linear group, and the inner product

$$\langle v,w\rangle = \omega(v,Jw) + i\omega(v,w)$$

defines a hermitian metric on M whose imaginary part coincides with ω by definition.

1.5 4-manifolds and the intersection form.

Poincaré duality on a closed oriented manifold of dimension n sets up an isomorphism PD between the groups $H_k(M;\mathbb{Z})$ and $H^{n-k}(M;\mathbb{Z})$. If N is a k-dimensional submanifold of M, we can associate to it a k-dimensional homology class $[N]$ and an $(n-k)$-cohomology class $PD[N]$, that is,

$$\begin{aligned} PD: \quad H_k(M;\mathbb{Z}) &\xrightarrow{\cong} H^{n-k}(M;\mathbb{Z})\\ [N] &\longmapsto PD[N]\end{aligned}$$

In particular, if the dimension of M is 4, Poincaré duality yields an isomorphism $H_2(M;\mathbb{Z}) \cong H^2(M;\mathbb{Z})$. Under this isomorphism, the intersection product of two homology classes $[N_1]$ and $[N_2]$ corresponds to evaluation of the cup product $PD[N_1]\cup PD[N_2]$ on the fundamental homology class of M, that is

$$[N_1]\cdot[N_2] = \langle PD[N_1]\cup PD[N_2],[M]\rangle$$

Definition 1.13. The above product is denoted by $Q_M([N_1],[N_2])$. The form Q_M is a symmetric and bilinear form on $H_2(M;\mathbb{Z})$ (equivalently, $H^2(M;\mathbb{Z})$), which goes under the name of **intersection form**. Since Q_M vanishes on torsion elements, we can regard it as a form on the free part of H_2. By choosing a basis of $H_2(M;\mathbb{Z})/\mathrm{Tor}$ we can represent Q_M by a matrix, called **intersection matrix**.

In particular, the matrix associated to the intersection form Q_M always has determinant ± 1. This can be seen by choosing a basis $x_1,\ldots,x_n$ of the free part of $H_2(M,\mathbb{Z})$. We denote by x_i^* the corresponding dual basis of the free part of $H^2(M,\mathbb{Z})$ with respect to the Kronecker product

$$\begin{array}{ccc} H^2(M)\times H_2(M) & \longrightarrow & \mathbb{Z} \\ (\alpha,x) & \longmapsto & \langle\alpha,x\rangle \end{array}$$

and set $y_i := PD(x_i^*)$. Then $Q_M(x_i,y_j)$ is the identity matrix and $Q_M(x_i,x_j)$ is equal to the matrix of the coordinate change from the y to the x basis. The latter has determinant ± 1, since it is invertible over $\mathbb{Z}$, hence so has $Q_M(x_i,x_j)$.

1.6 Pseudo-holomorphic curves.

Let M be a smooth closed manifold, J an almost complex structure on M.

Definition 1.14. A **pseudo-holomorphic curve** on M is a map from a compact Riemann surface Σ, with complex structure j, to M, such that

$$df\cdot j = J\cdot df : T\Sigma \longrightarrow TM.$$

(i.e., df is a complex linear bundle map).

Remark. If M is endowed with a symplectic structure ω, and J and ω are compatible, then smoothly embedded pseudo-holomorphic curves are also symplectically embedded (cf. Lemma 3.3 and 4.29).

Chapter 2

Construction of symplectic manifolds.

2.1 Introduction.

We have seen that a symplectic manifold with an integrable almost complex structure is a Kähler manifold and that, on the other hand, any complex manifold admits a Hermitian metric whose imaginary part, if closed, is a symplectic form.

The main classical examples of symplectic manifolds were indeed Kähler manifolds, in particular, nonsingular complex-projective algebraic varieties: in fact, complex projective spaces admit a Kähler structure and this induces a Kähler structure on any complex submanifold.

The most fruitful results in the direction of constructing new symplectic manifolds were actually achieved in the attempt of finding examples of non-Kähler symplectic manifolds.

2.2 Thurston's construction.

Definition 2.1. A **symplectic fibration** is by definition a fibration $\pi : M \to B$, where the fibre is a compact symplectic manifold (F, σ) and the structure group consists of symplectomorphisms of the fibre. A symplectic form ω on the total space M is said to be **compatible** with the fibration π if each fibre is a symplectic submanifold of (M, ω).

A necessary condition for M to admit a compatible form ω is the existence of a cohomology class $a \in H^2(M)$ which restricts to the cohomology class of the symplectic form on each fibre. Thurston has shown that, if M is compact and the base manifold B is a symplectic manifold, this condition is also sufficient.

Theorem 2.2 (Thurston). *Let $\pi : M \to B$ be a symplectic fibration with compact total space M, symplectic fibre (F, σ) and connected symplectic base (B, β). For $b \in B$, denote by i_b the* *inclusion of the fibre $F_b = \pi^{-1}(b)$ in M. Fix identifications $\varphi_b : F_b \xrightarrow{\cong} F$ and denote by σ_b the pullback $\varphi_b^*\sigma$ of the symplectic form on F. Suppose that there is a class $a \in H^2(M)$ such that $i_b^* a = [\sigma_b]$ for all b. Then for large enough $K \in \mathbb{R}$, the manifold M admits a symplectic form which is compatible with the fibration and represents the class $K[\pi^*\beta] + a$.*

Proof. The first step of the proof consists in finding a closed symplectic form τ which represents the class a and restricts to the canonical symplectic form in each fibre. This is done by choosing a closed two-form τ_0 representing a. Let $\{(U_\alpha, \phi_\alpha)\}$ be a trivialising atlas for the fibration $\pi : M \to B$. In particular, $\{U_\alpha\}$ is an open covering of B and by passing, if necessary, to a refinement, we may assume that each U_α is contractible and that the covering is locally finite. For each trivialisation $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$, denote by σ_α the pullback of σ along the projection $U_\alpha \times F \to F$. By the assumption on a, the form $\phi_\alpha^*(\sigma_\alpha - \tau_0)$ is exact, hence there exist one-forms λ_α such that $\phi_\alpha^*(\sigma_\alpha - \tau_0) = d\lambda_\alpha$. Choose a partition of unity ρ_α subordinate to $\{U_\alpha\}$ and define

$$\tau = \tau_0 + \sum d((\rho_\alpha \pi)\lambda_\alpha).$$

Then τ is closed, represents the class a and restricts to σ_b in each fibre. It is nondegenerate on the subbundle $\ker(d\pi)$, hence for large enough K the form $\omega_K = K\pi^*\beta + \tau$ is nondegenerate on M. □

Example 2.3. Let E be a complex rank $(n+1)$-bundle over a connected symplectic base B. Let $M = \mathbb{P}(E)$ be the projectified bundle, with projection $\rho : \mathbb{P}(E) \to B$, and $l_E \subset \rho^* E$ the canonical line subbundle. Observe that the induced bundle map $l_E \to B$ has fibres diffeomorphic to L, the canonical line bundle over $\mathbb{CP}^n$. The first Chern class $c_1(l_E^*) \in H^2(\mathbb{P}(E))$ has the properties required of the class a in the theorem. In fact,

$$c_1(l_E^*)|_{\mathbb{P}(E_p)} = c_1(l_E^*|_{\mathbb{P}(E_p)}) = c_1(L^*)$$

and the latter coincides with the class of the standard Kähler form on $\mathbb{CP}^n$.

Example 2.4. We are going to give a first example of a symplectic, non-Kähler manifold. Consider the T^2-bundle over S^1 defined as

$$[0,1] \times T^2 / \sim, \quad (0, y_1, y_2) \sim (1, y_1 + y_2, y_2),$$

that is, the ends of $[0,1]$ are indentified and the corresponding fibres glued with a Dehn twist. Then cross with S^1 to obtain the T^2 bundle over T^2

$$N = [0,1] \times S^1 \times T^2 / \sim, \quad (0, x, y_1, y_2) \sim (1, x, y_1 + y_2, y_2).$$

The projection over T^2 is given by $[t, x, y_1, y_2] \mapsto (t, x) \in T^2 = S^1 \times S^1$. The manifold N has by construction odd first Betti number, namely $b_1(N) = 3$, hence it cannot be Kähler: Hodge decomposition, in fact, implies that all odd Betti numbers of a Kähler manifold must be even. On the other hand, as a bundle over T^2 it admits a section

$$s(t, x) = [t, x, 1, 1], \quad \text{where} \quad (1, 1) = (e^{2\pi i 0}, e^{2\pi i 0}) \in S^1 \times S^1.$$

The structure group may be reduced from the group of orientation preserving diffeomorphisms of T^2 to the group of symplectomorphisms with respect to the standard Kähler form, that is, from $\mathrm{Diff}^+(T^2)$ to $\mathrm{Symp}(T^2,\sigma)$. Let β denote the Poincaré dual of $s(T^2)$ in N: then $[\beta]\in H^2(N)$ has the properties of the cohomology class a in the statement of Theorem 2.2 and this implies that N admits a symplectic structure. The symplectic form is the one induced by $dt\wedge dx+dy_1\wedge dy_2$.

Remark. The manifold M has an almost complex structure J_M, with respect to which π is J_M-holomorphic. This is constructed in [11] by choosing a metric, denoting by $H\subset TM$ the subbundle of orthogonal complements to the fibres of π with respect to this metric and setting $J|_H$ equal to the pullback of some almost complex structure on B, compatible with β. Since each fibre already has a canonical almost complex structure, J_M can be uniquely defined on TM by linearity. By construction, the form ω_K tames J_M for K large enough. Therefore, given two forms $\omega_K=K\pi^*\beta+\tau$ and $\omega_{K'}=K'\pi^*\beta+\tau'$, with τ and τ' closed two-forms representing the class a and restricting to a canonical symplectic form in each fibre, we can interpolate between them. Each $\omega_s=s\omega_K+(1-s)\omega_{K'}$ will be nondegenerate (and obviously closed), hence symplectic. If $K=K'$, all these forms will be cohomologous and ω_K and $\omega_{K'}$ will be isotopic. Moreover, by Moser stability (Theorem 1.5), they will be strongly isotopic, hence isomorphic.

In particular, Theorem 2.2 implies the following result for surface bundles.

Corollary 2.5. *Let F be a compact oriented Riemann surface of genus different from one. Then the total space of any oriented fibration with fibre F and compact symplectic base B admits a compatible symplectic form.*

Remark. This result applies for instance to S^2-bundles over compact symplectic manifolds. In that case, if s is a smooth section of the given bundle, we can always choose the symplectic form ω on M so that s is in fact a symplectic section. To see this, choose a closed two-form β on M such that $[\beta]=PD[s(B)]\in H^2(M,\mathbb{Z})$. Then for each fibre F of M with inclusion i we have

$$\langle i^*\beta,[F]\rangle=\langle[\beta]\cup PD[F],[M]\rangle=[s(B)]\cdot[F]=1.$$

On the other hand, if we let ω_F correspond in each fibre to the standard symplectic form with area 1 on S^2, then $\int_F i^*\beta=1=\int_F\omega_{S^2}$, hence $i^*[\beta]=[\omega_{S^2}]$ (because $H^2(M,\mathbb{Z})\cong\mathbb{Z}$). Then $[\beta]$ satisfies the conditions imposed on the class a by the statement of Theorem 2.2 and by the method of Thurston, it is possible to construct a closed two-form η' such that $i^*(\eta')=\omega_F$. The form $\eta:=\eta'-\pi^*s^*\eta'$ has the same properties and moreover $s^*\eta=0$. Also according to Thurston, the form $\omega_K:=K\pi^*\omega_B+\eta$ is then, for K large enough, symplectic and compatible with the fibration. It is immediate that, with respect to ω_K, $s(B)$ is a symplectic submanifold of M.

2.3 Symplectic blow-up

2.3.1 Blow-up at a point.

The operation of blowing up can be performed in the symplectic category as follows. We begin by describing the blow up of a symplectic manifold at a point. Let (M^{2n}, ω) be a closed symplectic manifold, $x \in M$. Then an open neighbourhood U of x can be identified (symplectically) with a neighbourhood V of the origin in $\mathbb{C}^n$, with its standard symplectic structure ω_0. Let $p : L \to \mathbb{CP}^{n-1}$ be the tautological line bundle over $\mathbb{CP}^{n-1}$. Then there is a projection $\Phi : L \to \mathbb{C}^n$, which is a diffeomorphism outside $\Phi^{-1}(0)$. The latter is a copy of $\mathbb{CP}^{n-1}$ and coincides with the zero section of p. Let $\widetilde{V} := \Phi^{-1}(V)$.

$$\begin{array}{ccc} L & \xrightarrow{\Phi} & \mathbb{C}^n \\ {\scriptstyle p}\downarrow & & \\ \mathbb{CP}^{n-1} & & \end{array}$$

Definition 2.6. The **blow-up of** M **at the point** x is defined as the sum

$$\widetilde{M} := \overline{M - U} \cup_{\partial U} \widetilde{V}$$

The manifold $\Phi^{-1}(0) \cong \mathbb{CP}^{n-1} \subset \widetilde{V}$ is called the exceptional divisor of the blow-up. In the case of a manifold of dimension 4, the exceptional divisor is a sphere. It is embedded in $\widetilde{V} \subset \widetilde{M}$ as the zero section of the tautological line bundle over $\mathbb{CP}^1$, hence its normal bundle coincides with the vertical bundle $p^*(L)$. From this we see that the square of this sphere is equal to -1. In fact:

$$\begin{aligned} \langle c_1(\nu_{\widetilde{M}} \Phi^{-1}(0)), [\Phi^{-1}(0)] \rangle &= \langle c_1(\nu_{\widetilde{V}} \Phi^{-1}(0)), [\Phi^{-1}(0)] \rangle \\ &= \langle p^* c_1(L), [\Phi^{-1}(0)] \rangle \\ &= \langle c_1(L), [\mathbb{CP}^1] \rangle \\ &= -1. \end{aligned}$$

Sometimes it will be useful to have the following description of blow-up at one point.

Lemma 2.7. *If M has dimension $2n$, then its blow-up at one point is diffeomorphic to the connected sum $M \# \overline{\mathbb{CP}^n}$, where $\overline{\mathbb{CP}^n}$ denotes the n-dimensional complex projective space with opposite orientation.*

Proof. By definition, we have

$$M \# \overline{\mathbb{CP}^n} = (M - B^{2n}) \cup_{\partial B^{2n}} (\overline{\mathbb{CP}^n} - B^{2n}).$$

Notice that $\overline{\mathbb{CP}^n} - B^{2n} \cong \overline{\mathbb{CP}^n} - \{[1 : \cdots : 0]\}$ admits a complex line bundle structure over $\mathbb{CP}^{n-1}$, with projection

$$[x_0 : \cdots : x_n] \mapsto \left[\frac{x_0}{x_n} : \cdots : \frac{x_{n-1}}{x_n} \right]$$

in local homogeneous coordinates in the neighbourhood $U_n = \{x_n \neq 0\}$. Then the zero section $\{x_0 = 0\} \cong \mathbb{CP}^{n-1} \subset \overline{\mathbb{CP}^n}$ has normal bundle isomorphic to the tautological line bundle L over $\mathbb{CP}^{n-1}$ and $\overline{\mathbb{CP}^n} - B^{2n}$ may be identified with a tubular neighbourhood of the zero section in L. The claim follows by comparing with the definition of $\widetilde{M}$. ☐

2.3.2 Symplectic form on the blow-up.

Let ω_0 and τ_0 denote the standard Kähler forms on $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$, respectively. The construction of a symplectic form $\widetilde{\omega}$ on $\widetilde{M}$ takes place in three steps.

(i) Prove that the form $\Phi^*\omega_0 + \varepsilon p^*\tau_0$ is nondegenerate on L for all $\varepsilon > 0$: let J_L be the canonical almost complex structure on L. Then $\Phi^*\omega_0(v, J_L v) \geq 0$ and $p^*\tau_0(v, J_L v) \geq 0$ for all vectors v (since p and Φ are J_L-holomorphic). Moreover, if v is a nonzero vector and $\Phi^*\omega_0(v, J_L v) = 0$, then necessarily $v \in T(\Phi^{-1}(0))$, since Φ_* is an isomorphism elsewhere. But then $p^*\tau_0(v, J_L v)$ cannot vanish, because $p_* : T(\Phi^{-1}(0)) \longrightarrow T\mathbb{CP}^{n-1}$ is an isomorphism. Hence $(\Phi^*\omega_0 + \varepsilon p^*\tau_0)(v, J_L v) > 0$ for all nonzero vectors and for all $\varepsilon > 0$.

(ii) Construct a symplectic form ρ on $\widetilde{V}$ which equals $\Phi^*\omega_0$ near $\partial\widetilde{V}$. Notice that the form $p^*\tau_0$ is exact outside $\Phi^{-1}(0)$, i.e., there exists a one-form β such that $p^*\tau_0 = d\beta$ on $L - \Phi^{-1}(0)$. Define a form ρ on $\widetilde{V}$ as follows:

$$\rho = \begin{cases} \Phi^*\omega_0 + \varepsilon p^*\tau_0 & \text{on} \quad \Phi^{-1}(0) \\ \Phi^*\omega_0 + \varepsilon d(\mu\beta) & \text{on} \quad \widetilde{V} - \Phi^{-1}(0), \end{cases}$$

with μ a smooth function which equals one near $\Phi^{-1}(0)$ and zero near $\partial\widetilde{V}$. Since $\Phi^*\omega_0$ is nondegenerate on $\widetilde{V} - \Phi^{-1}(0)$, ρ will be nondegenerate if we choose ε sufficiently small. Equivalently, since J_L tames $\Phi^*\omega_0$ on $\widetilde{V} - \Phi^{-1}(0)$ and the taming condition is open, J_L tames ρ.

(iii) Since $\rho = \Phi^*\omega_0$ outside a neighbourhood of $\Phi^{-1}(0)$ in the interior of $\widetilde{V}$, we may define $\widetilde{\omega}$ on $\widetilde{M}$ as follows: Define

$$\widetilde{\omega} = \begin{cases} \omega & \text{on} \quad M - U \\ \rho & \text{on} \quad \widetilde{V}. \end{cases}$$

Remark. With the form we have defined, the exceptional divisor $\Phi^{-1}(0)$ is a symplectic submanifold of $\widetilde{M}$. In fact, $\widetilde{\omega}_{|T\Phi^{-1}(0)} = \varepsilon p^*\tau_0$ and therefore

$$\langle [\widetilde{\omega}_{|T\Phi^{-1}(0)}]^{n-1}, [\Phi^{-1}(0)] \rangle = \varepsilon^{n-1} \langle \tau_0^{n-1}, [\mathbb{CP}^{n-1}] \rangle > 0.$$

2.3.3 Blow-up along a submanifold.

As to blow-up along a submanifold, consider a symplectic embedding $i : N \longrightarrow M$. Let E denote the normal bundle of this inclusion: since M and N carry almost complex structures, the vector bundle E also admits a complex structure, defined by the short exact sequence $0\longrightarrow TN\longrightarrow TM|_N\longrightarrow E\longrightarrow 0$. In other words, E may be identified with the symplectic orthogonal bundle of N, which carries a symplectic bundle structure, hence a complex structure as well. With respect to this structure we consider the projectivisation $\mathbb{P}(E)$. We choose a tubular neighbourhood U of N in M: this can be symplectically identified with a neighbourhood V of the zero section of E. Let l_E be the tautological line bundle over $\mathbb{P}(E)$, denote by p the bundle projecton $l_E\longrightarrow\mathbb{P}(E)$ and by Φ the projection $l_E\longrightarrow E$, so that we have the following diagram

$$\begin{array}{ccc} l_E & \xrightarrow{\Phi} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}(E) & \xrightarrow{\rho} & N \end{array}$$

Definition 2.8. Let $\widetilde{V} := \Phi^{-1}(V)$: we define the **blow up of** M **along** N to be the manifold

$$\widetilde{M} := \overline{M-U} \cup_{\partial U} \widetilde{V}.$$

Then a symplectic form on $\widetilde{M}$ may be defined as in the case of blow-up at a point: one also has to take care, though, of the normal direction. The precise construction in carried out in [19].

Example 2.9. It is also shown in [19] how blow-up along a symplectic submanifold can be used to generate examples of simply connected, symplectic, non-Kähler manifolds. One considers for example the T^2-bundle N over T^2 of Example 2.4: this can be symplectically embedded in $\mathbb{CP}^5$ with the standard symplectic structure. By blowing up $\mathbb{CP}^5$ along N we get a symplectic manifold, which is still simply connected because the fundamental group is invariant under blow-up. This manifold isn't Kähler: this can be detected, for example, by looking at the Betti numbers. It turns out, in fact, that $b_3 = 3$ (cf. Example 2.4).

2.4 Symplectic connected sum.

Gompf has shown in [9] how to construct the connected sum of two manifolds along diffeomorphic submanifolds, under the assumption that an orientation reversing diffeomorphism of the normal bundles of the submanifolds is given. Furthermore, he has proved this to be an operation in the symplectic category, namely: if we assume all manifolds and embeddings to be symplectic, then the result of the operation will also admit a symplectic structure.

Let $j_i : N\rightarrow M_i$, $i = 1,2$ be disjoint codimension two embeddings of closed oriented manifolds and denote by N_i the images $j_i(N)$ and by ν_i their normal bundles in M_i. Suppose moreover that there exists a fibre-orientation reversing bundle isomorphism $\psi : \nu_1\rightarrow\nu_2$. This condition can be also expressed by saying that the normal bundles have opposite Euler classes. By identifying each ν_i with a small tubular neighbourhood V_i of N_i and composing ψ with the diffeomorphism $z \mapsto z/\parallel z \parallel^2$ in each fibre, we obtain an orientation preserving diffeomorphism $\phi : V_1 - N_1\rightarrow V_2 - N_2$.

Definition 2.10. The **symplectic connected sum of M_1 and M_2 along N** is the manifold

$$(M_1 - N_1) \cup_\phi (M_2 - N_2),$$

and is denoted by $M_1 \#_\phi M_2$.

Its diffeomorphism type depends on the choice of the embeddings and of the orientation reversing bundle isomorphism ψ. Isotopic embeddings, though, still give rise to diffeomorphic manifolds, as do bundle isomorphisms which are connected by a fibre-preserving isotopy.

Now suppose that the manifolds M_i and N and the embeddings j_i are symplectic: then $M_1 \#_\phi M_2$ admits a symplectic structure. Assume for simplicity that the given embeddings have symplectically trivial normal bundles. Choose trivialisations $\nu_i \cong N \times \mathbb{C}$. By the Symplectic Neighbourhood Theorem there exist symplectic embeddings $f_i : N \times D_\varepsilon \hookrightarrow M_i$ such that $f_i(N \times \{0\}) = N_i$. Let $V_i = f_i(N \times D_\varepsilon)$, so that $V_i - N_i = f_i(N \times (D_\varepsilon - \{0\}))$. Consider the following automorphism of the punctured disk:

$$\begin{array}{rccc} \rho : & D_\varepsilon - \{0\} & \to & D_\varepsilon - \{0\} \\ & (r, \theta) & \mapsto & (\sqrt{\varepsilon^2 - r^2}, -\theta). \end{array} \tag{2.1}$$

This is in fact a symplectomorphism with respect to the standard area form $\omega = r\,dr\,d\theta$, since in polar coordinates on the punctured disc one has:

$$\omega_{(r_0,\theta_0)}(\partial_r, \partial_\theta) = r_0$$

and

$$\begin{aligned} \rho^*\omega_{(r_0,\theta_0)}(\partial_r, \partial_\theta) &= \omega_{\rho(r_0,\theta_0)}(\rho_{*(r_0,\theta_0)}\partial_r, \rho_{*(r_0,\theta_0)}\partial_\theta) = \\ &= \omega_{(\sqrt{\varepsilon^2 - r_0^2}, -\theta_0)}\left(-\frac{r_0}{\sqrt{\varepsilon^2 - r_0^2}}\partial_r, \partial_\theta\right) \\ &= \sqrt{\varepsilon^2 - r_0^2} \cdot \frac{r_0}{\sqrt{\varepsilon^2 - r_0^2}} = r_0. \end{aligned}$$

Let ϕ be defined by the commutative diagram

$$\begin{array}{ccc} N \times (D_\varepsilon - \{0\}) & \xrightarrow{id \times \rho} & N \times (D_\varepsilon - \{0\}) \\ f_1 \big\downarrow & & \big\downarrow f_2 \\ V_1 - N_1 & \xrightarrow[\phi]{} & V_2 - N_2 \end{array}$$

Then ϕ is a symplectomorphism and the manifold

$$M = (M_1 - N_1) \cup_\phi (M_2 - N_2)$$

admits a symplectic structure.

The following is a standard example of the dependence of the diffeomorphism type of a symplectic sum from the choice of gluing diffeomorphism.

Example 2.11. We consider trivial torus bundles over T^2 and S^2 and perform the symplectic connected sum along two fibres $T^2 \times \{\mathrm{pt}\}$, which are symplectically embedded tori of square zero. Topologically we obtain the sum

$$(T^2 \times T^2 - T^2 \times D^2) \cup_{T^2 \times \partial D^2} T^2 \times D^2$$

that is, we can think of the symplectic sum as obtained from the trivial T^2-bundle over T^2 by cutting out a tubular neighbourhood of one fibre and gluing it back in by an orientation preserving diffeomorphism of the boundary. Thus the manifold obtained is again a T^2-bundle over T^2, but from the classification of such bundles (compare with [6]), we know that different gluing diffeomorhisms give rise to different diffeomorphism types of the total space of the bundle. So if we choose standard framings for our symplectic sum, the gluing diffeomorphism will be just the identity map and we will recover the trivial bundle. We could make a different choice of framing, though: for instance, with a suitable choice of a twisted framing, it is possible to construct the torus bundle of Example 2.4.

2.5 Branched coverings.

2.5.1 Definitions.

Definition 2.12. Let M and N be smooth n-dimensional manifolds. A smooth map $f: M \to N$ is called a ***k*-fold branched covering** if it is a smooth proper map with critical set $B \subset N$ such that

(i) $f|M - f^{-1}(B) : M - f^{-1}(B) \to N - B$ is a covering map of degree k,

(ii) for every point $p \in f^{-1}(B)$ there are coordinate neighbourhoods $U, V \overset{\cong}{\to} \mathbb{C} \times \mathbb{R}^{n-2}$ around $f(p)$ and p, respectively, on which f is given by

$$(z,x) \mapsto (z^m, x)$$

for some integer m called the **branching index** of f at p.

The branching index is a local invariant, hence it is constant on each connected component of $f^{-1}(B)$.

The critical points of f are the points in $f^{-1}(B)$ at which the differential of f fails to be surjective: we will denote the set of these points by C. By definition, $f|M - f^{-1}(B)$ is an ordinary covering, hence in particular a local diffeomorphism, thus $T_p f$ is an isomorphism at all points of $M - f^{-1}(B)$. Given $p \in C$ and a neighbourhood U of p as in (ii), we see that $C \cap U$ can be identified locally with $\{0\} \times \mathbb{R}^{n-2} \subset \mathbb{C} \times \mathbb{R}^{n-2}$. This shows that C is a codimension 2 submanifold of M. Moreover, $f|_C$ is an immersion.

For each $p \in B$, the fibre $f^{-1}(p)$ is discrete. Since f is proper, $f^{-1}(p)$ is also compact, hence it must be a finite set, say $\{q_1, \ldots, q_n\}$. Then for every q_i there exist neighbourhoods U_i, V_i such that $f|V_i : V_i \to U_i$ is given by $(z,x) \mapsto (z^{m_i}, x)$. We may assume that $U_i = U$ for all i and that the V_i's are all disjoint.

In fact, by shrinking U we may assume that $\bigsqcup V_i = f^{-1}(U)$. For otherwise, denoting by U_j the $\frac{1}{j}$-neighbourhood of p, we would find $s_j \in f^{-1}(U_j)$ but not contained in V_i for any i and thus obtain a sequence $\{s_j\}$ with $f(s_j)$ converging to p. By passing to a subsequence we may assume that $\{s_j\}$ is convergent, say $s_j \to s$ with $f(s) = p$ by continuity. But then we would have $s \in f^{-1}(p)$, hence $s = q_i \in V_i$ for some i, which is a contradiction. Therefore we may assume that the V_i's are exactly the connected components of $f^{-1}(U)$.

For each p and U as above, the restriction $f|V_i - f^{-1}(B) : V_i - f^{-1}(B) \to N - B$ is an m_i-fold covering, whereas $f|V_i \cap f^{-1}(B) : V_i \cap f^{-1}(B) \to U \cap B$ is a diffeomorphism. Thus we have shown the following result:

Proposition 2.13. *The restriction $f|f^{-1}(B) : f^{-1}(B) \to B$ of a branched covering to the preimage of the branching set is an unbranched covering.*

We now briefly recall the classification of branched coverings with a given branching set in the differentiable setting.

Lemma 2.14. *Let $f : M \to N$ be a branched covering and suppose that the branching set B is an embedded submanifold of N: then f is determined, up to diffeomorphism, by the subgroup $f_*(\pi_1(M - f^{-1}(B))) < \pi_1(N - B)$.*

Remark. Notice that $B = f(C)$ is in general just an immersed submanifold.

Proof. Let νB be a tubular neighbourhood of B in N. Then νB has a D^2-bundle structure over B and its boundary $\partial \nu B$ a circle bundle structure. The structure group is in both cases $O(2)$. The preimages $f^{-1}(\nu B)$ and $f^{-1}(\partial \nu B)$ inherit analogous bundle structures over $f^{-1}(B)$.

The bundle projection $\sigma : f^{-1}(\partial \nu B) \to f^{-1}(B)$, for example, is defined as follows. For $q \in f^{-1}(\partial \nu B)$ with image $p \in \partial \nu B$, we choose a neighbourhood U of p as in (ii) of the definition of branched coverings and assume furthermore that, denoting by π the bundle projection $\partial \nu B \to B$, we have $\pi^{-1}(U \cap B) \subset U$. Suppose V_1 is the connected component of $f^{-1}(U)$ containing q: then $f|V_1 \cap f^{-1}(B) : V_1 \cap f^{-1}(B) \to U \cap B$ is a diffeomorphism, so there exists a unique $r \in V_1 \cap f^{-1}(B)$ such that $f(r) = \pi(p)$. Set $\sigma(q) = r$.

The S^1-bundle structure is uniquely determined by $f|M - f^{-1}(B)$ and this in turn determines the D^2-bundle that fills it (the correspondence is $1-1$ because the structure group is $O(2)$). □

2.5.2 Cyclic branched coverings.

Definition 2.15. A branched covering $f : M \to N$ with branching set $B \subset N$ is called **cyclic** if $f|M - f^{-1}(B)$ is a cyclic covering in the usual sense, that is, the cyclic group $\mathbb{Z}_k$ acts properly discontinuously on $M - f^{-1}(B)$ and $N - B \cong (M - f^{-1}(B))/\mathbb{Z}_k$.

By the previous Lemma and the classification of cyclic unbranched coverings, f is completely determined by the branching set B and a surjective homomorphism $\rho : \pi_1(N - B) \to \mathbb{Z}_k$. Notice that since $\mathbb{Z}_k$ is abelian, the epimorphism ρ factors through the abelianisation $H_1(N - B)$ of π_1, hence f is also uniquely determined by a surjective homomorphism $\tau : H_1(N - B) \to \mathbb{Z}_k$.

Example 2.16. Let the cyclic group $\mathbb{Z}_k$ act on M by orientation preserving diffeomorphisms. If N denotes the set of fixed points of this action, assume that every component of N has codimension 2 and that $\mathbb{Z}_k$ acts freely on $M-N$. Then $M_0 = M/\mathbb{Z}_k$ is a manifold and the quotient map $p: M \to M_0$ is a cyclic branched covering, which maps N diffeomorphically onto $p(N) \subset M_0$. There are local coordinates $(z, \xi) \in \mathbb{C} \times \mathbb{R}^{n-2}$ defined in a neighbourhood of N, with N described locally by $z = 0$, and $(z', \xi') \in \mathbb{C} \times \mathbb{R}^{n-2}$, defined in a neighbourhood of $p(N)$, with $p(N)$ described locally by $z' = 0$. With respect to these coordinates, the map p has the local description $z' = z^n$, $\xi' = \xi$.

We now turn our attention to the problem of the existence of (cyclic) branched coverings with a given branching set. Hirzebruch shows in [15] how to construct such coverings under the condition that the branching index divides the Poincaré dual of the branching set.

Proposition 2.17. *Suppose we are given a codimension* 2 *embedding of compact, oriented manifolds* $B \subset N$ *and an integer* $k \in \mathbb{Z}$ *satisfying the condition that* $k \,|\, PD[B]_N \in H^2(N;\mathbb{Z})$*. Then we can construct a k-fold cyclic branched cover of N with branching set B.*

Proof. Let $E' \to N$ be the complex line bundle with first Chern class equal to $PD[B]_N$. Since $k \,|\, PD[B]_N$, there exists $x \in H^2(N;\mathbb{Z})$ such that $PD[B]_N = kx$. Let E be the complex line bundle over N satisying $c_1(E) = x$. Then in particular $E^{\otimes k} \cong E'$. Notice that, if $H_1(N) = 0$, then $H^2(N)$ has no torsion and the isomorphism class of E is uniquely defined.

There exists a smooth section $s: N \to E'$ with the following properties:

(i) s vanishes on B;

(ii) s is everywhere nonzero on $N-B$;

(iii) s is transverse to the zero section of E'.

It suffices to construct such a section in a tubular neighbourhood U of B. The condition $c_1(E') = PD[B]_N$ is saying that the normal bundle of B in N is isomorphic to the pull-back $E'|_B$ of E' along B. Since E' is a complex bundle, we find a cover $\{V_\alpha\}$ of B and a trivialisation of $E'|_B$ of the form

$$E'|_{V_\alpha} \xrightarrow{\cong} V_\alpha \times \mathbb{C}.$$

By identifying U with a tubular neighbourhood of the zero section of E', we get a cover $\{(U_\alpha, \Phi_\alpha)\}$ of U with $\Phi_\alpha(U_\alpha) \cong \mathbb{R}^{n-2} \times \mathbb{C}$. If we compose Φ_α with projection onto the complex coordinate, we get a map $f_\alpha : U_\alpha \to \mathbb{C}$ such that $N \cap U_\alpha = f_\alpha^{-1}(0)$. Let $g_{\alpha\beta} := f_\alpha f_\beta^{-1}$ and denote by E' the bundle defined by the transitions functions $\{g_{\alpha\beta}\}$ with respect to the trivialising cover $\{U_\alpha\}$. By construction, $c_1(E') = e(E') = PD[B] = kx$.

Locally, there are sections $s_\alpha : U_\alpha \to U_\alpha \times \mathbb{C}$, $p \mapsto (p, f_\alpha(p))$. By definition of the transition functions, these sections glue together to give a global smooth section s of E', which vanishes on B and is everywhere nonzero on $U-B$. In fact, s can be extended to a section which is everywhere nonzero on $N-B$. Moreover, $s(N)$ intersects the zero section of E' transversely: this is also easily checked locally.

Define

$$\begin{array}{rccc} \tau: & E & \longrightarrow & E' \cong E^{\otimes k} \\ & v & \longmapsto & v \otimes \cdots \otimes v, \end{array}$$

and consider the diagram

$$\begin{array}{ccc} E & \xrightarrow{\tau} & E' \cong E^{\otimes k} \\ \pi \downarrow & & \downarrow \pi' \\ N & \xrightarrow{\equiv} & N \end{array}$$

We claim that $M := \tau^{-1}s(N) \subset E$ is a smooth manifold and that $f : M \to N$, where f is the restriction of the bundle projection π, is the desired cyclic k-fold branched covering of N, branched along B.

To see that the first claim holds, it is enough to observe that τ is transverse to $s(N)$.

As to the second one, notice first of all that since π is a bundle map, it is open and closed. The same holds for f, which moreover has compact fibres and is therefore proper. Of the definition of branched coverings, condition (i) is immediately verified: in fact, if $p \notin B$, then $s(p) \neq 0$ and $f^{-1}(p) = \tau^{-1}(s(p))$ consists of precisely k points (all k-fold roots of $s(p)$). Clearly, f is cyclic. Condition (ii) is easily verified for $\tau|M : M \to s(N)$; one just needs to take coordinate neighbourhoods around the zero section and let z be the fibre coordinate. This also proves that (ii) holds for f: it suffices to observe that τ and f, when restricted to M, only differ by the diffeomorphism s. By construction, the upstairs branching set is $f^{-1}(B) = \tau^{-1}(s(B))$ and $f \mid f^{-1}(B) : f^{-1}(B) \to B$ is a diffeomorphism. ☐

Remark. We can thus construct a cyclic branched covering as a submanifold of the total space of some complex line bundle over the base manifold, or rather of the associated sphere bundle. This admits a symplectic form (by Thurston's method) which, as we will see in the next section, restricts to a symplectic form on the branched covering.

2.5.3 Symplectic structures on branched coverings.

Proposition 2.18. *Let $f : M \to N$ be a k-fold branched covering, branched along a codimension 2 smooth submanifold $B \subset N$. Assume that N admits a symplectic form ω such that B is in fact a symplectic submanifold. Then M admits a symplectic form $\widetilde{\omega}$, which coincides with $f^*\omega$ outside an arbitrarily small neighbourhood of the upstairs branching set $f^{-1}(B)$.*

Remark. A proof of this result using symplectic cuts is given in [10].

Proof. Let $\{B_i\}$ be the connected components of $f^{-1}(B)$. The branching index on each component is constant, say k_i. The situation described in the statement is illustrated by the commutative diagram

$$\begin{array}{ccc} \bigcup_i B_i = f^{-1}(B) & \longrightarrow & M \\ \downarrow & & \downarrow f \\ B & \longrightarrow & N \end{array}$$

Denote by C the critical set of f, that is $C = \bigcup_{k_i \geq 2} B_i$. Since $f|_{M-C} : M - C \to N - B$ is a local diffeomorphism, the form $f^*\omega$ is symplectic along $M - C$. If $k_i \geq 2$, then $f|_{B_i} : B_i \to B$ is a connected (unbranched) covering, so the form $\sigma_i := (f|_{B_i})^*(\omega|_{TB})$ is symplectic along B_i, i.e., the manifold (B_i, σ_i) is symplectic.

Let νB be a tubular neighbourhood of B in N with its D^2-bundle structure over B (with projection p). This induces a D^2-bundle structure (over $f^{-1}(B)$) on $f^{-1}(\nu B)$ such that each connected component νB_i of $f^{-1}(\nu B)$ is a tubular neighbourhood of B_i, with projection $p_i : \nu B_i \to B_i$ making the following diagram commutative

$$\begin{array}{ccc} \nu B_i & \xrightarrow{f|_{\nu B_i}} & \nu B \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p} \\ B_i & \xrightarrow{f|_{B_i}} & B \end{array}$$

We are going to use the following result, which is contained in [9, Lemma 2.2]. We refer to the same paper for details of the proof.

Lemma 2.19. *Let (B, ω_B) be a closed symplectic manifold, $\pi : E \to B$ an $SO(2)$-vector bundle over B. Denote by E^0 and $S = E^0 \cup_{\partial E^0} \overline{E^0}$, respectively, the associated D^2- and S^2-bundles over B, with $\overline{E^0}$ denoting the bundle E^0 with opposite orientation. Let $i_0 : B \to S$ be the zero section of E^0. Then there is a closed 2-form η on S with $i_0^*\eta = 0$ and η restricting to a symplectic form of area 1 on each fibre. Moreover, η can be chosen so that $\eta|_{E^0}$ extends to a closed form on E that is symplectic on each fibre.*

Sketch of the proof. Let β be a closed 2-form on S, representing $PD[i_0(B)] \in H^2(S, \mathbb{Z})$. Then for each fibre F of S with inclusion i_F we have

$$\langle i_F^*[\beta], [F] \rangle = [i_0(B)] \cdot [F] = 1 = \langle [\omega_F], [F] \rangle$$

with ω_F denoting the standard symplectic form with area 1 on S^2. Hence $i_F^*[\beta] = [\omega_{S^2}] \in H^2(F, \mathbb{Z}) \cong \mathbb{Z}$, the last isomorphism given by evaluation on the class $[F]$.

Since its fibres are compact Riemann surfaces, the bundle map π has in fact a symplectic fibration structure and we may apply the first part of Thurston's theorem in order to get a closed 2-form η' on S which restricts to ω_F on each fibre. Let $\eta := \eta' - \pi^* i_0^* \eta'$. Then η also restricts to the standard symplectic form on each fibre and moreover $i_0^*\eta = 0$.

In order to prove the extension property, one needs to make some appropriate choices in the proof of the theorem of Thurston. □

We return to the proof of Proposition 2.18 and apply the above lemma to the manifold B_i and the normal bundle $p_i : E_i \to B_i$ of B_i in M. This yields a closed 2-form η_i on E_i, which is symplectic on each fibre of E_i.

By construction, $[\eta_i'] = [\beta]$, hence η_i is exact away from B_i and there exists a 1-form γ_i on $E_i - B_i$ such that $\eta = d\gamma_i$ on $E_i - B_i$.

At points of B_i, the form $f^*\omega$ is symplectic on TB_i, whereas η_i is nondegenerate in normal direction. This implies that $f^*\omega + t\eta_i$ is symplectic along B_i for all values of t smaller than or equal to a constant t_i.

Since $f^*\omega + t_i\eta_i$ is nondegenerate along B_i, it will be symplectic along a small tubular neighbourhood $\nu_0 B_i$ of B_i. We may assume that $\nu_0 B_i$ is contained in νB_i. In fact, $f^*\omega + t\eta_i$ will be symplectic along $\nu_0 B_i$ for all $t \leq t_i$.

Let λ_i be a smooth bump function in radial direction on νB_i, which is identically 1 near $\nu_0 B_i$ and equals 0 near the boundary of νB_i.

Define

$$\widetilde{\omega} := \begin{cases} f^*\omega + t_i\eta_i & \text{on} \quad \nu_0 B_i \\ f^*\omega + \sum t_i d(\lambda_i\gamma_i) & \text{on} \quad \overline{M - \cup\nu_0 B_i}. \end{cases}$$

Then $\widetilde{\omega}$ is smooth by construction. Moreover, by compactness it is symplectic on $\overline{M - \cup\nu_0 B_i}$ for a choice of t_i sufficiently small. Hence by reducing the t_i's, if necessary, $\widetilde{\omega}$ will be symplectic on M.

Finally, we observe that since $\eta_i|_{TB_i} = 0$, the following holds:

$$\widetilde{\omega}|_{TB_i} = f^*\omega|_{TB_i} + t_i\eta_i|_{TB_i} = f^*(\omega|_{TB}) = \sigma_i,$$

i.e., (B_i, σ_i) embeds symplectically in $(M, \widetilde{\omega})$. □

We now focus on cyclic branched coverings, constructed as in the proof of Proposition 2.17. In this situation we would like to regard M as a symplectic submanifold of the total space of the S^2-bundle S associated to the line bundle E.

Assume that N admits a symplectic form ω. Let ρ be the projection $S \to N$ $\omega_K = K\rho^*\omega + \eta$ the symplectic form on S, with η a closed 2-form restricting to a form of area 1 in each fibre and vanishing along the zero section of E^0. By Lemma 2.19, we may assume that $\eta_{|E^0}$ coincides with the restriction of a closed 2-form on E which is symplectic in each fibre. Then we have the following:

Proposition 2.20. *The restriction of the form ω_K to M is again a symplectic form, i.e., M is a symplectic submanifold of (S, ω_K).*

Proof. Let i denote the inclusion of M in $S = E^0 \cup_{\partial E^0} \overline{E^0}$. By compactness we may in fact assume that i maps M to the interior of E^0. Then we can write

$$\begin{aligned} \omega_K|_{TM} &= i^*\omega_K \\ &= i^*(K\rho^*\omega + \eta) \\ &= Kf^*\omega + i^*\eta. \end{aligned}$$

At points of $f^{-1}(B)$, the tangent space to M splits as

$$TM|_{f^{-1}(B)} \cong \nu_M f^{-1}(B) \oplus Tf^{-1}(B).$$

We take a closer look at $\nu_M f^{-1}(B)$: since by definition $f = \rho \circ i$, we see that $\nu_M f^{-1}(B) \cong f^*(E|_B) \cong (i^*\rho^*E)|_{f^{-1}(B)}$. As in Lemma 2.19, we may assume that $\eta_{|E^0}$ extends to a closed form on E that is symplectic on each fibre and from this we may conclude that it is non-degenerate on $i^*\pi^*E|_{f^{-1}(B)}$ (equivalently, symplectic in the fibres of $\nu_M f^{-1}(B)$, which are isomorphic to the fibres of E). On the other hand, $f^*\omega$ is nondegenerate on the symplectic complement of the vertical bundle with respect to $i^*\eta$, namely $Tf^{-1}(B)$, hence $Kf^*\omega + i^*\eta$ will be nondegenerate on $TM|_{f^{-1}(B)}$ if K is sufficiently large, say $K \geq K_1$. Hence it will be nondegenerate in a tubular neighbourhood $\nu f^{-1}(B)$ of $f^{-1}(B)$, again for all $K \geq K_1$.

On the complement of $\nu f^{-1}(B)$, the form $f^*\omega$ is symplectic, therefore, by compactness, $\omega_K|_{TM}$ is symplectic for K sufficiently large, say $K \geq K_2$. Hence for K larger than both K_1 and K_2, we conclude that $\omega_K|_{TM}$ defines a symplectic form on M. □

Chapter 3

Chern classes of symplectic manifolds.

3.1 Introduction.

We have already seen that the Chern classes of a symplectic manifold (M,ω) are well defined as the Chern classes of a tame almost complex structure. We would now like to answer the question: when do two symplectic forms induce the same Chern classes?

Definition 3.1. Two symplectic forms ω_0 and ω_1 on the same manifold M are called **deformation equivalent** if there exists a smooth family of symplectic forms $\{\omega_t\}$ on M connecting ω_0 and ω_1.

Remark. The symplectic forms ω_t need not be cohomologous. If that is the case, we say that ω_0 and ω_1 are isotopic and that is a much stronger condition: on a closed manifold M, in fact, it implies that (M,ω_0) and (M,ω_1) are symplectomorphic (compare Definition 1.4).

The answer to our initial question is that the Chern classes of a symplectic manifold (M,ω) only depend on ω up to deformation equivalence.

Lemma 3.2. *Let ω_0 and ω_1 be symplectic forms on the $2n$-dimensional manifold M. If they are deformation equivalent, then they induce the same Chern classes, that is,*

$$c_i(M,\omega_0) = c_i(M,\omega_1) \quad \textit{for all} \quad i = 1,\dots,n = \dim_{\mathbb{C}} M$$

Proof. Let $\{\omega_t\}$ be a smooth family of symplectic forms connecting ω_0 and ω_1. Then there exists a smooth family of bundle isomorphisms $\varphi_t : TM \to TM$ such that $\varphi_t^*\omega_t = \omega_0$ for all $t \in [0,1]$ (cf. [20, Ex. 2.4]). Then $\varphi_1 : (TM,\omega_0) \to (TM,\omega_1)$ is an isomorphism of symplectic vector bundles, hence an isomorphism of the underlying complex vector bundles. □

In handling the constructions introduced in the previous chapter, one would like to be able to speak of their Chern classes. It is necessary to understand, though, to which extent this can be done unambiguously. Here is one result which relies on the contractibility of the space of almost complex structures on a given symplectic manifold and which will be useful in handling this problem.

Lemma 3.3. *Let (M,ω) be a symplectic manifold, N a symplectic submanifold of M. Given any tame almost complex structure J_0 on the complement of N in M, one can find a tame almost complex structure J_M on M which coincides with J_0 outside a tubular neighbourhood of N and which is adapted to N, in the sense that TN is J_M-invariant and $J_M|_{TN}$ is again an almost complex structure, tame with respect to $\omega|_{TN}$.*

Proof. Let E be the normal bundle of N in M and U a tubular neighbourhood of N, symplectomorphic to a tubular neighbourhood of the zero section of E. Then U admits an almost complex structure J_U which is adapted to N: compare also with the Remark following Example 2.4.

The space $\mathcal{J}$ of almost complex structures on $U-N$ is contractible, i.e., the identity map is homotopic to the constant map sending any almost complex structure J to J_0, or rather its restriction to $U-N$. Let $F:\mathcal{J}\times I\rightarrow\mathcal{J}$ be the given homotopy. We may assume that $F(J,0)=J$ and $F(J,t)=J_0$ for all $t\geq 1-\varepsilon$ for some small positive ε.

If we let $0\leq t\leq 1$ denote the radial coordinate in U, so that $p\in U$ may be written as (x,v) in some bundle chart, with $x\in N$ and $\| v \|=t$, we can define an almost complex structure on J_M on M as follows:

$$\begin{cases} J_M(x,v)=F(J_U,t)(x,v) \quad \text{for} \quad (x,v)\in U \\ J_M\equiv J_0 \quad \text{on} \quad \mathrm{M-U} \end{cases}$$

Then $J_M|_{TN}=J_0|_{TN}$ is again a tame almost complex structure and $J_M=J_0$ outside U. □

3.2 Symplectic sphere bundles.

3.2.1 Chern classes of projective bundles.

Let $\rho:S\rightarrow N$ be an S^2-bundle with compact symplectic base (N,β) and fibre F. Recall that we consider on S the symplectic form $\omega_K=K\rho^*\beta+\eta$, with $K\in\mathbb{R}$ sufficiently large and η a closed 2-form, restricting to the standard symplectic form of area 1 on each fibre of ρ: compare with the Remark following Theorem 2.2. This definition involves some choices and two symplectic forms of this kind will not in general be symplectomorphic. One can show, though, that they induce the same Chern classes.

Given a tame almost complex structure J on N, following [11] we define an almost complex structure J_S on S. Let H be the subbundle of TS given pointwise as follows: for a point p in a fibre F of S, H_p is defined as the symplectic orthogonal complement of T_pF, that is,

$$H_p=\{v\in T_pS \mid \omega_K(v,w)=0 \text{ for all } w\in T_pF\}.$$

This definition makes sense because each fibre is a symplectically embedded submanifold of S. Then $\rho_*|_H:H\rightarrow TN$ is an isomorphism and $\omega_K(v,w)=0$ for all $v\in\ker\rho_*$ and $w\in H$. Let

j be the standard complex structure on S^2. Then we define J_S to be the pullback ρ^*J on H and j on $\ker\rho_*$, the vertical bundle. Finally, we extend by linearity, that is: we write a vector $v \in TS$ as $v_F + v_H$ and define $J_S v = jv_F + \rho_*^{-1}J\rho_* v_H$.

It was already shown that two forms ω_K and $\omega_{K'}$ are deformation equivalent, hence by Lemma 3.2, they induce the same Chern classes. Alternatively, one can show that they tame the same almost complex structure, namely J_S.

Lemma 3.4. *Any symplectic form ω_K as in Theorem 2.2 tames the almost complex structure J_S.*

Proof. Let $v = v_F + v_H \in TS$, $v \neq 0$.Then

$$\begin{aligned}
\omega_K(v, J_S v) &= \omega_K(v_F + v_H, jv_F + \rho_*^{-1}J\rho_* v_H) \\
&= \omega_K(v_F, jv_F) + \omega_K(v_H, \rho_*^{-1}J\rho_* v_H) \\
&= \eta(v_F, jv_F) + K\rho^*\beta(v_H, \rho_*^{-1}J\rho_* v_H) + \eta(v_H, \rho_*^{-1}J\rho_* v_H) \\
&= \eta(v_F, jv_F) + K\beta(\rho_* v_H, J\rho_* v_H) + \eta(v_H, \rho_*^{-1}J\rho_* v_H).
\end{aligned}$$

Notice that η restricts in each fibre to the canonical symplectic form on S^2: the latter tames j, so $\eta(v_F, jv_F) > 0$ unless $v_F = 0$. If v_H=0 we have thus $\omega_K(v, J_S v) = \eta(v_F, jv_F) > 0$. If $v_H \neq 0$, the term $K\beta(\rho_* v_H, J\rho_* v_H)$ is positive because $\rho_*|_H$ is an isomorphism and β tames J, whereas the term $\eta(v_H, \rho_*^{-1}J\rho_* v_H)$ may be negative, but by compactness it is bounded below by a constant. Therefore $K\beta(\rho_* v_H, J\rho_* v_H) + \eta(v_H, \rho_*^{-1}J\rho_* v_H)$ is positive for K sufficiently large. □

The Chern classes of (S, ω_K) are therefore well-defined as Chern classes of (S, J_S). With respect to this almost complex structure, we have the following complex vector bundle isomorphism:

$$TS \cong \rho^*TN \oplus \ker\rho_*.$$

In particular, we then have the Whitney formula

$$c(TS) = \rho^* c(TN) \cup c(\ker\rho_*).$$

Assume from now on that $S = \mathbb{P}(E)$ is obtained by projectifying a complex rank 2 bundle E over N and let $l_E \subset \rho^*E$ be the tautological line bundle over S.

$$\begin{array}{ccc}
l_E \subset \rho^*E & \longrightarrow & E \\
\big\downarrow & & \big\downarrow \pi \\
S = \mathbb{P}(E) & \xrightarrow{\ \rho\ } & N
\end{array}$$

If we set $c_1(l_E^*) =: \xi$, there exists a ring isomorphism

$$H^*(S;\mathbb{Z}) \xrightarrow{\cong} H^*(B;\mathbb{Z})[\xi]/\langle \xi^2 + \pi^* c_1(E)\xi + \pi^* c_2(E)\rangle, \tag{3.1}$$

see [3, p. 270]. Let $\mathbb{P}(E_p) \cong \mathbb{CP}^1$ be the fibre of ρ over a point $p \in N$ and let i_p denote the inclusion $\mathbb{P}(E_p) \hookrightarrow S$. The bundle l_E restricts over one such fibre to the canonical line bundle

L over $\mathbb{CP}^1$. Since $c_1(L^*)$ generates $H^2(\mathbb{CP}^1;\mathbb{Z}) \cong \mathbb{Z}$ and is compatible with the orientation (i.e., $\langle c_1(L^*), [\mathbb{CP}^1]\rangle = 1$), we conclude that $i_p^*\xi = i_p^* c_1(l_E^*) = c_1(L^*)$ must be equal to the class of the standard symplectic form of area 1 on $\mathbb{CP}^1$. Hence, according to Thurston's theorem, in the definition of ω_K we may choose η so that it represents the class ξ.

Consider the quotient bundle $E^{(1)} = \rho^* E/l_E$.

Lemma 3.5. *We have the following isomorphism of complex line bundles:*

$$\ker \rho_* \cong E^{(1)} \otimes l_E^*.$$

Proof. One needs to show that $E^{(1)} \otimes l_E^*$ fits into the short exact sequence

$$0 \longrightarrow E^{(1)} \otimes l_E^* \xrightarrow{\alpha} T\mathbb{P}(E) \xrightarrow{\beta} \rho^* TN \longrightarrow 0 \tag{3.2}$$

which defines $\ker \rho_*$, with the morphism β induced by ρ_*.

In order to check exactness, we need to define α. Restrict first to a fibre of S over a point $p \in N$, say $\mathbb{P}(E_p) \cong \mathbb{CP}^1$: then $E^{(1)}$ and l_E^* restrict to the canonical quotient bundle Q and the dual L^* of the tautological line bundle over $\mathbb{CP}^1$. We have there already an isomorphism (see, for example, [3, p. 281])

$$Q \otimes L^* \cong T\mathbb{CP}^1.$$

According to [23], the above isomorphism, followed by the inclusion $T\mathbb{P}(E_p) \subset T\mathbb{P}(E)_{|\mathbb{P}(E_p)}$, extends to a well-defined bundle map

$$\alpha : E^{(1)} \otimes l_E^* \to T\mathbb{P}(E).$$

Exactness of the sequence may now be checked fibrewise. After choosing a Riemannian metric, the canonical quotient bundle Q may be identified with the orthogonal complement of L. Thus $Q \otimes L^* \cong \mathrm{Hom}(L,Q) \cong \mathrm{Hom}(L,L^\perp)$. Given a point $l \in \mathbb{P}(E_p)$, the fibre of L at this point may be identified with l itself and the fibre of Q with $l^\perp$. At this level, then, α gives an isomorphism $\mathrm{Hom}(l,l^\perp) \cong T_l\mathbb{P}(E_p)$. On the other hand, given $(p,l) \in \mathbb{P}(E)$, we also have identifications $l_{E\,(p,l)} \simeq l$ and $Q_{(p,l)} \simeq l^\perp$. Since $T_{(p,l)}\mathbb{P}(E)$ splits as $T_l\mathbb{P}(E_p) \oplus T_pN$, the sequence at (p,l) has the form

$$0 \longrightarrow \mathrm{Hom}(l,l^\perp) \xrightarrow{\alpha} T_l\mathbb{P}(E_p) \oplus T_pN \xrightarrow{\beta} \rho^* TN \longrightarrow 0 \tag{3.3}$$

and by definition of the morphisms α and β it is obviously exact. □

Hence the Chern classes of S may be expressed as follows:

$$\begin{aligned} c(TS) &= \rho^* c(TN) \cup c(E^{(1)} \otimes l_E^*) && (3.4)\\ &= \rho^* c(TN) \cup c(\rho^* E \otimes l_E^*) && (3.5)\\ &= \rho^* c(TN) \cup \sum_{i=0}^{2} c_i(\rho^* E)(1+\xi)^{2-i}, && (3.6) \end{aligned}$$

where the second equivalence follows from the exact sequence

$$0 \longrightarrow l_E \longrightarrow \rho^* E \longrightarrow E^{(1)} \longrightarrow 0 \tag{3.7}$$

by tensoring with l_E^*, which gives

$$0 \longrightarrow \mathbb{C} \longrightarrow \rho^* E \otimes l_E^* \longrightarrow E^{(1)} \otimes l_E^* \longrightarrow 0, \tag{3.8}$$

thus showing that $c(E^{(1)} \otimes l_E^*) = c(\rho^* E \otimes l_E^*)$.

Example 3.6. If the dimension of N equals 4, for example, and E is a complex line bundle over N, the Chern classes of the projectified bundle $S = \mathbb{P}(E \oplus \mathbb{C})$ are given by:

$$\begin{aligned} c_1(S) &= \rho^*(c_1(N) + c_1(E)) + 2\xi, \\ c_2(S) &= \rho^*(c_1(N) \cup c_1(E) + c_2(N)) + 2\rho^* c_1(TN)\xi, \\ c_3(S) &= 2\rho^* c_2(N)\xi. \end{aligned} \tag{3.9}$$

3.2.2 Cyclic branched coverings.

Let $f : M \rightarrow N$ be a k-fold cyclic branched cover, constructed as in the previous chapter, with branching set $B \subset N$. Recall that $M = \tau^{-1}s(N) \subset E$ and $f = \pi|_M$, with $\pi : E \rightarrow N$ a complex line bundle over N satisfying $PD[B] = k\, c_1(E)$, s a section of $E^{\otimes k}$

$$\begin{array}{ccc} M & \xrightarrow{\ i\ } & E \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ N & \xrightarrow{\ \equiv\ } & N \end{array}$$

and $\tau : E \rightarrow E^{\otimes k} =: E'$ the k-fold tensor product map. The projection $E' \rightarrow N$ is denoted by π'.

We have seen in Proposition 2.20 that if the manifold N admits a symplectic form β, then M may be regarded as a symplectic submanifold of the S^2-bundle associated with the line bundle E, that is $S = \mathbb{P}(E \oplus \mathbb{C})$, with respect to the symplectic form $\omega_K = K\rho^*\beta + \eta$.

We would like to compute the Chern classes of M with respect to the restriction of this symplectic form, i.e., with respect to the restriction of a tame almost complex structure on S, say J_S, which is adapted to M: the existence of such an almost complex structure is guaranteed by Lemma 3.3, applied to the symplectic embedding of M in S. Then by the symplectic tubular neighbourhood theorem, this defines an almost complex structure in a tubular neighbourhood of the zero section of E. This almost complex structure restricts to an almost complex structure on M. To summarise, we may obtain the Chern classes of M, regarded as a symplectic submanifold of S, from the relation

$$\pi^*(c(E) \cup c(TN))|_M = c(TE)|_M = c(TM) \cup c(\nu_E M),$$

where $\nu_E M$ denotes the normal bundle of M in E and is described up to isomorphism by the following lemma.

Lemma 3.7. *The normal bundle of $s(N)$ in E' is isomorphic to the vertical bundle $(\pi')^* E'$ and the normal bundle of M in E to its pullback $\tau^*(\pi')^* E'|_M = f^* E'$.*

Proof. The first claim is immediate because s is a section.

To prove the second one, we need to use the transversality of τ and $s(N)$. Namely, there is a map ϕ, induced by τ_*, fitting into the commutative diagram

$$\begin{array}{ccc} \nu_E M \cong TE|_M/TM & \xrightarrow{\phi} & TE'|_{s(N)}/Ts(N) \cong \nu_{E'}s(N) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\tau|_M} & s(N) \end{array}$$

It is defined in the following way:

$$\phi[(x,v)] = [(\tau(x), \tau_*(v))]$$

for all $[(x,v)] \in TE|_M/TM$, which means $x \in M$, $v \in T_xE$ and $[(x,v)] = [(x,v')]$ if and only if $v - v' \in T_xM$. Then ϕ covers τ by definition.

If $[w] \in T_yE'/T_ys(N)$, with $y = \tau(x) \in s(N)$, by transversality of τ and $s(N)$ we may write w as $u+v$, with $u \in T_ys(N)$ and $v \in \tau_*T_xM$. Hence $[w] = [v] = [\tau_*s] = \phi_x[s]$ for some $s \in T_xM$ such that $\tau_*s = v$. This shows that ϕ is fibrewise surjective. Then, by dimension arguments, it is fibrewise an isomorphism.

Thus ϕ is a bundle map and the universal property of the pullback implies the claim. ☐

We have the following sequence of isomorphisms:

$$\nu_E M \cong \tau|_M^* \nu_{E'} s(N) \cong \tau|_M^* (\pi'|_{s(N)})^* E' \cong \pi|_M^* E' \cong (f^*E)^{\otimes k}.$$

Hence the total Chern class of M can be computed using the relation

$$c(TE)|_M = c(TM) \cup (1 + kf^*c_1(E)). \tag{3.10}$$

By induction one can now prove the following result.

Proposition 3.8. *Let $f : M \to N$ be a symplectic k-fold cyclic branched covering, obtained as above from the complex line bundle $\pi : E \to N$. Then the Chern classes of M are given by*

$$c_r(M) = f^*c_r(N) + (1-k)\sum_{i=1}^{r}(-1)^{i-1}k^{i-1} f^*c_{r-i}(N) \cup f^*c_1^i(E).$$

Proof. First of all, from the relation $c(TE) = \pi^*c(N) \cup \pi^*c(E)$ we get for the single Chern classes the expression

$$c_i(TE) = \pi^*c_i(N) + \pi^*(c_{i-1}(N) \cup c_1(E)).$$

If we denote by i the inclusion of M in E, equation (3.10) implies that

$$i^*c_1(TE) = c_i(M) + kf^*c_1(E) \cup c_{i-1}(M).$$

To finish the proof we compute

$$\begin{aligned}
c_{r+1}(M) &= i^*c_{r+1}(TE) - kf^*c_1(E)\cup c_r(M) \\
&= f^*c_{r+1}(N) + f^*c_r(N)\cup c_1(E) - kf^*c_1(E)\cup f^*c_r(N) + \qquad (3.11)\\
&\quad -k(1-k)f^*c_1(E)\sum_{i=1}^{r}(-1)^{i-1}k^{i-1}f^*c_{r-i}(N)\cup f^*c_1(E)^i \\
&= f^*c_{r+1}(N) + (1-k)f^*c_r(N)\cup c_1(E) + \\
&\quad +(1-k)\sum_{i=1}^{r}(-1)^i k^i f^*c_{r-i}(N)\cup f^*c_1(E)^{i+1} \\
&= f^*c_{r+1}(N) + (1-k)\sum_{i=0}^{r}(-1)^i k^i f^*c_{r-i}(N)\cup f^*c_1(E)^{i+1} \\
&= f^*c_{r+1}(N) + (1-k)\sum_{j=1}^{r+1}(-1)^{j-1}k^{j-1}f^*c_{(r+1)-j}(N)\cup f^*c_1(E)^j,
\end{aligned}$$

where (3.11) follows from inductive hypothesis. So the Proposition is proved. □

Example 3.9. (Compact Riemann surfaces as branched coverings of S^2.) Let a be the generator of $H^2(S^2)$ compatible with the orientation. Our aim is to construct a 2-fold branched covering of S^2, branched at $2g+2$ points. Consider the complex line bundle $\pi : L \to S^2$, defined by $c_1(L) = (g+1)a$. Then $c_1(L^{\otimes 2}) = 2(g+1)a = PD[(2g+2)\{\mathrm{pt}\}]$. By applying the cyclic branched covering construction, we get a compact surface $\Sigma_g \subset L$ and a 2-fold branched covering $f : \Sigma_g \to S^2$. Then one can compute

$$c_1(\Sigma_g) = f^*c_1(S^2) - f^*c_1(L) = (1-g)f^*a$$

and

$$\langle c_1(\Sigma_g), [\Sigma_g]\rangle = (1-g)\langle f^*a, [\Sigma_g]\rangle = 2(1-g)\langle a, [S^2]\rangle = 2(1-g)$$

that is, Σ_g is exactly the compact Riemann surface of genus g and the covering we have recovered is the well-known quotient map induced by rotation of Σ_g by π around a vertical axis with $2g+2$ fixed points.

3.2.3 Other submanifolds: sections.

We are considering the S^2-bundle $\rho : S \to N$, where $S = \mathbb{P}(E \oplus \mathbb{C})$ for a complex line bundle E over the symplectic manifold (N, β). Again we may also regard S as obtained by gluing along their boundary two copies of a closed disc subbundle of E, namely $S = E^0 \cup_{\partial E^0} \overline{E^0}$, where $\overline{E^0}$ has opposite orientation to E^0.

There are embeddings $i_+ : N \to S$ and $i_- : N \to S$, corresponding to the zero sections of E and $\overline{E}$, respectively. We denote their images in S by N_+ and N_-. Then the normal bundle of each of these sections coincides with the vertical bundle in E or $\overline{E}$ and we have $\nu_S N_+ = \rho^*E|_{N_+}$ and $\nu_S N_- = \rho^*E|_{N_-}$.

Notice that N_+ and N_- are symplectic submanifolds of $(S, \omega_K = K\rho^*\beta + \eta)$. In fact, $\omega_K|_{TN_+} = K\rho^*\beta|_{TN_+} + \eta|_{TN_+}$ and since $\rho^*\beta$ is nondegenerate on TN_+, also ω_K will be non-degenerate for K sufficiently large. The same argument applies to the restriction of ω_K to TN_-.

Remark. The restriction of the bundle map ρ to either section is not a symplectomorphism, but it is compatible with the almost complex structure, i.e., the restriction of its differential ρ_* to TN_+ or TN_- is a complex isomorphism. Hence $c(N_+) = \rho^*|_{N_+} c(N)$ and similarly for N_-.

3.2.4 Donaldson's theorem.

The following theorem is taken from [5].

Theorem 3.10. *Let (M, ω) be a compact symplectic manifold with integral cohomology class $[\omega] \in H^2(M, \mathbb{Z})$. Then for every sufficiently large integer λ there exists a connected codimension 2 submanifold $N_\lambda \subset M$ which represents the Poincaré dual of $\lambda[\omega]$.*

Remark. Writing $[\omega] \in H^2(M;\mathbb{Z})$ already denotes the choice of an integral lift of $[\omega] \in H^2(M;\mathbb{Z})$.

The integrality condition on $[\omega]$ is not a restrictive one for our purposes. In fact, we may always assume that it is satisfied.

Lemma 3.11. *Given a symplectic manifold (N, β), there exists an integral symplectic form $\overline{\beta}$ on N, inducing the same Chern classes as β.*

Proof. First we approximate β by a closed rational form β'. In order to do this, choose a basis $u_1, \ldots, u_m$ for $H^2(N;\mathbb{Z})$ and 2-forms $\alpha_j \in \Omega^2(N)$ representing the element of the basis, that is, $[\alpha_j] = u_j$. Then there exist coefficients $\lambda_j \in \mathbb{R}$ such that $[\beta] = \sum_{j=1}^m \lambda_j \alpha_j \in H^2(N;\mathbb{R})$. Now consider the form

$$\beta' = \beta + \sum_{j=1}^{m} (r_j - \lambda_j)\alpha_j,\ r_j \in \mathbb{Q}.$$

By choosing the r_j's to be rational we obtain a rational form. In fact,

$$[\beta'] = [\sum_{j=1}^{m} r_j \alpha_j] = \sum_{j=1}^{m} r_j u_j \in H^2(N;\mathbb{Q}).$$

The differences $(r_j - \lambda_j)$ can be made arbitrarily small and for a sufficiently small perturbation the form β' is still symplectic. Moreover, since we can obviously linearly interpolate between β and β', the two forms induce the same Chern classes and hence the same Chern numbers. Now choose a positive integer $x \in \mathbb{Z}_{>0}$ such that $x[\beta'] \in H^2(N;\mathbb{Z})$ and set $\overline{\beta} := x\beta'$. By construction, the form $\overline{\beta}$ is symplectic and represents an integral cohomology class. It is homotopic to β', hence also to the original form β, so it induces the same Chern classes. ☐

An immediate consequence of the above lemma is that given a symplectic sphere bundle (M, ω_K) with symplectic base (N, β), we may replace ω_K by $\overline{\omega}_K := K'\rho^*\overline{\beta} + \eta$, where K' is an integer larger than K, $\overline{\beta}$ is an integral symplectic form on N, satisfying the condition in the lemma, and η has been chosen among the representatives of $c_1(l_E^*)$. Replacing ω_K with the new symplectic form does not affect the Chern classes, and $[\overline{\omega}_K]$ is integral by construction.

Corollary 3.12. *We can always assume the form ω_K on M to be integral. In this situation Donaldson's theorem implies that for sufficiently large $\lambda \in \mathbb{Z}$, there exist symplectic submanifolds $X_\lambda \subset M$ satisfying the relation $PD_M[X_\lambda] = \lambda\,[\omega_K]$.*

The Chern classes of such submanifolds X_λ can be computed from the relation

$$\begin{aligned} c\,(M)_{|X_\lambda} &= c\,(X_\lambda) \cup c\,(\nu_M X_\lambda) \\ &= c\,(X_\lambda) \cup (1 + PD_M[X_\lambda]_{|X_\lambda}) \\ &= c\,(X_\lambda) \cup (1 + \lambda[\omega_K]_{|X_\lambda}). \end{aligned}$$

See Section A.2 in the Appendix for computations in one particular instance.

3.2.5 Branched coverings as submanifolds.

Proposition 2.20 shows the following: suppose S is an S^2-bundle over a symplectic base (N, β) with projection $\rho : S \to N$, obtained by compactifying the complex line bundle $E \to N$, that is, $S = \mathbb{P}(E \oplus \mathbb{C})$. If there exists a symplectic submanifold B of N such that $PD_N[B] = k\,c_1(E)$ for some $k \in \mathbb{N}$, then there also exists a symplectic submanifold M of S with $PD_S[M] = \rho^* c_1(E^{\otimes k})$: it is the k-fold branched covering of N along B. This is a much more elementary and explicit way of finding submanifolds of S^2-bundles over a symplectic base than the application of Donaldson's theorem. By numerical reasons, though, in order to include such submanifolds in the blow-up systems of Chapter 4, we need to find symplectically embedded curves in some 4-dimensional symplectic manifold, realising a multiple of a 2-dimensional homology class with negative square. In this way we come across the problem of solving symplectic singularities: we do not know how this can be achieved within a fixed homology class.

3.3 Chern classes of blow-up.

This section is essentially an adaptation of the results contained in [18] in a smooth topological setting.

We consider the symplectic embedding $i : N \to M$, where N and M are closed symplectic manifolds, and denote the corresponding normal bundle by E. Let $\widetilde{M}$ be the blow up of M along N. Then we have the commutative diagram

$$\begin{array}{ccc} \mathbb{P}(E) & \xrightarrow{\;j\;} & \widetilde{M} \\ {\scriptstyle \rho_E}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ N & \xrightarrow{\;i\;} & M \end{array}$$

which is referred to as the blow-up diagram.

One fundamental remark is the following: recall that $\widetilde{M}$ was defined as $\overline{M-U}\cup_{\partial U}\widetilde{V}$, with $\widetilde{V}$ a neighbourhood of the zero section of the canonical bundle l_E, and consider the inclusion $\mathbb{P}(E)\subset\widetilde{M}$. It is in fact an inclusion $\mathbb{P}(E)\subset\widetilde{V}\subset l_E$, which takes $\mathbb{P}(E)$ to the zero section of l_E, where l_E is regarded as a complex line bundle over $\mathbb{P}(E)$ itself. This implies that the normal bundle of $\mathbb{P}(E)$ in $\widetilde{M}$ is isomorphic to l_E. In other words, we have the following short exact sequence:

$$0\longrightarrow T\mathbb{P}(E)\longrightarrow T\widetilde{M}_{|\mathbb{P}(E)}\longrightarrow l_E\longrightarrow 0 \tag{3.12}$$

(Notice that $T\mathbb{P}(E)$ is a complex vector bundle).

The map f in the blow-up diagram also deserves a few words. It is defined as follows:

$$f=\begin{cases} id & \text{on} \quad M-U\\ \pi\Phi & \text{on} \quad \widetilde{V}\end{cases}$$

where Φ denotes the projection $l_E\to E$. With this definition, f is a diffeomorphism outside $\mathbb{P}(E)$ and, indeed, the blow-up diagram is commutative.

3.3.1 Some cohomological lemmas.

We start by proving some general results, which apply in particular to the blow-up situation. The general reference for the following section is [4].

Definition 3.13. For any map $f:N^{(n)}\to M^{(m)}$ of smooth, compact, oriented manifolds, one can define **"shriek" homomorphisms**

$$f^!:H^{n-p}(N,\partial N)\to H^{m-p}(M,\partial M)$$

and

$$f_!:H_{m-p}(M,\partial M)\to H_{n-p}(N,\partial N)$$

by $f^!=PD_M f_*(PD_N^{-1})$ and $f_!=(PD_N)^{-1}f^*PD_M$, respectively (see [4, p. 368] for more details).

Definition 3.14. If W is a k-disk bundle over a manifold N of dimension n, with projection $\pi:W\to N$, and $i:N\to W$ denotes the inclusion of N in W as zero section, the **Thom class** of W is $\tau=PD_W\, i_*[N]$. The Thom isomorphism

$$H^i(N)\overset{\pi^*}{\to}H^i(W)\overset{\cup\tau}{\to}H^{i+k}(W,\partial W)$$

coincides with $i^!$. If $i:N\to M$ is a smooth codimension k embedding of manifolds, possibly with boundaries that intersect transversely, the Thom class of the inclusion is

$$\tau_N^M=PD_M i_*[N]\in H^k(M).$$

If we denote by W a tubular neighbourhood of N in M and identify it with a k-disk subbundle of the normal bundle of N in M, the Thom class τ_N^M is the image of the Thom class of W under

$$H_k(W,\partial W)\overset{\text{exc}}{\cong}H^k(M,M-W)\to H^k(M).$$

The **Euler class** of the normal bundle is

$$\chi_N^M = i^*\tau_N^M \in H^k(N).$$

Proposition 3.15 (Excision Lemma). *Let $i : N \to M$ be an inclusion of smooth compact manifolds, $U := M - N$ the complement of N in M, with inclusion $u : U \hookrightarrow M$. Suppose $\lambda \in H^*(M)$ satisfies $u^*\lambda = 0$. Then there exists $\beta \in H^*(N)$ such that $i^!\beta = \lambda$.*

Proof. We need to show exactness of the sequence

$$H^*(N) \xrightarrow{i^!} H^*(M) \xrightarrow{u^*} H^*(U).$$

On the other hand, we know that the sequence

$$H^*(M,U) \xrightarrow{k^*} H^*(M) \xrightarrow{u^*} H^*(U)$$

is exact. Let W be a tubular neighborhood of N and consider inclusions $i_0 : N \to W$ and $i_1 : W \to M$ such that $i_1 i_0 = i$. Then $i_0^!$ is the Thom isomorphism. Denote by *exc* the excision isomorphism and by k the inclusion $(M,\emptyset) \hookrightarrow (M,U)$. Then there is a sequence of isomorphisms

$$H^*(N) \xrightarrow{i_0^!} H^*(W,\partial W) \xrightarrow{\equiv} H^*(W,W-N) \xrightarrow{(exc)^{-1}} H^*(M,U).$$

To prove our claim it will be enough to show that $k^*(exc)^{-1}i_0^! = i^!$. Notice that since $i^! = i_1^! i_0^!$ and $i_0^!$ is an isomorphism, we have further reduced our claim to the following

$$k^*(exc)^{-1} = i_1^!$$

for all elements of $H^*(W,\partial W)$ (and in fact, in this form we will apply the lemma later).

We therefore set out to prove that the following diagram is commutative:

$$\begin{array}{ccccc} H^*(W,\partial W) & \xrightarrow{(exc)^{-1}} & H^*(M,U) & \xrightarrow{k^*} & H^*(M) \\ \downarrow D_W^{-1} & & & & \downarrow \equiv \\ H^*(W) & \xrightarrow{i_{1*}} & H_*(M) & \xrightarrow{D_M} & H^*(M) \end{array}$$

Let $\alpha = [f] \in H^*(W,\partial W)$: then f is a cochain of W and it is zero on chains of ∂W. Since $(exc)^{-1} : H^*(W,\partial W) \cong H^*(M,M-N)$ is an isomorphism, we may extend f to a cochain $\tilde{f}$ of M which is zero on chains of $M-N$. Then by construction $\tilde{f} = k^*(exc)^{-1}f$, and $i_1^*\tilde{f} = f$ as chains on W.

We can choose chains c_W and c_M, representing $[W]$ and $[M]$, respectively, in such a way that $i_{1*}c_W = c_M$. Then on the (co)chains level we have

$$D_M i_{1*}(f \cap c) = D_M i_{1*}(i_1^*\tilde{f} \cap c) = D_M(\tilde{f} \cap i_{1*}c_W) = D_M(\tilde{f} \cap c_M) = \tilde{f}.$$

Therefore $PD_M i_{1*} PD_W^{-1}(\alpha) = [\tilde{f}] = k^*(exc)^{-1}(\alpha)$: this proves our claim and the lemma. □

The next is a corollary of the excision lemma which applies to blow-up diagrams.

Corollary 3.16. *Consider the blow-up diagram relative to the embedding $i : N \to M$. Denote by u the inclusion of $M - N$ in M. Assume $\lambda \in H^*(M)$ is such that $u^*(\lambda) = 0$. Then there exists $\mu \in H^*(\mathbb{P}(E))$ such that $(j)^!(\mu) = f^*(\lambda)$.*

Proof. Let v be the inclusion of $\widetilde{M} - \mathbb{P}(E)$ in $\widetilde{M}$. Then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{M} - \mathbb{P}(E) & \xrightarrow{v} & \widetilde{M} \\ f\downarrow & & \downarrow f \\ M - N & \xrightarrow{u} & M \end{array}$$

Hence $v^* f^*(\lambda) = f^* u^*(\lambda) = 0$. Applying the Excision Lemma to $f^*(\lambda)$ yields the claim. □

We would now like to show that for any $y \in H^*(N)$, with inclusion $i : N \to M$ of codimension $2r$ and normal bundle E, the following holds:

$$i^* i^!(y) = y \cup c_r(E),$$

where $c_r(E)$ denotes the top Chern class (i.e., the Euler class) of the bundle E. This equivalence goes under the name of Self Intersection Formula. Notice that if $y = 1 \in H^0(N)$, then the formula is easily verified:

$$i^* i^!(1) = i^* PD_M i_*[N] = i^* \tau_N^M = e(E) = 1 \cup c_r(E)$$

with τ_N^M and $e(E)$ denoting the Thom class and the Euler class, respectively.

Lemma 3.17 (Self-intersection formula). *For any $y \in H^*(N)$,*

$$y \cup c_r(E) = i^* i^!(y).$$

Proof. Let W be a tubular neighbourhood of N in M, with inclusions $i_0 : N \to W$ and $i_1 : W \to M$, so that $i_1 i_0 = i$. Let α be the first morphism in the short exact sequence

$$H^*(W, \partial W) \xrightarrow{\alpha} H^*(W) \to H^*(\partial W).$$

Suppose N has codimension $2r$ in M and denote by $\tau \in H^{2r}(W, \partial W)$ the Thom class $PD_W i_{0*}[N]$. Notice that $i_0^* \alpha\tau = c_r(E) \in H^{2r}(N)$.

We claim that $\alpha = i_1^* i_1^!$. In fact, if we choose $[f] \in H^*(W, \partial W)$, then f extends to a cochain $\tilde{f}$ on M which is zero on chains of $M - N$ (cf. Proposition 3.15). By definition, $i_1^*[\tilde{f}] = \alpha[f] \in H^*(W)$, hence it is enough to show that $i_1^![f] = [\tilde{f}]$. But

$$\begin{aligned} i_1^![f] = PD_M i_{1*} PD_W^{-1}[f] = \\ PD_M i_{1*}(i_1^*[\tilde{f}] \cap [W]) = PD_M([\tilde{f}] \cap [M]) = [\tilde{f}]. \end{aligned}$$

To finish the proof we just need to carry out one last computation:

$$\begin{aligned} i^* i^! y = i_0^* i_1^* i_1^! i_0^! y = i_0^* \alpha(\pi^* y \cup \tau) = \\ i_0^*(\pi^* y \cup \alpha\tau) \ = \ y \cup c_r(E). \end{aligned}$$

□

The Self Intersection Formula can be applied to the map j in the blow-up diagram to obtain the following corollary. Let ξ_E denote the first Chern class of the dual of the tautological line bundle l_E.

Corollary 3.18. *Suppose $\tilde{y} \in H^*(\widetilde{M})$ is such that $(j)^*(\tilde{y}) = -\xi_E \bar{y}$ for some $\bar{y} \in H^*(\mathbb{P}(E))$. Then $\tilde{y} = (j)^! \bar{y} + \lambda$ with $(j)^*\lambda = 0$.*

Proof. We have remarked in the previous section that the normal bundle of the inclusion $j : \mathbb{P}(E) \hookrightarrow \widetilde{M}$ is the tautological line bundle, whose first (top) Chern class is $-\xi_E$, hence by the previous lemma we have that $(j)^*(j)^! \bar{y} = -\xi_E \bar{y}$ for all $\bar{y} \in H^*(\mathbb{P}(E))$.

Rewrite $\tilde{y}$ as $(j)^! \bar{y} + \tilde{y} - (j)^! \bar{y}$ and take $\lambda = \tilde{y} - (j)^! \bar{y}$. Then

$$(j)^*\lambda = (j)^*(\tilde{y} - (j)^*(j)^! \bar{y}) = (j)^*\tilde{y} - (j)^*(j)^! \bar{y} = -\xi_E \bar{y} + \xi_E \bar{y} = 0.$$

□

3.3.2 Remark on the definition of Chern classes of blow-up.

The construction of a symplectic form on the blow-up of a symplectic manifold (M, ω) involves several choices and yields forms which are not necessarily isomorphic. Still, we would like to show that the Chern classes of such blown up manifolds are well defined. For this we need Lemma 3.3: it applies in particular to the blow-up situation and allows us to speak about the Chern classes of the blow-up without ambiguity.

Proposition 3.19. *Let $\widetilde{M}$ denote the blow-up of a symplectic manifold M along a symplectic submanifold N with normal bundle E. Let $\mathbb{P}(E)$ be the exceptional divisor of the blow-up. If J_M is a tame almost complex structure on M, there exists a tame almost complex structure on $\widetilde{M}$, adapted to $\mathbb{P}(E)$ and coinciding with f^*J_M outside a neighbourhood of $\mathbb{P}(E)$.*

Proof. By looking at the construction of a symplectic form on the blow-up of a symplectic manifold in [19], we notice that the inclusion $\mathbb{P}(E) \to \widetilde{M}$ is always symplectic. In fact, the form $\tilde{\omega}$ is defined on $\widetilde{V}$ as $p^*\omega_K + \varepsilon\Phi^*\alpha$, with ω a form on $\mathbb{P}(E)$ as given by the Example following Theorem 2.2 and α a closed 2-form on E. Since $p^*\omega$ is nondegenerate on $T\mathbb{P}(E)$, for ε sufficiently small $\tilde{\omega}$ will be nondegenerate as well. Moreover, since $f : \widetilde{M} \to M$ is a diffeomorphism away from the exceptional divisor $\mathbb{P}(E)$, it follows that f^*J_M is an almost complex structure on the complement of $\mathbb{P}(E)$ in $\widetilde{M}$. If we apply Lemma 3.3 to the symplectic manifold $\widetilde{M}$ and the symplectic submanifold $\mathbb{P}(E)$, we get an almost complex structure as in the statement. □

Corollary 3.20. *With respect to an almost complex structure as in Proposition 3.19, the sequence* (3.12) *gives in fact a complex splitting of vector bundles*

$$j^*T\widetilde{M} \cong T\mathbb{P}(E) \oplus l_E. \tag{3.13}$$

3.3.3 The blow-up formula.

From the isomorphism (3.13) of vector bundles over $\mathbb{P}(E)$ we get

$$j^*c(T\widetilde{M}) = c(T\mathbb{P}(E)) \cup c(l_E)$$

hence with $2r$ denoting the codimension of N in M, we can compute

$$\begin{aligned}
(j)^*(c(T\widetilde{M}) - f^*c(TM)) &= c(T\mathbb{P}(E))(1-\xi_E) - \rho_E^* i^* c(TM) \\
&= c(\ker \rho_*)\rho_E^* c(TN)(1-\xi_E) - \rho_E^* c(TN)\rho_E^* c(E) \\
&= \rho_E^* c(TN)\{(1-\xi_E)c(E') - \rho_E^* c(E)\} \\
&= \rho_E^* c(TN)\{\sum_{i=0}^{r-1} c_i(E^{(1)})(1+\xi_E)^{r-i-1}(1-\xi_E) - \rho_E^* c(E)\}.
\end{aligned}$$

We claim that the expression in brackets is of strictly positive degree in ξ_E. In fact, we might at most get constant terms (i.e., of degree zero in ξ_E) out of the terms

$$\begin{aligned}
\sum_{i=0}^{r-1} c_i(E^{(1)}) - \rho_E^* c(E) &= \sum_{i=0}^{r-1}\sum_{k=0}^{i} \rho_E^* c_k(E)\xi_E^{i-k} - \rho_E^* c(E) \\
&= \sum_{i=0}^{r-1}\sum_{k=0}^{i-1} \rho_E^* c_k(E)\xi_E^{i-k} + \sum_{k=0}^{r-1} \rho_E^* c_k(E) - \sum_{k=0}^{r} \rho_E^* c_k(E) \\
&= \sum_{i=0}^{r-1}\sum_{k=0}^{i-1} \rho_E^* c_k(E)\xi_E^{i-k} - \rho_E^* c_r(E) \\
&= \sum_{i=0}^{r-1}\sum_{k=0}^{i-1} \rho_E^* c_k(E)\xi_E^{i-k} + \sum_{k=1}^{r} \rho_E^* c_{r-k}(E)\xi_E^{k}
\end{aligned}$$

but this expression is of strictly positive degree in ξ_E (unless it vanishes).

Hence $j^*(c(T\widetilde{M}) - f^*c(TM)) = -\xi_E\gamma$, where

$$\gamma = -\frac{1}{\xi_E}\rho_E^* c(TN)\left\{\sum_{i=0}^{r-1} c_i(E^{(1)})(1+\xi_E)^{r-i-1}(1-\xi_E) - \rho_E^* c(E)\right\}.$$

That $j^*(c(T\widetilde{M}) - f^*c(TM))$ must be of positive degree in ξ_E can also be seen as follows. Let $J_{\widetilde{M}}$ be a compatible almost complex structure in $\widetilde{M}$ as in the statement of Proposition 3.19. Outside a tubular neighbourhood U of $\mathbb{P}(E)$, the difference $c(\widetilde{M}) - f^*c(M)$ is zero, because the almost complex structure there coincides with the pullback of a tame almost complex structure on M, that is, $J_{\widetilde{M}} = f^*J_M$. In fact, $c(\widetilde{M}) - f^*c(M)$ vanishes on the complement of $\mathbb{P}(E)$ in $\widetilde{M}$, since $H^*(U) \cong H^*(M)$. Then by Corollary 3.16, one finds $\gamma \in H^*(\mathbb{P}(E))$ such that $c(\widetilde{M}) - f^*c(M) = j^!\gamma$. By application of the Self-Intersection formula we obtain

$$j^*(c(\widetilde{M}) - f^*c(M)) = j^*j^!\gamma = -\gamma\xi_E.$$

We still need to show that the equation $j^*(c(\widetilde{M}) - f^*c(M)) = j^*j^!\gamma = -\gamma\xi_E$ implies $c(\widetilde{M}) - f^*c(M) = j^!\gamma$. By Corollary 3.18 following the proof of the Self-Intersection Formula, there exists λ such that $(j)^*\lambda = 0$ and $c(T\widetilde{M}) - f^*c(TM) = (j)^!\gamma + \lambda$.

Consider once more the inclusions $u : M - N \to M$ and $v : \widetilde{M} - \mathbb{P}(E) \to \widetilde{M}$, so that we have the commutative diagram

$$\begin{array}{ccc} \widetilde{M} - \mathbb{P}(E) & \xrightarrow{\quad v \quad} & \widetilde{M} \\ f|_{\widetilde{M}-\mathbb{P}(E)} \big\downarrow & & \big\downarrow f \\ M - N & \xrightarrow{\quad u \quad} & M \end{array}$$

Since the sequence $H^*(\mathbb{P}(E)) \stackrel{(j)^!}{\to} H^*(\widetilde{M}) \stackrel{v^*}{\to} H^*(\widetilde{M} - \mathbb{P}(E))$ is exact, we have

$$v^*(c(T\widetilde{M}) - f^*c(TM)) = v^*(j)^!\gamma + v^*\lambda = v^*\lambda.$$

On the other hand,

$$v^*(c(T\widetilde{M}) - f^*c(TM)) = c(T(\widetilde{M} - \mathbb{P}(E)) - v^*f^*c(TM) = 0$$

because $v^*f^*c(TM) = f^*|_{\widetilde{M}-\mathbb{P}(E)}u^*c(TM) = c(T(\widetilde{M} - \mathbb{P}(E)))$. Hence $v^*\lambda = 0$. We claim that this implies ultimately that $\lambda = 0$. By exactness of the sequence

$$H^*(\widetilde{M}, \widetilde{M} - \mathbb{P}(E)) \stackrel{k'^*}{\to} H^*(\widetilde{M}) \stackrel{v^*}{\to} H^*(\widetilde{M} - \mathbb{P}(E)),$$

the vanishing of $v^*\lambda$ implies $\lambda = k'^*\mu$ for some $\mu \in H^*(\widetilde{M}, \widetilde{M} - \mathbb{P}(E))$. Let W' be the preimage in $\widetilde{M}$ of a tubular neighbourhood W of N in M. Then W' is a tubular neighbourhood of $\mathbb{P}(E)$ and there are inclusions $j_0 : \mathbb{P}(E) \to W'$ and $j_1 : W' \to \widetilde{M}$, whose composition is equal to $j : \mathbb{P}(E) \to \widetilde{M}$. We denote by α' be the first morphism in the short exact sequence

$$H^*(W', \partial W') \stackrel{\alpha'}{\to} H^*(W') \to H^*(\partial W')$$

and consider the diagram

$$\begin{array}{ccc} H^*(\widetilde{M}, \widetilde{M} - \mathbb{P}(E))) & \xrightarrow{\quad (k')^* \quad} & H^*(\widetilde{M}) \\ exc \big\downarrow & & \big\downarrow j_1^* \\ H^*(W', \partial W') & \xrightarrow{\quad \alpha' \quad} & H^*(W') \cong H^*(\mathbb{P}(E)). \end{array}$$

We have

- $j_1^*k'^*\mu = j_1^*\lambda = 0 \in H^*(W')$ since $j^*\lambda = 0$ and j_0^* is an isomorphism;
- by commutativity of the diagram, $\alpha'\, exc(\mu) = j_1^*k'^*\mu = 0$;
- injectivity of α' implies that $exc(\mu) = 0$;
- finally, since the excision map is an isomorphism, we get $\mu = 0$ and therefore $\lambda = 0$.

Summarising, we are now able to write down the formula according to which the Chern classes behave under blow-up.

Theorem 3.21. *Let M be a symplectic manifold and let $\widetilde{M}$ denote its blow-up along a symplectic submanifold N of real codimension $2r$. If E denotes the normal bundle of N in M, so that we have the commutative blow-up diagram*

$$\begin{array}{ccc} \mathbb{P}(E) & \xrightarrow{\ j\ } & \widetilde{M} \\ {\scriptstyle \rho_E}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ N & \xrightarrow{\ i\ } & M \end{array}$$

the Chern classes of $\widetilde{M}$ are well defined and can be expressed as follows:

$$c(T\widetilde{M}) - f^*c(TM) = \tag{3.14}$$
$$j^! \left[-\frac{1}{\xi_E}\rho_E^* c(TN) \left\{ \sum_{i=0}^{r-1} c_i(E^{(1)})(1+\xi_E)^{r-i-1}(1-\xi_E) - \rho_E^* c(E) \right\} \right]$$

where ξ_E denotes the first Chern class of the dual of the tautological line bundle over $\mathbb{P}(E)$.

In order to gain some confidence about the correctness of this formula, we derive from it the expression for the first Chern class of the blow-up and compare it with the existing formula for algebraic manifolds (see, for example, [12]). In the following we drop the subscript E from the notation.

Corollary 3.22. *The first Chern class of the blow-up of a symplectic manifold M along a submanifold of codimension $2r$ has the expression*

$$c_1(T\widetilde{M}) = f^*c_1(TM) - (r-1)\, j^! 1.$$

Proof. It is enough to observe that $j^!$ maps $H^0(\mathbb{P}(E))$ to $H^2(\widetilde{M})$, so one needs to look for terms of degree one in ξ inside the braces. Recall that $c_i(E^{(1)}) = \sum_{k=0}^{i} \rho^* c_k(E)\, \xi^{i-k}$. Then it is enough to find in the following expression the terms which are linear in ξ.

$$\begin{aligned} &\sum_{i=0}^{r-1} c_i(E^{(1)})(1+\xi)^{r-i-1}(1-\xi) = \\ &\qquad (1+\xi)^{r-1}(1-\xi) + c_1(E^{(1)})(1+\xi)^{r-2}(1-\xi) + \ldots = \\ &\qquad (1+\xi)^{r-1}(1-\xi) + (\xi + \rho^* c_1(E))(1+\xi)^{r-2}(1-\xi) + \ldots = \\ &\qquad \xi(-1+r-1+1) + \ldots = (r-1)\xi + \ldots \end{aligned}$$

Taking into account the $-\frac{1}{\xi}$ factor, we obtain

$$c_1(\widetilde{M}) - f^*c_1(M) = j^! \left[-\frac{1}{\xi}(r-1)\xi \right] = -(r-1) j^! 1.$$

□

Remark. Observe that $j^! 1$ is exactly the Poincaré dual of the exceptional divisor in $\widetilde{M}$: in fact, by definition, $j^! 1 = PD\, j_*[\mathbb{P}(E)]$.

3.4 Symplectic sums.

3.4.1 Symplectic sum along surfaces with trivial normal bundle.

If two symplectic manifolds are symplectically glued along a codimension 2 submanifold with trivial normal bundle, the Chern numbers of the symplectic sum can be expressed in terms of the Chern numbers of the original manifolds and those of a trivial projective bundle over the given submanifold. More precisely we have the following statement (cf. [13]):

Lemma 3.23. *Let X and Y be symplectic $2n$-dimensional manifolds. Suppose N is another symplectic submanifold, of dimension $2n-2$, and that there exists symplectic embeddings of N in X and Y with trivial normal bundle. Then we may consider the symplectic connected sum of X and Y along N, defined as in Section 2.4 and denoted by $W := X\#_N Y$: this has Chern numbers given by*

$$c_I[W] = c_I[X] + c_I[Y] - c_I[N \times S^2]$$

where I stands for any arbitrary partition of n.

Proof. Let D_ε denote the 2-dimensional disk with radius ε. By the Symplectic Tubular Neighbourhood Theorem, there exist symplectically embedded neighbourhoods of N of the form $N \times D_\varepsilon$ both in X and Y, such that N is identified with $N \times \{0\} \subset N \times D_\varepsilon$. Recall that the symplectic connected sum of X and Y is obtained by cutting out N from both manifolds and identifying the tubular shell neighbourhoods $N \times D^*_\varepsilon = N \times (D_\varepsilon \setminus \{0\})$ via the map

$$id \times \rho : (p,(r,\theta)) \longmapsto (p,(\sqrt{\varepsilon^2 - r^2}, -\theta)).$$

Notice that the circle S_0 with radius $r_0 = \varepsilon/\sqrt{2}$ is fixed by ρ, so $N \times S_0$ is identified with itself. Denoting by D_0 the disc with radius r_0, so that $\partial D_0 = S_0$, we can describe W (topologically) as

$$X - N \times D_0 \cup_{N \times S_0} Y - N \times D_0.$$

Let θ be a top-dimensional form on W. The restriction of θ to $X - N \times D_0 =: X^0$ and to $Y - N \times D_0 =: Y^0$ extend to forms θ_1 and θ_2 on X and Y. In turn, the restrictions of θ_1 and θ_2 to the tubular neighbourhoods $N \times D_0$ may be glued together to obtain a form $\widetilde{\theta}$ on $N \times D_0 \cup_{N \times S_0} N \times D_0 = N \times S^2$. Then if we integrate θ on W we get

$$\begin{aligned}
\int_W \theta &= \int_{X^0} \theta|_{X^0} + \int_{Y^0} \theta|_{Y^0} \\
&= \int_X \theta_1 - \int_{N \times D_0} \theta_1|_{N \times D_0} + \int_Y \theta_2 - \int_{N \times D_0} \theta_2|_{N \times D_0} \\
&= \int_X \theta_1 + \int_Y \theta_2 - \int_{N \times S^2} \widetilde{\theta}.
\end{aligned} \tag{3.15}$$

Let σ_i denote the invariant polynomial defined by

$$\det(I + tA) = 1 + t\sigma_1(A) + \ldots + t^n \sigma_n(A)$$

for every square matrix A, and for a partition $I = (i_1 \dots i_r)$ of n let σ_I denote the product $\sigma_{i_1} \dots \sigma_{i_r}$. Then given connections ∇_X and ∇_Y on X and Y with curvature tensors K_{∇_X} and K_{∇_Y}, the polynomials $\sigma_I(K_{\nabla_X}) =: \theta_1$ and $\sigma_I(K_{\nabla_Y}) =: \theta_2$ represent some product of Chern classes of X and Y, namely $c_I(X)$ and $c_I(Y)$, respectively. By using a partition of unity, we may assume that ∇_X and ∇_Y coincide with a product connection over $N \times D_0$. Then they can be pasted toghether to yield a connection ∇ on W with $\sigma_I(K_\nabla) =: \theta$ representing $c_I(W)$. Moreover, the product connections over the tubular neighbourhoods $N \times D_0$ give a product connection $\widetilde{\nabla}$ on $N \times S^2$ such that $\sigma_I(K_{\widetilde{\nabla}})$ coincides with the form $\widetilde{\theta}$ obtained by gluing the restrictions of θ_1 and θ_2 along $N \times S_0$. By applying (3.15) to the situation just described we get

$$\begin{aligned} c_I[W] &= \int_W \theta = \int_X \theta_1 + \int_Y \theta_2 - \int_{N\times S^2} \widetilde{\theta} \\ &= c_I[X] + c_I[Y] - c_I[N \times S^2]. \end{aligned}$$

□

3.4.2 Symplectic sums along tori in dimension 4.

Now let $W = X\#_{T^2}Y$ denote the symplectic sum of two 4-dimensional manifolds X and Y along symplectically embedded tori with square zero. The latter condition means that the first Chern class of the normal bundle of each torus, evaluated on its fundamental homology class, is zero: that is, $\langle c_1(\nu), [T^2] \rangle = 0$. Since Kronecker product with the fundamental class defines an isomorphism between $H^2(T^2;\mathbb{Z})$ and $\mathbb{Z}$, we see that the square of T^2 being zero is equivalent to its normal bundle having vanishing first Chern class or, in other words, to its being trivial. Notice that this by no means implies that the torus is homologically trivial.

We may assume that $PD^{-1}c_1(X)$ and $PD^{-1}c_1(Y)$ admit representatives which are disjoint from the tori along which the sum is performed. In fact, let $K_X = \Lambda^2 T^*X$ be the canonical bundle over X. Then $c_1(K_X^*) = c_1(X)$ and in particular $c_1(K^*_{X|T^2}) = c_1(T^2) + c_1(\nu_X T^2) = 0$ (the first summand is zero because the tangent bundle to any oriented torus is trivial, the second because T^2 is embedded with square zero). Hence there exists a section s of K_X^*, which is nonvanishing over T^2. Let Z_s denote the zero locus of s and identify X with the zero section of K_X^*: then $c_1(X) = PD[Z_s]$ and $Z_s \cap T^2 = \emptyset$.

Hence $PD^{-1}c_1(X)$ and $PD^{-1}c_1(Y)$ represent homology classes in W and we may consider the Poincaré duals of these, which are again denoted by $c_1(X)$ and $c_1(Y)$. Then we can state the following result, which is taken from [25].

Lemma 3.24. *Let X and Y be symplectic manifolds of dimension 4, containing symplectically embedded tori with square zero. If we denote by W their sympletic connected sum along these tori, we have for the first Chern class of W the expression:*

$$c_1(W) = c_1(X) + c_1(Y) - 2PD[T^2]. \tag{3.16}$$

Proof. Consider K_X^*, the anticanonical bundle of X. There is a symplectically embedded neighbourhood $N \cong T^2 \times D_\varepsilon$ of T^2 in X. The tangent bundle of X splits over such a neighbourhood as $L_1 \oplus L_2$, where L_1 and L_2 are line bundles corresponding to the tangent and normal direction to the torus. More precisely, the bundles L_1 and L_2 represent the pull-back of the bundles parallel and normal to T^2 along the symplectomorphism $N \cong T^2 \times D_\varepsilon$. Then the restriction of the anticanonical bundle of X to N has the form

$$K_X^*|_N \cong (\Lambda^2 T^* X)^* \cong (\Lambda^2(L_1^* \oplus L_2^*))^* \cong (L_1^* \otimes L_2^*)^* \cong L_1 \otimes L_2.$$

In particular, a section

$$\sigma_X : N \longrightarrow K_X^*|_N \cong N \times \mathbb{C}$$

can be written as a product $\sigma_X^1 \cdot \sigma_X^2$ of sections of L_1 and L_2, respectively. This implies for the zero sets the relation $Z_{\sigma_X} = Z_{\sigma_X^1} \cup Z_{\sigma_X^2}$. Analogous considerations hold for $T^2 \subset Y$.

So over $T^2 \times (D_\varepsilon - \{0\})$ we also have trivial line bundles L_1 and L_2. Recall that the symplectic sum was obtained by performing the identification of (x,z) with $(x,\rho(z))$, with ρ the symplectic automorphism of the punctured disk defined in (2.1). After identification, then, in the direction parallel to the torus we still have the trivial bundle L_1. In normal direction, though, the effect of the identification is that of a connected sum in each fibre. If we denote by F the fibre of the normal bundle of T^2, we see that L_2 may be regarded as the pullback of the tangent bundle of this fibre along the projection $T^2 \times F \to F$. After identification, that is, we have the following picture

$$\begin{array}{ccc} \widetilde{L_2} & \longrightarrow & T(F\#F) \\ \big\downarrow & & \big\downarrow \\ T^2 \times (F\#F) & \longrightarrow & F\#F \end{array}$$

and there is a contribution to the first Chern class of TW coming from the zero set of a section of the bundle $\widetilde{L_2}$. Consider a section $\sigma : F\#F \to T(F\#F)$: this has zero set $Z_\sigma \subset F$ which represents the Poincaré dual of the element $-2 \in H^2(F\#F) \cong \mathbb{Z}$. It pulls back to a section $\widetilde{\sigma}$ of $\widetilde{L_2}$, defined by $\widetilde{\sigma}(x,y) = (x,y,\sigma(y))$. Then the zero section $Z_{\widetilde{\sigma}}$ is equal to $T^2 \times Z_\sigma \subset T^2 \times (F\#F)$ and represents the class $[Z_{\widetilde{\sigma}}] = -2[T^2] \in H_2(W)$. This implies for the first Chern class of the symplectic connected sum the equivalence

$$\begin{aligned} c_1(W) &= c_1(X) + c_1(Y) + PD_W[Z_{\widetilde{\sigma}}] \\ &= c_1(X) + c_1(Y) - 2PD_W[T^2]. \end{aligned}$$

□

Chapter 4

Symplectic geography.

4.1 Cobordism ring and Chern numbers.

4.1.1 Stable equivalence.

Definition 4.1. A **stable complex structure** on a real vector bundle ξ is a complex structure on $\xi \oplus \varepsilon^k$, where ε denotes the trivial line bundle. Two stable complex structures are **isomorphic** if there is a stable isomorphism of the defining complex vector bundles. A **stably almost complex manifold** is a smooth manifold M with a stable complex structure on the tangent bundle.

Remark. Recall that a bundle ξ over a manifold is orientable if and only if $w_1(\xi) = 0$, that is, its first Stiefel-Whitney class vanishes [3, Ex. 12-A]. This, together with the equivalence $w_1(\xi \oplus \varepsilon) = w_1(\xi)$, implies that any stably almost complex manifold is orientable.

Equivalently, a stably almost complex manifold can be defined as a smooth manifold together with an embedding into some large euclidean space whose normal bundle admits a complex structure. We prove that the two definitions are equivalent in the next lemma.

Lemma 4.2. *Let M be a smooth manifold of dimension m. Then the following two conditions are equivalent:*

(i) the tangent bundle TM admits a stable complex structure;

(ii) there exists an embedding $M \rightarrow \mathbb{R}^N$, with $m + N \equiv 0\,(mod\,2)$, whose normal bundle $\nu(M, \mathbb{R}^N)$ is a complex bundle of rank $\frac{N-m}{2}$.

Proof. First of all recall that if ξ is a real vector bundle, the Whitney sum $\xi \oplus \xi$ is isomorphic to the complexification $\xi \otimes \mathbb{C}$ of ξ and admits a complex structure defined by $J(v, w) = (w, -v)$. Suppose (i) holds, i.e., that for some integer k the bundle $TM \oplus \varepsilon^k$ admits a complex structure. Consider an embedding of M in $\mathbb{R}^n$ for some large n (such an embedding always exists for $n > 2m + 1$) and compose it with the inclusion

$$\mathbb{R}^n \rightarrow \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k.$$

The resulting embedding $M \to \mathbb{R}^{n+k}$ has normal bundle

$$\nu(M,\mathbb{R}^{n+k}) \cong \nu(M,\mathbb{R}^n) \oplus \varepsilon^k$$

and we can write

$$\varepsilon^{n+k} \cong T\mathbb{R}^{n+k}|_M \cong TM \oplus \nu(M,\mathbb{R}^{n+k}) \cong TM \oplus \nu(M,\mathbb{R}^n) \oplus \varepsilon^k.$$

By taking the Whitney sum with $\nu(M,\mathbb{R}^n)$ on both the left and the right hand side we get:

$$\nu(M,\mathbb{R}^n) \oplus \varepsilon^{n+k} \cong \nu(M,\mathbb{R}^n) \oplus \nu(M,\mathbb{R}^n) \oplus TM \oplus \varepsilon^k.$$

By the remark at the beginning of the proof, the sum $\nu(M,\mathbb{R}^n) \oplus \nu(M,\mathbb{R}^n)$ admits a complex structure and so does $TM \oplus \varepsilon^k$ by assumption. Hence $\nu(M,\mathbb{R}^n) \oplus \varepsilon^{n+k}$ admits a complex structure (which implies, in particular, that $\nu(M,\mathbb{R}^n)$ has a stable complex structure). Now the required embedding is

$$M \to \mathbb{R}^{2n+k} = \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n,$$

whose normal bundle $\nu(M,\mathbb{R}^{2n+k}) \cong \nu(M,\mathbb{R}^n) \oplus \varepsilon^{n+k}$ is indeed a complex bundle.

If on the other hand (ii) holds and we are given an embedding $M \to \mathbb{R}^N$ with complex normal bundle $\nu(M,\mathbb{R}^N)$, then we have

$$\varepsilon^N \cong T\mathbb{R}^N|_M \cong TM \oplus \nu(M,\mathbb{R}^N).$$

By summing with TM we obtain

$$TM \oplus \varepsilon^N \cong TM \oplus TM \oplus \nu(M,\mathbb{R}^N).$$

On the right hand side, $TM \oplus TM$ again admits a complex structure as the complexification of TM, whereas $\nu(M,\mathbb{R}^N)$ is a complex bundle by assumption. This implies that $TM \oplus \varepsilon^N$ admits a complex structure as well, i.e., (i) holds. □

The Chern classes of the bundle ξ may be defined as the Chern classes of the complex bundle $\xi \oplus \varepsilon^k$ and the Chern classes of a stably almost complex manifold M as the Chern classes of its tangent bundle. This definition only depends on the stable complex structure up to stable isomorphism. In fact, if $\xi \oplus \varepsilon^k$ and $\xi \oplus \varepsilon^h$ are complex vector bundles defining isomorphic stable complex structures on ξ, that is, such that there is an isomorphism of complex vector bundles

$$\xi \oplus \varepsilon^k \oplus \varepsilon^{N-k} \cong \xi \oplus \varepsilon^h \oplus \varepsilon^{N-h},$$

then we have that $c(\xi \oplus \varepsilon^k) = c(\xi \oplus \varepsilon^h)$. If M is compact of dimension $2m$ and stably almost complex we may also speak of the Chern numbers of its tangent bundle. In this case, $c_m[M]$ does not necessarily coincide with the Euler number.

Example 4.3. Let $M = S^{2n} \subset \mathbb{R}^{2n+1}$: the Whitney sum of the normal and tangent bundle of S^{2n} is trivial, hence S^{2n} admits a stably almost complex structure with $c_n[S^{2n}] = 0$.

Lemma 4.4. *Let M and W be stably almost complex manifolds and suppose that there is a smooth embedding of M in W. Let ν denote the normal bundle of M in W. Then ν is orientable.*

Proof. From the decomposition $TM \oplus \nu = TW|_M$ we get $w_1(TM) + w_1(\nu) = w_1(TW)|_M$. The manifolds M and W are stably almost complex, hence orientable, so $w_1(TM) = w_1(TW) = 0$. This implies that $w_1(\nu)$ must be zero, that is, ν is orientable. □

Let M and W be as in Lemma 4.4 and assume furthermore that the embedding of M in W has codimension 1. Let $W \to \mathbb{R}^N$ and $M \to \mathbb{R}^k$ be the embeddings defining stable almost complex structures on W and M. The inclusion of M in $\mathbb{R}^{N+1}$ resulting from the composition

$$M \to W \to \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R} = \mathbb{R}^{N+1}$$

has normal bundle

$$\nu(M, \mathbb{R}^{N+1}) \cong \nu(M, W) \oplus \nu(W, \mathbb{R}^N) \oplus \varepsilon$$

which is also a complex bundle: in fact, $\nu(W, \mathbb{R}^N)$ is complex by definition and $\nu(M, W) \oplus \varepsilon$ is a real orientable rank 2 bundle, hence it admits a unique complex structure. In this way $\nu(M, \mathbb{R}^{N+1})$ defines a second stably almost complex structure on M, which we call the stably almost complex structure induced by W. We say that it is equivalent to the given stable almost complex structure if there is a stable isomorphism between $\nu(M, \mathbb{R}^{N+1})$ and $\nu(M, \mathbb{R}^k)$.

On the set of stably almost complex manifolds we can introduce an equivalence relation as follows.

Definition 4.5. Two stably almost complex manifolds M_1 and M_2 of equal dimension are called **stably equivalent** if they bound a stably almost complex manifold W in such a way that the stably almost complex structures induced by W on its boundary components, namely M_1 and M_2, are equivalent to the given ones. The equivalence classes of n-dimensional stably almost complex manifolds with respect to stable equivalence form an abelian group, denoted by Ω_n^U. The topological product of manifolds turns the direct sum $\Omega^U = \oplus \Omega_n^U$ into a graded ring, called the **complex cobordism ring**.

The notion of stable equivalence will be from now on referred to as (complex) cobordism.

Definition 4.6. Let E be a $\mathrm{U}(n)$-bundle over a manifold M, with projection π and let E^0 be the associated bundle with fibre $D^{2n} \subset \mathbb{C}^n$: its boundary ∂E^0 is the bundle associated to E with fibre $S^{2n-1} \subset \mathbb{C}^n$. The **Thom space** of E, denoted by $M(E)$, is the quotient $E^0/\partial E^0$: it is homeomorphic to the one-point compactification of E. Let $\mathrm{BU}(n)$ denote the classifying space for the unitary group $\mathrm{U}(n)$. If ζ_n is the universal $\mathrm{U}(n)$-bundle over $\mathrm{BU}(n)$, we write MU_n for the Thom space $M(\zeta_n)$. This gives rise to the so called **Thom complex** MU.

Then by general theory of cobordism (generalized Pontrjagin-Thom theorem), the groups Ω_n^U may be identified with the stable homotopy groups $\pi_n(MU)$. The latter may be calculated by applying homotopy theoretical methods such as Adams' spectral sequence. See [21] for more details.

Lemma 4.7. *The complex cobordism groups Ω_n^U are zero for n odd and for $n = 2m$ they are free abelian of rank equal to the number of partitions of m.*

Milnor also showed that Ω^U is a polynomial ring on even dimensional generators and was able to produce explicit representatives for these generators, which in particular turn out to be, together with their inverses, complex projective algebraic varieties. In particular, Milnor's result involves so called hypersurfaces of bidegree $(1,1)$.

4.1.2 Hypersurfaces of bidegree $(1,1)$.

Definition 4.8. We call **hypersurface of degree** $(1,1)$ a subset $\mathbb{H}_{i,j}$ of the product $\mathbb{CP}^i \times \mathbb{CP}^j$, defined by

$$\mathbb{H}_{i,j} = \{([w_0 : \cdots : w_i], [z_0 : \cdots : z_j]) \in \mathbb{CP}^i \times \mathbb{CP}^j \mid \\ w_0 z_0 + \cdots + w_k z_k = 0, k = \min(i,j)\}.$$

Lemma 4.9. *Hypersurfaces of degree $(1,1)$ are simply connected algebraic varieties of dimension $i+j-1$.*

Proof. Suppose $i \leq j$. There is an obvious projection

$$\begin{array}{ccc} \mathbb{H}_{i,j} & \xrightarrow{\pi} & \mathbb{CP}^i \\ ([w_0 : \cdots : w_i], [z_0 : \cdots : z_j]) & \longmapsto & [w_0 : \cdots : w_i]. \end{array}$$

The fibre over a point $w = [w_0 : \cdots : w_i] \in \mathbb{CP}^i$ is given by

$$\pi^{-1}(w) = \{z = [z_0 : \cdots : z_j] \in \mathbb{CP}^j \,|\, w_0 z_0 + \cdots + w_i z_i = 0\} \subset \mathbb{CP}^j.$$

Since not all coefficients in the defining equation can be zero, we have $\pi^{-1}(w) \cong \mathbb{CP}^{j-1}$. It is not difficult to see that the projection π is locally trivial, so we conclude that $\mathbb{H}_{i,j}$ has a $\mathbb{CP}^{j-1}$-bundle structure over $\mathbb{CP}^i$ and for this reason it is an algebraic variety of the required dimension.

□

If we denote by f the inclusion $\mathbb{H}_{i,j} \to \mathbb{CP}^i \times \mathbb{CP}^j$ and by a and b the 2-dimensional generators of the cohomology rings of $\mathbb{CP}^i$ and $\mathbb{CP}^j$, respectively, the Poincarè dual of $\mathbb{H}_{i,j}$ in $\mathbb{CP}^i \times \mathbb{CP}^j$ is $a+b$, hence in $H^*(\mathbb{H}_{i,j})$ the following relation holds:

$$\begin{aligned} c(\mathbb{H}_{i,j}) \cup f^*(1+a+b) &= f^* c(\mathbb{CP}^i \times \mathbb{CP}^j) \\ &= f^*(1+a)^{i+1} \cup (1+b)^{j+1}. \end{aligned}$$

Example 4.10. We compute the Chern numbers of $\mathbb{H}_{2,2} \subset \mathbb{CP}^2 \times \mathbb{CP}^2$. We start with

$$\begin{aligned} c(\mathbb{H}_{2,2}) \cup f^*(1+a+b) &= f^*(1+a)^3 \cup f^*(1+b)^3 \\ &= f^*(1+3(a+b)+3(a^2+b^2+3ab)+9(a^2b+ab^2)), \end{aligned}$$

where we have disregarded terms which exceed the dimension of $\mathbb{H}_{2,2}$, namely 3. By comparing the left- and right-hand side of the above equivalence, we get that the Chern classes of $\mathbb{H}_{2,2}$ are

$$\begin{aligned} c_1 &= 2f^*(a+b), \\ c_2 &= f^*(a^2+b^2+5ab), \\ c_3 &= 3f^*(a^2b+ab^2). \end{aligned}$$

Recalling that the Poincarè dual of $\mathbb{H}_{2,2}$ in $\mathbb{CP}^2 \times \mathbb{CP}^2$ is $a+b$, we can compute the Chern numbers. For example,

$$\begin{aligned} c_1^3[\mathbb{H}_{2,2}] &= \langle 8f^*(3a^2b+3ab^2), [\mathbb{H}_{2,2}]\rangle \\ &= \langle 8(3a^2b+3ab^2)\cup(a+b), [\mathbb{CP}^2\times\mathbb{CP}^2]\rangle \\ &= \langle 48a^2b^2, [\mathbb{CP}^2\times\mathbb{CP}^2]\rangle = 48. \end{aligned}$$

Similarly,

$$\begin{aligned} c_1c_2[\mathbb{H}_{2,2}] &= \langle 12f^*(a^2b+ab^2), [\mathbb{H}_{2,2}]\rangle \\ &= \langle 12(a^2b+ab^2)\cup(a+b), [\mathbb{CP}^2\times\mathbb{CP}^2]\rangle \\ &= \langle 24a^2b^2, [\mathbb{CP}^2\times\mathbb{CP}^2]\rangle = 24 \end{aligned}$$

and

$$\begin{aligned} c_3[\mathbb{H}_{2,2}] &= \langle 3f^*(a^2b+ab^2), [\mathbb{H}_{2,2}]\rangle \\ &= \langle 3(a^2b+ab^2)\cup(a+b), [\mathbb{CP}^2\times\mathbb{CP}^2]\rangle \\ &= \langle 6a^2b^2, [\mathbb{CP}^2\times\mathbb{CP}^2]\rangle = 6. \end{aligned}$$

4.1.3 The complex cobordism ring.

Now we can completely describe the structure of the complex cobordism ring.

Proposition 4.11. *There exists an isomorphism of graded rings*

$$\begin{aligned} \phi: \quad (\mathbb{Z}[x_1,x_2,\ldots],\cdot,+) &\longrightarrow (\Omega^U,\times,\sqcup) \\ x_n &\mapsto [K_n]. \end{aligned}$$

Moreover, the manifold K_n in the definition of ϕ may be chosen in a class of manifolds generated, under Cartesian product and disjoint sum, by complex projective spaces, hypersurfaces of double degree $\mathbb{H}_{i,j}\subset\mathbb{CP}^i\times\mathbb{CP}^j$ and their inverses.

Let s_n denote the unique polynomial in the elementary symmetric functions σ_j of variables t_i satisfying

$$s_n(\sigma_1,\ldots,\sigma_n) = \sum_{i=1}^n t_i^n.$$

What Milnor shows, then, is that K_n may be taken as $2n$-dimensional generator for the ring Ω^U provided it satisfies the condition

$$\begin{aligned} s_n[K_n] &= \langle s_n(c_1(K_n),\ldots,c_n(K_n)), [K_n]\rangle \\ &= \begin{cases} \pm 1 & \text{if} \quad n+1\neq q^r \quad \text{for any prime } q \\ \pm q & \text{if} \quad n+1 = q^r \quad \text{for some prime } q. \end{cases} \end{aligned}$$

If we assume that a bundle E splits as a Whitney sum of complex line bundles $L_1\oplus\ldots\oplus L_n$, we see that each Chern class $c_k(E)$ coincides with the k-th elementary symmetric function in the variables $c_1(L_1),\dots,c_1(L_n)$, so that $s_n(E) = s_n(c_1(E),\dots,c_n(E)) = \sum_{i=1} c_1(L_i)^n$. In particular, for a Whitney sum $E\oplus F$, the characteristic polynomial satisfies then the relation $s_n(E\oplus F) = s_n(E)+s_n(F)$. By the splitting principle, this relation holds for arbitrary bundles. If we consider the manifold $M = K\times L$, we also have that $s_n(M) = s_n(K)+s_n(L)$ and if K and L are both of dimension strictly smaller than M, this implies that $s_n[K\times L] = 0$. The characteristic number $s_n[M]$ does not vanish for all manifolds, though. If $M = \mathbb{CP}^n$, for example, we know that there is an isomorphism $T\mathbb{CP}^n\oplus\varepsilon \cong (L^*)^{\oplus n+1}$, with ε denoting the trivial complex line bundle and L the tautological line bundle over $\mathbb{CP}^n$. Then

$$c(T\mathbb{CP}^n) = c(T\mathbb{CP}^n\oplus\varepsilon) = c(L^*)^{\oplus n+1}$$

and

$$\begin{aligned} s_n[\mathbb{CP}^n] &= \langle s_n(c_1(L^*),\dots,c_1(L^*),[\mathbb{CP}^n]\rangle \\ &= \langle \sum_{i=1}^{n+1} c_1(L^*)^n, [\mathbb{CP}^n]\rangle = n+1. \end{aligned}$$

For $n = 1,\dots,4$ the characteristic polynomial s_n is given by:

$$\begin{aligned} s_1 &= \sigma_1 \\ s_2 &= \sigma_1^2 - 2\sigma_2 \\ s_3 &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\ s_4 &= \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4. \end{aligned}$$

Since $s_n(\mathbb{CP}^n) = n+1$ and $s_3(\mathbb{H}_{2,2}) = -6$, we see that it is possible to choose

$$\begin{aligned} K_1 &= \mathbb{CP}^1 \\ K_2 &= \mathbb{CP}^2 \\ K_3 &= \mathbb{CP}^3 \sqcup \mathbb{H}_{2,2} \\ K_4 &= \mathbb{CP}^4. \end{aligned} \tag{4.1}$$

4.1.4 The notion of C-equivalence.

We would like to describe now the relation between complex cobordism and Chern numbers of (stably) almost complex manifolds.

Definition 4.12. We introduce the quotient Γ_n, which is obtained from the set of $2n$-dimensional almost complex manifolds by identifying two manifolds M_1 and M_2 if and only if they have the same Chern numbers. In the literature, this equivalence relation also goes under the name of **C-equivalence**.

It is shown in [29] by induction that each Γ_n is a group (in fact, an abelian group), since each element admits an inverse.

Lemma 4.13. *For each almost complex manifold M of dimension $2n$, the class $[M] \in \Gamma_n$ admits an additive inverse.*

Proof. As announced, the proof is by induction. In dimension 2 ($n = 1$) we have $-[\mathbb{CP}^1] = [F_2]$, with F_2 a Riemann surface of genus 2.

Now consider the products

$$\mathbb{CP}^I = \mathbb{CP}^{i_1} \times \cdots \times \mathbb{CP}^{i_k},$$

with $I = (i_1, \ldots, i_k)$ a partition of n. By a theorem of Thom (cf. [22, Thm. 16.7]) the matrix $(c_J[\mathbb{CP}^I])$, where I and J run over all possible partitions of n, is non-singular. If M is an almost complex manifold of dimension $2n$, we can then find rational coefficients a_I such that

$$c_J[M] = \sum_I a_I \, c_J[\mathbb{CP}^I].$$

Let b be an integer number such that $b\,a_I \in \mathbb{Z}$ for all I. Then

$$c_J[bM] = \sum_I b\,a_I \, c_J[\mathbb{CP}^I]$$

and the linear combination on the right hand side has integer coefficients (although not necessarily positive).

We need to distinguish two cases.

Suppose $s_n[M] = 0$. We claim that the coefficient a_n must be zero. In fact, $s_n[M] = \sum a_I s_n[\mathbb{CP}^I]$ and $s_n[\mathbb{CP}^I] = 0$ unless $I = \{n\}$, in which case $s_n[\mathbb{CP}^n] = n+1$. This means that bM is C-equivalent to a disjoint union of products of complex projective spaces of dimension strictly less than n. By inductive hypothesis these products admit an inverse, hence we have an equivalence in Γ^n, namely

$$[bM] = \sum_I b\,a_I \, [\mathbb{CP}^I].$$

This shows that $[bM]$ admits an inverse, i.e., there exists N such that $-[bM] = [N]$. Since we can write bM as $(b-1)M \sqcup M$, we see that M also admits an inverse, namely

$$-[M] = [(b-1)M] + [N].$$

If $s_n[M] \neq 0$, on the other hand, we may find M' and integer numbers c and d such that $s_n[cM + dM'] = 0$, then apply the argument of the previous case. In particular, we may choose M' to be a complex projective space or a regular hypersurface, depending on the sign of $s_n[M]$.

□

We would like to show that the groups Γ_n may be identified with the cobordism groups Ω_n^U. Then an identity such as $-[\mathbb{CP}^1] = [F_2]$ in the proof of Lemma 4.13 can be geometrically visualised as a stably almost complex 3-dimensional manifold N, inducing the (stabilisation of the) standard almost complex structure on its boundary components $M_1 = S^2 \sqcup F_2$ and $M_2 = T^2$, a 2-dimensional torus.

Since each complex cobordism class admits a complex representative (generators do), we can define maps

$$\phi_n : \Omega_n^U \longrightarrow \Gamma_n \tag{4.2}$$

by sending an element $[M] \in \Omega_n^U$ to the class in Γ_n determined by the Chern numbers of such a representative of $[M]$. Since Chern numbers behave additively with respect to disjoint sum, we immediately see that (4.2) is a homomorphism for all n. To show that it is surjective, one just needs to observe that almost complex manifolds are in particular stably almost complex. The map is well defined and injective because of the following result (see [28]).

Proposition 4.14. *Two stably almost complex manifolds are cobordant if and only if they have the same Chern numbers.*

Proof. We start by showing that cobordant manifolds have the same Chern numbers. By additivity, it is enough to show that the Chern numbers of a boundary vanish. So suppose we are considering stably almost complex manifolds W and its boundary ∂W. Let i denote the inclusion of ∂W in W. If $x \in H^n(BU;\mathbb{Z})$ is a top dimensional product of universal Chern classes, compatibility of the stable almost complex structures implies that $x(\partial W) = i^* x(W)$. Let ∂_* denote the connecting homomorphism of the exact homology sequence of the pair $(W,\partial W)$:

$$\ldots \to H_{n+1}(W,\partial W;\mathbb{Z}) \xrightarrow{\partial_*} H_n(\partial W;\mathbb{Z}) \xrightarrow{i_*} H_n(W;\mathbb{Z}) \to \ldots$$

Since $[\partial W] = \partial_*[W]$, we have

$$\langle x(\partial W), [\partial W]\rangle = \langle i^* x(W), \partial_*[W]\rangle = \langle x(W), i_*\partial_*[W]\rangle = 0 \tag{4.3}$$

by exactness of the sequence above. Hence all Chern numbers of ∂W are zero.

To show the converse, consider the composition of the Hurewicz homomorphism

$$\pi_n(MU) \longrightarrow H_n(MU;\mathbb{Z})$$

with the Thom isomorphism

$$H_n(MU;\mathbb{Z}) \xrightarrow{\cong} H_n(BU;\mathbb{Z}).$$

One gets a homomorphism

$$\tau : \Omega_n^U \cong \pi_n(MU) \longrightarrow H_n(BU;\mathbb{Z}).$$

Recall that $H_n(BU;\mathbb{Z})$ is generated as a $\mathbb{Z}$-module by the n-dimensional universal Chern classes c_I, with I a partition of n. The kernel of the Hurewicz homomorphism is a finite group by a theorem of Serre (see [24]), hence it is trivial becase Ω_n^U is torsion-free. Thus we may regard Ω_n^U as a subgroup of $H_n(BU;\mathbb{Z})$ (in fact, for reasons of rank, it is a maximal free subgroup).

We have then the homology-cohomology pairing

$$H_n(BU;\mathbb{Z}) \otimes H^n(BU;\mathbb{Z}) \to H_0(\{\text{pt}\};\mathbb{Z}) \cong \mathbb{Z}.$$

If $[M] \in \Omega_n^U$, the above pairing coincides with the evaluation of the top dimensional characteristic classes on the fundamental class of M. In terms of the generators c_I, the pairing sends $([M], c_I)$ to $c_I[M]$, the I-th Chern number of M. Since the pairing is nondegenerate, each element in Ω_n^U is determined by its pairing with the generators of $H^n(BU;\mathbb{Z})$ and this shows that complex cobordism is completely determined by the Chern numbers. □

The topological product of manifolds gives a well-defined operation on $\Gamma = \oplus_n \Gamma_n$, which turns it into a ring. This multiplicative structure coincides under the homomorphisms ϕ_n with the multiplicative structure on the complex cobordism ring Ω^U. In view of the identification of the groups Γ_n with the complex cobordism groups and the identification of the corresponding ring structures, Proposition 4.11 gives a complete description of the ring Γ, including representatives for the generating equivalence classes.

Proposition 4.15. *The C-equivalence ring Γ is a polynomial ring with generators K_n in dimension $2n$ which may be chosen in a class of manifolds consisting of disjoint sums and products of complex projective spaces and hypersurfaces of bidegree $(1,1)$.*

4.2 The geography problem.

The geography problem for manifolds endowed with an additional structure (which allows the definition of their Chern classes) aims at determining which systems of integer numbers can be realised as the system of Chern numbers of such a manifold of suitable dimension.

Let $\pi(n)$ denote the cardinality of the set of partitions of n. Proposition 4.15 implies for the Chern numbers of almost complex manifolds the following result, see [14].

Theorem 4.16 (Milnor). *A system of $\pi(n)$ integer numbers occurs as the system of Chern numbers of a $2n$-dimensional almost complex manifold M if and only if it occurs as the system of Chern numbers of a $2n$-dimensional algebraic manifold X, which belongs to the class $\mathcal{M}$ of algebraic manifolds generated under cartesian product and disjoint sum by projective spaces, hypersurfaces of degree $(1,1)$ and their negatives.*

Knowing the Chern numbers of the manifolds generating $\mathcal{M}$ and their behaviour under the operations of cartesian product and disjoint sum, it is in principle possible to write down a set of necessary and sufficient congruence relations for a given system of integer numbers to occur as the system of Chern numbers of an almost complex manifold.

Example 4.17. For K_1 and K_2 we may choose $\mathbb{CP}^1$ and $\mathbb{CP}^2$, respectively. Hence we see that Ω^2 is generated by $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2$ and their negatives $-(\mathbb{CP}^1 \times \mathbb{CP}^1) = \mathbb{CP}^1 \times F_2$, with F_2 a compact Riemann surface of genus two, and $-\mathbb{CP}^2 = X(3)\#12\overline{\mathbb{CP}^2}$, where $X(3)$ denotes a hypersurface of degree 3 in $\mathbb{CP}^3$. Then a pair of integer numbers (p,q) is realised as the system of Chern numbers of an almost complex manifold M of the form $a(\mathbb{CP}^1 \times \mathbb{CP}^1) \sqcup b\mathbb{CP}^2$ if and only if the system

$$\begin{cases} 8a + 9b = p \\ 4a + 3b = q \end{cases}$$

admits a solution in $\mathbb{Z}^2$ and this happens if and only if the given numbers satisfy the congruence relation $p + q \equiv 0 \pmod{12}$.

4.2.1 The theorem of Riemann-Roch.

Some of the congruence relations among Chern numbers of almost complex manifolds arise from algebraic geometry, namely the theorem of Riemann-Roch. Let M be an almost complex manifold of dimension $2n$ and denote by $c_i \in H^{2i}(M;\mathbb{Z})$ the Chern classes of its tangent bundle. In the direct product $H^{**}(M;\mathbb{Z})$ of the cohomology groups of M, one writes these classes as elementary symmetric functions in variables $x_j \in H^2(M;\mathbb{Z})$ and defines e_i to be the i-th elementary symmetric function of the variables $e^{x_j}-1$ [27].

Definition 4.18. Let T_n be the multiplicative sequence of polynomials belonging to the power series $f(t) = t/(1-e^{-t})$ (cf. [22, p. 221]). Then the **Todd class** of M is defined by

$$T(M) = \sum_{j=0}^{\infty} T_j(c_1,\ldots,c_j) \in H^{**}(M;\mathbb{Z}),$$

where the first T-polynomials are

$$\begin{aligned} T_1 &= \frac{1}{2}\,c_1 \\ T_2 &= \frac{1}{12}\,(c_1^2+c_2) \\ T_3 &= \frac{1}{24}\,c_1c_2 \\ T_4 &= \frac{1}{720}\,(-c_4+c_1c_3+3c_2^2+4c_1^2c_2-c_1^4) \\ &\vdots \end{aligned}$$

(compare [27] and [16, p. 14]).

In $H^{**}(M;\mathbb{Z})$ we now consider the subring S_U, consisting of formal power series over $\mathbb{Z}$ in the elements e_i. If α is an element of $H^{**}(M;\mathbb{Z})\otimes\mathbb{Q}$, denote by α^n the $2n$-dimensional component of α. The Riemann-Roch theorem implies that

$$\langle (z\cdot T)^n, [M]\rangle \in \mathbb{Z} \text{ for all } z \in S_U. \tag{4.4}$$

Since the number $\langle (z\cdot T)^n, [M]\rangle$ can be expressed as a linear combination of Chern numbers of M with linear coefficients, this indeed implies some congruence relations among the Chern numbers of M.

It was conjectured by Atiyah and Hirzebruch [1] and proved by Stong [27] that (4.4) actually implies all relations.

Theorem 4.19 (Stong). *The Riemann-Roch theorem*

$$\langle (z\cdot T)^n, [M]\rangle \in \mathbb{Z} \textit{ for all } z \in S_U$$

gives all relations among the Chern numbers of $2n$*-dimensional almost complex manifolds.*

As long as we do not impose any connectedness condition, then, we know necessary and sufficient conditions for a given sequence of integer numbers to appear as the system of Chern numbers of an almost complex manifold (in fact, even Kähler), and we can conclude that the problem of the geography of almost complex or Kähler manifolds has a well known solution. If we want to consider connected manifolds, though, the problem immediately becomes more difficult. In fact, one would be tempted to consider connected rather than disjoint sums, but connected sums of almost complex manifolds do not always admit an almost complex structure. The classical example is given by the connected sum of two copies of $\mathbb{CP}^2$ (cf. [2], [17]).

Suppose, for example, that we were interested in determining the geography of connected almost complex manifolds. The Chern numbers of such manifolds still satisfy the same necessary relations and one might ask whether these are sufficient in order to guarantee the existence of a connected realisation. In fact it turns out they are, as shown in [7].

Proposition 4.20. *Let k denote the cardinality of the set of all partitions of n. If a given k-tuple of integer numbers is realised by a disjoint union $M_1 \sqcup \ldots \sqcup M_r$ of $2n$-dimensional almost complex manifolds, then the same k-tuple is realised by the connected almost complex manifold*

$$M_1 \# \ldots \# M_r \# (r-1) S^2 \times S^{2n-2}.$$

Remark. The main issue is of course to show that such a connected sum admits an almost complex structure, see [7].

4.2.2 Geography of symplectic manifolds.

Since symplectic manifolds admit in particular a compatible almost complex structure, their Chern numbers must also satisfy the congruence relations which necessarily hold in the almost complex case. Let $\pi(n)$ denote once more the cardinality of the set of all partitions of n and consider congruence relations among the Chern numbers of an almost complex manifold of dimension $2n$.

Definition 4.21. We call **admissible** those systems of $\pi(n)$ integer numbers which satisfy the above congruence relations and ask ourselves which admissible $\pi(n)$-tuples may in fact be realised by a connected symplectic manifold of dimension $2n$. We refer to the problem of determining admissible $\pi(n)$-tuples which admit a connected symplectic realisation as to the **symplectic geography problem**.

We list some results about symplectic geography in low dimension. In dimension 2, the Chern number c_1 coincides with the Euler number and satisfies $c_1 \equiv 0 \pmod 2$. Moreover, connectedness implies $c_1 \leq 2$. Any integer number satisfying the above relations can be realised by considering a compact Riemann surface of suitable genus.

In dimension 4, one has the necessary relation $c_1^2 + c_2 \equiv 0 \pmod{12}$. Any admissible pair (p,q) satisfying the inequality $p \leq 2q$ admits a closed, connected, symplectic (in fact, even Kähler) realisation. More precisely, admissible pairs with $p = 2q$ may be realised by a product of compact Riemann surfaces of suitable genus. Any admissible pair (p,q) with $p < 2q$, that is, below the line $p = 2q$, can be obtained by blowing up such a product a suitable number of times. This is possible because blow-up at one point adds $(-1,+1)$ to the pair of Chern numbers of the original manifold. The picture below should explain the use of the word geography.

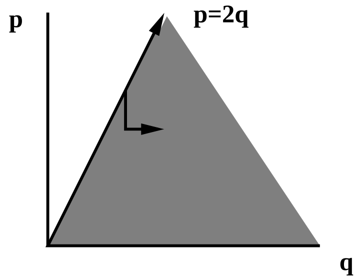

Figure 4.1: Geography in dimension 4: the admissible pairs correspond to points of intersection of the $\mathbb{Z}^2$-lattice with the lines $p = -q + 12K$. The arrow illustrates the effect of a one point blow-up.

In general, though, the problem of the geography of of symplectic 4-dimensional manifolds is still open. Several authors have addressed in particular the problem of geography of simply connected minimal symplectic 4-manifolds, that is, simply connected symplectic 4-manifolds which do not contain symplectically embedded spheres with square -1. In this situation one can show an interesting feature, namely that many pairs (p,q) may be realised by such surfaces which do not admit a complex simply connected minimal realisation. For more details, see [26].

There are also pairs which certainly do not admit a simply connected symplectic realisation. Suppose for example that M is a simply connected symplectic 4-manifold. We want to show that the sum of the Chern numbers of M cannot be negative.

Proposition 4.22. *Suppose M is a closed, connected symplectic 4-manifold and that $c_1^2[M] + c_2[M] < 0$. Then M cannot be simply connected.*

Proof. Suppose we have $c_1^2[M] + c_2[M] < 0$. One can make M minimal by blowing down a finite number of disjoint exceptional spheres. Denote by $\hat{M}$ the minimal manifold resulting from the blow-down of say n such spheres. Then $c_1^2[\hat{M}] = c_1^2[M] + n$, $c_2[\hat{M}] = c_2[M] - n$ and $\hat{M}$ is still simply connected. Since any simply connected, closed, symplectic 4-manifold must satisfy $c_2 > 2$, and $c_1^2[\hat{M}] + c_2[\hat{M}] = c_1^2[M] + c_2[M] < 0$, we see that we must have $c_1^2[\hat{M}] < c_1^2[M] + c_2[M] - 2 < -2$. By a theorem of Liu (cf. [20], Theorem 13.39), $\hat{M}$ is then a ruled surface over a base of genus greater than 1, hence it cannot be simply connected. ☐

Finally, in dimension six, one has the following system of relations:

$$\begin{aligned} c_1^3 &\equiv 0 \pmod 2 \\ c_1 c_2 &\equiv 0 \pmod{24} \\ c_3 &\equiv 0 \pmod 2. \end{aligned}$$

The standard form for an admissible triple is thus $(2a, 24b, 2c)$. The symplectic geography in this case is completely determined by the following theorem [13].

Proposition 4.23. *Every admissible triple* $(2a, 24b, 2c)$ *is realised as the system of Chern numbers of a closed, connected, simply connected, symplectic six-manifold.*

Having obtained a simply connected symplectic realisation for each admissible triple, one might ask whether it is possible to find a realisation with arbitrary fundamental group. We will address this problem in Section 4.7.2.

4.3 The eight-dimensional case.

4.3.1 Congruence relations in dimension eight.

The group Ω_8^U is generated by the equivalence classes of K_4, $K_1 \times K_3$, $K_2^2 = K_2 \times K_2$, $K_1^2 \times K_2 = K_1 \times K_1 \times K_2$, $K_1^4 = K_1 \times K_1 \times K_1 \times K_1$ (compare (4.1)). The corresponding Chern numbers are shown in Table 4.1.

	c_4	c_1c_3	c_2^2	$c_1^2c_2$	c_1^4
K_4	5	50	100	250	625
$K_1 \times K_3$	20	116	192	416	896
K_2^2	9	54	99	216	486
$K_1^2 \times K_2$	12	60	96	204	432
K_1^4	16	64	96	192	384

Table 4.1: Chern numbers of the generators of the group Ω_8^U.

Because of the identification of Ω_8^U with Γ_4, any almost complex manifold M of dimension 8 must be C-equivalent to a disjoint union of a suitable number of copies of the above generators and their inverses. This means that we are able to express the Chern numbers of M as linear combinations with integer coefficients of the Chern numbers of the generators. More precisely, suppose

$$M \sim_C x_1K_4 \sqcup x_2K_1K_3 \sqcup x_3K_2^2 \sqcup x_4K_1^2K_2 \sqcup x_5K_1^4, \tag{4.5}$$

where xK denotes the disjoint union of x copies of K if x is positive of and the disjoint union of $|x|$ copies of the inverse of K if x is negative. Then the Chern numbers of M are given by:

$$\begin{aligned}
c_4[M] &= 5x_1 + 20x_2 + 9x_3 + 12x_4 + 16x_5 \\
c_1c_3[M] &= 50x_1 + 116x_2 + 54x_3 + 60x_4 + 64x_5 \\
c_2^2[M] &= 100x_1 + 192x_2 + 99x_3 + 96x_4 + 96x_5 \\
c_1^2c_2[M] &= 250x_1 + 416x_2 + 216x_3 + 204x_4 + 192x_5 \\
c_1^4[M] &= 625x_1 + 896x_2 + 486x_3 + 432x_4 + 384x_5.
\end{aligned}$$

Then if we compute the expressions

$$\begin{aligned}
&-c_4[M]+c_1c_3[M]+3c_2^2[M]+4c_2c_1^2[M]-c_1^4[M] \\
&\quad = \quad 720x_1+1440x_2+720x_3+720x_4+720x_5 \\
&2c_1^4[M]+c_1^2c_2[M] = 1500x_1+2208x_2+1188x_3+1068x_4+960x_5 \\
&\quad = \quad 12\cdot 125x_1+12\cdot 184x_2+12\cdot 99x_3+12\cdot 89x_4+12\cdot 80x_5 \\
&-2c_4[M]+c_1c_3[M] = 40x_1+76x_2+36x_3+36x_4+32x_5 \\
&\quad = \quad 4\cdot 10x_1+4\cdot 19x_2+4\cdot 9x_3+4\cdot 9x_4+4\cdot 8x_5
\end{aligned}$$

we immediately see that the Chern numbers of any almost complex 8-dimensional manifold necessarily satisfy the following congruence relations:

$$\begin{aligned}
-c_4+c_1c_3+3c_2^2+4c_2c_1^2-c_1^4 &\equiv 0 \pmod{720} \\
2c_1^4+c_1^2c_2 &\equiv 0 \pmod{12} \\
-2c_4+c_1c_3 &\equiv 0 \pmod 4.
\end{aligned} \tag{4.6}$$

We claim that these relations are also sufficient. Suppose in fact that we are given a quintuple of integer numbers $(c_4, c_1c_3, c_2^2, c_1^2c_2, c_1^4)$ satisfying them. Then there exist integers (a, j, k, m, b) such that

$$\begin{aligned}
a &= c_4 \\
720j &= -c_4+c_1c_3+3c_2^2+4c_2c_1^2-c_1^4 \\
12k &= 2c_1^4+c_1^2c_2 \\
4m &= -2c_4+c_1c_3 \\
b &= c_1^4
\end{aligned} \tag{4.7}$$

and the above system is equivalent to

$$\begin{aligned}
c_4 &= a \\
c_1c_3 &= 4m+2a \\
c_1^4 &= b \\
c_1^2c_2 &= 12k-2b \\
3c_2^2 &= 720j-a-4m-48k+9b.
\end{aligned}$$

From this we see that there is a one-to-one correspondence between quintuples of integers satisfying (4.6) and quintuples (a, b, j, k, m) subject to the condition $a+m \equiv 0 \pmod 3$.

In Table 4.2 we collect the values of the parameters (a, b, j, k, m) for the group generators in dimension 8.

In order to prove our claim that the relations (4.6) represent a sufficient condition for a quintuple of integer numbers to appear as the system of Chern numbers of an almost complex manifold of dimension 8, we have to show that given a quintuple (a, b, j, k, m) satisfying the condition $a+m \equiv 0 \pmod 3$ there exist integer coefficients $x_1, \dots, x_5$ such that the parameters

	a	m	j	k	b
K_4	5	10	1	125	625
$K_1 \times K_3$	20	19	2	184	896
K_2^2	9	9	1	99	486
$K_1^2 \times K_2$	12	9	1	89	432
K_1^4	16	8	1	80	384

Table 4.2: Parameters for the generators of the group Ω_8^U.

of the manifold $x_1K_4 \sqcup x_2K_1K_3 \sqcup x_3K_2^2 \sqcup x_4K_1^2K_2 \sqcup x_5K_1^4$ coincide with the given quintuple. For this to happen, in turn, the system

$$\begin{aligned}
a &= 5x_1 + 20x_2 + 9x_3 + 12x_4 + 16x_5 \\
m &= 10x_1 + 19x_2 + 9x_3 + 9x_4 + 8x_5 \\
j &= x_1 + 2x_2 + x_3 + x_4 + x_5 \\
k &= 125x_1 + 184x_2 + 99x_3 + 89x_4 + 80x_5 \\
b &= 625x_1 + 896x_2 + 486x_3 + 432x_4 + 384x_5
\end{aligned}$$

has to admit an integer solution. This is indeed the case, namely the system admits a solution

$$\begin{aligned}
x_1 &= \frac{2}{3}a + 96j + \frac{8}{3}m + 3b - 16k \\
x_2 &= \frac{8}{3}a + 600j + \frac{41}{3}m + 17b - 91k \\
x_3 &= -a - 80j - 4m - 3b + 16k \\
x_4 &= -\frac{25}{3}a - 1920j - \frac{124}{3}m - 54b + 289k \\
x_5 &= \frac{10}{3}a + 705j + \frac{46}{3}m + 20b - 107k
\end{aligned}$$

and the condition $a + m \equiv 0 \pmod 3$ guarantees that the x_i's are in fact integer.

We would like to show how the relations can also be written down starting from the Riemann-Roch theorem. In dimension 8,

$$e^{x_j} - 1 = x_j + \frac{x_j^2}{2} + \frac{x_j^3}{3!} + \frac{x_j^4}{4!}.$$

Then comparing with [22, p.188], we get for instance

$$\begin{aligned} e_1 &= \sum_{j=1}^{4}(e^{x_j}-1) \\ &= \sum x_j + \frac{1}{2}\sum x_j^2 + \frac{1}{3!}\sum x_j^3 + \frac{1}{4!}x_j^4 \\ &= s_1 + \frac{1}{2}s_2 + \frac{1}{3!}s_3 + \frac{1}{4!}s_4 \\ &= c_1 - \frac{1}{2}\,(c_1^2 - 2c_2) + \frac{1}{6}\,(c_1^3 - 3c_1c_2 + 3c_3) \\ &\quad -\frac{1}{24}\,(c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3 - 4c_4). \end{aligned}$$

Similarly, the other classes e_i are given by

$$\begin{aligned} e_2 &= c_2 - \frac{1}{2}(c_1c_2 - c_3) + \frac{1}{12}\,(14c_4 - 8c_1c_3 - c_2^2 + 2c_1^2c_2) \\ e_3 &= c_3 - \frac{1}{2}\,(c_1c_3 - 4c_4) \\ e_4 &= c_4. \end{aligned}$$

Notice that the components of e_i have all dimension greater than or equal to $2i$. Congruence relations for the Chern numbers of M arise from integrality of the numbers

$$\begin{aligned} \langle T_4, [M]\rangle &= \frac{1}{720}\,(-c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4) \\ \langle (e_1\cdot T)^4, [M]\rangle &= \frac{1}{12}\,c_1c_3 + \frac{1}{6}\,c_4 \\ \langle (e_1^2\cdot T)^4, [M]\rangle &= \frac{1}{12}\,c_1^2c_2 + \frac{1}{6}\,c_1^4 \\ \langle (e_1^3\cdot T)^4, [M]\rangle &= \frac{1}{4}\,(-c_1^4 - 6c_1^2c_2) \\ \langle (e_2\cdot T)^4, [M]\rangle &= \frac{1}{12}\,(c_1c_3 + 14c_4) \\ \langle (e_2^2\cdot T)^4, [M]\rangle &= c_2^2 \\ \langle (e_3\cdot T)^4, [M]\rangle &= 2c_4 \\ \langle (e_4\cdot T)^4, [M]\rangle &= c_4. \end{aligned}$$

This leads to the following system of congruence relations

$$\begin{aligned} -c_4 + c_1c_3 + 3c_2^2 + 4c_2c_1^2 - c_1^4 &\equiv 0 \pmod{720} \\ 2c_1^4 + c_1^2c_2 &\equiv 0 \pmod{12} \\ 2c_4 + c_1c_3 &\equiv 0 \pmod{12}, \end{aligned}$$

which is easily seen to be equivalent to (4.6).

4.3.2 The symplectic case.

We now turn our attention to the geography of symplectic 8-dimensional manifolds. We will keep working with the parameters (a, j, k, m, b) rather than with the Chern numbers themselves. The result which we want to prove can be summarised by saying that all quintuples (a, b, j, k, m) satisfying the condition $a + m \equiv 0 \pmod 3$ admit a symplectic realisation.

Theorem 4.24. *Given a quintuple of integer numbers* (a, j, k, m, b)*, subject to the additional condition* $a + m \equiv 0$ $(mod\ 3)$*, there exists a closed, connected, symplectic* 8*-dimensional manifold* M *such that the given parameters are related to the Chern numbers of* M *by the system of equations* (4.7).

In view of the correspondence between quintuples (a, j, k, m, b) satisfying the condition $a + m \equiv 0 \pmod 3$ and admissible quintuples, the Theorem immediately implies that the congruence relations (4.6) are not only necessary, but also sufficient for a given quintuple of integer numbers to occur as the system of Chern numbers of a closed connected 8-dimensional symplectic manifold.

Corollary 4.25. *Given a quintuple of integer numbers* $(n_1, n_2, n_3, n_4, n_5)$ *which is admissible, i.e., which satisfies the system of congruence relations*

$$\begin{aligned} -n_1 + n_2 + 3n_3 + 4n_4 - n_5 &\equiv 0\ (mod\ 720) \\ 2n_5 + n_4 &\equiv 0\ (mod\ 12) \\ -2n_1 + n_2 &\equiv 0\ (mod\ 4) \end{aligned}$$

there exists a closed, connected, symplectic 8*-dimensional manifold* M *realising the given quintuple as its system of Chern numbers, namely* $(c_4[M],\ c_1c_3[M],\ c_2^2[M],\ c_1^2c_2[M],\ c_1^4[M]) = (n_1, n_2, n_3, n_4, n_5)$.

4.3.3 Behaviour of the parameters under blow-up.

From the blow up formulae (A.3), (A.5), (A.6) for the Chern numbers, we obtain the following expressions for the transformation of the parameters of an eight dimensional manifold M under blow-up.

- Blow-up at a point:

$$\begin{aligned} a' &= a + 3 \\ 4m' &= 4m \\ 720j' &= 720j \\ 12k' &= 12k - 180 \\ b' &= b - 81 \end{aligned} \tag{4.8}$$

- Blow-up along a symplectically embedded curve C of genus g and with normal bundle

νC:

$$\begin{aligned} a' &= a+4(1-g) \\ 4m' &= 4m-4(1-g) \\ 720j' &= 720j \\ 12k' &= 12k-144(1-g)-36\langle c_1(\nu_C),[C]\rangle \\ b' &= b-64(1-g)-16\langle c_1(\nu_C),[C]\rangle \end{aligned} \tag{4.9}$$

- Blow up along a symplectic four-dimensional submanifold X, with normal bundle νX:

$$\begin{aligned} a' &= a+c_2[X] \\ 4m' &= 4m+c_1^2[X]-3c_2[X] \\ 720j' &= 720j \\ 12k' &= 12k-13c_1^2[X]-c_2[X]-18\langle c_1(X)c_1(\nu_X),[X]\rangle \\ &\quad -6\langle c_1^2(\nu_X),[X]\rangle \\ b' &= b-6c_1^2[X]-8\langle c_1(X)c_1(\nu_X),[X]\rangle-3\langle c_1^2(\nu_X),[X]\rangle \\ &\quad +\langle c_2(\nu_X),[X]\rangle \end{aligned} \tag{4.10}$$

With reference to the proof of Proposition 4.24, notice that the parameter j defined in (4.7) is invariant under blow up.

Following the strategy of Halic [13], we will start by producing examples of closed symplectic manifolds realising any given value of j and admitting enough symplectic submanifolds. Then we will show that by blowing up along these submanifolds a suitable number of times, the other parameters may also be varied so as to obtain any prescribed quintuple.

4.4 Building blocks.

4.4.1 Elliptic surfaces.

Definition 4.26. An **elliptic surface** is by definition a complex surface together with a holomorphic map to a complex curve C whose fibres are tori, except for a finite number of so-called singular fibres.

By abuse of terminology, such a map is also called a fibration and we are particularly interested in the case where the base of the fibration is the complex projective line $\mathbb{CP}^1$. Consider for example two distinct nondegenerate cubic curves C_0 and C_1 in $\mathbb{CP}^2$, given as the zero sets of some homogeneous cubic polynomials P_i, $i=0,1$. We may assume that C_0 and C_1 intersect transversely in exactly nine positive points $\{q_1,\ldots,q_9\}$. For any other point $p\in\mathbb{CP}^2-\{q_1,\ldots,q_9\}$, there exists a unique $\mu=[w_0:w_1]\in\mathbb{CP}^1$ such that p lies on the cubic curve

$$C_\mu=\{[z_0:z_1:z_2]\in\mathbb{CP}^2\,|\,(w_0P_0+w_1P_1)(z_0,z_1,z_2)=0\}.$$

In fact, if $p=[x:y:z]$, we may assume that $C_0(x,y,z)\neq 0$ and let

$$w_0=-\frac{C_1(x,y,z)}{C_0(x,y,z)}w_1.$$

Then obviously $(w_0C_0+w_1C_1)(x,y,z)=0$. In this way one gets a well defined map

$$f:\mathbb{CP}^2-\{q_1,\dots,q_9\}\longrightarrow\mathbb{CP}^1,$$

which maps a point p to the element $[w_0:w_1]\in\mathbb{CP}^1$ satisfying the condition that p belongs to the curve defined by $w_0C_0+w_1C_1$. The map f extends, after blowing up the nine intersection points of C_0 and C_1, to a map

$$f:\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}\longrightarrow\mathbb{CP}^1$$

such that the fibre $f^{-1}(\mu)=C_\mu$, $\mu\in\mathbb{CP}^1$, is a cubic curve. Each exceptional sphere is a section of f. All but a finite number of fibres are smooth and have genus $g=\frac{1}{2}(d-1)(d-2)=1$, that is, except for a finite number of singular fibres, they are topologically tori. That there need to be singular fibres as well can be seen from the fact that $c_2(\mathbb{CP}^2\#9\overline{\mathbb{CP}^2})$ is different from zero. The types of singular fibres depend on the choice of polynomials $P_0$and P_1. The manifold $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ admits a symplectic (in fact, even Kähler) form, namely the one obtained by blow-up from the standard Kähler form on $\mathbb{CP}^2$. The regular fibres of f are complex submanifolds, hence again Kähler, hence in particular symplectic submanifolds. A neighbourhood of a regular fibre coincides with a neighbourhood of the 9-times blow-up of a smooth cubic curve in $\mathbb{CP}^2$: therefore we see that these regular fibres have trivial normal bundle.

From now on we will denote by $E(1)$ precisely the manifold $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$, equipped with such an elliptic fibration. We assume that this fibration has only cusp fibres as singular fibres (these are fibres obtained by blowing up curves ambiently isotopic to $\{[z_0:z_1:z_2]\in\mathbb{CP}^2\,|\,z_2z_1^2=z_0^3=0\}$).

The homology of $E(1)$ is the homology of a connected sum and it is given by

$$H_2(E(1))\cong H_2(\mathbb{CP}^2\#9\overline{\mathbb{CP}^2})\cong H_2(\mathbb{CP}^2;\mathbb{Z})\oplus 9H_2(\overline{\mathbb{CP}^2};\mathbb{Z})$$

Let h be the generator of $H_2(\mathbb{CP}^2;\mathbb{Z})$, e_i the exceptional sphere of the i-th blow up, $i=1,\dots,9$. Then $\langle h,e_1,\dots,e_9\rangle$ is a basis for $H_2(E(1))$ with intersection matrix $\langle 1\rangle\oplus 9\langle -1\rangle$.

A regular fibre represents the class $f=3h-\sum_{i=1}^9 e_i$: another basis for $H_2(E(1))$ is given by $\langle f,e_9,e_1-e_2,\dots,e_7-e_8,-h+e_6+e_7+e_8\rangle$ and the intersection matrix with respect to this basis (cf. Section 1.5) is

$$\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\oplus(-E_8).$$

The first Chern class of $E(1)$ is $PD(3h-\sum e_i)$ (cf. Corollary 3.22 and following Remark), which gives for the Chern numbers the values $c_1^2[E(1)]=0$ and $c_2[E(1)]=12$.

Let F be a regular fibre in $E(1)$: we can perform the symplectic sum of two copies of $E(1)$ along two such fibres: the resulting manifold $E(2):=E(1)\#_F E(1)$ still fibres over

$\mathbb{CP}^1$ (regarded as the connected sum of two copies of itself) with fibres which are tori, i.e., it is again an elliptic surface. The second homology group $H_2(E(2))$ admits a basis with intersection matrix

$$2(-E_8) \oplus 3 \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

where the elements of square 0 are tori and the other spheres (cf. [8, p. 72]). By the expression for the first Chern class of a symplectic sum along tori with trivial normal bundle, (3.16), $c_1(E(2))$ vanishes. In fact

$$c_1(E(1)\#_F E(1)) = c_1(E(1)) + c_1(E(1)) - 2PD[F] = 2f - 2f = 0.$$

Inductively, one can define $E(n+1) := E(n)\#_F E(1) = \#_F^{n+1} E(1)$ and find a basis for its second homology with corresponding intersection matrix

$$n(-E_8) \oplus 2(n-1) \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & -n \end{bmatrix} \tag{4.11}$$

We denote the elements of this basis by

$$\langle \tau_{ij}, i = 1, \ldots, n;\ j = 1, \ldots, 8;\ \alpha_k, \beta_k, k = 1, \ldots, 2(n-1);\ f, \sigma \rangle,$$

where σ denotes the class of a section of $E(n)$, which is obtained by pasting together n sections of $E(1)$.

The first Chern class is $c_1(E(n)) = (2-n)f$, so the Chern numbers are given by

$$\begin{cases} c_1^2[E(n)] = 0 \\ c_2[E(n)] = 12n. \end{cases}$$

Definition 4.27. The **nucleus** $N(n)$ of the elliptic surface $E(n)$ consists of a neighbourhood of the union of a singular fibre and a section of the fibration.

If we consider the nucleus of the elliptic surface $E(n)$, we have that $H_2(N(n)) \cong \mathbb{Z}^2$ and the corresponding intersection matrix is the last summand in $Q_{E(n)}$, namely $\begin{bmatrix} 0 & 1 \\ 1 & -n \end{bmatrix}$.

4.4.2 Other building blocks.

Other "building blocks" are obtained as follows [9]:

- In $T^2 \times T^2$ consider the union of the two tori $T^2 \times \{p\} \cup \{p\} \times T^2$. The intersection point is positive and transverse: in a neighbourhood of this point, the union of the two tori looks like the set $\{z_1 z_2 = 0\}$ near the origin of $\mathbb{C}^2$, that is, like two 2-dimensional discs intersecting in one point of the 4-dimensional ball. We may replace it then with the annulus $\{z_1 z_2 = \varepsilon\}$, performing the gluing so that the boundary of the annulus comes to coincide with that of the two discs. This eliminates the intersection point, without changing the ambient manifold. The process increases the Euler number by 2, and the resulting submanifold is still symplectically embedded in $T^2 \times T^2$, in the same homology class as $T^2 \times \{p\} \cup \{p\} \times T^2$. After this, we blow up twice to obtain a symplectic genus 2 surface F_2 with square 0 in $Q = T^4 \# 2\overline{\mathbb{CP}^2}$. Then Q contains a symplectic torus F, disjoint from F_2. This F is obtained from

$$F' = \{(x_1, x_2, x_3, x_4) \in \mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4 \,|\, x_2 = x_4 = 0\},$$

which is in fact Lagrangian, by perturbing the symplectic form. Then F will be disjoint from F_2 provided $p = (0, c)$ with $c \neq 0$.

- Let p and q be distinct points in T^2 and consider the two tori $T^2 \times p$ and $q \times T^2$ in $T^4 = T^2 \times T^2$. Blow up the intersection point (q, p) to obtain two disjoint tori with square -1, then blow up 16 more times to reduce the square of both to -9. Now take the symplectic sum of the resulting manifold $T^4 \# 17\overline{\mathbb{CP}^2}$ with 2 copies of $\mathbb{CP}^2$ along cubic curves: denote the final result of these operations by S. Then S is a simply connected, symplectic 4-manifold, containing disjoint symplectically embedded surfaces of genus 1 and 2 with trivial normal bundle.

- Consider a curve of degree 4 with one transverse double point in $\mathbb{CP}^2$. The genus of such a curve is given by $g = \frac{(d-1)(d-2)}{2} - \#(\text{nodes})$ and in our case this number is 2. We can get a smooth surface by blowing up the the double point: this surface represents the homology class $4h - 2e$, hence it has square 12. We thus need to blow up 12 more times to get a smooth submanifold with genus 2 and square 0 in $\mathbb{CP}^2 \# 13\overline{\mathbb{CP}^2}$. Finally blow up three extra points, away from F_2, to get the manifold $P \cong \mathbb{CP}^2 \# 16\overline{\mathbb{CP}^2}$. P is a symplectic simply connected manifold with Chern numbers $c_1^2 = -7$ and $c_2 = 19$.

4.5 Construction of the examples.

4.5.1 Symplectic sphere bundles, Part II.

Let (N, β) be a closed symplectic 4-dimensional manifold, for example, one of the above building blocks, and $E \to N$ a complex line bundle over N. We take a step back and consider once more the bundle $\rho : S \to N$ with fibre S^2 over N, obtained by projectifying the complex rank two bundle $E \oplus \mathbb{C}$ over N.

Then (3.4) gives us the expression for the Chern classes of S: from this and the ring structure (3.1) we can compute the corresponding Chern numbers, which are given by

$$\begin{aligned} c_1^3[S] &= 6c_1^2[N] + 2\langle c_1^2(E), [N] \rangle \\ c_1 c_2[S] &= 2(c_1^2[N] + c_2[N]) \\ c_3[S] &= 2c_2[N]. \end{aligned}$$

The examples we will consider are 8-dimensional symplectic manifolds of the form $M = S \times F$, with F a compact Riemann surface of genus g. Using a product formula we can easily compute the Chern numbers of M.

$$\begin{aligned}
c_4[M] &= 2(1-g)c_3[S] = 4(1-g)c_2[N] \\
c_1c_3[M] &= 2(1-g)(c_1c_2[S]+c_3[S]) = 4(1-g)(c_1^2[N]+2c_2[N]) \\
c_2^2[M] &= 4(1-g)c_1c_2[S] = 8(1-g)(c_1^2[N]+c_2[N]) \\
c_1^2c_2[M] &= 2(1-g)(c_1^3[S]+2c_1c_2[S]) \\
&= 4(1-g)(5c_1^2[N]+2c_2[N]+\langle c_1^2(E),[N]\rangle) \\
c_1^4[M] &= 8(1-g)c_1^3[S] = 16(1-g)(3c_1^2[N]+\langle c_1^2(E),[N]\rangle).
\end{aligned}$$

The next step will be to consider symplectic submanifolds of M which are of the form $B\times\{\text{pt}\}$, with B a symplectic submanifold of S and $\{\text{pt}\}\in F$. For such submanifolds, the normal bundle in M coincides with the Whitney sum of the normal bundle in S and a copy of the trivial line bundle, which we denote by $\mathbb{C}$. This implies in particular an equivalence of Chern classes

$$c(\nu_M B) = c(\nu_S B\oplus\mathbb{C}) = c(\nu_S B).$$

We consider for example sections N_+ and N_- of S, corresponding to the embeddings of N in $S = E^0\cup_{\partial E^0}\overline{E^0}$ (cf. Lemma 2.19) as the zero section of E and $\overline{E}$, respectively. In this case, the characteristic numbers which appear in the blow-up formulae are given by

$$\begin{aligned}
c_1^2[N_\pm] &= c_1^2[N] \\
c_2[N_\pm] &= c_2[N] \\
\langle c_1^2(\nu_M N_\pm),[N_\pm]\rangle &= \langle c_1^2(E),[N]\rangle \\
\langle c_1(\nu_M N_\pm)c_1(N_\pm),[N_\pm]\rangle &= \pm\langle c_1(N)c_1(E),[E]\rangle.
\end{aligned}$$

Let s be any such section and assume that F is a symplectically embedded curve in N: then it lifts along s to a symplectically embedded curve in S. Moreover, the square of F will change by an amount equal to the product $\langle c_1(E),[F]\rangle$. By stretching the terminology, we call here square of F also the number resulting from evaluating the first Chern class of the normal bundle of F (or rather, $s(F)$) in S on its fundamental homology class. More precisely we have:

Lemma 4.28. *In the situation described above, the square of the lift of an embedded curve F is given by*

$$\langle c_1(\nu(s(F),S)),[s(F)]\rangle = \langle c_1(\nu(F,N)),[F]\rangle + \langle c_1(E),[F]_N\rangle,$$

where $\nu(\cdot,\cdot)$ denotes the normal bundle of an embedding and s is the section under consideration.

Proof. We refer to the following commutative diagram for the notation:

$$\begin{array}{ccc}
s(F) & \xrightarrow{\;j\;} & S \\
\cong\downarrow & & \downarrow\rho \\
F & \xrightarrow{\;i\;} & N
\end{array}$$

We have an isomorphism of vector bundles:

$$\begin{aligned}\nu(s(F),S) &\cong \nu(s(F),s(N)) \oplus \nu(s(N),S)|_{s(F)}\\ &\cong \rho^*\nu(F,N) \oplus \rho^* E|_{s(F)},\end{aligned}$$

which implies a corresponding equivalence on the level of cohomology classes, namely:

$$c_1(\nu(s(F),S)) = \rho^* c_1(\nu(F,N)) + \rho^* i^* c_1(E).$$

We now evaluate on $[s(F)]$ and get

$$\begin{aligned}\langle c_1(\nu(s(F),S)),[s(F)]\rangle &= \langle \rho^* c_1(\nu(F,N)),[s(F)]\rangle + \langle \rho^* i^* c_1(E),[s(F)]\rangle\\ &= \langle \rho^* c_1(\nu(F,N)), s_*[F]\rangle + \langle \rho^* i^* c_1(E), s_*[F]\rangle\\ &= \langle s^*\rho^* c_1(\nu(F,N)),[F]\rangle + \langle s^*\rho^* i^* c_1(E),[F]\rangle\\ &= \langle c_1(\nu(F,N)),[F]\rangle + \langle i^* c_1(E),[F]\rangle\\ &= \langle c_1(\nu(F,N)),[F]\rangle + \langle c_1(E),[F]_N\rangle.\end{aligned}$$

Notice that, in particular, if $c_1(E)\cap[F]=0$, then the lifted curve has the same square in S as the original one in N, i.e.,

$$\langle c_1(\nu(s(F),S)),[s(F)]\rangle = \langle c_1(\nu(F,N)),[F]\rangle.$$

□

Recall that we may assume the symplectic form $\omega_K = K\rho^*\beta+\eta$ on S to be integral and to represent the class $K\rho^*[\beta]+\xi$, with $[\beta]\in H^2(N;\mathbb{Z})$ (cf. Lemma 3.11). As far as the last condition is concerned, we have already seen that it can always be realised without changing the Chern classes of N. In fact, given any symplectic form ω', we are able to find an integral symplectic form ω, which is obtained by first approximating ω' and then rescaling. Suppose ω' tames an almost complex structure J: since the taming condition is open and an invariant with respect to rescaling, we see that J is also a tame almost complex strcuture for the integral form ω'.

This, together with the following lemma, implies that symplectically embedded curves (2-dimensional submanifolds) also remain symplectically embedded with respect to the new integral symplectic form.

Lemma 4.29. *A smooth* 2*-dimensional submanifold F of a symplectic* 4*-manifold (N,ω) is symplectically embedded if and only if it is a J-holomorphic curve with respect to some tame almost complex structure J on (N,ω).*

Proof. Suppose first that the inclusion $i:(F,j)\to(N,J)$ is a J-holomorphic map, that is, $di\circ j = J\circ di$. Let $v\in TF$. If $i^*\omega(v,w)=0$ for all $w\in TF$, then in particular

$$\begin{aligned}0 &= i^*\omega(v,jv) = \omega(di(v), di\circ j(v))\\ &= \omega(di(v), J\circ di(v)),\end{aligned}$$

but the latter is strictly positive by the taming condition unless $v = 0$. Hence $i^*\omega$ is nondegenerate on TF, i.e., it is a symplectic form.

Conversely, suppose F is a symplectic submanifold. Then by Lemma 3.3 there exists a tame almost complex structure J_N on N such that its restriction to TF is again an almost complex structure: in fact, $J_N|_{TF}$ must be homotopic to j, since there exists only one homotopy class of almost complex structures on every orientable surface ($SO(2) = U(1)$). It is then easy to perturb J_N so that in fact $J_N|_{TF} = j$. □

With an integral symplectic form at our disposal, we may apply Donaldson's existence theorem and obtain a whole family of symplectic submanifolds $\{X_\lambda\}$ of S which, for sufficiently large $\lambda \in \mathbb{Z}$, realise the Poincaré dual of $\lambda[\omega_K]$, i.e., $PD_S[X_\lambda] = \lambda(\xi + K[\pi^*\beta])$ in $H^2(S;\mathbb{Z})$.

Let i denote the inclusion X_λ in M. We have the relation

$$i^*c(S) = c(X_\lambda) \cup c(\nu_S X_\lambda)$$

and since $c_1(\nu_S X_\lambda) = e(\nu_M X_\lambda) = i^*PD_M[X_\lambda]$, we can rewrite it as

$$i^*c(M) = c(X_\lambda) \cup (1 + i^*\lambda(\xi + K[\pi^*\beta])).$$

From this relation, using

$$\langle i^*y, [X_\lambda]\rangle = \langle y \cup (\lambda\xi + \lambda K[\pi^*\beta]), [M]\rangle \text{ for all } y \in H^4(M)$$

and the cohomology ring structure of M, we can compute the invariants of X_λ (see the Appendix for explicit computations). They are:

$$\begin{aligned}
c_2[X_\lambda] &= \lambda c_2[N] + \lambda(\lambda-1)\langle c_1(N)c_1(L), [N]\rangle \\
&\quad -2\lambda K(\lambda-1)\langle c_1(N)[\beta], [N]\rangle + \lambda^2(\lambda-1)\langle c_1^2(E), [N]\rangle \\
&\quad +\lambda^2 K(2-3\lambda)\langle c_1(E)[\beta], [N]\rangle \\
&\quad +\lambda^2 K^2(3\lambda-2)\langle [\beta]^2, [N]\rangle \\
c_1^2[X_\lambda] &= \lambda c_1^2[N] + 2\lambda(\lambda-1)\langle c_1(N)c_1(L), [N]\rangle \\
&\quad +4\lambda K(1-\lambda)\langle c_1(N)[\beta], [N]\rangle \\
&\quad +\lambda(\lambda^2 - 2\lambda + 1)\langle c_1^2(E), [N]\rangle \\
&\quad +\lambda^2 K(4-3\lambda)\langle c_1(E)[\beta], [N]\rangle \\
&\quad +\lambda^2 K^2(3\lambda-4)\langle [\beta]^2, [N]\rangle \\
\langle c_1^2(\nu_M X_\lambda), [X_\lambda]\rangle &= \lambda^3\langle c_1^2(E), [N]\rangle - 3\lambda^3 K\langle c_1(E)[\beta], [N]\rangle \\
&\quad +3\lambda^3 K^2\langle [\beta]^2, [N]\rangle \\
\langle c_1(X_\lambda)c_1(\nu), [X_\lambda]\rangle &= -\lambda^2\langle c_1(N)c_1(L), [N]\rangle + 2\lambda^2 K\langle c_1(N)[\beta], [N]\rangle \\
&\quad +\lambda^2(1-\lambda)\langle c_1^2(E), [N]\rangle + \lambda^2 K(3\lambda-2)\langle c_1(E)[\beta], [N]\rangle \\
&\quad +\lambda^2 K^2(2-3\lambda)\langle [\beta]^2, [N]\rangle.
\end{aligned} \tag{4.12}$$

4.6 The blow-up systems.

Now we have introduced all the elements necessary for the proof of Theorem 4.24. The proof itself follows that of Halic for the case of dimension 6. For any given j, that is, we will show that it is possible to construct a symplectic manifold realising j and with enough symplectic submanifolds so that we can vary the other parameters and eventually realise all admissible quintuples. We construct our examples distinguishing three main cases.

4.6.1 Realising sets of parameters with $j \geq 1$.

We start by considering examples of symplectic 8-dimensional manifolds for which the parameter j is greater than or equal to 1. In order to produce such examples, we perform the symplectic sum of the manifolds Q and $E(n)$ of paragraph 4.4.2 along embedded tori with square zero. Before doing so, though, we blow up one extra point in Q, away from the torus along which we intend to perform the sum.

The result of these operations is the manifold

$$X_n := Q^* \#_F E(n),$$

which is a simply connected symplectic manifold, with Chern numbers $c_1^2 = -3$ and $c_2 = 3 + 12n$.

Following our discussion of the first Chern class of such a sum, we may write

$$c_1(X_n) = c_1(Q^*) + c_1(E(n)) - 2PD([F]).$$

Assume that the torus $F \subset E(n)$ is actually contained in $N(n)$, the nucleus of the elliptic surface (cf. Definition 4.27), and recall that we denote by τ_{ij}, $i = 1, \dots, n$, $j = 1, \dots, 8$, the elements of the basis of $H_2(E(n))$ corresponding to the n copies of the $(-E_8)$-block in the intersection matrix (4.11). The τ_{ij}'s are represented by submanifolds of the complement of $N(n)$, hence disjoint from F: for this reason they represent homology classes in X_n (which we still denote by τ_{ij}) and we may consider their Poincaré duals, which will be elements of $H^2(X_n)$.

According to this interpretation of the elements $\tau_{ij} \in H^2(X_n)$ we consider the complex line bundle L over X_n, specified by its first Chern class

$$c_1(L) = \sum_{i=1}^{n-1} 2PD(\tau_{i1}) + \sum_{j=1}^{3} PD(\tau_{nj}) + PD(\tau_{n8}) \in H^2(X_n; \mathbb{Z}),$$

and compute

$$\begin{cases} \langle c_1^2(L), [X_n] \rangle = -8(n-1) - 12 \\ \langle c_1(L) c_1(X_n), [X_n] \rangle = 0. \end{cases}$$

Now consider the manifold $S = \mathbb{P}(L \oplus \mathbb{C})$. Let $s : X_n \to S$ be the section which embeds X_n in S as $\mathbb{P}(L \oplus \{0\}) = (X_n)_-$. Denote by E the exceptional sphere of the last blow-up in Q^* and by $e = [E]$ its fundamental homology class. Similarly, if $E_- = s(E)$ is the lift of E along the section s, let $e_- = [E_-] = [s(E)]$ be the corresponding homology class, so that we have the following commmutative diagram:

$$\begin{array}{ccc} E_- & \longrightarrow & S \\ \rho \downarrow \cong & & \downarrow \rho \\ E & \longrightarrow & X_n \end{array}$$

Then $\langle c_1(L_{|E}), e\rangle = PD^{-1}c_1(L)\cdot e = 0$, so by Lemma 4.28 we have

$$\langle c_1(\nu_M E_-), e_-\rangle = e^2 + \langle c_1(L_{|E}), e\rangle = e^2.$$

Observe that if F_2 denotes the symplectically embedded surface of genus 2 in Q^*, it is disjoint from any representative of the classes $[\tau_{ij}]$ and therefore it also lifts to $F_{2-} \subset X_{n-}$ with

$$\langle c_1(\nu_M F_{2-}), [F_2]_-\rangle = 0.$$

Finally recall that S admits an integral symplectic form ω_K and hence, for λ large enough, symplectic submanifolds X_λ, realising the Poincaré dual of $\lambda[\omega_K]$, whose invariants are given by the expressions in (4.12).

Take the product of S with S^2 to obtain the simply connected symplectic 8-dimensional manifold $M = S\times S^2$. The parameter j (recall: j was defined by the congruence relation $-c_4 + c_1c_3 + c_2^2 + 4c_2c_1^2 - c_1^4 = 720j$) in this case takes on precisely the value n. The other parameters are

$$\begin{aligned} a &= 48n+12 \\ 4m &= -12 \\ 12k &= -192n-468 \\ b &= -128n-208. \end{aligned}$$

Now blow up M at x points, y copies of E, z copies of F_2, u copies of X_{n-} and v copies of X_λ. Denote by $\widetilde{M}$ the manifold obtained after performing these blow-ups and let (a', m', j', k', b') be the parameters associated to $\widetilde{M}$. Then $j' = j$ (recall that we already remarked that j is invariant under blow-up), whereas by applying the blow-up formulae (4.8), (4.9) and (4.10) we find the following expressions for the other parameters:

$$\begin{aligned} a' &= 48n+12+3x+4y-4z+(12n+3)u+b_1v \\ 4m' &= -12-4y+4z+(-36n-12)u+b_2v \\ 12k' &= -192n-468-180x-108y+144z+(36n+60)u+b_3v \\ b' &= -128n-208-81x-48y+64z+(24n+30)u+b_4v. \end{aligned} \tag{4.13}$$

The coefficients coming from blowing up along the submanifold X_λ, whose invariants are computed in the Appendix, are given by:

$$\begin{aligned}
b_1 &= \lambda c_2(N) + (\lambda^3 - \lambda^2)\, c_1^2(E) + (3\lambda^3 K^2 - 2\lambda^2 K^2)\, [\beta]^2 \\
&\quad + (\lambda^2 - \lambda)\, c_1(N) \cup c_1(E) + (2\lambda K - 2\lambda^2 K)\, c_1(N) \cup [\beta] \\
&\quad + (2\lambda^2 K - 3\lambda^3 K)\, c_1(E) \cup [\beta] \\
b_2 &= \lambda c_1^2(N) - 3\lambda c_2(N) + (\lambda^2 - 2\lambda^3 + \lambda)\, c_1^2(E) + (2\lambda^2 - 6\lambda^3 K^2)\, [\beta]^2 \\
&\quad + (\lambda - \lambda^2)\, c_1(N) \cup c_1(E) + 2(\lambda^2 K - \lambda K)\, c_1(N) \cup [\beta] \\
&\quad + (6\lambda^3 K - 2\lambda^2 K)\, c_1(E) \cup [\beta] \\
b_3 &= -13\lambda c_1^2(N) - \lambda c_2(N) + (9\lambda^2 - 2\lambda^3 - 13\lambda)\, c_1^2(E) \\
&\quad + (18\lambda^2 K^2 - 6\lambda^3 K^2)\, [\beta]^2 + (27\lambda - 9\lambda^2)\, c_1(N) \cup c_1(E) \\
&\quad + (18\lambda^2 K - 54\lambda K)\, c_1(N) \cup [\beta] + (6\lambda^3 K - 18\lambda^2 K)\, c_1(E) \cup [\beta] \\
b_4 &= -6\lambda c_1^2(N) + (4\lambda^2 - 6\lambda - \lambda^3)\, c_1^2(E) + (8\lambda^2 K^2 - 3\lambda^3 K^2)\, [\beta]^2 + \\
&\quad (12\lambda - 4\lambda^2)\, c_1(N) \cup c_1(E) + (8\lambda^2 K - 24\lambda K)\, c_1(N) \cup [\beta] \\
&\quad + (3\lambda^3 K - 8\lambda^2 K)\, c_1(E) \cup [\beta],
\end{aligned}$$

where we have suppressed evaluation on the fundamental class of $[N]$ from the notation.

We regard (4.13) as a linear system in the variables x, y, z, u, v. If we can prove that for arbitrary parameters a', m', k', b', satisfying the additional condition $a' + m' \equiv 0 \,(\mathrm{mod}\ 3)$, the system admits a quintuple of positive, integer solutions, then we will have shown that we can realise all such parameters, precisely by performing on M the sequence of blow-ups corresponding to the solutions (x, y, z, u, v).

The solutions of system (4.13) are:

$$\begin{aligned}
x &= [(32n+13)\lambda^3 K^2 [\beta]^2 + r_1(\lambda, K)]v + \big(8n + \frac{10}{3}\big)a' + \\
&\quad \big(32n + \frac{40}{3}\big)m' + (-128n - 48)k' + (24n+9)b' + \\
&\quad 640n^2 + 224n \\
y &= \big[\big(12n + \frac{9}{2}\big)\lambda^3 K^2 [\beta]^2 + r_2(\lambda, K)\big]v + (3n+3)a' + \\
&\quad (12n+8)m' + (-48n - 21)k' + (9n+4)b' + \\
&\quad 240n^2 + 32n + 1 \\
z &= [(48n+18)\lambda^3 K^2 [\beta]^2 + r_3(\lambda, K)]v + (12n+6)a' + \\
&\quad (48n+21)m' + (-192n - 69)k' + (36n+13)b' + \\
&\quad 960n^2 + 272n + 4 \\
u &= [4\lambda^3 K^2 [\beta]^2 + r_4(\lambda, K)]v + a' + 4m' - 16k' + \\
&\quad 3b' + 80n.
\end{aligned}$$

First of all notice that these solutions are, indeed, integer, because of the additional condition $a' + m' \equiv 0 \ (\mathrm{mod}\ 3)$ and our freedom of choice of the variable v. Moreover, one can observe that the $r_i(\lambda, K)$ are polynomials of degree at most 2 in λ and 2 in K, with coefficients depending on $c_1^2(N), c_2(N), c_1^2(E), [\beta]^2, c_1(N)\cup c_1(E), c_1(N)\cup[\beta], c_1(E)\cup[\beta]$, evaluated on $[N]$; recall also that $\langle[\beta]^2,[N]\rangle$ is strictly positive, because $\beta^2=\beta\wedge\beta$ is a volume form on N: by the previous two remarks we conclude that, by choosing λ large enough, we may ensure the positivity of the v-coefficients and consequently, again by a choice of v sufficiently large, positivity of all variables.

4.6.2 The case $j=0$.

In order to show that all quintuples of parameters with $j=0$ admit a symplectic realisation, we start again by constructing a manifold with $j=0$. We consider the 4-manifold Q^* and the complex line bundle L defined by $c_1(L)=-2e_3$, with e_3 denoting the exceptional divisor of the last blow-up. Since

$$c_1(Q^*)=c_1(\mathbb{T}^4)-\sum_{i=1}^{3}e_i,$$

we see that

$$\begin{cases}\langle c_1^2(L),[Q^*]\rangle=-4\\ \langle c_1(L)c_1(X_n),[Q^*]\rangle=-2.\end{cases}$$

We proceed to construct $S=\mathbb{P}(L\oplus\mathbb{C})$ and $M=S\times S^2$ as in the previous section. Then M realises $j=0$, as required, and the other parameters are

$$\begin{aligned} a &= 12\\ 4m &= -24\\ 12k &= -488\\ b &= -218. \end{aligned}$$

We let Q^*_- be the image of the embedding of Q^* in M as $\mathbb{P}(L\oplus\{0\})$. Then Q^*_- contains a sphere E with square -1 (the exceptional sphere of either the first or the second blow-up) and a genus 2 surface F_2. These curves intersect in S, but since the cup product of their Poincaré duals and $c_1(L)$ vanishes, by Lemma 4.28 they provide disjoint submanifolds $E\times\{\mathrm{pt}\}$ and $F_2\times\{\mathrm{pt}\}$ of $M=S\times S^2$ with the same genus and square.

Together with the submanifold Q^*_-, we consider as in the previous cases submanifolds X_λ, realising the Poincaré duals of multiples $\lambda\omega_K$ of some integral symplectic form ω_K on M.

We are now able to write down the blow-up system for M, where we blow up at x points, y copies of E, z copies of F_2, u copies of $Q^*{}_-$ and v copies of X_λ. Then the parameters of our 8-dimensional manifold transform according to the following expressions

$$\begin{aligned} a' &= a+3x+4y-4z+3u+b_1v\\ 4m' &= 4m-4y+4z-12u+b_2v\\ 12k' &= 12k-180x-108y+144z+24u+b_3v\\ b' &= b-81x-48y+64z+14u+b_4v, \end{aligned}\tag{4.14}$$

where the b_i's are once again the coefficients corresponding to blow-up along submanifolds belonging to the family $\{X_\lambda\}$.

The solutions in this case are

$$\begin{aligned}
x &= [(13\lambda^3K^2[\beta]^2+r_1(\lambda,K)]v+\frac{10}{3}a'+\frac{40}{3}m'-48k'+9b' \\
y &= \big[\frac{17}{2}\lambda^3K^2[\beta]^2+r_2(\lambda,K)\big]v+4a'+12m'-37k'+7b'+1 \\
z &= [22\lambda^3K^2[\beta]^2+r_3(\lambda,K)]v+7a'+25m'-85k'+16b'+4 \\
u &= [4\lambda^3K^2[\beta]^2+r_4(\lambda,K)]v+a'+4m'-16k'+3b'.
\end{aligned}$$

Once again, we may observe that all solutions are integer and that by a suitable choice of λ and v we may assume them to be positive, as well. Hence all admissible quintuples with $j=0$ admit a symplectic realisation.

4.6.3 Negative values of j.

We are left with only the case $j<0$ to take care of. For this we construct a 6-dimensional manifold S as in the cases of positive values of the parameter j and then define M to be the product of S with a compact Riemann surface of genus two. Notice that in this case the realisation will not be simply connected.

The only difference in the blow-up system occurs in the parameters corresponding to the manifold Σ which is blown up: these have in fact opposite sign. Therefore the blow-up system has the form

$$\begin{aligned}
a' &= -48n-12+3x+4y-4z+(12n+3)u+b_1v \\
4m' &= 12-4y+4z+(-36n-12)u+b_2v \\
12k' &= 192n+468-180x-108y+144z+(36n+60)u+b_3v \\
b' &= 128n+208-81x-48y+64z+(24n+30)u+b_4v
\end{aligned}$$

and the solutions are given by

$$\begin{aligned}
x = {} & [(32n+13)\lambda^3K^2[\beta]^2+r_1(\lambda,K)]v+(8n+\frac{10}{3})a'+ \\
& (32n+\frac{40}{3})m'+(-128n-48)k'+(24n+9)b'+ \\
& -640n^2-224n
\end{aligned}$$

$$
\begin{aligned}
y &= [(12n+\frac{9}{2})\lambda^3K^2[\beta]^2+r_2(\lambda,K)]v+(3n+3)a'+ \\
&\quad (12n+8)m'+(-48n-21)k'+(9n+4)b'+ \\
&\quad -240n^2-32n-1 \\
z &= [(48n+18)\lambda^3K^2[\beta]^2+r_3(\lambda,K)]v+(12n+6)a'+ \\
&\quad (48n+21)m'+(-192n-69)k'+(36n+13)b'+ \\
&\quad -960n^2-272n-4 \\
u &= [4\lambda^3K^2[\beta]^2+r_4(\lambda,K)]v+a'+4m'-16k'+ \\
&\quad 3b'-80n.
\end{aligned}
$$

The same considerations as to positivity and integrality apply as in the case of positive j.

4.7 Some final remarks.

4.7.1 Kähler manifolds.

We have shown that, in dimension 8, the geography of symplectic manifolds coincides with that of almost complex manifolds. It is then natural to ask whether this is true also for the geography of Kähler manifolds. Our work unfortunately does not provide a positive answer to this question. The examples we construct, in fact, cease to be Kähler at the point where they are blown up along submanifolds of Donaldson' s type. This is because the latter are not necessarily complex submanifolds.

4.7.2 Geography with fundamental group.

We would like to conclude by briefly addressing the question of geography with fundamental group, that is: to which extent is it possible to prescribe Chern numbers and fundamental group of a symplectic manifold at the same time.

Observe that Halic' result yields in dimension 6 simply connected realisations for all admissible triples. In dimension 4, on the other hand, if (p,q) is an admissible pair with $p+q<0$, there exists no simply connected symplectic manifold with $(c_1^2,c_2)=(p,q)$, as was shown in Proposition 4.22.

If G is any finitely presentable group, Gompf has shown that there exists a closed symplectic 4-manifold with fundamental group G. This manifold, moreover, may be assumed to satisfy certain additional properties.

Theorem 4.30 (Gompf). *Let G be any finitely presentable group. Then there is a closed symplectic 4-manifold M_G with $\pi_1(M_G)\cong G$. Furthermore we may assume:*

(i) $c_1^2[M_G]=0$, $c_2[M_G]=12r>0$;

(ii) *M_G contains a symplectic torus T with square 0 and inclusion $i:T\to M_G$ inducing the trivial map on π_1.*

The proof can be found in [9] and relies in fact on the symplectic connected sum construction.

Remark. The existence of a four-dimensional symplectic manifold M_G with $\pi_1(M_G) \cong G$ for every finitely presentable group G is another feature that distinguishes symplectic from Kähler manifolds. The abelianisation of the fundamental group of a Kähler manifold, in fact, necessarily has even rank, since $(\pi_1)_{\text{ab}} = \pi_1/[\pi_1, \pi_1] \cong H_1$ (cf. Example 2.4).

Theorem 4.30 can be applied to improve partially Halic' result in dimension 6 and show that some admissible triples may be realised by a closed connected symplectic manifold M having a given finitely presentable fundamental group G.

Proposition 4.31. *For every admissible triple $(2a, 24b, 2c)$ with $b \leq -2$ and every finitely presentable group G there exists a closed symplectic 6-manifold M such that*

$$\begin{aligned} c_1^3[M] &= 2a \\ c_1c_2[M] &= 24b \\ c_3[M] &= 2c \\ \pi_1(M) &\cong G. \end{aligned}$$

Proof. Let S be as in Section 4.4.2 and denote by X the symplectic connected sum of S with E_n^*, the blow-up at one point of the elliptic surface E_n. In order to realise all admissible triples with $b \leq -2$, Halic considers the manifold:

$$M' = X \times F_2 \#_{F_1 \times F_2} S \times F_1.$$

For a given finitely presented group G, let M be the manifold obtained by taking the connected sum of M' with the product $M_G \times F_1$, where M_G is as in Theorem 4.30 and F_1 is a surface of genus 1. In other words we have

$$M = M' \#_{F_1 \times F_1} M_G \times F_1,$$

where the sum is performed on the left-hand side along $F_1 \times F_1 \subset S \times F_1$ (this is possible because F_1 and F_2 are disjoint in S) and $F_1 = T \subset M_G$ is also as in (ii) of Theorem 4.30. The Chern numbers of M are the same as those of M' by Lemma 3.23. Moreover, one can show that the fundamental group $\pi_1(M)$ is isomorphic to G. To see that, let

$$U = M' - F_1 \times F_1 \text{ and } V = M_G \times F_1 - F_1 \times F_1.$$

Then $\{U, V\}$ is an open covering of M and $U \cap V = F_1 \times F_1 \times A$. Observe that we may rewrite U as

$$X \times F_2 \#_{F_1 \times F_2} (S - F_1) \times F_1.$$

Since the complement of F_1 in S is simply connected, the argument in [13, p. 379] still goes through and shows that U is simply connected. By Seifert-van Kampen, the fundamental group of M is given by

$$\pi_1(M) = \pi_1(V)/\langle \pi_1(U \cap V) \rangle = \pi_1(M_G - F_1)/\langle \pi_1(F_1) \times \pi_1(A) \rangle. \tag{4.15}$$

The epimorphism

$$i_* : \pi_1(M_G - F_1) \rightarrow \pi_1(M_G),$$

is surjective because of the codimension of the embedding $F_1 \subset M_G$. Moreover, the kernel of i_* is generated by a meridian of F_1 and can be identified with $\pi_1(A)$, so we have

$$\pi_1(M_G - F_1)/\langle \pi_1(F_1 \times A) \rangle \cong \pi_1(M_G)/\langle \pi_1(F_1) \rangle \cong G,$$

which implies, together with (4.15), that $\pi_1(M) \cong G$. By blowing up M we get symplectic realisations with fundamental group G for all admissible triples with $b \leq -2$. □

Appendix A

Some computations.

A.1 Chern numbers of blow-up in dimension 8.

Recall that this is the situation we are considering: M is a closed, connected, symplectic 8-dimensional manifold, N a symplectic submanifold of M, E denotes the normal bundle of N in M, $\mathbb{P}(E)$ its projectivisation and $\widetilde{M}$ the symplectic blow-up of M along N. Notice that if N is a submanifold of dimension six (i.e., codimension two), then $\mathbb{P}(E) \cong N$ and $\widetilde{M} \cong M$. We consider maps as in the blow-up diagram

$$\begin{array}{ccc} \mathbb{P}(E) & \xrightarrow{j} & \widetilde{M} \\ \rho_E \downarrow & & \downarrow f \\ N & \xrightarrow{i} & M \end{array}$$

To carry out the computations, we are going to make use of the Projection Formula, which states that for $a \in H^*(X)$, $b \in H^*(Y)$ and $f : Y \to X$ one has

$$f^!(f^*(a) \cup b) = a \cup f^!(b).$$

This formula is easily proved:

$$\begin{aligned} f^!(f^*(a)\cup b) &= D_M f_* D_N^{-1}(f^*(a)\cup b) \\ &= D_M f_*((f^*(a)\cup b)\cap [N]) \\ &= D_M f_*(f^*(a)\cap (b\cap [N])) \\ &= D_M(a\cap (f^!(b)\cap [M])) \\ &= D_M((a\cup f^!(b))\cap [M]) \\ &= a\cup f^!(b). \end{aligned}$$

The Projection Formula immediately implies the following relations:

- For $a \in H^k(M)$ and $b' \in H^l(\mathbb{P}(E))$ we have

$$\begin{aligned} f^*a \cup j^! b' &= j^!(j^* f^* a \cup b') && \text{Projection Formula} \\ &= j^!(\rho^* i^* a \cup b'). && \text{by commutativity of the blow-up diagram.} \end{aligned}$$

Since $i^*a \in H^k(N)$ and the latter is zero for all $k > \dim N$, we get

$$f^*a \cup j^! b' = 0 \text{ for all } a \in H^k(M) \text{ with } k > \dim N \tag{A.1}$$

and similarly, if $b' = \rho^* b$ for some $b \in H^l(N)$,

$$f^*a \cup j^! \rho^* b = 0 \text{ for all } a \in H^k(M), b \in H^l(N) \text{ with } k + l > \dim N. \tag{A.2}$$

- Let ξ denote the first Chern class of the dual of the tautological line bundle over $\mathbb{P}(E)$. Then

$$\begin{aligned} j^!\xi^k \cup j^!\xi^h &= j^!(\xi^k \cup j^* j^! \xi^h) && \text{Projection Formula} \\ &= j^!(\xi^k \cup (-\xi^{h+1})) && \text{Self-Intersection formula} \\ &= -j^!\xi^{k+h+1}. \end{aligned}$$

Moreover, one can prove by induction that

$$(j^!\xi)^k = (-1)^{k+1} j^! \xi^{2k-1}.$$

- If we let η denote the Poincaré dual of $[\mathbb{P}(E)]$ in $\widetilde{M}$, i.e. $\eta = j^! 1 = PD_{\widetilde{M}}[\mathbb{P}(E)]$, then we see that

$$j^*\eta = j^* j^! 1 = -\xi$$

by the Self-Intersection formula, and hence, in particular, $j^!\xi^k = (-1)^k \eta^{k+1}$ and $j^*\eta^k = (-1)^k \xi^k$.

A.1.1 Blow-up at a point.

If M is blown up at a point, from the blow-up formula (3.14) we get

$$j^*(c(\widetilde{M}) - f^*c(M)) = -\xi(3\xi^3 + 2\xi^2 - 2\xi - 3),$$

hence

$$c(\widetilde{M}) - f^*c(M) = j^!(3\xi^3 + 2\xi^2 - 2\xi - 3).$$

The latter we may rewrite as the following system of equations for the individual Chern classes:

$$\begin{aligned} c_4(\widetilde{M}) &= f^*c_4(M) + 3j^!\xi^3 \\ c_3(\widetilde{M}) &= f^*c_3(M) + 2j^!\xi^2 \\ c_2(\widetilde{M}) &= f^*c_2(M) - 2j^!\xi \\ c_1(\widetilde{M}) &= f^*c_1(M) - 3\eta. \end{aligned}$$

The top-dimensional products of Chern classes are

$$
\begin{aligned}
c_4(\widetilde{M}) &= f^*c_4(M) + 3j^!\xi^3 \\
c_1c_3(\widetilde{M}) &= (f^*c_1(M) - 3\eta)(f^*c_3(M) + 2j^!\xi^2) \\
&= f^*c_1c_3(M) + 2f^*c_1(M)\cup j^!\xi^2 - 3f^*c_3(M)\cup\eta - 6j^!\xi^2\cup\eta \\
&= f^*c_1c_3(M) - 6j^!\xi^2\cup\eta \\
c_2^2(\widetilde{M}) &= (f^*c_2(M) - 2j^!\xi)^2 \\
&= f^*c_2^2(M) - 4f^*c_2(M)\cup j^!\xi + 4(j^!\xi)^2 \\
&= f^*c_2^2(M) - 4j^!\xi^3 \\
c_1^2c_2(\widetilde{M}) &= (f^*c_1(M) + -3\eta)^2 \cup (f^*c_1(M) - 2j^!\xi) \\
&= f^*c_1^2c_2(M) - 2f^*c_1^2(M)\cup j^!\xi + 9f^*c_2(M)\cup\eta^2 - 18\eta^2\cup j^!\xi \\
&\quad -6f^*c_1c_2(M)\cup\eta + 12f^*c_1(M)\cup\eta\cup j^!\xi \\
&= f^*c_1^2c_2(M) - 18j^!\xi^3 \\
c_1^4(\widetilde{M}) &= (f^*c_1(M) - 3\eta)^4 \\
&= f^*c_1^4(M) - 12f^*c_1^3(M)\cup\eta + 54f^*c_1^2(M)\cup\eta^2 \\
&\quad -108f^*c_1\cup\eta^3 - 81\eta^4 \\
&= f^*c_1^4(M) - 81\eta^4.
\end{aligned}
$$

We now proceed to compute the Chern numbers. Recall that in the case of blow-up at a point, $\mathbb{P}(E)\cong\mathbb{CP}^3$. Since $\xi_E = c_1(l_E^*)$ is the generator of $H^*(\mathbb{CP}^3)$ which is compatible with the orientation, we have

$$\langle j^!\xi^3, [\widetilde{M}]\rangle = -\langle\eta^4, [\widetilde{M}]\rangle = -\langle j^*\eta^3, [\mathbb{P}(E)]\rangle = \langle\xi^3, [\mathbb{P}(E)]\rangle = 1.$$

Hence we have:

$$
\begin{aligned}
c_4[\widetilde{M}] &= \langle c_4(\widetilde{M}), [\widetilde{M}]\rangle \\
&= \langle f^*c_4(M), [\widetilde{M}]\rangle + 3\langle j^!\xi_E^3, [\widetilde{M}]\rangle \\
&= c_4[M] + 3 \\
c_1c_3[\widetilde{M}] &= \langle c_1c_3(\widetilde{M}), [\widetilde{M}]\rangle \\
&= \langle f^*c_1c_3(\widetilde{M}), [M]\rangle + 6\langle j^!\xi^3, [\widetilde{M}]\rangle \\
&= c_1c_3[M] + 6 \\
c_2^2[\widetilde{M}] &= \langle c_2^2(\widetilde{M}), [\widetilde{M}]\rangle \\
&= \langle f^*c_2^2(\widetilde{M}), [M]\rangle - 4\langle j^!\xi^3, [\widetilde{M}]\rangle \\
&= c_2^2[M] - 4 \\
c_1^2c_2[\widetilde{M}] &= \langle c_1^2c_2(\widetilde{M}), [\widetilde{M}]\rangle \\
&= \langle f^*c_1^2c_2(\widetilde{M}), [M]\rangle - 18\langle j^!\xi^3, [\widetilde{M}]\rangle \\
&= c_1^2c_2[M] - 18
\end{aligned}
\tag{A.3}
$$

$$\begin{aligned} c_1^4[\widetilde{M}] &= \langle c_1^4(\widetilde{M}), [\widetilde{M}]\rangle \\ &= \langle f^* c_1^4(\widetilde{M}), [M]\rangle - 81\langle j^! \xi^3, [\widetilde{M}]\rangle \\ &= c_1^4[M] - 81. \end{aligned}$$

A.1.2 Blow-up along a curve.

If M is blown up along a submanifold of dimension 2

$$\begin{aligned} j^*(c(\widetilde{M}) - f^*c(M)) &= \rho^* c(N)\{\sum_{i=0}^{r-1} c_i(E^{(1)})(1+\xi)^{r-i-1}(1-\xi) - \rho^* c(E)\} \\ &= \rho^* c(N)\{(1+\xi)^2(1-\xi) + c_1(E^{(1)})(1+\xi)(1-\xi) \\ &\quad + c_2(E^{(1)})(1-\xi) - 1 - \rho^* c_1(E) - \rho^* c_2(E) \\ &\quad - \rho^* c_3(E)\} \\ &= \rho^* c(N)\{-2\xi^3 - \rho^* c_1(E)\xi^2 + \rho^* c_1(E)\xi + 2\xi\} \\ &= -\rho^* c(N)\{2\xi^2 + \rho^* c_1(E)\xi - \rho^* c_1(E) - 2\}\xi. \end{aligned}$$

Remark. In the computations above we have used the fundamental relation

$$\xi^3 + \rho^* c_1(E)\xi^2 + \rho^* c_2(E)\xi + \rho^* c_3(E) = 0, \tag{A.4}$$

which in our situation, namely $\dim N = 1$, reduces to $\xi^3 + \rho^* c_1(E)\xi^2 = 0$.

Hence

$$\begin{aligned} c(\widetilde{M}) - f^*c(M) &= j^! \left[\rho^* c(N)(2\xi^2 + \rho^* c_1(E)\xi - \rho^* c_1(E) - 2)\right] \\ &= j^! \left[2\rho^* c_1(N)\xi^2 + \rho^* c_1(E)\xi + 2\xi^2 \right. \\ &\quad \left. - 2\rho^* c_1(N) - \rho^* c_1(E) - 2\right], \end{aligned}$$

that is,

$$\begin{aligned} c_4(\widetilde{M}) &= f^* c_4(M) + 2j^!(\rho^* c_1(N)\xi^2) \\ c_3(\widetilde{M}) &= f^* c_3(M) + j^! \rho^* c_1(E)\xi + 2j^!\xi^2 \\ c_2(\widetilde{M}) &= f^* c_2(M) - 2j^! \rho^* c_1(N) - j^! \rho^* c_1(E) \\ c_1(\widetilde{M}) &= f^* c_1(M) - 2\eta. \end{aligned}$$

The top-dimensional products of Chern classes are:

$$\begin{aligned} c_4(\widetilde{M}) &= f^* c_4(M) + 2j^! \rho^* c_1(N)\xi^2 \\ c_1c_3(\widetilde{M}) &= (f^* c_1(M) - 2\eta)(f^* c_3(M) + j^! \rho^* c_1(E)\xi + 2j^!\xi^2)) \\ &= f^* c_1c_3(M) + f^* c_1(M) \cup j^!(\rho^* c_1(E)\xi) + 2f^* c_1(M) \cup j^!\xi^2 \\ &\quad - 2f^* c_3(M) \cup \eta - 2\eta \cup j^!(\rho^* c_1(E)\xi) - 4\eta \cup j^!\xi^2 \end{aligned}$$

$$\begin{aligned}
c_2^2(\widetilde{M}) &= (f^*c_2(M) - 2j^!\rho^*c_1(N) - j^!\rho^*c_1(E))^2 \\
&= f^*c_2^2(M) + 4(j^!\rho^*c_1(N))^2 + (j^!\rho^*c_1(E))^2 \\
&\quad -4f^*c_2(M)\cup j^!\rho^*c_1(N) - 2f^*c_2(M)\cup j^!\rho^*c_1(E) \\
&\quad +4j^!\rho^*c_1(N)\cup j^!\rho^*c_1(E) \\
c_1^2c_2(\widetilde{M}) &= (f^*c_1(M)-2\eta)^2(f^*c_1(M) - 2j^!\rho^*c_1(N) - j^!\rho^*c_1(E)) \\
&= f^*c_1^2c_2(M) - 2f^*c_1^2(M)\cup j^!\rho^*c_1(N) - f^*c_1^2(M)\cup j^!\rho^*c_1(E) \\
&\quad -4f^*c_1c_2(M)\cup\eta + 8f^*c_1(M)\cup j^!\rho^*c_1(N)\cup\eta \\
&\quad +4f^*c_1(M)\cup\eta\cup j^!\rho^*c_1(E) + 4f^*c_2(M)\cup\eta^2 \\
&\quad -8\eta^2\cup j^!\rho^*c_1(N) - 4\eta^2\cup j^!\rho^*c_1(E) \\
c_1^4(\widetilde{M}) &= (f^*c_1(M)-2\eta)^4 \\
&= f^*c_1^4(M) - 8f^*c_1^3(M)\cup\eta + 24f^*c_1^2(M)\cup\eta^2 \\
&\quad -32f^*c_1(M)\cup\eta^3 + 16\eta^4.
\end{aligned}$$

Expressions (A.1) and (A.2) immediately imply

$$\begin{aligned}
f^*a\cup j^!\rho^*b &= 0 \text{ for all } a\in H^4(M), b\in H^2(N) \\
f^*a\cup j^!(\rho^*b\xi) &= 0 \text{ for all } a\in H^2(M), b\in H^2(N) \\
f^*a\cup j^!\rho^*b\cup\eta &= 0 \text{ for all } a\in H^2(M), b\in H^2(N) \\
f^*a\cup\eta &= 0 \text{ for all } a\in H^6(M) \\
f^*a\cup\eta^2 &= 0 \text{ for all } a\in H^4(M).
\end{aligned}$$

We now consider the other terms, or rather their evaluation on $[\widetilde{M}]$. We obtain:

$$\begin{aligned}
\langle j^!(\rho^*b\xi^2),[\widetilde{M}]\rangle &= \langle j^!(\rho^*b\,j^*\eta^2),[\widetilde{M}]\rangle = \langle j^!\rho^*b\eta^2,[\widetilde{M}]\rangle \\
&= \langle j^*j^!\rho^*b(-\xi),[\mathbb{P}(E)]\rangle = \langle\rho^*b\xi^2,[\mathbb{P}(E)]\rangle \\
&= \langle b,[N]\rangle \quad \text{for all } b\in H^2(N) \\
\langle j^!(\rho^*b\xi)\cup\eta,[\widetilde{M}]\rangle &= \langle j^*j^!(\rho^*b\xi),[\mathbb{P}(E)]\rangle = \langle -\rho^*b\xi^2,[\mathbb{P}(E)]\rangle \\
&= \langle b,[N]\rangle \quad \text{for all } b\in H^2(N) \\
\langle j^!\rho^*b\cup\eta^2,[\widetilde{M}]\rangle &= \langle j^*j^!\rho^*b\,(-\xi),[\mathbb{P}(E)]\rangle = \langle\rho^*b\xi^2,[\mathbb{P}(E)]\rangle \\
&= \langle b,[N]\rangle \quad \text{for all } b\in H^2(N) \\
\langle f^*a\cup\eta^3,[\widetilde{M}]\rangle &= \langle j^*f^*a\cup j^*\eta^2,[\mathbb{P}(E)]\rangle = \langle\rho^*i^*a\xi^2,[\mathbb{P}(E)]\rangle \\
&= \langle a,[N]\rangle \quad \text{for all } a\in H^2(M) \\
\langle f^*a\cup j^!\xi^2,[\widetilde{M}]\rangle &= \langle f^*a\cup j^!(j^*\eta^2\cup 1),[\widetilde{M}]\rangle = \langle f^*a\cup\eta^2\cup\eta),[\widetilde{M}]\rangle \\
&= \langle f^*a\cup\eta^3,[\widetilde{M}]\rangle = \langle j^*f^*a\cup j^*\eta^2,[\mathbb{P}(E)]\rangle \\
&= \langle\rho^*i^*a\xi^2,[\mathbb{P}(E)]\rangle = \langle i^*a,[N]\rangle \quad \text{for all } a\in H^2(M)
\end{aligned}$$

and by the fundamental relation (A.4) we also have

$$\begin{aligned}
\langle j^!\xi^2\cup\eta,[\widetilde{M}]\rangle &= \langle j^*j^!\xi^2,[\mathbb{P}(E)]\rangle = \langle\xi^3,[\mathbb{P}(E)]\rangle \\
&= \langle -\rho^*c_1(E)\xi^2,[\mathbb{P}(E)]\rangle = \langle c_1(E),[N]\rangle.
\end{aligned}$$

Then the Chern numbers are easily seen to be:

$$\begin{aligned}
c_4[\widetilde{M}] &= \langle c_4(\widetilde{M}), [\widetilde{M}]\rangle \\
&= c_4[M] + 2c_1[N] \\
c_1c_3[\widetilde{M}] &= \langle c_1c_3(\widetilde{M}), [\widetilde{M}]\rangle \\
&= c_1c_3[M] + 2c_1[N] \\
c_2^2[\widetilde{M}] &= \langle c_2^2(\widetilde{M}), [\widetilde{M}]\rangle \\
&= c_2^2[M] \\
c_1^2c_2[\widetilde{M}] &= \langle c_1^2c_2(\widetilde{M}), [\widetilde{M}]\rangle \\
&= c_1^2c_2[M] - 8c_1[N] - 4\langle c_1(E), [N]\rangle \\
c_1^4[\widetilde{M}] &= \langle c_1^4(\widetilde{M}), [\widetilde{M}]\rangle \\
&= c_1^4[M] - 32c_1[N] - 16\langle c_1(E), [N]\rangle.
\end{aligned} \tag{A.5}$$

A.1.3 Blow-up along a four-dimensional submanifold.

Similarly, if M is blown up along a submanifold of dimension 4,

$$j^*(c(\widetilde{M}) - f^*c(M)) = -\rho^*c(N)(\xi - 1)\xi,$$

hence

$$c(\widetilde{M}) - f^*c(M) = j^!\left(\rho^*c_2(N)\xi + (\rho^*c_1(N)\xi - \rho^*c_2(N)) + (\xi - \rho^*c_1(N)) - 1\right)$$

and the Chern classes of the blown up manifold satisfy the following relations

$$\begin{aligned}
c_4(\widetilde{M}) &= f^*c_4(M) + j^!(\rho^*c_2(N)\xi) \\
c_3(\widetilde{M}) &= f^*c_3(M) + j^!(\rho^*c_1(N)\xi) - j^!\rho^*c_2(N) \\
c_2(\widetilde{M}) &= f^*c_2(M) + j^!\xi_E - j^!\rho^*c_1(N) \\
c_1(\widetilde{M}) &= f^*c_1(M) - \eta.
\end{aligned}$$

In this case we have the fundamental relation

$$\xi^2 + \rho^*c_1(E)\xi + \rho^*c_2(E).$$

For the top-dimensional products of Chern classes we get the expressions:

$$
\begin{aligned}
c_4(\widetilde{M}) &= f^*c_4(M) + j^!(\rho^*c_2(N)\xi) \\
c_1c_3(\widetilde{M}) &= (f^*c_1(M) - \eta)(f^*c_3(M) + j^!(\rho_E^*c_1(N)\xi) - j^!\rho^*c_2(N)) \\
&= f^*c_1c_3(M) + f^*c_1(M) \cup j^!(\rho^*c_1(N)\xi) - f^*c_1(M) \cup j^!\rho^*c_2(N) \\
&\quad -\eta \cup f^*c_3(M) - \eta \cup j^!(\rho^*c_1(N)\xi) + \eta \cup j^!\rho^*c_2(N) \\
c_2^2(\widetilde{M}) &= (f^*c_2(M) + j^!\xi - j^!\rho^*c_1(N))^2 \\
&= f^*c_2^2(M) + (j^!\xi)^2 + (j^!\rho^*c_1(N))^2 + 2f^*c_2(M) \cup j^!\xi \\
&\quad -2f^*c_2(M) \cup j^!\rho^*c_1(N) - 2j^!\xi \cup j^!\rho^*c_1(N) \\
c_1^2c_2(\widetilde{M}) &= (f^*c_1(M) - \eta)^2(f^*c_2(M) + j^!\xi - j^!\rho^*c_1(N)) \\
&= (f^*c_1^2(M) - 2f^*c_1(M) \cup \eta + \eta^2)(f^*c_2(M) + j^!\xi - j^!\rho^*c_1(N)) \\
&= f^*c_1^2c_2(M) + f^*c_1^2(M) \cup j^!\xi - f^*c_1^2(M) \cup j^!\rho^*c_1(N) \\
&\quad -2f^*c_1c_2(M) \cup \eta - 2f^*c_1(M) \cup \eta \cup j^!\xi \\
&\quad +2f^*c_1(M) \cup \eta \cup j^!\rho^*c_1(N) + f^*c_2(M) \cup \eta^2 \\
&\quad +j^!\xi \cup \eta^2 - j^1\rho^*c_1(N) \cup \eta^2 \\
c_1^4(\widetilde{M}) &= (f^*c_1(M) + j^!(-1))^4 \\
&= f^*c_1^4(M) - 4f^*c_1^3(M) \cup \eta + 6f^*c_1^2(M) \cup \eta^2 \\
&\quad -4f^*c_1(M) \cup \eta^3 + \eta^4.
\end{aligned}
$$

Again by (A.1) and (A.2) we immediately see that some of the terms which appear above vanish, namely

$$
\begin{aligned}
f^*a \cup j^!(\rho^*b) &= 0 \quad \text{for all } a \in H^2(M) \text{ and } b \in H^4(N); \\
f^*a \cup \eta &= 0 \quad \text{for all } a \in H^6(M).
\end{aligned}
$$

We evaluate the other terms on the homology class $[\widetilde{M}]$:

$$
\begin{aligned}
\langle f^*a \cup j^!(\rho^*b\xi), [\widetilde{M}]\rangle &= \langle f^*a \cup j^!(\rho^*b(-j^*\eta)), [\widetilde{M}]\rangle \\
&= \langle f^*a \cup (-j^!\rho^*b) \cup \eta, [\widetilde{M}]\rangle \\
&= \langle j^*f^*a \cup (-j^*j^!\rho^*b), [\mathbb{P}(E)]\rangle \\
&= \langle \rho^*i^*a \cup \rho^*b\xi, [\mathbb{P}(E)]\rangle \\
&= \langle i^*a \cup b, [N]\rangle \\
&\quad \text{for all } a \in H^2(M), b \in H^2(N) \\
\langle f^*a \cup j^!\rho^*b \cup \eta, [\widetilde{M}]\rangle &= \langle j^*f^*a \cup j^*j^!\rho^*b, [\mathbb{P}(E)]\rangle = \langle \rho^*i^*a \cup (-\rho^*b\xi), [\mathbb{P}(E)]\rangle \\
&= \langle -\rho^*(i^*a \cup b)\xi, [\mathbb{P}(E)]\rangle = -\langle i^*a \cup b, [N]\rangle \\
&\quad \text{for all } a \in H^2(M), b \in H^2(N)
\end{aligned}
$$

$$
\begin{aligned}
\langle j^!(\rho^* b\xi)\cup\eta,[\widetilde{M}]\rangle &= \langle j^* j^!(\rho^* b\xi),[\mathbb{P}(E)]\rangle = \langle -\rho^* b\xi^2,[\mathbb{P}(E)]\rangle \\
&= \langle \rho^*(b\cup c_1(E))\xi,[\mathbb{P}(E)]\rangle = \langle b\cup c_1(E),[N]\rangle \\
&\ \text{for all } b\in H^2(N) \\
\langle j^!\rho^* b\cup\eta,[\widetilde{M}]\rangle &= \langle -\rho^* b\xi,[\mathbb{P}(E)]\rangle = -\langle b,[N]\rangle \\
&\ \text{for all } b\in H^4(N) \\
\langle (j^!\rho^* b)^2,[\widetilde{M}]\rangle &= \langle j^!(j^* j^!\rho^* b\cup\rho^* b),[\widetilde{M}]\rangle = \langle j^!(-\rho^* b\xi\cup\rho^* b),[\widetilde{M}]\rangle \\
&= \langle -j^!(\rho^* b^2\cup j^*\eta),[\widetilde{M}]\rangle = \langle -j^!\rho^* b\cup\eta,[\widetilde{M}]\rangle \\
&= \langle i^* a\cup b,[N]\rangle \\
&\ \text{for all } b\in H^2(N) \\
\langle f^* a\cup\eta^2,[\widetilde{M}]\rangle &= \langle j^* f^* a\cup j^*\eta,[\mathbb{P}(E)]\rangle = \langle \rho^* i^* a\cup(-\xi),[\mathbb{P}(E)]\rangle \\
&= -\langle i^* a,[N]\rangle \\
&\ \text{for all } a\in H^4(M) \\
\langle j^!\rho^* b\cup\eta^2,[\widetilde{M}]\rangle &= \langle j^* j^!\rho^* b\cup j^*\eta,[\mathbb{P}(E)]\rangle = \langle -\rho^* b\xi\cup(-\xi),[\mathbb{P}(E)]\rangle \\
&= \langle -\rho^*(b\cup c_1(E))\xi,[\mathbb{P}(E)]\rangle = -\langle b\cup c_1(E),[N]\rangle \\
&\ \text{for all } b\in H^2(N) \\
\langle f^* a\cup\eta^3,[\widetilde{M}]\rangle &= \langle j^* f^* a\cup j^*\eta^2,[\mathbb{P}(E)]\rangle = \langle \rho^* i^* a\xi^2,[\mathbb{P}(E)]\rangle \\
&= \langle -\rho^*(i^* a\cup c_1(E))\xi,[\mathbb{P}(E)]\rangle = -\langle i^* a\cup c_1(E),[N]\rangle \\
&\ \text{for all } a\in H^2(M) \\
\langle \eta^4,[\widetilde{M}]\rangle &= \langle j^*\eta^3,[\mathbb{P}(E)]\rangle = \langle -\xi^3,[\mathbb{P}(E)]\rangle \\
&= \langle \xi(\rho^* c_1(E)\xi+\rho^* c_2(E)),[\mathbb{P}(E)]\rangle = \langle -\rho^* c_1^2(E)\xi+\rho^* c_2(E)\xi,[\mathbb{P}(E)]\rangle \\
&= \langle -c_1^2(E)+c_2(E),[N]\rangle.
\end{aligned}
$$

As to the remaining terms, we simply observe that

$$
\begin{aligned}
(j^!\xi)^2 &= \eta^2 \\
f^* a\cup j^!\xi &= -f^* a\cup\eta^2 \\
j^!\rho^* b\cup j^!\xi &= -j^!\rho^* b\cup\eta^2 \\
f^* a\cup j^!\xi\cup\eta &= -f^* a\cup\eta^3 \\
j^!\xi\cup\eta^2 &= -\eta^4.
\end{aligned}
$$

Using once again the relation $\langle f^* i^! a,[\widetilde{M}]\rangle = \langle a,[N]\rangle$, we see that the Chern numbers can be expressed as:

$$
\begin{aligned}
c_4[\widetilde{M}] &= \langle c_4(\widetilde{M}),[\widetilde{M}]\rangle \\
&= c_4[M]+c_2[N] \\
c_1c_3[\widetilde{M}] &= \langle c_1c_3(\widetilde{M}),[\widetilde{M}]\rangle \\
&= c_1c_3[M]+c_1^2[N]-c_2[N]
\end{aligned}
$$

$$\begin{aligned}
c_2^2[\widetilde{M}] &= \langle c_2^2(\widetilde{M}), [\widetilde{M}]\rangle \\
&= c_2^2[M] - c_1^2[N] + 2c_2[N] - \langle c_1^2(E), [N]\rangle + 3\langle c_2(E), [N]\rangle \\
c_1^2c_2[\widetilde{M}] &= \langle c_1^2c_2(\widetilde{M}), [\widetilde{M}]\rangle \\
&= c_1^2c_2[M] - c_1^2[N] - c_2[N] - 2\langle c_2(E), [N]\rangle - 2\langle c_1(N)\cup c_1(E), [N]\rangle \\
c_1^4[\widetilde{M}] &= \langle c_1^4(\widetilde{M}), [\widetilde{M}]\rangle \\
&= c_1^4[M] - 6c_1^2[N] - 3\langle c_1^2(E), [N]\rangle + \langle c_2(E), [N]\rangle + \\
&\quad -8\langle c_1(N)\cup c_1(E), [N]\rangle.
\end{aligned} \tag{A.6}$$

A.2 Submanifolds from Donaldson's theorem.

We refer to the following situation: (S, ω_K) is a symplectic sphere bundle with compact symplectic base (N, β), obtained by projectifying the bundle $E \oplus \mathbb{C}$, with E a complex line bundle over N. The form β is integral and the form ω_K is defined as $K\rho^*\beta + \eta$, where ρ denotes the bundle map from S to N and η represents the class $\xi \in H^2(S;\mathbb{Z})$, which is the first Chern class of the dual of the canonical line bundle over S. Then by Theorem 3.10, for large enough $\lambda \in \mathbb{Z}$ there exist symplectic submanifolds N_λ of S, realising the Poincaré dual of the classes $\lambda[\omega_K]$. Our aim in this section is to compute the characteristic numbers of the submanifolds N_λ in the case where the dimension of N equals 4.

Recall from Example 3.6 that the total Chern class of S can be written as:

$$\begin{aligned}
c(S) &= 1 + \rho^*(c_1(N) + c_1(E)) + 2\xi \\
&\quad + \rho^*(c_1(N)\cup c_1(E) + c_2(N)) + 2\rho^*c_1(N)\xi \\
&\quad + 2\rho^*c_2(N)\xi.
\end{aligned}$$

The Chern classes of S and N_λ are related by the Whitney product formula

$$c(N_\lambda)\cup c(\nu_S N_\lambda) = i^*c(S), \tag{A.7}$$

where $\nu_S N_\lambda$ denotes as usual the normal bundle of N_λ in S, and i the inclusion $N_\lambda \to S$. The bundle $\nu_S N_\lambda$ is a complex line bundle and its first Chern class coincides with the Euler class, which in turn is equal to the restriction to N_λ of the Poincaré dual of N_λ. Since we know $PD[N_\lambda]$ to be $\lambda[\omega_K] = \lambda K[\rho^*\beta] + \lambda\xi$, we can rewrite equation (A.7) as

$$c(N_\lambda)\cup i^*(\lambda K[\rho^*\beta] + \lambda\xi) = i^*c(S) \tag{A.8}$$

and by substituting the expression for the total Chern class of S we get

$$\begin{aligned}
&c(N_\lambda)\cup i^*(\lambda K[\rho^*\beta] + \lambda\xi) = \\
&\quad i^*[1 + \rho^*(c_1(N) + c_1(E)) + 2\xi + \rho^*(c_1(N)\cup c_1(E) + c_2(N)) \\
&\quad + 2\rho^*c_1(N)\xi + 2\rho^*c_2(TN)\xi].
\end{aligned} \tag{A.9}$$

By comparing the terms of equal degree on the two sides of the above equation, we get the following two identities:

$$c_1(N_\lambda) + \lambda K i^*[\rho^*\beta] + \lambda i^*\xi = i^*\rho^*c_1(N) + i^*\rho^*c_1(E) + 2i^*\xi$$

and

$$\begin{aligned} & c_1(N_\lambda) \cup (\lambda K i^*[\rho^*\beta] + \lambda i^*\xi) + i^* c_2(N_\lambda) = \\ & \quad i^*\rho^* c_1(N) \cup i^*\rho^* c_1(E) + i^*\rho^* c_2(N) + 2i^*(\rho^* c_1(N)\xi). \end{aligned}$$

Using these identities we can compute the Chern classes of N_λ, namely

$$\begin{aligned} c_1(N_\lambda) &= i^*\rho^* c_1(N) + i^*\rho^* c_1(E) + (2-\lambda)\, i^*\xi - \lambda K i^*[\rho^*\beta] \\ c_2(N_\lambda) &= i^*\rho^* c_2(N) + i^*\rho^*(c_1(N) \cup c_1(E)) + (2-\lambda) i^*(\rho^* c_1(N)\xi) \\ & \quad -\lambda K i^*(\rho^* c_1(N) \cup [\rho^*\beta]) - (\lambda^2 - \lambda)\, i^*(\rho^* c_1(E)\xi) \\ & \quad -\lambda K i^*(\rho^* c_1(E) \cup [\rho^*\beta]) + (2\lambda^2 K - 2\lambda K)\, i^*([\rho^*\beta]\xi) \\ & \quad +\lambda^2 K^2 i^*[\rho^*\beta]^2. \end{aligned}$$

In particular, we have

$$\begin{aligned} c_1^2(N_\lambda) &= i^*\rho^* c_1^2(N) + i^*\rho^* c_1^2(E) - (\lambda^2 - 2\lambda) i^*(\rho^* c_1(E)\xi) \\ & \quad +\lambda^2 K^2 i^*[\rho^*\beta]^2 + 2i^*\rho^*(c_1(N) \cup c_1(E)) \\ & \quad +2(2-\lambda)\, i^*(\rho^* c_1(N)\xi) - 2\lambda K i^*(\rho^* c_1(N) \cup [\rho^*\beta]) \\ & \quad -2\lambda K i^*(\rho^* c_1(E) \cup [\rho^*\beta]) - 2(2-\lambda)\lambda K i^*([\rho^*\beta]\xi). \end{aligned}$$

The top-dimensional Chern classes c_1^2 and c_2 can be evaluated on the fundamental homology class of N_λ. Using Poincaré duality we can compute these values. In fact, notice that $c_1^2(N_\lambda)$ and $c_2(N_\lambda)$ are contained in the image of i^*, i.e., they can be written as i^*x_1 and i^*x_2, respectively, where each x_i is a class of S. But for each $x \in H^4(S;\mathbb{Z})$, the product $\langle i^*x, [N_\lambda]\rangle$ has the form

$$\langle i^*x, [N_\lambda]\rangle = \langle x \cup PD[N_\lambda], [S]\rangle = \langle x \cup (\lambda K[\rho^*\beta] + \lambda\xi), [S]\rangle.$$

In order to compute the Chern numbers of N_λ, then, we start by computing the products

$$\begin{aligned} x_1 \cup (\lambda K[\rho^*\beta] + \lambda\xi) &= \lambda\rho^* c_1^2(N)\xi + \lambda(\lambda^2 - 2\lambda + 1)\, \rho^* c_1^2(E)\xi \\ & \quad +2\lambda(\lambda-1)\, \rho^*(c_1(N) \cup c_1(E))\xi \\ & \quad +4\lambda K(1-\lambda)\rho^* c_1(N) \cup [\rho^*\beta]\xi \\ & \quad +\lambda^2 K(4-3\lambda)\rho^* c_1(E) \cup [\rho^*\beta]\xi \\ & \quad +\lambda^2 K^2(3\lambda - 4)[\rho^*\beta]^2\xi \\ x_2 \cup (\lambda K[\rho^*\beta] + \lambda\xi) &= \lambda\rho^* c_2(N)\xi + \lambda^2(\lambda-1)\, \rho^* c_1^2(E)\xi \\ & \quad +\lambda(\lambda-1)\, \rho^*(c_1(N) \cup c_1(E))\xi \\ & \quad -2\lambda K(\lambda-1)\rho^* c_1(N) \cup [\rho^*\beta]\xi \\ & \quad +\lambda^2 K(2-3\lambda)\rho^* c_1(E) \cup [\rho^*\beta]\xi \\ & \quad +\lambda^2 K^2(3\lambda-2)[\rho^*\beta]^2\xi \end{aligned}$$

Notice that we have used the fundamental relation

$$\xi^2 + \rho^* c_1(E)\xi = 0 \in H^*(S;\mathbb{Z})$$

in order to reduce the products to this form. From the ring structure of the cohomology of S we also get

$$\langle \rho^* y\xi, [S]\rangle = \langle y, [N]\rangle \tag{A.10}$$

for all $y \in H^4(N;\mathbb{Z})$. We can thus immediately write down the Chern numbers of N_λ, which are

$$\begin{aligned}
c_1^2[N_\lambda] &= \lambda c_1^2[N] + \lambda(\lambda^2 - 2\lambda + 1)\langle c_1^2(E), [N]\rangle \\
&\quad +2\lambda(\lambda-1)\langle (c_1(N)\cup c_1(E)), [N]\rangle \\
&\quad +4\lambda K(1-\lambda)\langle c_1(N)\cup[\rho^*\beta], [N]\rangle \\
&\quad +\lambda^2 K(4-3\lambda)\langle c_1(E)\cup[\rho^*\beta], [N]\rangle + \lambda^2K^2(3\lambda-4)\langle [\rho^*\beta]^2, [N]\rangle \\
c_2[N_\lambda] &= \lambda c_2[N] + \lambda^2(\lambda-1)\langle c_1^2(E), [N]\rangle \\
&\quad +\lambda(\lambda-1)\langle c_1(N)\cup c_1(E), [N]\rangle - 2\lambda K(\lambda-1)\langle c_1(N)\cup[\rho^*\beta], [N]\rangle \\
&\quad +\lambda^2K(2-3\lambda)\langle c_1(E)\cup[\rho^*\beta], [N]\rangle + \lambda^2K^2(3\lambda-2)\langle[\rho^*\beta]^2, [N]\rangle.
\end{aligned}$$

We are also interested in other invariants of N_λ, namely $\langle c_1^2(\nu_S N_\lambda), [N_\lambda]\rangle$ and $\langle c_1(N_\lambda)\, c_1(\nu_S N_\lambda), [N_\lambda]\rangle$. We use the same strategy as for c_1^2 and c_2. Recall that $c_1(\nu_S N_\lambda) = i^*(\lambda\xi + \lambda K[\rho^*\beta])$: then

$$\begin{aligned}
\langle c_1^2(\nu_S N_\lambda), [N_\lambda]\rangle &= \langle i^*(\lambda\xi+\lambda K[\rho^*\beta])^2, [N_\lambda]\rangle = \langle(\lambda\xi+\lambda K[\rho^*\beta])^3, [S]\rangle \\
&= \langle \lambda^3\xi^3 + 3\lambda^3K[\rho^*\beta]\xi^2 + 3\lambda^3K^2[\rho^*\beta]^2\xi, [S]\rangle \\
&= \langle \lambda^3\rho^* c_1^2(E)\xi + 3\lambda^3Kc_1(E)\cup[\rho^*\beta]\xi \\
&\quad +3\lambda^3K^2[\rho^*\beta]^2\xi, [S]\rangle \\
&= \lambda^3\langle c_1^2(E), [N]\rangle + 3\lambda^3K\langle c_1(E)\cup[\beta], [N]\rangle \\
&\quad +3\lambda^3K^2\langle[\beta]^2, [S]\rangle.
\end{aligned}$$

Similarly we compute the last invariant:

$$\begin{aligned}
\langle c_1(N_\lambda)\cup c_1(\nu_S N_\lambda), [N_\lambda]\rangle &= \langle i^*(\rho^*c_1(N) + \rho^*c_1(E) + (2-\lambda)\xi - \lambda K[\rho^*\beta]) \\
&\quad \cup i^*(\lambda\xi+\lambda K[\rho^*\beta]), [N_\lambda]\rangle \\
&= \langle(\rho^*c_1(N)+\rho^*c_1(E)+(2-\lambda)\xi-\lambda K[\rho^*\beta]) \\
&\quad \cup(\lambda\xi+\lambda K[\rho^*\beta])^2, [S]\rangle \\
&= \langle \lambda^2c_1(N)\xi^2 + \lambda^2c_1(E)\xi^2 + \lambda^2(2-\lambda)\xi^3 \\
&\quad -\lambda^3K[\rho^*\beta]\xi^2 + 2\lambda^2Kc_1(N)[\rho^*\beta]\xi \\
&\quad +2\lambda^2Kc_1(E)[\rho^*\beta]\xi + 2(2-\lambda)\lambda^2K[\rho^*\beta]\xi^2 \\
&\quad +\lambda^2K^2(2-\lambda)[\rho^*\beta]^2\xi - 2\lambda^3K^2[\rho^*\beta]^2\xi, [S]\rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle -\lambda^2 c_1(N) \cup c_1(E)\xi - \lambda^2 c_1^2(E)\xi \\
&\quad +\lambda^2(2-\lambda)c_1^2(E)\xi - \lambda^3 K c_1(E) \cup [\rho^*\beta]\xi \\
&\quad +2\lambda^2 K c_1(N)[\rho^*\beta]\xi + 2\lambda^2 K c_1(E)[\rho^*\beta]\xi \\
&\quad +2(2-\lambda)\lambda^2 K c_1(E) \cup [\rho^*\beta]\xi \\
&\quad +\lambda^2 K^2(2-\lambda)[\rho^*\beta]^2\xi - 2\lambda^3 K^2[\rho^*\beta]^2\xi, [S]\rangle \\
&= -\lambda^2\langle c_1(N) \cup c_1(E), [N]\rangle \\
&\quad +\lambda^2(1-\lambda)\langle c_1^2(E), [N]\rangle \\
&\quad +\lambda^2 K(3\lambda-2)\langle c_1(E) \cup [\beta], [N]\rangle \\
&\quad +2\lambda^2 K\langle c_1(N)[\beta], [N]\rangle \\
&\quad +\lambda^2 K^2(2-3\lambda)\langle [\beta]^2, [N]\rangle.
\end{aligned}$$

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Samenvatting.

Over de geografie van symplectische variëteiten.

Het onderwerp van dit proefschrift betreft constructies en invarianten van symplectische variëteiten.

Een symplectische variëteit is een gladde variëteit met een symplectische structuur. Dat wil zeggen een gesloten, niet-gedegenereerde 2-vorm. Een belangrijk voorbeeld van een dergelijke variëteit komt uit de Hamiltoniaanse mechanica, namelijk de totaalruimte van de coraakbundel van een gladde variëteit. Dit kan men ook beschouwen als de faseruimte van een dynamisch systeem over de gegeven gladde variëteit. Een diffeomorfisme tussen twee symplectische variëteiten dat de symplectische structuur behoudt, heet symplectomorfisme. Deze symplectomorfismen behouden tevens de klasse van Hamiltoniaanse differentiaalvergelijkingen.

Twee van de meest opvallende vragen in verband met symplectische meetkunde betreffen het bestaan van symplectische structuren en de definitie van symplectische invarianten. Aangezien volgens de stelling van Darboux alle symplectische structuren lokaal isomorf zijn, moeten deze invarianten globaal van aard zijn.

Dit proefschrift gaat over enkele numerieke invarianten van symplectische variëteiten. Gegeven een gesloten, $2n$-dimensionale variëteit met een symplectische vorm ω, dan bestaat er over haar raakbundel een complexe structuur. Ten opzichte van deze structuur is het mogelijk een systeem van $\pi(n)$ gehele getallen te bepalen, waarbij $\pi(n)$ de cardinaliteit van de verzameling van alle partities van n is. Deze getallen zijn alleen van de symplectische structuur afhankelijk en worden Chern getallen genoemd, naar de Chinees-Amerikaans wiskundige Shiing-Shen Chern, geboren in 1911.

Het specifieke probleem waarmee wij ons in dit proefschrift bezighouden is dat van welke systemen van gehele getallen mogen optreden als systeem van Chern getallen van een gesloten, samenhangende, symplectische variëteit. Dit probleem is bekend in de literatuur onder de naam "symplectische geografie".

De Chern getallen van een willekeurige symplectische variëteit voldoen aan bepaalde voorwaarden, die uit de algebraische meetkunde zijn ontstaan, namelijk een aantal congruentie-relaties die de stelling van Riemann-Roch impliceert. Het hoofdresultaat van dit proefschrift laat zien, dat in dimensie 8 deze voorwaarden ook voldoende zijn. Dat wil zeggen, gegeven een vijftal van gehele getallen $(a_1, \ldots, a_5)$ die aan de congruentie-relaties voldoen, dan bestaat er een gesloten, samenhangende, symplectische 8-dimensionale variëteit M, zo dat $(a_1,\ldots,a_5)$ met het vijftal van Chern getallen van M overeenstemt.

De inhoud van het proefschrift is als volgt. Hoofdstuk 1 bevat de belangrijkste definities en feiten uit de literatuur. Hoofdstuk 2 gaat over verschillende manieren om symplectische variëteiten te construeren. In het derde hoofdstuk beschouwen wij de symplectische invarianten en we beschrijven, hoe zij efficiënt te berekenen zijn voor de variëteiten die wij in hoofdstuk 2 hebben geconstrueerd. De details van de berekeningen zijn in de appendix uitgevoerd. In hoofdstuk 4 construeren wij voorbeelden van 8-dimensionale symplectische variëteiten en we realiseren zo alle vijftallen die door de congruentie-relaties niet zijn uitgesloten. Daarmee is ons hoofdresultaat bewezen.