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Order-convergence in partially ordered vector spaces

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Abstract

In this bachelor thesis we will be looking at two different definitions of convergent nets in partially ordered vector spaces. We will investigate these different convergences and compare them. In particular, we are interested in the closed sets induced by these definitions of convergence. We will see that the complements of these closed set form a topology on our vector space. Moreover, the topologies induced by the two definitions of convergence coincide. After that we characterize the open sets this topology in the case that the ordered vector spaces are Archimedean. Futhermore, we how that every set containg 0, which is open for the topology of order convergence, contains a neighbourhood of 0 that is full.

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1 Introduction

In ordered vector spaces there are several natural ways to define convergence using only the ordering. We will refer to them as 'order-convergence'. Order-convergence of nets is widely used. In for example, the study of normed vector lattices, it is used for order continuous norms [1, 3]. It is also used in theory on operators between vector lattices to define order continuous operators which are operators that are continuous with respect to order-convergence [3]. The commonly used definition of order-convergence for nets [3] originates from the definition for sequences. Unfortunately, it has a problem. Namely, the order-convergence of a net is not only dependent on its tail but also on the beginning of a net, this will be elaborated in Example 1.9 below. Abramovich and Sirotkin [2] proposed, in the setting of vector lattices, a new and improved definition for order-convergence of nets. This definition is also used in [1], where some relations are given in the case of vector lattices.
We are going to study both definitions of order-convergence for nets in general partially ordered vector spaces. Our main result is that both definitions induce the same closed sets and, therefore, the same topology.
We will start off with giving the basic definitions, which can be found in, e.g., [6].
Definition 1.1. A real vector space $(X, \leq)$ equipped with a partial ordering such that

(i) $x \leq y \Rightarrow x + z \leq y + z$

(ii) $x \leq y \Rightarrow \lambda x \leq \lambda y$

hold for all $x, y, z \in X$ and $\lambda \in \mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$ is called an **ordered vector space**.

In ordered vector spaces we want to consider intervals, which are naturally defined as follows.
Definition 1.2. If $a, b \in X$ and $a \leq b$ we define the **order interval** to be

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

As mentioned earlier we want to look at order-convergence of nets. A net is basically a generalisation of a sequence as it is map from a directed set into a space, and $\mathbb{N}$ is in fact such a directed set.
Definition 1.3. A non-empty set I with a pre-order $\leq$ is called a **directed set** if $(I, \leq)$ is inductive, i.e.:

$$\forall x, y \in I : \exists z \in I : z \geq x, z \geq y.$$

Definition 1.4. A map $\varphi : I \to X : \alpha \mapsto x_\alpha$ is called a **net** if I is a directed set.

Notation: $(x_\alpha)_{\alpha \in I}$ or just (x_α) if convenient.

We will now give the definitions leading to the two different definitions of order-convergence for nets as given in [2]. Let $(X, \leq)$ be a partially ordered vector space.
Definition 1.5. If $Y \subseteq X$ and $y \in X$ then $y = \inf(Y)$ (y is the **infimum** of Y) iff

(1) $\forall x \in Y : y \leq x$

(2) $\forall x \in Y : z \leq x \Rightarrow z \leq y$

Analogously we can define a **supremum** $\sup(Y)$ of Y.
Definition 1.6. A net $(x_\alpha)_{\alpha \in I}$ in $(X, \leq)$ **descends to** 0 if

(i) $\alpha \geq \beta \Rightarrow x_\alpha \leq x_\beta$

(ii) $0 = \inf\{x_\alpha : \alpha \in I\}$

Notation: $(x_\alpha)_{\alpha \in I} \downarrow 0$.

Definition 1.7. A net $(x_\alpha)_{\alpha\in I}$ is O_1**-convergent** to x if there exists a net $(y_\alpha)_{\alpha\in I}$ such that:

- $(y_\alpha) \downarrow 0$
- $\forall \alpha \in I : -y_\alpha \leq x - x_\alpha \leq y_\alpha$

Notation: $(x_\alpha)_{\alpha\in I} \overset{O_1}{\to} x$

Definition 1.8. A net $(x_\alpha)_{\alpha\in I}$ is O_2**-convergent** to x if there exists a net $(y_\beta)_{\beta\in J}$ such that:

- $(y_\beta) \downarrow 0$
- $\forall \beta \in J : \exists \alpha_0 \in I : \alpha \geq \alpha_0 \Rightarrow\ -y_\beta \leq x - x_\alpha \leq y_\beta$

Notation: $(x_\alpha)_{\alpha\in I} \overset{O_2}{\to} x$

The definition of O_1-convergence is the commonly used generalisation of order-convergence for sequences as used in [5]. The second definition of order-convergence, O_2, is the one proposed in [2]. We will now give an example which shows a problem with O_1-convergence, which is taken from [2].

Example 1.9. If x is a positive real number then the net $(x_n)_{n\in\mathbb{N}}$ defined by $x_n := \frac{1}{n}x$ descends to zero and is therefore both O_1- and O_2-convergent to zero. Let us now add all negative integers and place them between 1 and 2 in the ordering. Define for all negative integers $x_n := |n|x$. Then the net (x_n) is still O_2-convergent to zero but is not O_1-convergent anymore.

This example also shows that O_1- and O_2-convergence are not equivalent to each other. It is important to note, though, that O_2-convergence implies O_1-convergence by definition. As mentioned earlier, a reason to study order-convergence of nets is to apply it in the study norms and operators. So a natural question we want to ask is whether O_1- and O_2-convergence induce the same continuous norms or operators. In order to answer this question we are going to look at the topologies that both O_1 and O_2 induce.

Definition 1.10. A subset $A \subseteq X$ is said to be O_i**-closed** if and only if

$$(x_\alpha)_{\alpha\in I} \overset{O_i}{\to} x, x_\alpha \in A \Rightarrow x \in A$$

(for $i = 1, 2$).

Note that it is not contained in this definition that the complements of these O_i-closed sets, which we will call O_i-open, form a topology on X. Later on we will show that they do. Before we go into that, we will first prove some results that follow immediately from the definitions given above, in the next section.

2 Elementary Observations

This section consists of a collection of useful lemma's, which are all easily verified. We will start off with looking how the addition in our vector spaces behaves with respect to the ordering.

Lemma 2.1. *If $(X, \leq)$ is an ordered vector space and $w, x, y, z \in X$, then we have that*

$$w \leq x,\ y \leq z \Rightarrow w + y \leq x + z.$$

(Note: If $z \in X^+$ then we have $x \leq y \Rightarrow x \leq y + z$, which is the most common use of this lemma.)

Proof. From $w \leq x$ we get $w + y \leq x + y$ and from $y \leq z$ we get $y + x \leq z + x$. If we put these together we get $w + y \leq x + z$. □

We know that all subsequences of a convergent sequence in a metric space are themselves again convergent to the same point. We are interested in having a similar result for convergent nets in ordered vector spaces. There exists a formal definition of a subnet which generalizes the idea of a subsequence. For these subnets one can prove the desired property. In practice, however, the following approach is much more convenient for us. We begin with defining a cofinal subset of a directed set as done in [4].

Definition 2.2. If $(I, \leq)$ is a directed set and $J \subseteq I$, then J is cofinal in I iff

$$\forall \alpha \in I : \exists \beta \in J : \beta \geq \alpha.$$

The following property regarding cofinal subsets of directed sets is easily verified.

Lemma 2.3. *If $(I, \leq)$ is a directed set and $J \subseteq I$, then either J or $I \backslash J$ is cofinal.*

Proof. If J is cofinal there is nothing to prove, so let us assume that it is not cofinal. Then there exists an $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ implies that $\alpha \notin J$ and so $\alpha \in I \backslash J$. So for every $\beta \in I$ there exists an $\alpha \in I$ such that $\alpha \geq \beta$ and $\alpha \geq \alpha_0$. The latter inequality means that $\alpha \in I \backslash J$. So we can conclude that $I \backslash J$ is cofinal. □

The above lemma immediately yields the statement that we wanted. It will allow us to construct new nets out of old ones, by restricting to a cofinal subset of the directed set of our original net, and guarantee that it still order-converges to the same element. Hopefully the entries of the net share an extra property on this cofinal subset. Let $(X, \leq)$ be a partially ordered vector space.

Lemma 2.4. *If $(x_\alpha)_{\alpha \in I} \xrightarrow{O_i} x$ in X holds and $J \subseteq I$ is cofinal, then also $(x_\alpha)_{\alpha \in J} \xrightarrow{O_i} x$ holds. (for $i = 1, 2$)*

Proof. We will show that the above statement holds for $i = 1$. Let $(x_\alpha)_{\alpha \in I} \xrightarrow{O_1} x$. Then there exists a net $(y_\alpha)_{\alpha \in I}$ such that $(y_\alpha) \downarrow 0$ and $\forall \alpha \in I : -y_\alpha \leq x - x_\alpha \leq y_\alpha$. Now let $J \subseteq I$ be cofinal. We want to show that $(x_\alpha)_{\alpha \in J} \xrightarrow{O_1} x$ holds. It is clear that $-y_\alpha \leq x - x_\alpha \leq y_\alpha$ holds for all $\alpha \in J$ and that for $\alpha, \beta \in J$ it holds that $\alpha \geq \beta \Rightarrow y_\alpha \leq y_\beta$. So we want to show that $\inf\{y_\alpha : \alpha \in J\} = 0$. For all $\alpha \in J$ we know that $y_\alpha \geq 0$. Now assume $z \leq y_\alpha$ holds for all $\alpha \in J$ and let $\alpha_1 \in I$. Then there exists an $\alpha_2 \in J$ with $\alpha_2 \geq \alpha_1$ which means that $y_{\alpha_2} \leq y_{\alpha_1}$. So we see that $z \leq y_{\alpha_2} \leq y_{\alpha_1}$. Because $\alpha_1 \in I$ was chosen arbitrarily and $(y_\alpha)_{\alpha \in I} \downarrow 0$, it follows that $z \leq 0$. Therefore we see that $(y_\alpha)_{\alpha \in J} \downarrow 0$ and thus $(x_\alpha)_{\alpha \in J} \xrightarrow{O_1} x$ must hold.
In exactly the same way one can show the statement for $i = 2$. □

A statement that holds for a convergent sequence in $\mathbb{R}$ is the so called Squeeze Theorem. It tells us that, if two sequences both converge to a point and a third net lies between them, then the third net will also be convergent to the same point. We will now give an analog of the Squeeze Theorem in the setting of ordered vector spaces and order-convergence of nets.

Lemma 2.5. *If $(x_\alpha), (y_\alpha), (z_\alpha)_{\alpha \in I}$ are nets in X such that $(x_\alpha) \xrightarrow{O_i} y, (z_\alpha) \xrightarrow{O_i} y$ for a $y \in X$ and $\forall \alpha \in I : x_\alpha \leq y_\alpha \leq z_\alpha$, then also $(y_\alpha) \xrightarrow{O_i} y$ (for $i = 1, 2$).*

Proof. Let $i = 1$ and $(x_\alpha), (y_\alpha), (z_\alpha)_{\alpha \in I}$ be as stated above. Then there exist nets $(a_\alpha), (b_\alpha)_{\alpha \in I} \downarrow 0$ such that $\forall \alpha \in I : -a_\alpha \leq x_\alpha - y \leq a_\alpha$ and $-b_\alpha \leq z_\alpha - y \leq b_\alpha$. In particular we find

$$-(a_\alpha + b_\alpha) \leq -a_\alpha \leq x_\alpha - y \leq y_\alpha - y \leq z_\alpha - y \leq b_\alpha \leq a_\alpha + b_\alpha.$$

The previous lemma says us that $(a_\alpha + b_\alpha) \downarrow 0$ so we conclude that $y_\alpha \xrightarrow{0_1} y$.
With a similar argument we can provide a proof for the case $i = 2$. □

Note that if we have a net $(x_\alpha)_{\alpha\in I} \downarrow 0$, some element $x \in X$, and another net $(y_\alpha)_{\alpha\in I}$ such that $\forall \alpha \in I : y_\alpha \in [x, x + x_\alpha]$, then $(y_\alpha) \overset{O_i}{\to} x$ as follows by the previous lemma.

In the last part of this section we will look at O_i-open sets. We are in particular interested in translating and scaling O_i-open sets and establishing that they remain O_i-open after this. It turns out that this is in fact true. It follows easily from the fact that nets that descend to zero will still do so after being translated or scaled by a positive scalar.

Lemma 2.6. *If $(x_\alpha), (y_\alpha)_{\alpha\in I}$ are nets such that $(x_\alpha), (y_\alpha) \downarrow 0$ and $\lambda, \mu \in \mathbb{R}^+$, then $(\lambda x_\alpha + \mu y_\alpha)_{\alpha\in I} \downarrow 0$.*

Proof. We need to check the two properties given by definition 1.7:

(i) $\alpha \geq \beta \Rightarrow x_\alpha \leq x_\beta$ and $y_\alpha \leq y_\beta \Rightarrow \lambda x_\alpha \leq \lambda x_\beta$ and $\mu y_\alpha \leq \mu y_\beta \Rightarrow \lambda x_\alpha + \mu y_\alpha \leq \lambda x_\beta + \mu y_\beta$

(ii) $\forall \alpha \in I$ we have that $x_\alpha, y_\alpha \geq 0$ holds, and therefore $\lambda x_\alpha + \mu y_\alpha \geq 0$.
Now let $z \leq \lambda x_\alpha + \mu y_\alpha$ and $\eta := \max\{\lambda, \mu\}$ then we have that $z \leq \eta(x_\alpha + y_\alpha)$ and thus $\frac{1}{\eta}z \leq x_\alpha + y_\alpha$ which means that $\frac{1}{\eta}z - y_\alpha \leq x_\alpha$. If we now fix $\beta \in I$, then we obtain $\forall \alpha \geq \beta : \frac{1}{\eta}z - y_\beta \leq \frac{1}{\eta}z - y_\alpha \leq x_\alpha$. So from the fact that $\inf\{x_\alpha : \alpha \geq \beta\} = 0$ we see that $\frac{1}{\eta}z - y_\beta \leq 0$. And because $\beta \in I$ was chosen arbitrary we have that $\frac{1}{\eta}z - y_\beta \leq 0$ holds for all $\beta \in I$. Therefore $\frac{1}{\eta}z \leq y_\beta$ must hold for all $\beta \in I$ and because $\inf\{y_\beta : \beta \in I\} = 0$ we find that $\frac{1}{\eta}z \leq 0$. From which we can conclude that $z \leq 0$.

□

Lemma 2.7. *If $(x_\alpha)_{\alpha\in I}$ is a net in X, $(x_\alpha) \overset{O_i}{\to} x$, $y \in X$ and $\lambda \in \mathbb{R}$ then we have $(\lambda x_\alpha + y) \overset{O_i}{\to} \lambda x + y$. (for $i = 1, 2$)*

Proof. Let $(x_\alpha)_{\alpha\in I}$ be a net, $y \in X$, $\lambda \in \mathbb{R}$ and $(x_\alpha) \overset{O_1}{\to} x$. Then there exists a net $(y_\alpha)_{\alpha\in I}$ such that $(y_\alpha) \downarrow 0$ and $\forall \alpha \in I : -y_\alpha \leq x_\alpha - x \leq y_\alpha$. From this we see that $\forall \alpha \in I : -|\lambda| y_\alpha \leq \lambda x_\alpha - \lambda x \leq |\lambda| y_\alpha$ and therefore $-|\lambda| y_\alpha \leq \lambda(x_\alpha + y) - (\lambda x - y) \leq |\lambda| y_\alpha$ and so by the above Lemma $(\lambda x_\alpha + y) \overset{O_1}{\to} \lambda x + y$.
With the same argument we can prove this for the case $i = 2$. □

Corollary 2.8. *If $A \subseteq X$ is O_i-open, $y \in X$ and $\lambda \in \mathbb{R}$ then $A + y := \{a + y : a \in A\}$ and $\lambda A := \{\lambda a : a \in A\}$ are O_i-open.*

Proof. Let $A \subseteq X$ be O_i-open, $y \in X$, $\lambda \in \mathbb{R}$ and $(x_\alpha) \overset{O_i}{\to} x$ an convergent net in $(A + y)^c$. Then we see that $(x_\alpha - y) \overset{O_i}{\to} x - y$ holds by the previous lemma and $\forall \alpha \in I : x_\alpha - y \in A^c$. And because A^c is O_i- closed it follows that $x - y \in A^c$. This means that $x \in (A + y)^c$ and so $(A + y)$ is O_i-open. If we now take a net $(z_\alpha) \overset{O_i}{\to} z$ in $(\lambda A)^c$, then $(\frac{1}{\lambda}z_\alpha)_{\alpha\in I} \overset{O_i}{\to} \frac{1}{\lambda}z$ holds by the previous Lemma and this is a net in A^c. So again because A^c is O_i-closed we find that $\frac{1}{\lambda}z \in A^c$ and therefore $z \in (\lambda A)^c$. And this means that λA is O_i-open. □

3 Topologies induced by order convergence

We are now going to look at the topologies induced by O_1- and O_2-convergence in ordered vector spaces. Luxemburg and Zaanen study the topology induced by order convergence of sequences in Archimedean Riesz spaces in section 16 of [5]. They show that the complements of the sets that are closed under order convergence of sequences form a topology. As convergence of sequences is not rich enough to capture non-metrizable topologies, it is natural to consider topologies induced by order convergence of nets. We will conclude this chapter with proving that the topologies induced by O_1- and O_2-convergence do in fact coincide. Before we can start working up to that we will first need to prove the following theorem.

Theorem 3.1. *For an ordered vector space $(X, \leq)$ the set*

$$T_i := \{U \subseteq X : U \text{ is } O_i - \text{open}\}$$

is a topology on X. (for $i = 1, 2$)

Proof. There is nothing to check in order to conclude that $\emptyset, X \in T_i$. Now let A, B be O_i-closed and $(x_\alpha)_{\alpha\in I} \overset{O_i}{\to} x$ a convergent net with $x_\alpha \in A \cup B$ for all $\alpha \in I$. Define the set $J := \{\alpha \in I : x_\alpha \in A\}$. We know from Lemma 2.3 that either J or $I\backslash J$ is cofinal. If J is cofinal then by Lemma 2.4 $(x_\alpha)_{\alpha\in J} \overset{O_i}{\to} x$ holds and because A is O_i-closed it follow that $x \in A \subseteq A \cup B$. If $K := I\backslash J$ is cofinal, then we get that $(x_\alpha)_{\alpha\in K} \overset{O_i}{\to} x$ holds, because of Lemma 2.4, while $(x_\alpha)_{\alpha\in K}$ is a net in $B\backslash A$, which is a subset of B, which is O_i-closed. So now it follows that $x \in B \subseteq A \cup B$. So in either situation we find that $x \in A \cup B$ must hold and therefore $A \cup B$ is O_i-closed. Now let $\mathfrak{B}$ be a collection of O_i-closed subsets of X and $(x_\alpha)_{\alpha\in I} \overset{O_i}{\to} x$ a convergent net with $x_\alpha \in \cap\mathfrak{B}$ for all $\alpha \in I$. For each $B \in \mathfrak{B}$ $(x_\alpha)_{\alpha\in I}$ is a net in B, which is O_i-closed, so $x \in B$ must hold. Therefore $x \in \cap\mathfrak{B}$ must hold, which means that $\cap\mathfrak{B}$ is O_i-closed. We conclude that T_i is a topology for $i = 1, 2$. □

Now that we know that T_1 and T_2 are in fact topologies we can start proving that they are equal. We will do so by first defining another type of convergence, namely, the one induced by the topology T_1. The next definition is taken from [4] and is a natural generalisation for the case of sequences.

Definition 3.2. We say a net $(x_\alpha)_{\alpha\in I}$ in X T_1-**converges** to x in X iff

$$\forall U \in T_1, x \in U \ \exists\alpha_0 \in I : \alpha \geq \alpha_0 \Rightarrow x_\alpha \in U.$$

Notation: $(x_\alpha)_{\alpha\in I} \overset{T_1}{\to} x$

Our goal is now to show that O_2-convergence implies T_1-convergence. Once we know this it is easy to show that $T_1 = T_2$. It turns out that proving this implication is the hardest part of the proof. If we take an O_2-convergent net we know there exists a net that descends to zero that bounds our original net. This means that the net is contained within order intervals where the borders descend/ascend to the limit. Unfortunately this is not enough to conclude that the net enters and remains in each open set around the limit point. So we need to add structure to our open set to make sure it 'contains enough order intervals'. It turns out that the following definition captures this structure.

Definition 3.3. A subset $U \subseteq X$ is called an **O-neighbourhood** of $x \in X$ if the following holds

$$\forall(x_\alpha)_{\alpha\in I} \downarrow 0 \ \exists\alpha_0 \in I : [x - x_{\alpha_0}, x + x_{\alpha_0}] \subseteq U.$$

We will now show a relation between O_1-open sets and O-neighbourhoods. Which will allow us to give O_1-open sets the desired property spoken of before.

Lemma 3.4. *Let $U \subseteq X$ be O_1-open with $0 \in U$, then U is an O-neighbourhood of 0.*

Proof. Let $(x_\alpha)_{\alpha\in I} \downarrow 0$ be a net in X. Assume that $\forall\alpha \in I : [-x_\alpha, x_\alpha] \nsubseteq U$, then we can pick $y_\alpha \in [-x_\alpha, x_\alpha]\backslash U$ for all $\alpha \in I$. Then it is clear that $(y_\alpha)_{\alpha\in I} \overset{O_1}{\to} 0$ and because this is a net contained in U^c, which is O_1-closed, it follows that $0 \in U^c$. This contradicts our assumption and therefore there exists an $\alpha_0 \in I$ such that $[-x_{\alpha_0}, x_{\alpha_0}] \subseteq U$. □

Lemma 3.5. *For a net $(x_\alpha)_{\alpha\in I}$ in $(X,\leq)$ we have*

$$x_\alpha \overset{O_2}{\to} 0 \Rightarrow x_\alpha \overset{T_1}{\to} 0.$$

Proof. Let $(x_\alpha)_{\alpha\in I} \overset{O_2}{\to} 0$ be a net in X and let $U \subseteq X$ be O_1-open with $0 \in U$. Then there exists a net $(y_\beta)_{\beta\in J} \downarrow 0$ in X such that $\forall\beta \in J\ \exists\alpha_0 \in I : \alpha \geq \alpha_0 \Rightarrow -y_\beta \leq x_\alpha \leq y_\beta$. We know that U is an O-neighbourhood of 0 and therefore there exists a $\beta_0 \in J$ such that $[-y_\beta, y_\beta] \subseteq U$. Also there exists an $\alpha_0 \in I$ such that $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in [-y_\beta, y_\beta] \Rightarrow x_\alpha \in U$. Therefore the net (x_α) is eventually in U which was chosen arbitrarily. We conclude that $x_\alpha \overset{T_1}{\to} 0$ holds. □

As we have claimed in the beginning of this section the previous lemma enables us to prove that $T_1 = T_2$ holds. It is more convenient to restate that equality in an equivalence of closed sets.
Theorem 3.6. *For a subset $A \subseteq X$ we have that A is O_1-closed $\Leftrightarrow$ A is O_2- closed.*

Proof. $\Leftarrow$ is trivial because $x_\alpha \overset{O_1}{\to} x \Rightarrow x_\alpha \overset{O_2}{\to} x$.
$\Rightarrow$: Let $A \subseteq X$ be O_1-closed and $(x_\alpha)_{\alpha\in I} \overset{O_2}{\to} x$ be a net in A for some $x \in X$. Then by Lemma 2.7 we see that $(x_\alpha - x)_{\alpha\in I} \overset{O_2}{\to} 0$. From Lemma 3.5 it follows that $(x_\alpha - x)_{\alpha\in I} \overset{T_1}{\to} 0$ and thus, by Corollary 2.8, that $(x_\alpha)_{\alpha\in I} \overset{T_1}{\to} x$. Now assume that $x \notin A$ then A^c is O_1-open and $x \in A^c$. Thus by defintion of T_i-convergence $\exists\alpha_0 \in I : \alpha \geq \alpha_0 \Rightarrow x_\alpha \in A^c$. But this is a contradiction because (x_α) is a net in A. Thus we know that $x \in A$ must hold and with that we can conclude that A is O_2-closed. □

Corollary 3.7. $T_1 = T_2$. *That is, O_1- and O_2-convergence induce the same topology.*

4 Characterization of order-open sets

Now that we know that O_1- and O_2-convergence both induce the same topology, we can try to gain more information on the open sets of this topology. Because every O_1-open set appears to be O_2-open and vice versa we will call these sets **order-open sets** from now on. In this section we will give insight in the properties of an order-open set. Our first result will be that a set $U \subseteq X$ is order-open if and only if it is an open neighbourhood of all $x \in U$.
Let $(X,\leq)$ be a partially ordered vector space. First we will observe that open neighbourhoods are preserved under translation.
Lemma 4.1. *If $U \subseteq X$ is an O-neighbourhood of $x \in X$, $y \in X$, then $U + y$ is an O-neighbourhood of $x + y$.*

Proof. Let $U \subseteq X$ be an O-neighbourhood of $x \in X$, y be an element of X and $(x_\alpha)_{\alpha\in I} \downarrow 0$ a net in X. Then we know that by definition there exists an $\alpha \in I$ such that $[x - x_\alpha, x + x_\alpha] \subseteq U$. Therefore it follows that $[y + x - x_\alpha, y + x + x_\alpha] \subseteq U + y$. From this we obtain that $U + y$ is an O-neighbourhood of $x + y$. □

Theorem 4.2. *For a subset $U \subseteq X$ the following equivalence holds: U is order-open $\Leftrightarrow$ U is an O-neighbourhood of all $x \in U$.*

Proof. $\Rightarrow$: Let $U \subseteq X$ be order-open, $x \in U$ and assume that U is not an O-neighbourhood of x. Then by definition we know that $\exists(x_\alpha)_{\alpha\in I} \downarrow 0 : \forall\alpha \in I\ [x - x_\alpha, x + x_\alpha] \nsubseteq U$. So we can pick for every $\alpha \in I$ a $y_\alpha \in [x - x_\alpha, x + x_\alpha]\backslash U$. Now $(y_\alpha)_{\alpha\in I}$ is a net in U^c with $(y_\alpha) \overset{O_1}{\to} x$. Because U^c is an order-closed set, it follows that $x \in U^c$ must hold, which is a contradiction. Therefore U must be an O-neighbourhood of x.
$\Leftarrow$: Let $U \subseteq X$ be such that U is an O-neighbourhood of all $x \in U$. Let $(x_\alpha)_{\alpha\in I} \overset{O_1}{\to} x$ be a net in

U^c for some $x \in X$. Then there exists a net $(y_\alpha)_{\alpha\in I} \downarrow 0$ such that for all $\alpha \in I : -y_\alpha \leq x - x_\alpha \leq y_\alpha$. Assume that $x \in U$ holds, then we know that $\exists \alpha \in I$ such that $[x - y_\alpha, x + y_\alpha] \subseteq U$. Note that $x_\alpha \in [x - y_\alpha, x + y_\alpha]$ and therefore $x_\alpha \in U$, which is in contradiction with our assumption. We conclude that $x \in U^c$ must hold and therefore U must be order-open. □

The rest of the results that we will discuss in this section only hold in partially ordered vector spaces that satisfy an additional property.

Definition 4.3. An ordered vector space $(X, \leq)$ is called **Archimedean** if it satisfies the property

$$\forall n \in \mathbb{N} : ny \leq x \Rightarrow y \leq 0.$$

On first sight it might not be apparent why this property is useful to us. The next Lemma will give us a very useful property of Archimedean ordered vector spaces.

Lemma 4.4. *If $(X, \leq)$ is Archimedean, $y \in X^+$ and $Y \subseteq \mathbb{R}$ then we have the following implication*

$$\inf(Y) = 0 \Rightarrow \inf\{\alpha y : \alpha \in Y\} = 0.$$

Note that the infima are in different spaces.

Proof. Assume that $\inf(Y) = 0$ holds and note that that means that for all $n \in \mathbb{N}$ there exists an $\alpha \in Y$ such that $\alpha \leq \frac{1}{n}$. So if we take $y \in X^+$ then the latter inequality yields $\alpha y \leq \frac{1}{n}y$. Now let $z \in X$ such that $\forall \alpha \in Y$ we have $z \leq \alpha y$, then we also have that $\forall n \in \mathbb{N} : z \leq \frac{1}{n}y$. Therefore the Archimedean property of X tells us that $z \leq 0$ must hold. So it is sufficient to note that for all $\alpha \in Y$ $\alpha y \geq 0$ holds, because $\alpha \geq 0$, to conclude that indeed $\inf\{\alpha y : \alpha \in Y\} = 0$. □

Let for the rest of this section $(X, \leq)$ be an Archimedean ordered vector space.
We have the following consequence of Theorem 4.2.

Corollary 4.5. *For an order-open subset $U \subseteq X$ the following property holds*

$$\forall x \in U : \forall y \in X^+ : \exists \lambda \in \mathbb{R}^+ \text{ such that } [x - \lambda y, x + \lambda y] \subseteq U.$$

Proof. Let $U \subseteq X$ be order-open, then by Theorem 4.2 it is an O-neighbourhood of all $x \in U$. Let $x \in U$ and $y \in X^+$ be given. We define $I := \mathbb{R}^+$ and order it as follows: $\lambda \geq_I \mu \Leftrightarrow \lambda \leq_{\mathbb{R}} \mu$. Define for all $\alpha \in I$ $y_\alpha := \alpha y$ then is $(y_\alpha)_{\alpha\in I}$ a net that descends to 0 by Lemma 4.4. Therefore there exists an $\alpha \in I$ such that $[x - y_\alpha, x + y_\alpha] \subseteq U$, which yields that $[x - \lambda y, x + \lambda y] \subseteq U$. □

In case of a metric space open sets have the useful characterization that for every element x in the set there exists a radius $r > 0$ such that the open ball around x with radius r is still contained within the original set. We will now try to find a similar property for subsets of X can satisfy which is equivalent to being an order-open set.

We turn back to the convergence that is induced by the toplogies that were induced by O_1- and O_2-convergence, which we called T_1- and T_2-convergence. Knowing that the topologies coincide, we will now just speak of $T-$convergence. In the last part of this section we will show that this convergence is in its turn equivalent to O_2-convergence, under the extra condition that $(X, \leq)$ has a so called strong unit, which we will assume for the rest of the section.

Definition 4.6. An element $e \in X$ is a **strong unit** of $(X, \leq)$ if

(i) $e \geq 0$

(ii) $\forall x \in X : \exists \lambda \in \mathbb{R} : x \leq \lambda e$

Note that $(ii) \Rightarrow \forall x \in X : \exists \lambda \in \mathbb{R}^+$ such that $x \geq -\lambda e$, hence
$\forall x \in X : \exists \lambda \in \mathbb{R}^+$ such that $-\lambda e \leq x \leq \lambda e$ holds.

With this strong unit we construct a radius of $U \subseteq X$, which will be the smallest number λ that satisfies $U \subseteq [-\lambda e, \lambda e]$. Our aim is to get a similar situation as in metric spaces, where open balls centered at x form a neighbourhood base at x of the topology. But obviously we can only find such a radius for bounded set which we will define next.

Definition 4.7. A subset $U \subseteq X$ is called **bounded** if there exists a $\lambda \in \mathbb{R}^+$ such that

$$\forall x \in U : -\lambda e \leq x \leq \lambda e.$$

Let U be a non-empty subset of X, not necessarily bounded, and fix $x \in U$. We can define for all $y \in U$ the real number $\lambda_y := \inf\{\lambda \in \mathbb{R}^+ : -\lambda e \leq x - y \leq \lambda e\}$. The infimum exists because the set is non-empty by the defintion of a strong unit and is bounded from below by 0. If U is bounded we can also define $\lambda_U := \sup\{\lambda_y : y \in U\}$. To see that this supremum exists, first note that $\{\lambda_y : y \in U\}$ is not empty. Secondly, there exists a $\lambda \in \mathbb{R}^+$ such that $\forall y \in U : -\lambda e \leq y \leq \lambda e$, so this also holds for y equal to the fixed $x \in U$. From this we obtain that $\forall y \in U$ we have $-2\lambda e \leq x - y \leq 2\lambda e$, so 2λ is an upper bound of the set. Note that λ_U depends on the choice of $x \in U$. Since we will be using this radius to show equivalence of two types of convergences, the point x will just be the limit point of the given arbitrary net. In the following lemmas we will construct a net that maps order-open, bounded sets U to $\lambda_U e$. We will show that this net descends to zero.

Lemma 4.8. *If $I := \{U \subseteq X : U$ is order-open, bounded and $x \in U\}$ ordered by $U \geq V \Leftrightarrow U \subseteq V$, then $(I, \leq)$ is a directed set.*

Proof. It is clear that $\geq$ is transitive in I, because $\subseteq$ is transitive on the power set of X. If $U, V \in I$ then we have that $W := U \cap V \in I$, because W is order-open by the fact that $\{U \subseteq X : U$ is order-open$\}$ is a topology and W is bounded with bounding scalar $\lambda_W := \min\{\lambda_U, \lambda_V\}$. Because $W \subseteq U, V$ it follows that $W \geq U, V$ in I. □

Lemma 4.9. *If U is an order-open and bounded subset of X containing x and $n \in \mathbb{N}$, then there exists an order-open and bounded subset U_n of X containing x such that $\lambda_{U_n} \leq \frac{1}{n}\lambda_U$.*

Proof. Let $U \subseteq X$ be bounded, order-open, containing x and let $n \in \mathbb{N}$. Then by applying Corollary 2.8 multiple times we find that $U - x$ is order-open, $\frac{1}{n}(U - x)$ is order-open and finally that $U_n := \frac{1}{n}(U - x) + x$ is order-open. Note that $x \in U_n$ holds. We want to show that U_n is bounded and satisfies $\lambda_{U_n} \leq \frac{2}{n}\lambda_U$. Let $y \in U_n$ then there exists a $z \in \frac{1}{n}(U - x)$ such that $y = z + x$. This means $\exists u \in U$ such that $z = n(u - x)$ and therefore $y = nu + (1 - n)x$. From this we obtain that $u = \frac{1}{n}(y - (1 - n)x)$ and we know that $-\lambda_U \leq u - x \leq \lambda_U$ so we get $-\lambda_U \leq \frac{1}{n}(y - (1 - n)x) - x \leq \lambda_U$. We can write this as $-\lambda_U \leq \frac{1}{n}(y - x) \leq \lambda_U$ and so $-n\lambda_U \leq y - x \leq n\lambda_U$. From the above observation we can conclude that indeed $\lambda_{U_n} \leq \frac{1}{n}\lambda_U$ holds. □

Theorem 4.10. *If we let $\forall U \in I$ $y_U := \lambda_U e$ then we get $(y_U)_{U \in I} \downarrow 0$.*

Proof. Let us note first that for all $U, V \in I$ we have $U \geq V \Rightarrow U \subseteq V \Rightarrow \lambda_U \leq \lambda_V$. Secondly we observe that $\lambda_U \geq 0$ for all $U \in I$ by construction. Let us now fix a $U \in I$. Then for all $n \in \mathbb{N}$ there exists an $U_n \in I$ with $\lambda_{U_n} \leq \frac{1}{n}\lambda_U$ by the previous lemma. This means that $\inf\{\lambda_U : U \in I\} = 0$ and because $(X, \leq)$ is Archimedean and with the aid of Lemma 4.4 we get that $\inf\{y_U : U \in I\} = 0$. □

Corollary 4.11. *For a net $(x_\alpha)_{\alpha \in J}$ in $(X, \leq)$ we have the following implication*

$$x_\alpha \xrightarrow{T} x \Rightarrow x_\alpha \xrightarrow{O_2} x.$$

Proof. Let $(x_\alpha)_{\alpha\in J}$ be a net in $(X,\leq)$ with $x_\alpha \overset{T}{\to} x$. Then by definition for all $U \in I$ there exists an $\alpha_0 \in J$ such that $\alpha \geq \alpha_0 \Rightarrow x_\alpha \in U$. We know that $\forall x_\alpha \in U : -\lambda_U e \leq x_\alpha - x \leq \lambda_U e$. So we have that $\forall U \in I : \exists \alpha_0 \in J$ such that $\alpha \geq \alpha_0 \Rightarrow -y_U \leq x_\alpha - x \leq y_U$. Therefore we can conclude that $x_\alpha \overset{O_2}{\to} x$ holds, by the previous theorem. □

Note that by Lemma 3.5 we find that O_2- and T-convergence are equivalent. This means that if we start with O_2-convergence, then induce the topology T_2 and then go back to $T-$convergence we end up with the same convergence.

5 Full Neighbourhoods

In the previous section we have seen that every subset U of a partially ordered vector space is order-open if and only if it is an $O-$neighbourhood of each element $x \in U$. When studying convergence of nets one is interested in the local properties of a space around a point. Therefore we are going to investigate $O-$neighbourhoods further in this section. It will help us in better understanding the local topological properties of a partially ordered vector space around a point. Without loss of generality, we will only consider the point 0 for convenience, because all concepts defined so far are translation invariant. Let $(X,\leq)$ be a partially ordered vector space.

Our goal is to add structure to order-open sets $U \subseteq X$ containing 0. We will construct a subset of U that will satisfy a useful additional property defined below. Unfortunately, this subset will not itself be order-open but it will be an $O-$neighbourhood of 0.
Definition 5.1. A subset $A \subseteq X$ is called **full** if the following holds

$$a \in A^+ \Rightarrow [-a,a] \subseteq A.$$

Recall that $A^+ = \{a \in A : a \geq 0\}$.

Let U be a subset of X with $0 \in U$. We construct the following set:

$$U' := \{x \in U^+ : [-x,x] \subseteq U\}$$

$$V(U) := \bigcup_{x\in U'} [-x,x].$$

As announced earlier, this set $V(U)$ will be a full $O-$neighbourhood of 0. Note, though, that an element $y \in V(U)$ need to be comparible with 0. We only know there exists a $z \in U^+$ such that $y \in [-z,z]$.
Lemma 5.2. *Let U be a subset of X. Then the set $V(U)$ is full.*

Proof. Assume that $z \in V(U)^+$ holds. We know that $z \geq 0$ and there exists a $y \in U^+$ such that $z \in [-y,y] \subseteq U$. We obtain $-y \leq -z \leq 0 \leq z \leq y$ and therefore $[-z,z] \subseteq [-y,y] \subseteq U$ holds. By construction of U' we see that $z \in U'$. This yields that $[-z,z] \subseteq V(U)$ from which we can conclude that $V(U)$ is full. □

Lemma 5.3. *Let $U \subseteq X$ be order-open with $0 \in U$. The set $V(U)$ is an $O-$neighbourhood of 0.*

Proof. Let $U \subseteq X$ be order-open with $0 \in U$ and $(x_\alpha)_{\alpha \in I} \downarrow 0$ be a net in X. From Theorem 4.2 we obtain that U is an $O-$neighbourhood of 0. Therefore there exists an $\alpha_0 \in I$ such that $[-x_{\alpha_0}, x_{\alpha_0}] \subseteq U$. The net (x_α) descends to 0 so in particular $x_{\alpha_0} \geq 0$. So by construction of $V(U)$ we see that $[-x_{\alpha_0}, x_{\alpha_0}] \subseteq V(U)$ holds. We conclude that $V(U)$ is an $O-$neighbourhood of 0. □

We can conclude that every order-open set $U \subseteq X$ with $0 \in U$ contains a subset $V(U)$ that is a full $O-$neighbourhood of 0. Furthermore, we can define when a set U is full around x with $x \in U$ and adjust our construction of $V(U)$ to be centered around the same element x. Because these properties are translation invariant, we obtain that every order-open set $U \subseteq X$ contains a full $O-$neighbourhood of any element $x \in U$.

Recall that a $T-$neighbourhood of $x \in X$ is a set U that contains an order-open set that contains x. Thus we arrive at the following proposition.

Proposition 5.4. *Let $x \in X$. Each $T-$neighbourhood of x contains a full $O-$neighbourhood of x.*

Conclusion

In this thesis we have seen that O_1- and O_2-convergence in a partially ordered vector space induce the same topology. This result holds in full generality; no conditions like being Archimedean or being directed are needed. Consequently, they also induce the same order continuous operators with respect to the convergence and the same order continuous norms as mentioned in [1, 3]. This means that, if one wants to prove a result about O_2-convergence in one of these contexts, one can restrict oneself to the more simple O_1-convergence. A question that I have not been able to answer is the following. Is an O_2-convergent net eventually O_1-convergent, i.e.

$$(x_\alpha)_{\alpha \in I} \overset{O_2}{\to} x \Rightarrow \exists \alpha_0 \in I : (x_\alpha)_{\alpha \in I_{\geq \alpha_0}} \overset{O_1}{\to} x?$$

This implication is stronger than the statement that a set is O_1-open if and only if it is O_2-open.

We have also studied the topology, T, induced by O_1- and O_2-convergence. It turned out that each order-open set is an $O-$neighbhourhood around each point in this set. Furthermore, the convergence induced by the topology T is equivalent to O_2-convergence in Archimedean ordered vector space with a strong unit.

Finally, we have investigated $O-$neighbourhoods in partially ordered vector spaces. It turns out that every order-open set $U \subseteq X$ contains a full $O-$neighbourhood of any element $x \in U$. In the proving this statement we have given a construction of the set $V(U)$ which satisfies the desired properties. It is still left to investigate whether this set $V(U)$ is order-open whenever U is order-open.

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