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Composition Multiplication Operators on pre-Riesz spaces.

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Contents

Abstract

In the theory of operators on Riesz spaces an important result states that Riesz homomorphisms on a $C(\Omega)$-space are composition multiplication operators. Our aim is to extend this theorem to, not necessarily Riesz, subspaces of such a $C(\Omega)$-space. The main result entails the following, Riesz* homomorphisms on a pointwise order dense subspace X of $C(\Omega)$ are composition multiplication operators. Furthermore, we use this result to find additional results on Riesz* homomorphisms on these subspaces. We will exhibit, for example, that the inverse of a bijective Riesz* homomorphism on X is again a Riesz* homomorphism. As another corollary of the result we characterize which Riesz* homomorphisms on X are even complete Riesz homomorphisms. Results developed on pointwise order dense subspaces of $C(\Omega)$ can be applied in Sobolev space theory.
As an analogy of the above we will develop a similar theory on subspaces of L^p for a finite measure space. Most results carry over easily from the $C(\Omega)$ case. We will investigate difference in structure of Riesz* homomorphisms between these two type of space.

Introduction

In the theory of Riesz spaces different classes of operators and their properties have been studied extensively. In particular, Riesz homomorphisms are a main focus in this study. On well-known function spaces these operators are composition multiplication operators. Contents of this result are given in [1]. We cite Theorem 4.25 from this book adopted in notation and terminology that will be used in this text.

Theorem. *Let Ω and S be compact Hausdorff spaces. A positive operator $T : C(\Omega) \to C(S)$ is a Riesz homomorphism if and only if there exist a map $\pi : \Omega \to S$ and some weight function $\eta \in C(S)$ such that we have*

$$(Tf)(s) = \eta(s)f(\pi(s)) \qquad f \in C(\Omega), s \in S. \tag{1}$$

Moreover, in this case, $\eta = T\mathbb{1}_\Omega$ and the map π is uniquely determined and continuous on the set $\{\eta > 0\}$.

An operator $T : C(\Omega) \to C(S)$ is called a *composition multiplication operator* if it satisfies (1). Our aim is to generalize the above characterization theorem to subspaces of $C(\Omega)$. We do not restrict our study to Riesz subspaces Y of $C(\Omega)$. On such a space Y we do not have Riesz homomorphisms.

One approach to avoid this, is to instead study *disjointness preserving operators.* Here two elements f and g of a Riesz space are said to be *disjoint* if $|f| \wedge |g| = 0$. A positive operator between two Riesz spaces is a Riesz homomorphism if and only if it is disjointness preserving. Therefore, the above theorem tells us that all positive disjointness preserving operators between $C(\Omega)$ and $C(S)$ are of composition multiplication type. The notion of disjointness elements has been extended to partially ordered vector spaces that are not necessarily Riesz spaces in [8]. Are positive disjointness preserving operators on subspaces of $C(\Omega)$ also of composition multiplication type? Given two positive elements f and g in a Riesz space E one can define $u = f - (f \wedge g)$ and $v = g - (f \wedge g)$. This u and v are now positive elements of E that are also disjoint. The existence of this construction gives disjointness preserving operators between Riesz space a lot of structure, for example that all positive disjointness preserving operators are Riesz homomorphisms. It is not clear that disjointness preserving operators enjoy the same structure on subspaces of $C(\Omega)$, as the above construction makes critical use of taking the infimum of two arbitrary elements.

Therefore, our approach is to look at operators that are extendable to a Riesz homomorphism on a Riesz space. Van Haandel has developed a theory on this topic in his PhD. thesis, [9]. He introduces the notion of *pre-Riesz* spaces which are spaces that can be embedded as an order dense subspace of a Riesz space, X^ρ, the *Riesz completion.* On these spaces he has characterized which operators extend to a Riesz homomorphism on the Riesz completion and he has called them *Riesz* homomorphisms.* These definitions and results will be discussed in the Preliminaries section. Our main result, as discussed in chapter 3, is that Riesz* homomorphisms on *pointwise order dense* subspaces of $C(\Omega)$ are composition multiplication operators. Here a subset $X \subset C(\Omega)$ is called pointwise order dense if for all $f \in C(S)$ and $s \in S$ we have $\inf\{g(s) : g \in X, g \geq f\} = f(s)$. Such a space X is, in particular, a pre-Riesz space so we can apply the theory of Van Haandel. It turns out that for any operator being extendable to a Riesz homomorphism on $C(\Omega)$ is equivalent with being of composition multiplication type. In the remainder of this text we will exhibit consequences and analogies, in other function spaces, of this result.

An operator $T : X \to Y$ between partially ordered vector spaces is called a *complete Riesz homomorphism* if for all $A \subset X$ with $\inf A = 0$ one has $\inf T(A) = 0$ in Y. Obviously all complete Riesz homomorphisms are in particular Riesz* homomorphisms. Using that all Riesz* homomorphisms are of composition multiplication type we can impose conditions on the operator

under which the converse holds aswell. If T is a compostion multiplication operator with maps η and π as given in (1), then if π is *semi open* on $\{\eta > 0\}$ the operator T is a complete Riesz homomorphism, where semi open is a weaker condition than being open. We exhibit interactions between properties of the composition map on $\{\eta > 0\}$ and the operator T being a complete Riesz homomorphism.
As an application of this theory we analyse operators on Sobolev Spaces. We are able to embed these spaces in the frame of our theory. In the sense that we can view them as pre-Riesz subspaces of a particular $C(\Omega)$ space which are pointwise order dense. As these Sobolev spaces are Riesz spaces we can conclude by the theory developed that all Riesz homomorphisms on these spaces are of composition multiplication type.
Similar results as the above theorem also hold for L^p spaces. Which is another example of a function space that is a Riesz space. One has to be a bit more careful with composition multiplication operators compared to the continuous function case. This is due to the fact that elements of a L^p are not determined in every point. Due to results from Rodriguez-Salinas in [13] we are able to give a sensible definition of composition multiplication operators on L^p spaces. Several results from the continuous function case then have an obvious analogy for the L^p spaces. Moreover, in this case the notions of order dense and pointwise order dense coincide. So all of the theory holds on order dense subspaces of L^p.

1 Preliminaries

We are interested in partially ordered vector spaces X that can be embedded as an order dense subspace in a Riesz space, as this will allow us to extend the operators of interest to Riesz homomorphisms on the Riesz space in which the space lies order-dense. Recall the following terminology on partially ordered vector spaces as, for example, given in [6].

Definition 1.1. Let Y be a Riesz space and $X \subset Y$ a partially ordered vector space.

(i) X is called **directed** if for all $f, g \in X$ there exists an $h \in X$ with $h \geq f$ and $h \geq g$.

(ii) X is called **Archimedean** if for all $f, g \in X$ with $nx \leq y$ for all $n \in \mathbb{N}$ ones has $f \leq 0$.

(iii) X is called **order-dense in** Y if for all $f \in Y$ one has $f = \inf\{g \in X : g \geq f\}$.

Van Haandel has characterized exactly which subspaces can be embedded in a Riesz space in the way described above in [9] and has called these spaces pre-Riesz.

Definition 1.2. X is called a **pre-Riesz** space if for every $f, g, h \in X$ such that $\{f+g, f+h\}^u \subset \{g, h\}^u$ one has $f \geq 0$.

Theorem 1.3 (Van Haandel)**.** *The following statements are equivalent.*

(i) X is pre-Riesz.

(ii) There exists a vector lattice Y and a bipositive linear map $i : X \to Y$ such that $i(X)$ is order dense in Y and generates Y as a Riesz space.

Moreover, all spaces Y in (ii) are isomorphic as Riesz spaces.

We call the space Y in the above theorem the *Riesz completion* of X. As all spaces Y with this property are isomorphic we can talk about *the* Riesz completion and denote it by X^ρ. As these pre-Riesz space are of great important in this text we want a convenient way of showing when a space is pre-Riesz. This result is due to Van Haandel.

Theorem 1.4 (Van Haandel)**.** *Every pre-Riesz space is directed and every directed Archimedean partially ordered vector space is pre-Riesz.*

For the rest of this section let X be a pre-Riesz space. As mentioned earlier, we want to look at operators on X that we can extend to its Riesz completion so that we can apply the known theory of operators on Riesz space. Van Haandel has shown exactly which operators can be extended this way to Riesz homomorphisms.

Definition 1.5. Let X be a partially ordered vector space and $T : X \to X$ an operator. Then T is called a **Riesz* homomorphism** if for all $f, g \in X$ one has $T(\{f, g\}^{ul}) \subseteq \{Tf, Tg\}^{ul}$.

We observe here that every Riesz* homomorphism is in particular positive, which is a necessary property to be extendable to a Riesz homomorphism.

Theorem 1.6 (Van Haandel [9], Theorem 5.6, p.29)**.** *Let X and Y be pre-Riesz spaces and $T : X \to Y$ a linear operator. Then there exists a linear Riesz homomorphism $T^\rho : X^\rho \to Y^\rho$ that extend T if and only if T is a Riesz* homomorphism.*

Let X be a pre-Riesz subspace of $C(S)$ with S a compact Hausdorff space and $T : X \to X$ a Riesz* homomorphism. We can then extend T to a Riesz homomorphism T_ρ on the Riesz completion X^ρ. Next we want to extend this T_ρ to a Riesz homomorphism on $C(S)$ so we can apply the theorem stated in the introduction to conclude that T is a composition-multiplication operator. In order to make this extension we need X^ρ to be a Riesz subspace of $C(S)$. So we will show below that this is the case if X is order-dense in $C(S)$.

Lemma 1.7. *Let Y be a Riesz space and $X \subset Y$ an order-dense pre-Riesz subspace. Then the Riesz completion of X is a Riesz subspace of Y, i.e., $X^\rho \subset Y$.*

Proof. Let Z be the Riesz subspace of Y generated by X. As X is order-dense in Y it is, in particular, order-dense in Z. The unicity of the Riesz completion in Theorem 1.3(ii) shows now that $X^\rho = Z \subset Y$. □

As we have discussed above, it is important that our space X is order-dense in $C(S)$. In the case that X is a Riesz subspace of $Y = C(S)$, X is usually said to be order-dense in Y if for all $f \in Y$, $f > 0$, there exists a $g \in X$ with $0 < g \leq f$. (Here we use the notation $f > g$ for $f \geq g$ and $f \neq 0$.) It is then shown in [4, p. 34] that if Y is Archimedean, which is true in the case when $Y = C(\Omega)$, that the above property is equivalent to each positive $f \in Y$ being the supremum of all positive $g \in X$ below it. The latter is equivalent to X being order-dense in Y as in the sense of Definition 1.1. In our case, however, we only assume X to be pre-Riesz. We take the usual formulation of order-denseness of Riesz subspaces as the definition of being pervasive in Y. This corresponds to the definition in [7].

Definition 1.8. Let X be a partially ordered subspace of a Riesz space Y. X is called **pervasive** in Y if for all positive $f \in Y$, $f \neq 0$, there exists a $g \in X$ such that $0 < g \leq f$.

We can now adopt the proof of Theorem 1.34 on page 31 of [4] to give the following characterization of pervasive subspaces. The only issue in adopting the proof that needs some consideration is the one supremum that is taken. However, it is the supremum of two elements of Y which is still a Riesz space. This yields the proof of the following theorem.

Theorem 1.9. *Let Y be an Archimedean Riesz space and $X \subset Y$ a partially ordered vector subspace. Then X is pervasive in Y if and only if for all positive $f \in Y$ we have*

$$f = \sup\{g \in X : 0 \leq g \leq f\}.$$

If, in addition, Y is a Dedekind complete Riesz space we have a slightly stronger result for pervasive subspaces of Y which deals with the existence of infima.

Lemma 1.10. *Let Y be a Dedekind complete Riesz space and $X \subset Y$ a pervasive partially ordered vector subspace. For any $A \subset X$ where $\inf A = f$ holds in X we also have that $\inf A$ exists in Y and equals f.*

Proof. Let X and Y be as stated above and $A \subset X$ satisfying $\inf A = f$, $f \in X$. As Y is Dedekind complete, the infimum of A exists in Y as it is bounded from below by f. Let g be a lower bound of A in Y. Suppose that $g \leq f$ does not hold. Let $e := f \vee g$ in Y, then it is a lower bound of A and $e > f$ holds. We can find an $h \in X$, $f < h \leq e$ as $f \in X$ and X is pervasive. ($\exists h' \in X : 0 < h' \leq h - f, h = h' + f \in X$.) The existence of such an $h \in X$ contradicts that $\inf A = f$ holds in X. Therefore, $g \leq f$ must hold and we can conclude that $\inf A = f$ holds in Y. □

2 Spaces of continuous functions

For the rest of this chapter let Ω and S be compact Hausdorff spaces, $C(\Omega)$ and $C(S)$ the spaces of all real-valued continuous functions on Ω and S respectively with the usual ordering. Unless stated otherwise $X \subseteq C(\Omega)$ and $Y \subset C(S)$ are partially ordered vector subspaces. Our main goal of this section is to generalize the characterization theorem of Riesz homomorphisms from [1, Theorem 4.25] to subspaces of $C(\Omega)$. We are looking for a contidition that we can impose on X that will guarantee that all Riesz* homomorphisms on X satisfy equation (1) for all $f \in X$. It turns out that the following property is exactly what we are looking for.

Definition 2.1. X is called **pointwise order dense** if for all $s \in S$ and $f \in C(\Omega)$ we have

$$\inf\{g(s) : g \in X, g \geq f\} = f(s).$$

For the rest of this section we are only interested in subspaces $X \subset C(\Omega)$ which are pointwise order dense. So we start out with a lemma that helps us find subspaces of $C(\Omega)$ which are pointwise order dense.

Lemma 2.2. *If X is norm dense in $(C(\Omega), \|.\|_\infty)$ and contains the constant functions, it is also pointwise orde dense.*

Proof. Suppose X is norm dense and let $f \in C(\Omega)$ be given. We can now find $f_n \in X$ with $\|f_n - f\|_\infty \to 0$, say $\|f_n - f\|_\infty \leq \frac{1}{n}$. Now we define $g_n := f_n + \frac{1}{n}$, as X contains the constant functions, these g_n are in X as well. Also, we have $g_n \geq f$ and

$$\|f - g_n\|_\infty \leq \|f - f_n\|_\infty + \|f_n - g_n\|_\infty \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.$$

Hence, we have found a sequence (g_n) in X above f which converges uniformly, so in particular pointwise, to f. So X is pointwise order-dense. □

Example 2.3. The following subspaces of $C[0,1]$ are pointwise order dense due to the above lemma.

(i) $C^1[0,1]$, the space of continuously differentiable functions.

(ii) Polyn$[0,1]$, the polynomials on $[0,1]$.

(iii) The space of all piecewise-linear functions on $[0,1]$.

All of the above examples are norm-dense subspace of $C[0,1]$. The three-dimensional space Polyn$_2[0,1]$ of all polynomials of degree up to 2 is not norm-dense in $C[0,1]$, however, it is pointwise order-dense, as follows from arguments in Example 4.4 in [6].

It is interesting to note that in the proof of the above mentioned Example 4.4 in [6] it is shown that Polyn$_2[0,1]$ is order dense in $C[0,1]$ through showing it is pointwise order dense. As obviously the former implies the latter.

2.1 Characterization of Riesz* homomorphisms

We wish to characterize Riesz* homomorphisms on pointwise order-dense subspaces of $C(\Omega)$. So we start with a lemma linking this property to other useful properties of partially order vector subspaces of a Riesz space.

Lemma 2.4. *If X is pointwise order dense, then X is a pre-Riesz space and X is majorizing and order-dense in $C(\Omega)$.*

Proof. Suppose that X is pointwise order dense. It follows immediately that X must be majorizing. This means X is also directed. As it is also Archimedean it follows that X is pre-Riesz by Theorem 1.4. As discussed earlier, it follows from Definition 2.1 that pointwise order-dense implies order-dense which concludes the proof. $\square$

Due to Lemma 2.4 if X is pointwise order dense in $C(\Omega)$, then it is, in particular, a pre-Riesz space. Hence, X has a Riesz completion, X^ρ which by Lemma 1.7 is a Riesz subspace of $C(\Omega)$. Next we will show that all Riesz* homomorphism on a pointwise order dense subspace of $C(\Omega)$ are composition multiplication operators. Our strategy is to use the Dedekind completeness of $\mathbb{R}$ to be able to extend our operator to a Riesz homomorphism on $C(\Omega)$ and apply the Theorem stated in the introduction.

Theorem 2.5. *Suppose X and Y are both pre-Riesz spaces and X a majorizing and order-dense subspace of $C(\Omega)$. If $T : X \to Y$ is a Riesz* homomorphism, then there exist $\eta : S \to \mathbb{R}_+$ and $\pi : S \to \Omega$ such that*

$$(Tf)(s) = \eta(s)f(\pi(s)) \qquad f \in X, s \in S.$$

Moreover, if X is pointwise order dense in $C(\Omega)$, then we know in addition that η is continuous and π is continuous on $\{\eta > 0\}$.

Proof. Let X and Y be given as in the first statement and $T : X \to Y$ a Riesz* homomorphism. We extend T to a Riesz homomorphism $T^\rho : X^\rho \to X^\rho$ using Theorem 1.6. We fix some $s \in S$ and define the operator $T_s : X^\rho \to \mathbb{R}$ by

$$T_s f := (T^\rho f)(s),$$

which is obviously a Riesz homomorphism. As $C(\Omega)$ and $\mathbb{R}$ are both Riesz spaces, $\mathbb{R}$ is Dedekind complete and X^ρ is a majorizing Riesz subspace of $C(\Omega)$ we can apply Corollary 4.36 of [1, p. 153] to extend T_s to a Riesz homomorphism

$$\hat{T}_s : C(\Omega) \to \mathbb{R}.$$

Riesz homomorphisms from $C(\Omega)$ to $\mathbb{R}$ are characterized in Lemma 4.23 of [1, p. 144]. It tells us that there exist $\eta(s) \in \mathbb{R}_+$ and $\pi(s) \in \Omega$ such that

$$\hat{T}_s f = \eta(s)f(\pi(s)), \qquad f \in C(\Omega).$$

So, in particular, we have that

$$(Tf)(s) = \eta(s)f(\pi(s)), \qquad f \in X, s \in S.$$

In order to show the second part of the statement, suppose that X is pointwise order-dense. We fix an $s \in S$ and take a net (s_α) in S which converges to $s \in S$. For any $f \in X$ we know that Tf is continuous, hence we get

$$\eta(s)f(\pi(s)) = (Tf)(s) = \lim_\alpha (Tf)(s_\alpha) = \lim_\alpha \eta(s_\alpha)f(\pi(s_\alpha)).$$

We can find an $f \in X$ above the constant one function. Its image Tf is a bounded function, so we see that η is bounded aswell on S as T is positive. As S is compact there exists a subnet of (s_α), which we will again denote by (s_α), which converges to some $x \in \mathbb{R}$, which can be done as S is compact. So we get

$$\eta(s)f(\pi(s)) = x \lim_\alpha f(\pi(s_\alpha)) = xf(\lim_\alpha \pi(s_\alpha)),$$

for every $f \in X$. Here the first equality follows from the fact that f is bounded and the second one because it is continuous. We know the limit $\lim_\alpha \pi(s_\alpha)$ exists because the limit of $(Tf)(s_\alpha)$

exists as Tf is continuous. We take a further subnet, again denoted by (s_α), such that $\pi(s_\alpha)$ converges to some $\omega \in \Omega$.
Now either $\eta(s) = 0$ holds, in which case wee see that $x = 0$ holds and hence that η is continuous in s. Or we have that $\eta(s) > 0$, in which case $x > 0$ as we can find a $f \in X$ above the constant one function. So we know that for all $f \in X$ we have $f(\omega) = cf(\pi(s))$ with $c = \frac{\eta(s)}{x}$. Suppose that $\pi(s) \neq \omega$ holds. Then we can find a $g \in C(\Omega)$ with $g(\omega) > cg(\pi(s))$ and apply the pointwise order denseness of X to $\pi(s)$. We find an $f \in X$ with $f(\omega) > cf(\pi(s))$ which yields a contradiction with the above equality. Therefore, we get that $\omega = \pi(s) = \lim_\alpha \pi(s_\alpha)$ holds and then we also get

$$\eta(s)f(\pi(s)) = \lim_\alpha \eta(s_\alpha)f(\pi(s_\alpha)) = \lim_\alpha \eta(s_\alpha)f(\pi(s)),$$

for all $f \in X$, so $\eta(s) = \lim_\alpha \eta(s_\alpha)$ holds and we are done. □

From now on let $T_{\eta,\pi}$ denote the composition muliplication operator with multiplication map η and composition map π on a suitable $C(\Omega)$ space, i.e.,

$$(T_{\eta,\pi}f)(s) = \eta(s)f(\pi(s)), \qquad f \in C(\Omega), s \in S.$$

A converse of the above theorem holds as well under the assumption that X is pointwise order dense. Before we can show this, however, we prove the following lemma that tells us when an operator is a Riesz* homomorphism.

Lemma 2.6. *Let Y be a pre-Riesz space. If an operator $T : X \to Y$ satisfies*

$$\forall f, g \in X : \ \inf\{Th : h \in X, h \geq f, g\} = Tf \vee Tg,$$

then it is a Riesz homomorphism, where the infimum en supremum are both taken in the Riesz completion, Y^ρ, of Y.*

Proof. Let $T : X \to Y$ be an operator satisfying the above property. Let us observe that T is positive. Suppose that T is not a Riesz* homomorphism. Then there exist $f, g, h \in X$ such that

$$h \in \{f, g\}^{ul} \text{ and } Th \notin \{Tf, Tg\}^{ul}.$$

If for this f and g the infimum $\inf\{Th : h \in X, h \geq f, g\}$ does not exist in Y^ρ we are done. Suppose that is does exist. There exists an $h_0 \in Y^\rho$ with $h_0 \in \{Tf, Tg\}^u$ while $Th \not\leq h_0$. Therefore, we also have that $Th \not\leq Tf \vee Tg$ holds in Y^ρ.
On the other hand, for any $k \in \{f, g\}^u$ we have $h \leq k$. So by positivity of T we get $Th \leq Tk$. Hence, we get $Th \leq \inf\{Tk : k \in X, k \geq f, g\}$. Combining this with the above we get that $\inf\{Th : h \in X, h \geq f, g\} \neq Tf \vee Tg$, by contrapositivity this proves the claim. □

With all work done previously, we arrive at the following characterization theorem of Riesz* homomorphism on pointwise order dense subspace of $C(\Omega)$.

Theorem 2.7. *Let X be pointwise order-dense in $C(\Omega)$ and let Y be a order-dense pre-Riesz subspace of $C(S)$. Let $T : X \to Y$ be a linear operator. Then the following are equivalent:*

(i) T is a Riesz homomorphism.*

(ii) There exist $\eta : S \to \mathbb{R}_+$ continuous and $\pi : S \to \Omega$ continuous on $\{\eta > 0\}$ such that $T = T_{\eta,\pi}$ on X, i.e.,

$$(Tf)(s) = \eta(s)f(\pi(s)), \qquad f \in X, s \in S.$$

(iii) $\forall f, g \in X : \ \inf\{Th : h \in X, h \geq f, g\} = Tf \vee Tg$, in Y^ρ.

Proof. Let X, Y and T be as given in the statement. Theorem 2.5 immediately yields the implication $(i) \Rightarrow (ii)$. For the converse, suppose that $T = T_{\eta,\pi}$ on X for some $\eta \in C(S)^+$ and $\pi : S \to \Omega$ continuous $\{\eta > 0\}$ and note that $T_{\eta,\pi} : X^\rho \to Y^\rho$ defines a Riesz homomorphism, as by 1.7 we can view X^ρ and Y^ρ as Riesz subspaces of $C(\Omega)$ and $C(S)$ respectively. Theorem 1.6 now yields the desired result that T is a Riesz* homomorphism.
Implication $(iii) \Rightarrow (i)$ follows immediately from Lemma 2.6. Suppose that (i) holds and let η and π be as given in that statement. Let $f, g \in X$ be given note that since T is positive, $Tf \vee Tg$ is a lower bound of $\{Th : h \in \{f,g\}^u\}$ in Y^ρ. For any lower bound k of $\{Th : h \in \{f,g\}^u\}$ in Y^ρ, which by Lemma 1.7 we can view as an element of $C(S)$, and any $s \in S$ we now get that

$$
\begin{aligned}
k(s) &\leq \inf\{(Th)(s) : h \in \{f,g\}^u\} \\
&\leq \inf\{\eta(s)h(\pi(s)) : h \in \{f,g\}^u\} \\
&= \eta(s)(\inf\{h(\pi(s)) : h \in \{f \vee g\}^u\}) \\
&= \eta(s)(f \vee g)(\pi(s)) = (Tf \vee Tg)(s),
\end{aligned}
$$

hence (iii) holds. Here we have used that X is pointwise order dense in $C(\Omega)$ in the second last equality on $\pi(s) \in \Omega$ and $f \vee g \in C(\Omega)$.

□

In the rest of this text we will often use the equivalence between (i) and (ii) in both direction. Property (iii) gives further insight in the order structure of a Riesz* homomorphism. Using the equivalence between (i) and (ii) we will show a result on the inverse of a bijective Riesz* homomorphism. Observe that we are going to use the above theorem in both spaces X and Y so we require both spaces to be pointwise order dense.

Theorem 2.8. *Let X and Y be pointwise order dense in $C(\Omega)$ and $C(S)$ respectively, X pervasive in $C(\Omega)$ and let $T : X \to Y$ be a bijective Riesz* homomorphism. Then the inverse T^{-1} is also a Riesz* homomorphism.*

Proof. By Theorem 2.7 we know there exist an $\eta : S \to \mathbb{R}_+$ continuous and a $\pi : S \to \Omega$ continuous on $\{\eta > 0\}$ such that $T = T_{\eta,\pi}$ on X. Suppose that $\eta(s) = 0$ for some $s \in S$. Then we get that $(Tf)(s) = 0$ for all $f \in \mathrm{Im}(T)$. As Y is majorizing so we can find a $g \in Y$ greater than 1, which can not be in the image of T. This contradicts the bijectivity of T, so η is non-zero everywhere. Suppose π is not injective, then there exist $s_1, s_2 \in S$ such that $s_1 \neq s_2$ and $\pi(s_1) = \pi(s_2)$. Now we get for any $f \in X$ that

$$
(Tf)(s_1) = \eta(s_1)f(\pi(s_1)) = \eta(s_2)\frac{\eta(s_1)}{\eta(s_2)}f(\pi(s_2)) = \frac{\eta(s_1)}{\eta(s_2)}(Tf)(s_2).
$$

So there exists some $\lambda \geq 0$ such that $g(s_1) = \lambda g(s_2)$ for all $g \in \mathrm{Im}(T)$. We can find an $f \in C(S)$ with $f(s_1) > \lambda f(s_2)$ and as Y is pointwise order-dense, we can find a $g \in Y$ above f with $\lambda g(s_2) < f(s_1) \leq g(s_1)$. This g is not in the image of T, contradicting its bijectivity. Now we suppose that π is not surjective and let $K := \pi(S)$. As π is continuous everywhere, because we have $\eta > 0$, we get that K is compact. So K is a closed proper subset of Ω. By Urysohn's lemma we can find an $f \in C(\Omega)$, positive, $f \neq 0$, with the support of f contained in $S \backslash \Omega$. As X is pervasive, we can find a $g \in X$ with the same properties. Notice now that $Tg = 0$ holds by construction while $g \neq 0$. This contradicts the injectivity of T. So we have shown that π is bijective. Let us now define the operator $R : X \to X$ by

$$
(Rf)(s) := \eta^{-1}(\pi^{-1}(s))f(\pi^{-1}(s)).
$$

Now we have for all $f \in X$ and $s \in S$ that

$$
(TRf)(s) = \eta(s)(\eta^{-1}(\pi^{-1}(\pi(s))))f(\pi^{-1}(\pi(s))) = f(s) = (If)(s) = (RTf)(s).
$$

Hence, R is the inverse of T. As η^{-1} and π^{-1} are both continuous and Y is pointwise order dense, we can apply Theorem 2.7 again to find that $R = T^{-1}$ is a Riesz*-homomorphism. □

The above theorem does not hold without assuming that X is pervasive. This is illustrated by the following example.

Example 2.9. Let $S = [0,1]$, $\eta = 1$, $\pi(s) = \frac{1}{2}s$ and X the set of polynomials on S. Notice that X is pointwise order dense by Example 2.3 and is not pervasive. For all $p \in X$ we define

$$(Tp)(s) = \eta(s)p(\pi(s)) = p(\frac{1}{2}s).$$

Notice that Tp is again a polynomial, so T defines an operator on X. Also, by Theorem 2.7, this T is a Riesz*-homomorphism. Observe that T is injective. Let $g \in X$ be of the form $g(s) = \alpha_n s^n + ... + \alpha_1 s + \alpha_0$. Then we can find $\beta_i := 2^i \alpha_i$ and define $f(s) = \sum_{i=0}^{n} \beta_i s^i$. Hence, we have that $Tf = g$, so T is surjective. Suppose now that there exist a $\theta : S \to \mathbb{R}$ continuous and $\tau : S \to S$ continuous on $\{\theta > 0\}$ such that $(T^{-1}f)(s) = \theta(s)f(\tau(s))$. Let us observe that $T\mathbb{1} = \mathbb{1}$ holds and hence $T^{-1}\mathbb{1} = T^{-1}T\mathbb{1} = \mathbb{1}$. For any $s \in S$ we thus have $\theta(s) = 1$. Now let f be the identity map on S for any $s \in S$ we get

$$s = (T^{-1}Tf)(s) = (Tf)(\tau(s)) = f(\frac{1}{2}(\tau(s))) = \frac{1}{2}\tau(s).$$

So we get that $\tau(s) = 2s$ must hold for all $s \in S$, but $s \mapsto 2s$ is not a well-defined map on S. So T^{-1} can not be of that form, hence, by Theorem 2.7 T^{-1} is not a Riesz*-homomorphism.

2.2 Complete Riesz homomorphism

We have seen that the Riesz*-homomorphism are exactly the composition multiplication operators on pointwise order dense subspaces of a $C(\Omega)$ space. We will now look at a stronger type of homomorphism, namely the complete Riesz homomorphisms (see [9]). We are going to investigate how being a complete Riesz homomorphism relates to properties of the multiplication and composition maps η and π of the operator.

Definition 2.10. Let X and Y be partially ordered vector spaces. An operator $T : X \to Y$ is called **complete Riesz homomorphism** if for all $A \subset X$ we have

$$\inf\ A = 0\ \Rightarrow \inf\ T(A) = 0.$$

It holds generally that complete Riesz homomorphisms are also Riesz* homomorphisms (see [9]). We will now define two properties that we can impose on π resulting in interesting properties of the operator $T_{\eta,\pi}$ for any postive $\eta \in C(S)$.

Definition 2.11. A continuous function $\pi : S \to \Omega$ is called

(i) **semi-open** if, for all non-empty $U \subset S$ open, the image $\pi(U)$ has a non-empty interior.

(ii) **nowhere constant** if, for all non-empty $U \subset S$ open, the image $\pi(U)$ is not a singleton.

For a continuous map $\pi : S \to \Omega$ we have the following chain of implications:

$$\pi \text{ is injective} \Rightarrow \pi \text{ is open} \Rightarrow \pi \text{ is semi-open} \Rightarrow \pi \text{ is nowhere constant.}$$

It turns out that if π is semi-open and η is continuous and positive that $T_{\eta,\pi}$ will be a complete Riesz homomorphism. We exhibit an example that shows that semi-open is strictly weaker than open.

Example 2.12. Let $\Omega = [-1, 1]$ and $\pi : \Omega \to \Omega : \omega \mapsto \omega^2$. Obviously this π is continuous and not injective. Also, π is not open as $\pi((-1, 1)) = [0, 1)$ is not open. If we take some $U \subset [-1, 1]$ non-empty and open, then either $U \cap (-1, 0)$ or $U \cup (0, 1)$ is also non-empty. Suppose without loss of generality that $V := U \cap (0, 1) \neq \emptyset$. We then see that $\pi(U)$ contains $\pi(V)$ which has a non-empty interior, so π is semi-open.

Note that we do not require X to be pointwise order dense in the next two lemmas which together deal with complete Riesz homomorphisms on X.

Lemma 2.13. *For any $A \subset X^+$ we have*

$$\inf\ A = 0 \Longleftrightarrow \forall U \subset S \text{ open}, \epsilon > 0\ \exists f \in A, s \in U : f(s) \leq \epsilon.$$

Proof. Let $A \subset X^+$ be given with $\inf\ A = 0$ and suppose the converse of the right-hand side holds. Then there is some $U \subset S$ open and $\epsilon > 0$ such that for all $f \in A$ and $s \in U$ we have $f(s) > \epsilon$. So we can find a $f \in C(S)^+$, $f \neq 0$, which is a lower bound of A. As X is assumed to be order-dense, we can find a $g \in X$ with $g \nleq 0$ and $g \leq f$. This contradicts that $\inf\ A = 0$ holds in X.
For the converse, suppose that $\inf\ A = 0$ does not hold. Then we can find some $g \in X$ a lower bound of A with $g \nleq 0$. We can take the positive part g^+ of g which is a positive element of $C(S)$, still a lower bound of A which is not zero. As g^+ is continuous, we can find an $\epsilon > 0$ and $U \subset S$ open with $f(s) \geq g^+(s) > \epsilon$ for all $s \in U, f \in A$. □

Lemma 2.14. *Let $\eta \in C(S)^+$ and $\pi : S \to \Omega$ semi-open be given. The operator $T_{\eta,\pi} : X \to Y$ is then a complete Riesz-homomorphism.*

Proof. Let η and π be as given and $A \subset X$ with $\inf\ A = 0$. As η is bounded, $M := \sup_s \eta(s)$ exists. Suppose that $V \subset S$ is open and $\delta > 0$ is given. Let us put $U := \pi(V)$ and $\epsilon = \delta(M + 1)^{-1} > 0$. As π is semi-open we know that U has a non-empty interior. So by the above lemma we can find $f \in A$, $s \in U^\circ$ with $f(s) \leq \epsilon$, where U° denotes the interior of U. Now there is some $t \in V$ with $\pi(t) = s$ and we get

$$(Tf)(t) = \eta(t)f(\pi(t)) \leq Mf(\pi(t)) = Mf(s) \leq M\epsilon \leq \delta.$$

We can apply the above lemma again to see that $\inf T(A) = 0$ holds, so T is a complete Riesz-homomorphism. □

We now combine the above result with Theorem 2.8 on bijective Riesz* homomorphisms that gives us an alternative way to show when an operator is a complete Riesz homomorphism.

Corollary 2.15. *Let X and Y both be pointwise order dense, X pervasive and $T : X \to Y$ a bijective Riesz* homomorphism. Then both T and T^{-1} are complete Riesz homomorphisms.*

Proof. Let η and π be as in the proof of Theorem 2.8. Now π is continuous and bijective on the compact space S, so π is even a homeomorphism. In particular, π and π^{-1} are semi-open. So by the above lemma T and T^{-1} are complete Riesz-homomorphisms. □

We have seen that requiring that the composition map π to be semi-open suffices to make an operator $T_{\eta,\pi}$ a complete Riesz homomorphism. The converse, however, does not generally hold. It turns out that if we restrict to a bounded and closed interval of $\mathbb{R}$, we can characterize exactly which composition-multiplication operators are complete Riesz homomorphisms and which are not. In order to do so, we need the following lemma, which concerns the extendability of complete Riesz homomorphism.

Lemma 2.16. *If X is pointwise order dense and $T_{\eta,\pi} : X \to Y$ is a complete Riesz homomorphism, then $T_{\eta,\pi}$ extends to a complete Riesz homomorphism from $C(\Omega)$ to $C(S)$ being a composition-multiplication operator with the same maps η and π.*

Proof. Let $T_{\eta,\pi} : X \to Y$ be a complete Riesz homomorphism, then obviously we can extend it to $\hat{T}_{\eta,\pi} : C(\Omega) \to C(S)$, for simplicity we dnote these operators by T and $\hat{T}$ respectively. We want to show that this operator is a complete Riesz homomorphism. To this end let $A \subset C(\Omega)$ be given with $\inf\ A = 0$ and let us fix some $s \in S$. We define the set

$$B := \{g \in X : \exists f \in A,\ g \geq f\} \subset X.$$

Observe that $\inf\ B = 0$ holds as X is pointwise order dense and hence order dense in $C(\Omega)$. We have assumed T to be a complete Riesz homomorphism so we get $\inf\ T(B) = 0$. As $\hat{T}$ is positive, zero is a lower bound of $\hat{T}(A)$. Let h be a lower bound of $\hat{T}(A)$, then by positivity of $\hat{T}$ we get

$$\begin{aligned} h(s) &\leq \inf\{(\hat{T}f)(s) : f \in A\} \\ &\leq \inf\{(Tg)(s) : g \in B\}. \end{aligned}$$

This shows that $h \leq Tg$ for all $g \in B$ and, therefore, $h \leq 0$ as $\inf T(B) = 0$ holds as T is a complete Riesz homomorphism. We conclude the proof by noting that we have shown that $\inf \hat{T}(A) = 0$ holds. □

Using the above lemma we conclude this section by giving a full characterization of complete Riesz homomorphisms $T_{\eta,\pi}$ between closed, bounded intervals on $\mathbb{R}$ in terms of the composition map π.

Theorem 2.17. *Let Ω and S be bounded and closed intervals in $\mathbb{R}$ and $\eta \in C(S)^+$ and $\pi : S \to \Omega$ a continuous map on $\{\eta > 0\}$. Then $T_{\eta,\pi} : X \to Y$ is a complete Riesz homomorphism if and only if π is nowhere constant on $\{\eta > 0\}$.*

Proof. Suppose that π is nowhere constant on $\{\eta > 0\}$. It suffices to show that π is semi-open so that we can apply Lemma 2.14 and conclude that T is a complete Riesz homomorphism. To this end let $U \subset S$ be open and non-empty. Suppose first that U is contained in $\{\eta = 0\}$. We can find a π' which equals π on $S \backslash U$ and such that $\pi'(U)$ is semi-open, simply by choosing a small open interval inside U and mapping it onto an open interval in S. As U is contained in $\{\eta = 0\}$ we see that $T_{\eta,\pi} = T_{\eta,\pi'}$ where the right operator is a complete Riesz homomorphism by Lemma 2.14. So we assume that U is not contained in $\{\eta = 0\}$ and let $V \subset U$ be a non-empty open subset which is disjoint from $\{\eta = 0\}$. As π is nowhere constant on $\{\eta > 0\}$, $\pi(V)$ contains some x and y with $x \neq y$. Now there exist $a, b \in V$ with $a < b$, $\pi(a) = x$ and $\pi(b) = y$ or the other way around. Without loss of generality we can assume that $x < y$ holds aswell. We restrict π to $\hat{\pi} : [a, b] \to \Omega$ which is continuous. For any $x < z < y$ we can find, by the Intermediate-Value-Theorem, a $c \in (a, b)$ with $\pi(c) = z$. Hence, we get $(x, y) \subset \pi(V)$ which shows that $\pi(U)$ has a non-empty interior so π is semi-open.
In order to show the converse, let us suppose that π is not nowhere constant on $\{\eta > 0\}$. Then there exists some $U \subset \{\eta > 0\}$ open and non-empty such that $\pi(s) = s_0$ for some $s_0 \in \Omega$ and all $s \in U$. We now shrink U, keeping it non-empty and open, such that $\eta(s) \geq \epsilon$ for some $\epsilon > 0$ and all $s \in U$. By Urysohn's lemma we know there exists a sequence of continuous $f_n : C(\Omega) \to [0, 1]$ which are equal to one in s_0 and whose supports are contained in the open balls $B_{n^{-1}}(s_0)$. It is then clear that $\inf\{f_n : n \in \mathbb{N}\} = 0$. Suppose that $\inf\{Tf_n : n \in \mathbb{N}\}$ does not exist, then we are done. So we assume that the infimum does exist. Note that for any $n \in \mathbb{N}$ and $s \in U$ we get

$$(Tf_n)(s) = \eta(s) f_n(\pi(s)) = \eta(s) f_n(s) \geq \epsilon.$$

As this holds for all $n \in \mathbb{N}$ on an open set U we see that $\inf\{Tf_n : n \in \mathbb{N}\} = 0$ does not hold. This shows that $T_{\eta,\pi}$ is not a complete Riesz homomorphism. □

2.3 An Application: Sobolev spaces

As an application of the above theory about Riesz*-homomorphisms on pointwise order-dense subspace of $C(S)$, we are going to look at which operators on the Sobolev space $W^{1,p}(\Omega)$ are composition-multiplication operators. For the rest of this section, let $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ open and bounded both be given. In order to be able to apply our theory we need that $W^{1,p}(\Omega)$ is pointwise order dense in $C(S)$ for some compact and Hausdorff space S. The obvious choice for S is the topological closure of Ω, $S = \bar{\Omega}$. We need to impose an extra condition on Ω for this to be true. More precisely, Ω need to be a Lipschitz domain, which means that its boundary can be viewed as being locally the graph of a Lipschitz continuous function. The precise definition and properties of Lipschitz domains can be found on pages 66 and 67 in [2].

Theorem 2.18. *If Ω is a Lipschitz domain and $p > d$, then $W^{1,p}(\Omega) \subset C(\bar{\Omega})$ is pointwise order dense.*

Proof. We use Lemma 2.2 to show that $W^{1,p}(\Omega)$ is pointwise order dense. Before we can begin to show that $W^{1,p}(\Omega)$ is norm-dense in $C(\bar{\Omega})$, we must first show it is even a linear subspace of $C(\bar{\Omega})$. We use a Sobolev imbedding theorem by Adams, [2] p.97, which says that if Ω is a Lipschitz domain in $\mathbb{R}^d$ we have

$$W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\bar{\Omega})$$

for any $0 \leq \lambda \leq m - \frac{p}{d}$. In this notation the space $C^{j,\lambda}(\bar{\Omega})$ is the space of all functions having continuous derivatives up to order j and such that the jth partial derivatives are Hölder continuous with exponent λ, where $0 \leq \lambda \leq 1$. We apply this theorem with $j = 0$ and $m = 1$. We have assumed that $d < p$, so there exists a $0 \leq \lambda \leq 1 - \frac{d}{p}$ and Adams gives us

$$W^{1,p}(\Omega) \subset C^{0,\lambda}(\bar{\Omega}).$$

As Hölder continuity implies continuity we get $W^{1,p}(\Omega) \subset C(\bar{\Omega})$. Now let $f : \bar{\Omega} \to \mathbb{R}$ be a continuous function.

Claim. *There exist $f_n : \bar{\Omega} \to \mathbb{R}$ continuous which are smooth on Ω and converge uniformly to f on $\bar{\Omega}$.*

We will show later that these f_n must then be elements of $W^{1,p}(\Omega)$, which will conclude the proof of the norm denseness of $W^{1,p}(\Omega)$ in $C(\hat{\Omega})$.

Proof. First we will extend f to a continuous $g : \mathbb{R}^d \to \mathbb{R}$ with compact support. In order to find this g we will use a result by Tietze see Theorem 2.47 on page 45 in [3], which says that any continuous map on a closed subset of a compact Hausdorff space can be continuously extended to the entire space without increasing the norm. As $\bar{\Omega}$ is bounded, we can find an open $U \supset \bar{\Omega}$ and a $S \supset U$ compact. We now let $A := \bar{\Omega} \cup (S \backslash U)$ which is closed in S. Let us define the function $\tilde{f} : A \to \mathbb{R}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if} \quad x \in \bar{\Omega} \\ 0 & \text{if} \quad x \in S \backslash U. \end{cases}$$

Now $\tilde{f}$ is continuous as $\bar{\Omega}$ and $S \backslash U$ are disjoint. By the theorem of Tietze we can find a continuous extension $g : S \to \mathbb{R}$ of $\tilde{f}$. As g now equals zero on $S \backslash U$, we can continuously extend it to $\mathbb{R}^d$ by putting g equal to zero on $\mathbb{R}^d \backslash S$. Note that g now also has compact support.
Now we define the *standard mollifier function* $\varphi : \mathbb{R}^d \to [0, 1]$ by

$$\varphi(x) := \begin{cases} e^{-\frac{1}{1-\|x\|^2}} & \text{if} \quad \|x\| \leq 1 \\ 0 & \text{if} \quad \|x\| > 1. \end{cases}$$

For any $n \in \mathbb{N}$ we also define $\varphi_n(x) := n^{-d}\varphi(\frac{x}{n})$. Note that these φ_n are positive C^∞-functions with compact support that satisfy $\int_{\mathbb{R}^d} \varphi_n(x)dx = 1$. As g is locally integrable and the φ_n are smooth, distribution theory tells us that all convolutions $(g \star \varphi_n)$ are also smooth. Next we will show that these convolutions converge uniformly to g. As g is continuous and has compact support, it is uniformly continuous. Let $\epsilon > 0$ be given, then there exists a $\delta > 0$ such that $\|x - y\| < \delta$ implies $|g(x) - g(y)| < \epsilon$. Now let n be large enough such that supp $\varphi_n \subset B_\delta(0)$. For any $x \in \mathbb{R}^d$ we now get

$$|(g \star \varphi)(x) - g(x)| = \left| \int_{\mathbb{R}^d} g(x-y)\varphi(y)dy - g(x) \right| = \left| \int_{\mathbb{R}^d} (g(x-y) - g(x))\varphi_n(y)dy \right|$$

$$= \left| \int_{\text{supp } \varphi_n} (g(x-y) - g(x))\varphi_n(y)dy \right| \leq \int_{\text{supp } \varphi_n} \epsilon\varphi_n(y)dy = \epsilon.$$

This shows that indeed $\|(g \star \varphi_n) - g\|_\infty \to 0$. We now let f_n be the restrictions of $(g \star \varphi)$ to $\bar{\Omega}$ and we are done. □

We are done if we can show that all the f_n from the above claim are elements of $W^{1,p}(\Omega)$. We use a result, the so called ACL characterization of Sobolev spaces, where ACL stand for Absolute Continuous on Lines, which says that $W^{1,p}(\Omega) = ACL^p(\Omega)$ holds for all p and Ω. This result is the content of Theorem 2.1.4 in [14]. Here $ACL^p(\Omega)$ is the class of L^p functions on Ω which are absolutely continuous on $L \cap \Omega$ for almost every line L parallel to any coordinate axis, whose classical first order partial derivatives all lie in $L^p(\Omega)$. So, in particular, we see that the space of all C^1-functions on $\bar{\Omega}$ is contained in $W^{1,p}(\Omega)$. Our sequence (f_n) in the claim which converges to f in supremum norm is, therefore, contained in $W^{1,p}(\Omega)$ which concludes the proof. □

As a corollary of the ACL characterization of Sobolev spaces described in the proof, we see that $W^{1,p}(\Omega)$ is a Riesz space. As for any $u \in W^{1,p}(\Omega)$ we see that $u^+ = u \vee 0$ is an element of $ACL^p(\Omega)$ again. All Riesz homomorphism are, in particular, Riesz* homomorphism so we can apply our developed theory in the previous sections to Riesz homomorphisms on $W^{1,p}$. Most notably this yields us the following result.

Theorem 2.19. *If $d \in \mathbb{N}, \Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, $p > d$ and $T : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is a Riesz homomorphism, then there exist $\eta \in C(\bar{\Omega})^+$ and $\pi : \Omega \to \Omega$ continuous on $\{\eta > 0\}$ such that*

$$(Tf)(\omega) = \eta(\omega)f(\pi(\omega)), \qquad f \in W^{1,p}(\Omega), a.e.\ \omega \in \Omega.$$

Proof. We have shown in Theorem 2.18 that $W^{1,p}(\Omega)$ is pointwise order-dense in $C(\bar{\Omega})$. Any Riesz-homomorphism on a Riesz space is in particular also a Riesz*-homomorphism, so we can apply Theorem 2.7 on T and we get the desired result. □

This result is similar to Theorem 4.4 in [5] by Biegert. It shows that Riesz homomorphisms on the Sobolev space $W_0^{1,p}(\Omega)$ have a composition multiplication representation with Ω a non-emtpy open subset of $\mathbb{R}^d$. Serveral such results are discussed in his paper. Interestingly, Biegert almost exclusively uses the norm structure of $W^{1,p}$ to get his result. While in contrast, our theory developed in this section mostly relies on the order structure of the function spaces. Yet both theories get similar results on Riesz homomorphisms on $W^{1,p}$.

3 Spaces of measurable functions

For the rest of this section let (Ω, Σ) be a measurable space and $\mathfrak{L}$ the space of all measurable functions on this space. Our aim is to develop a characterization of Riesz homomorphism on $\mathfrak{L}$ similar as in the continuous function space case. This results in the following theorem.

Theorem 3.1. *If $T : \mathfrak{L} \to \mathfrak{L}$ is a Riesz homomorphism, then there exist $\eta \in \mathfrak{L}_+$ and $\pi : \Omega \to \Omega$, such that*

$$(T\mathbb{1}_A)(\omega) = \eta(\omega)\mathbb{1}_A(\pi(\omega)) \qquad A \in \Sigma, \text{a.e. } \omega \in \Omega. \tag{2}$$

Proof. Let T be a Riesz homomorphism. We first define $\eta := T\mathbb{1} \in \mathfrak{L}$ which is positive as T is. The construction of π takes more work and we will need to proof two claims first.

Claim. *There exists a map $\tau : \Sigma \to \Sigma$ such that for all $A \in \Sigma$ we have*

$$T\mathbb{1}_A = \eta\mathbb{1}_{\tau(A)}. \tag{3}$$

Proof. Let $A \in \Sigma$ be given. T respects infima and suprema, so we get

$$T\mathbb{1}_A \wedge T\mathbb{1}_{A^c} = 0 \tag{4}$$

$$T\mathbb{1}_A \vee T\mathbb{1}_{A^c} = \eta. \tag{5}$$

As $T\mathbb{1}_A \in \mathfrak{L}$ holds, the set $\tau(A) := [T\mathbb{1}_A > 0]$ is measurable. From (4) it follows that $T\mathbb{1}_A$ and $T\mathbb{1}_{A^c}$ are disjoint and as they are also positive, (5) gives us

$$T\mathbb{1}_A = \eta\mathbb{1}_{[T\mathbb{1}_A > 0]} = \eta\mathbb{1}_{\tau(A)}.$$

$\square$

In particular, (3) yields us a map τ from Ω to Σ. Let us also note that this τ is increasing, i.e., $A \subset B \Rightarrow \tau(A) \subset \tau(B)$, as the operator T is positive. For any $s \in \Omega$ we use the notation $\tau(s)$ for $\tau(\{s\})$. We want to show that this τ gives rise to a map from Ω to Ω which is the inverse of τ on all singleton sets. This map will be the desired π.

Claim. *If $\omega, s, t \in \Omega$ with $s \neq t$ are such that $\omega \in \tau(s) \cap \tau(t)$, then $\eta(\omega) = 0$ holds.*

Proof. Using that T is a Riesz homomorphism, we get

$$T(\mathbb{1}_{\{s,t\}})(\omega) = T(\mathbb{1}_s \vee \mathbb{1}_t)(\omega) = T\mathbb{1}_s(\omega) \vee T\mathbb{1}_t(\omega) = \eta(\omega)$$
$$T(\mathbb{1}_{\{s,t\}})(\omega) = T(\mathbb{1}_s + \mathbb{1}_t)(\omega) = T\mathbb{1}_s(\omega) + T\mathbb{1}_t(\omega) = 2\eta(\omega).$$

$\square$

So now we know that for all $\omega \in \Omega$, either $\eta(\omega) = 0$ holds in which case we let $\pi(\omega) := \omega$, or there is exactly one $s \in \Omega$ with $\omega \in \tau(s)$ in which case we let $\pi(\omega) := s$. We obtain a map $\pi : \Omega \to \Omega$ which satisfies

$$T\mathbb{1}_s(\omega) = \eta\mathbb{1}_{\tau(s)}(\omega) = \eta\mathbb{1}_s(\pi(\omega)) \qquad s \in \Omega, \text{a.e. } \omega \in \Omega. \tag{6}$$

We want to show that this equality holds for all indicator functions instead of only the indicator functions of singletons. To this end we prove the following claim.

Claim. *For any $A \in \Sigma$ we have, $\tau(A) = \bigcup_{s \in A} \tau(s)$.*

Proof. As we have noted earlier, τ is increasing, so the inclusion $\bigcup_{s\in A}\tau(s)\subset\tau(A)$ is immediate. For the converse, suppose $\omega\in\tau(A)$ is given. By definition of τ it follows that $\eta(\omega)>0$. So there exists an unique $t\in\Omega$ with $\omega\in\tau(t)$, hence $\omega\in\tau(t)\cap\tau(A)$ holds and we get

$$\begin{aligned}\omega\in\tau(t)\cap\tau(A) &= [T\mathbb{1}_t>0]\cap[T\mathbb{1}_A>0]\\ &= [(T\mathbb{1}_t\wedge T\mathbb{1}_A)>0]\\ &= [T(\mathbb{1}_t\wedge\mathbb{1}_A)>0].\end{aligned}$$

This last set is empty if $t\notin A$, hence

$$\omega\in\tau(t)\subset\bigcup_{s\in A}\tau(s).$$

□

Let $A\in\Sigma$ be given. We show that (2) holds for this A. For any $\omega\in\Omega$ we have

$$\begin{aligned}T\mathbb{1}_A(\omega) &\overset{(3)}{=} \eta(\omega)\mathbb{1}_{\tau(A)}(\omega)=\eta(\omega)\mathbb{1}_{\cup_{s\in A}\tau(s)}(\omega)\\ &= \eta(\omega)\vee_{s\in A}\mathbb{1}_{\tau(s)}(\omega)\overset{(6)}{=}\eta(\omega)\vee_{s\in A}\mathbb{1}_s(\pi(\omega))\\ &= \eta(\omega)\mathbb{1}_A(\pi(\omega)).\end{aligned}$$

□

Corollary 3.2. *If T, in addition to being a Riesz homomorphism, respects coutable suprema we have that equation* (2) *holds for all measurable functions.*

Proof. By linearity of T equation (2) is satisfied for all positive simple functions. As T respects countable suprema we get that (2) holds for all potisive measurable functions. As T is both positive and linear equation (2) holds for all measurable functions. □

We have exhibited a weakness of $\mathcal{L}$, in particular also that of all $\mathcal{L}^p(\Omega,\Sigma,\mu)$ spaces with finite measure space, that not all Riesz homomorphism respect countable suprema or infima. If we want to show that an analogy of 2.5 for the $\mathcal{L}^p$ case holds, we have the problem that $\mathcal{L}$ is not Dedekind complete hence it is not clear how we can extend a Riesz homomorphism on the Riesz completion to the entire space. To solve both these problems we will restrict our study to the $L^p(\Omega)$ spaces. For the rest of this chapter let (Ω,Σ,μ) be a finite measure space and $1\le p<\infty$ be given. $L^p(\Omega)$ is now a Dedekind complete space and as we will show later all Riesz homomorphisms on this space respect countable suprema. Even more notably, L^p is *super Dedekind complete.* This result is proved in [12], example 23.3(iv) on page 126.

Theorem 3.3. *Let (Ω,Σ,μ) be a finite measure space and $1\le p\le\infty$. Then $L^p(\Omega,\Sigma,\mu)$ is a super Dedekind complete Riesz space, i.e.*

$$A\subset L^p,\inf A=0\Rightarrow\exists f_n\in A:\inf_n f_n=0.$$

We will use all these nice properties of L^p in the next sections to investigate the composition multiplication operators on these spaces.

3.1 Composition Multiplication Operators on L^p

We have seen why it is advantageous to look at the L^p-spaces instead of $\mathcal{L}^p$. One downside, however, is that we have to be careful with a lot of properties that are defined pointwise as we are looking at equivalence classes of measurable functions. So first we need to build a framework in whichh we can define a composition multiplication operators in a useful manner. The following ideas are taken from a paper by Rodriguesz-Salinas, [13].
We call a function $\tau : \Sigma \to \Sigma$ a $\mu-$**homomorphism** if it satisfies the following two properties

1. $\mu(\tau(\bigcup A_n) \triangle \bigcup \tau(A_n)) = 0$ for every sequence (A_n) in Σ.
2. $\mu(\tau(\bigcap A_n) \triangle \bigcap \tau(A_n)) = 0$ for every sequence (A_n) in Σ.

From now on let τ be such a $\mu-$homomorphism on (Σ, μ). Let $g = \sum_{n=1}^{m} \alpha_n \mathbb{1}_{A_n}$ be some positive simple function then we define

$$\tau^{-1}(g) = g \circ \tau^{-1} = \sum_{n=1}^{m} \alpha \mathbb{1}_{\tau(A_n)}.$$

For any positive, measurable function f we can find a non-decreasing sequence (f_n) of $\Sigma-$simple functions such that $f = \sup_n f_n$. So we can also define

$$f \circ \tau^{-1} = \sup_n f_n \circ \tau^{-1}.$$

In the paper [13] it is shown that the map $f \mapsto f \circ \tau^{-1}$ is well-defined, linear and preserves both $\wedge$ and $\vee$.

We will now proof that $\mu-$homomorphisms satisfy additional properties that will be used later in order to characterize the Riesz*-homomorphisms on subspaces of L^p.

Lemma 3.4. *Let $\tau : \Sigma \to \Sigma$ be a $\mu-$homomorphism, then*

1. $f, g \in \Sigma \Rightarrow (f \circ \tau^{-1}) \cdot (g \circ \tau^{-1}) = (fg \circ \tau^{-1})$.
2. $(f_n) \in \Sigma,\ f = \inf_n f_n \Rightarrow (f \circ \tau^{-1}) = \inf_n (f_n \circ \tau^{-1})$.

Proof. 1. Let f and g both be positive simple functions of the form $f = \sum_n \alpha \mathbb{1}_{A_n}$ and $g = \sum_m \beta_m \mathbb{1}_{B_m}$. Then we have fg is also a simple function and

$$\begin{aligned}(fg \circ \tau^{-1}) &= \sum_{n,m} \alpha_n \beta_m \mathbb{1}_{\tau(A_n \cap B_m)} \\ &= \sum_{n,m} \alpha_n \beta_m \mathbb{1}_{\tau(A_n)} \mathbb{1}_{\tau(B_m)} \\ &= (f \circ \tau^{-1})(g \circ \tau^{-1}).\end{aligned}$$

Now let f, g be positive measurable functions and $f_n, g_n \in \Sigma$ positive simple functions such that $f = \sup_n f_n, g = \sup_n g_n$. Then we get

$$\begin{aligned}(f \circ \tau^{-1})(g \circ \tau^{-1}) &= \sup_n (f_n \circ \tau^{-1}) \sup_m (g_m \circ \tau^{-1}) \\ &= \sup_{n,m} (f_n \circ \tau^{-1})(g_m \circ \tau^{-1}) \\ &= \sup_{n,m} (f_n g_m \circ \tau^{-1}) \\ &= (fg) \circ \tau^{-1}.\end{aligned}$$

2. If (f_n) is such a sequence in Σ with $f = \inf_n f_n$, we can let $g_n = f_n - f$. Then we get $\inf_n g_n = 0$, so by the paper $\inf_n(g_n \circ \tau^{-1}) = 0$. So we see that

$$\begin{aligned}\inf_n(g_n \circ \tau^{-1}) &= \inf_n((f_n - f) \circ \tau^{-1}) \\ &= \inf_n((f_n \circ \tau^{-1}) - (f \circ \tau^{-1})) = 0.\end{aligned}$$

It follows that indeed $f \circ \tau^{-1} = \inf_n(f_n \circ \tau^{-1})$. □

In the space $L^p(\Omega, \Sigma, \mu)$ we have identified with each other any two functions that are euqal μ-almost everywhere. In a similar way we want to identify two measurable sets with each other if their symmetric difference have measure zero. This will be useful in the next section when we are looking at bijective μ-homomorphisms on Σ. The following ideas are based on observations made in [10] on pages 166 and 167. For any $A, B \in \Sigma$ we define the equivalence relation

$$A \sim B :\Longleftrightarrow \mu(A \triangle B) = 0.$$

Now let $\Sigma' := \Sigma/\sim$ be the quotient and note that is a σ-algebra. It is obvious by the definition of $\sim$ that μ is a well-defined measure on (Ω, Σ'). Additionally, if $\tau : \Sigma \to \Sigma$ is a μ-homomorphism then it induces a map $\tau' : \Sigma' \to \Sigma'$ on the quotient with $\tau(A) = \tau'(A')$ for all $A \in \Sigma'$ and representative A of A'. It is then obvious that τ' is also a μ-homomorphism.
So from now on if we have some measure space (Ω, Σ, μ) and μ-homomorphism $\tau : \Sigma \to \Sigma$ we can carry out the above construction. Then we have a measure space such that for all $A, B \in \Sigma$ either $A = B$ or $\mu(A \triangle B) > 0$ and τ is still a μ-homomorphism.

We have seen in the introduction of this section that it is important that Riesz homomorphsims on L^p respect countable suprema in showing that they are of composition multiplication type.

Lemma 3.5. *Let $T : L^p \to L^p$ be a Riesz homomorphism, $1 \le p < \infty$ and $f, f_n \in L^p$ be such that $f = \sup_n f_n$. We then get $Tf = \sup_n Tf_n$.*

Proof. Let T, f, f_n all be as in the statement. We can assume that the f_n are increasing by replacing f_n be the supremum of all previous elements in the sequence. So then $g_n = f - f_n$ is a sequence satisfying, $g_n \downarrow 0$. By the dominated convergence theorem we get $\|g_n\|^p = \int_\Omega f_n^p d\mu \to 0$. Theorem 4.3 in [4] tells us that all postive operators between a Banach lattice is norm-continuous. As T is, in particular, positive and L^p is a Banach lattice we can conclude that T is norm-continuous, hence $\|Tg_n\|_p \to 0$. So we can thin our sequence until $(Tg_n)(\omega) \to 0$ holds μ-a.e. If h is a lower bound of all Tg_n, then we have $h(\omega) \le Tg_n(\omega)$ for all n, μ-a.e. $\omega \in \Omega$. As the union of countably many sets of measure zero has again measure zero, we conclude that $h \le Tg_n$ holds. So we get that $h \le 0$ holds and, therefore, $\inf_n Tg_n = 0$. As T is linear we get

$$0 = \inf_n(Tg_n) = \inf_n(Tf - Tf_n) = Tf - \sup_n Tf_n.$$

Which proves the statement. □

In particular, this lemma shows that the Riesz homomorphisms on L^p coincide with the complete Riesz homomorphisms, which is certainly not true in the continuous function case. This result is contained in the following corollary.

Corollary 3.6. *Let $1 \le p < \infty$ and an operator $T : L^p \to L^p$ both be given. Then T is a Riesz homomorphism if and only if it is a complete Riesz homomorphism.*

Proof. In order to show the non-trivial implication, let $T : L^p \to L^p$ be a Riesz homomorphism and $A \subset L^p$ be such that $\inf A = 0$. As L^p is super Dedekind complete by Theorem 3.3, we can find $f_n \in A$ with $\inf_n f_n = 0$. By the above theorem and the fact that T is positive and linear we get

$$0 \leq \inf T(A) \leq \inf_n T f_n = T \inf_n f_n = T0 = 0.$$

□

Before we start looking at Riesz* homomorphisms on L^p it is important to note the following. We have assumed above that our measure space is finite. We need this in the following section so all indicator functions are in our space. It is possible, however, to generalize this to σ-finite measure spaces. Start with a σ-finite measure space (Ω, Σ, μ) one can construct a finite measure on the same space and an isometric Riesz isomorphism between the two measure spaces. Then one can apply all the theory to the latter space and get the result for the former one. We will give a sketch of this construction below. As our space is σ-finite we can find disjoint and measurable S_n which cover the space have positive finite measure. We know define a measure for $A \in \Sigma$ by

$$\nu(A) = \sum_{n=1}^{\infty} \frac{1}{2^n \mu(S_n)} \mu(A \cap S_n).$$

Note that this ν is finite and absolutely continuous with respect to μ. The Radon-Nikodym Theorem now tells us there exists a measurable $h \geq 0$ such that

$$\nu(A) = \int_A h d\mu.$$

As $\nu(\Omega)$ is finite we see that h is μ-integrable we also know that $h > 0$ almost surely. Now we can define an operator $J : L^p(\nu) \to L^p(\mu)$ by $Jf := hf$. This operator is is linear, bijective, isometric and a Riesz homomorphism and we are done.

3.2 Riesz* homomorphisms on L^p-spaces

Let (Ω, Σ, μ) be a finite measure space and $L^p := L^p(\Omega, \Sigma, \mu)$ with $1 \leq p < \infty$ for the rest of this section. Let τ be a μ-homomorphism and f a measurable function. In the previous section we have defined $(f \circ \tau^{-1})$ which we interpret as a composition of the two functions. Some caution is advised though, as it is not a true composition. However, it does allow us to talk about composition multiplication operators. If η is a measurable function and τ is a μ-homomorphism, then we can define

$$T_{\eta,\tau} f = \eta (f \circ \tau^{-1}) \qquad f \in L^p. \tag{7}$$

The aim of this section will be to show that if $X \subset L^p$ is order-dense and $T : X \to X$ is positive, that then T is a Riesz* homomorphism if and only if it is of the form as in (7). In order to apply results from the previous section we need the following to hold.

Lemma 3.7. *Let $T : L^p \to L^p$ be a Riesz homomorphism then $\tau(A) := [T\mathbb{1}_A > 0]$ is a $\mu-$homomorphism.*

Proof. 1. Let (A_n) be a sequence in Σ. Then we have that

$$\tau(\cup_n A_n) = [T\mathbb{1}_{\cup_n A_n} > 0] = [T(\vee_n \mathbb{1}_{A_n}) > 0].$$

As T is a Riesz homomorphism Lemma 3.5 tell us that $T\mathbb{1}_{\cup_n A_n} = T(\vee_n \mathbb{1}_{A_n}) = \vee_n T\mathbb{1}_{A_n}$ holds. So we get

$$[T(\mathbb{1}_{\cup_n A_n}) > 0] = [\vee_n T\mathbb{1}_{A_n} > 0] = \cup_n [T\mathbb{1}_{A_n} > 0] = \cup_n \tau(A_n)$$

which proves the claim. Analogously, one can proof property 2., so τ is indeed a $\mu-$homomorphism. □

Before we can characterize Riesz* homomorphisms, we must proof that Riesz homomorphisms on the L^p are of the desired composition multiplication form. Later we can then extend our Riesz* homomorphisms on an order dense subspace of L^p to a Riesz homomorphism on L^p and apply the following theorem.

Theorem 3.8. *If $T : L^p \to L^p$ is a Riesz homomorphism, then there exist $\eta \in L^p$ and a $\mu-$homomorphism $\tau : \Sigma \to \Sigma$ such that $T = T_{\eta,\tau}$, i.e.*

$$Tf = \eta(f \circ \tau^{-1}), \qquad f \in L^p.$$

Proof. Let $T : L^p \to L^p$ be a Riesz homomorphism. We define η and τ in the same way as in the proof of Theorem 3.1, by the above lemma this τ is a $\mu-$homomorphism and we have

$$T\mathbb{1}_A = \eta\mathbb{1}_{\tau(A)} = \eta(\mathbb{1}_A \circ \tau^{-1}). \tag{8}$$

Now we use the standard machinery of L^p-spaces to show that (8) holds for all $f \in L^p$. Let $g = \sum_{n=1}^m \alpha_n \mathbb{1}_{A_n}$ be some positive simple function. Then we get

$$\begin{aligned} Tg = T\sum_{n=1}^m \alpha_n \mathbb{1}_{A_n} &= \sum_{n=1}^m \alpha_n T\mathbb{1}_{A_n} = \eta \sum_{n=1}^m \alpha_n(\mathbb{1}_{A_n} \circ \tau^{-1}) \\ &= \eta((\sum_{n=1}^m \alpha_n \mathbb{1}_{A_n}) \circ \tau^{-1}) = \eta(g \circ \tau^{-1}). \end{aligned}$$

Now let f be a positive measurable function and (f_n) a sequence of positive simple functions such that $f = \sup_n f_n$. Then we get

$$Tf = T\sup_n f_n = \sup_n Tf_n = \sup_n \eta(f_n \circ \tau^{-1}) = (f \circ \tau^{-1}).$$

Where the second equality follows from Lemma 3.5. For an arbitrary measurable function f, we can write $f = f_+ - f_-$ with f_+, f_- positive and measurable. By linearity of T and $f \mapsto (f \circ \tau^{-1})$ we get the desired result. □

In the previous section we showed that Riesz*-homomorphisms on pointwise order-dense subspaces of $C(S)$ are exactly the composition-multiplication operators. We try to get an analogous result in the L^p case. As mentioned earlier we do not have a lot pointwise structure in our space. So we only require our composition-multiplication operators to have a composition map defined on the σ-algebra instead of on the measure space. This does mean, however, that our result holds on a larger group of subspaces of L^p. Namely, all the order-dense subspaces. This will be the contents of Theorem 3.10. For the rest of this section let X be a partially ordered order-dense subspace of L^p.

Theorem 3.9. *If $T : X \to X$ is positive such that $T = T_{\eta,\pi}$ on X for some $\eta \in \Sigma$ and $\mu-$homomorphism $\tau : \Sigma \to \Sigma$, then T satisfies*

$$\inf\{Th : h \in \{f, g\}^u\} = Tf \vee Tg, \qquad f, g \in X.$$

Proof. Let $f, g \in X$ be given. As T is positive we immediately get the following inequality

$$\inf\{Th : h \in \{f, g\}^u\} \geq Tf \vee Tg.$$

Note here that part of the statement is that the infimum exists. By positivity of T the set is bounded by below and L^p is Dedekind complete. As we have assumed X to be order dense in

L^p we can find a sequence $h_n \in X$ with $h_n \geq f \vee g$ and $f \vee g = \inf_n h_n$. Here we also use that the space L^p is super Dedekind-complete. Now we let $B := \tau(\Omega)$ and show that on this set the other inequality holds. The references (1) and (2) below refer to Lemma 3.4.

$$\begin{aligned} \inf\{Th : h \in \{f,g\}^u\}\mathbb{1}_B &\geq \inf_n (Th_n)\mathbb{1}_B \\ &= \inf_n (\eta(h_n \circ \tau^{-1})\mathbb{1}_B) \\ &\overset{(1)}{=} \eta \inf_n ((h_n \mathbb{1}_A) \circ \tau^{-1}) \\ &\overset{(2)}{=} \eta((f \vee g)\mathbb{1}_A \circ \tau^{-1}) \\ &= \eta((f \vee g) \circ \tau^{-1})\mathbb{1}_B \\ &= \eta((f \circ \tau^{-1}) \vee (g \circ \tau^{-1}))\mathbb{1}_B \\ &= (Tf \vee Tg)\mathbb{1}_B. \end{aligned}$$

All that is left to show is that the equality holds outside of $B = \tau(\Omega)$. It suffices to show that Tf is zero outside of $\tau(\Omega)$ for all $f \in X$, as then both sides of the equality are zero on $\Omega \backslash B$. Suppose there exists a $S \in \Sigma$, disjoint from $\tau(\Omega)$ such that $Tf > 0$ on S for some $f \in X$. We then have $(Tf)\mathbb{1}_S = \eta(f \circ \tau^{-1})\mathbb{1}_S = 0$ as $(f \circ \tau^{-1})$ is supported on $\tau(\Omega)$, this follows from the definition of $(f \circ \tau^{-1})$ and the fact that τ is increasing. So we know that $\mu(S) = 0$ must hold and we are done. □

We now arrive to the main result of this section that uses all of the work we have done so far.

Theorem 3.10. *Suppose $X \subset L^p$ is order-dense and $T : X \to X$ is some operator, then the following are equivalent*

(i) $\forall f, g \in X : \ \inf\{Th : h \in \{f,g\}^u\} = Tf \vee Tg$

(ii) T is a Riesz homomorphism*

(iii) There exist $\eta \in \Sigma$ positive and $\mu-$homomorphism $\tau : \Sigma \to \Sigma$ such that

$$Tf = \eta(f \circ \tau^{-1}), \qquad f \in X.$$

Proof. Implication $(i) \Rightarrow (ii)$ follows from a similar argument as used in Lemma 2.6, replacing the open set $U \subset S$ by a set $S \in \Sigma$ with $\mu(S) > 0$. Now note that X is marjorizing as it is order dense. Also this means that it is a directed subspace of an Archimedean space, hence it is pre-Riesz. So its Riesz completion X^ρ exists and is a Riesz subspace of L^p. So if (ii) holds we can extend T to a Riesz homomorphism $T_\rho : X^\rho \to X^\rho$. As L^p is Dedekind complete and X^ρ is majorizing in L^p we can extend T to a lattice homomorphism $\hat{T}$ on L^p which by Theorem 3.8 has the desired form, which shows (iii). The last implication $(iii) \Rightarrow (i)$ is the same statement as Theorem 3.9.

□

We have already seen that on L^p spaces all Riesz homomorphisms are, in particular, also complete Riesz homomorphisms. Using the above theorem we get the following result for pervasive subspaces of L^p.

Corollary 3.11. *Let $X \subset L^p$ be pervasive and order dense. Any Riesz* homomorphism $T : X \to X$ is a complete Riesz homomorphism.*

Proof. Let X and T be given as in the statement. By the above theorem T extends to a Riesz homomorphism $\hat{T} : L^p \to L^p$. Let $A \subset X$ be such that $\inf A = 0$. As L^p is Dedekind complete and X is pervasive, Lemma 1.10 tells us that $\inf A = 0$ also holds in L^p. Corollary 3.6 now tells us that $\hat{T}$ is a complete Riesz homomorphism, hence $\inf \hat{T}(A) = 0$. As A is a subset of X we have $\hat{T}(A) = T(A)$ and as X is a subspace of L^p we have that $\inf \hat{T}(A) = \inf T(A) = 0$ holds in X. □

Analogous to the case of continuous functions we can use this theorem to try to show that bijective Riesz* homomorphism have an inverse that is again a Riesz* homomorphism. Similarly, it requires X to be pervasive. The structure of the proof is nearly the same as that of Theorem 2.8 but a bit more involved.

Theorem 3.12. *If X is in addition pervasive and $T : X \to X$ is a bijective Riesz* homomorphism, then so is T^{-1}.*

Proof. Let T be as in the statement, then we know by Theorem 3.10 that there exist $\eta \in \Sigma$ positive and μ-homomorphism τ such that $T = T_{\eta,\tau}$. For notation purposes let $\hat{T}$ be the extension of T.
Let us define $S = \{\eta = 0\}$. Then we see that for all $f \in X$ one has $(Tf)\mathbb{1}_S = 0$. Suppose $\mu(S) > 0$ holds. As X is order-dense we can find a $f \in X$ with $f \geq \mathbb{1}_S$ so this f can not be in the range of T which contradicts the bijectivity of T. hence, $\mu(S) = 0$ holds and the map $s \mapsto \frac{1}{\eta(s)}$ is defined almost everywhere. Next we show that τ is bijective. To this end let $A, B \in \Sigma$ be given with $A \neq B$ and $\tau(A) = \tau(B)$. From the observation of the previous section we have $\mu(A \triangle B) > 0$, so $\mathbb{1}_{A\setminus B} > 0$ or $\mathbb{1}_{B\setminus A} > 0$ we assume the first. As X is pervasive we can find a $f \in X$ positive, $f \neq 0$, with $f \leq \mathbb{1}_{A\setminus B}$. As T is both positive and injective we get

$$0 < Tf \leq \hat{T}\mathbb{1}_{A\setminus B}.$$

This shows that $\mu([\hat{T}1_{A\setminus B} > 0]) > 0$ holds. This is by definition the same as $\mu(\tau(A\setminus B)) > 0$. As τ is a μ-homomorphism, it preserves the set-theoretic operations $(\cup, \cap, \setminus, \triangle)$. So we get

$$0 < \mu(\tau(A\setminus B) \cup \tau(B\setminus A)) = \mu(\tau(A) \triangle \tau(B)).$$

Hence, we get $\tau(A) \neq \tau(B)$ which gives us a contradiction, so τ is injective. In order to show that τ is surjective we need the following claim.

Claim. *If $f \in X_+$, then $[Tf > 0] \in \mathcal{R}(\tau)$, the range of τ.*

Proof. Let $f \in X_+$ be given and let (f_n) be a sequence of positive, simple functions with $f = \sup_n f_n$ and where we have $f_n = \sum_m \alpha_{n,m}\mathbb{1}_{A_{n,m}}$, with $\alpha_{n,m} > 0$ and $A_{n,m} \in \Sigma$. Then we get

$$\begin{aligned}
[\hat{T}f_n > 0] &= [\sum_m \hat{T}\alpha_{n,m}\mathbb{1}_{A_{n,m}} > 0] \\
&= \cup_m[\alpha_{n,m}\hat{T}\mathbb{1}_{A_{n,m}} > 0] \\
&= \cup_m[\hat{T}\mathbb{1}_{A_{n,m}} > 0] \\
&= \cup_m\tau(A_{n,m}).
\end{aligned}$$

We can now apply Lemma 3.5 to get

$$\begin{aligned}
[Tf > 0] &= [\hat{T}(\sup_n f_n) > 0] \\
&= [(\sup_n \hat{T}f_n) > 0] \\
&= \cup_n[\hat{T}f_n > 0] \\
&= \cup_n \cup_m \tau(A_{n,m}) = \tau(\cup_n \cup_m A_{n,m}).
\end{aligned}$$

□

Now let $A \in \Sigma$ be given, by pervasiveness of X, Theorem 1.9 gives us then

$$\mathbb{1}_A = \sup\{f \in X : 0 \leq f \leq \mathbb{1}_A\}.$$

As L^p is super Dedekind complete by Theorem 3.3 we can find a sequence (f_n) in X_+, $f_n \leq \mathbb{1}_A$ with $\mathbb{1}_A = \sup_n f_n$. Observe that we then have $\cup_n[f_n > 0] \subset A$. Now let $B := A\setminus \cup_n [f_n > 0]$ and suppose $\mu(B) > 0$. Then all f_n must be zero on B and we get for all $n \in \mathbb{N}$ that

$$f_n \leq \mathbb{1}_{A\setminus B} < \mathbb{1}_A.$$

As $\mu(B) > 0$ this shows that $\sup_n f_n < \mathbb{1}_A$ which gives us a contradiction so we get $A = \cup_n[f_n > 0]$. As T is bijective we can find $g_n \in X$ with $Tg_n = f_n$. We want to apply the previous claim on these g_n so we have to show that they are positive. We prove the following more general case.

Claim. *If X is pervasive, $T : X \to X$ an injective Riesz* homomorphism, then T is bipositive.*

Proof. Suppose there exists a $g \in X$, $g \ngeq 0$ such that $Tg \geq 0$. We then have $g^- > 0$ so by the pervasiveness of X there exists an $h \in X$ with $0 < h \leq g^-$. So we get

$$0 \leq Th = \hat{T}h \leq \hat{T}g^- = (\hat{T}g)^- = (Tg)^- = 0.$$

This shows that $Th = 0$ holds which now contradicts the injectivity of T. □

So our g_n are positive and the first claim gives us $[Tg_n > 0] = [f_n > 0] \in \mathcal{R}(\tau)$, therefore, we also have $A \in \mathcal{R}(\tau)$ and τ is surjective. Now we can define the operator $S : X \to X$ by

$$Sf = ((\frac{1}{\eta}f) \circ (\tau^{-1})^{-1})$$

which is well-defined by the work done above. Similarly as in the proof of Theorem 2.8 we can show that $TS = \mathrm{Id}_X = ST$. In the proof of Lemma 3.7 we have seen that τ preserves countable unions and intersections. It is then clear that the same holds for τ^{-1} which then shows that it is also a μ-homomorphism. So we can invoke Theorem 3.10 and conclude that T^{-1} is a Riesz* homomorphism. □

In all of the above theorems, we have looked at operators $T : X \to Y$ that are of the form $Tf = \eta(f \circ \tau^{-1})$. It is interesting to note that, in the paper [11] of Halmos and Neumann, it is shown that μ-homomorphisms on the σ-algebra induce maps on the underlying space Ω under certain conditions imposed on the measure space, in their terminology *normal.* A measure space is normal if it is point isomorphic to the unit interval. The formal definitions are given in [11] as Definitions 1 and 2 on page 336. On normal measure spaces, for example bounded subspaces of $\mathbb{R}^d$, the operator T as above can in fact be given as a 'true' composition multiplication operator, i.e. there exists a measurable map $\pi : \Omega \to \Omega$ such that

$$(Tf)(s) = \eta(\omega)f(\pi(\omega))$$

holds for all $f \in X$ and μ-almost every $\omega \in \Omega$.

Discussion

We have investigated composition multiplication operators on pre-Riesz subspaces of $C(\Omega)$. We witnessed a close relation between these operators and the Riesz* homomorphims. On a wide class of pre-Riesz subspaces, namely the pointwise order dense ones, these two type of operators coincide. This result, in a way, is a generalization of the theorem discussed in the introduction. On one hand it uses the theorem in its proof and on the other hand it extends the statement to not necessarily Riesz subspaces. Where naturally the Riesz* homomorphisms take on the role of the Riesz homomorphisms, as they are exactly the operators that extend to one.
All of this takes place on pointwise order dense subspaces of $C(\Omega)$. One could naturally ask whether the same result holds on a wider class of subspaces. Given a Riesz* homomorphism on a pre-Riesz subspace of $C(\Omega)$, it seems inevitable to have to extend it to a Riesz homomorphism on $C(\Omega)$ in order to show it is of composition multiplication type. Therefore, one needs to require that subspace to be at least order-dense in $C(\Omega)$. If we want all Riesz* homomorphisms to satisfy a pointwise structure, namely being of composition multiplication type, on this subspace it seems natural to impose pointwise order denseness on it. Whether this is indeed necessary is still an open problem.
When considering the space of measurable functions $L^p(\Omega, \Sigma, \mu)$ on some finite measure space, it turns out that requiring a subspace to be order-dense is in fact sufficient to guarantee that all Riesz* homomorphisms on it are composition multiplication operators. In this case the concepts of order dense and pointwise order dense coincide.
A clear link between Riesz* homomorphisms and composition multiplication operators on (pointwise) order dense subspaces of (Ω) and $L^p(\Omega, \mu)$ has been exhibited. Reflecting on oberservations made in the introduction, one could naturally ask, whether the two above mentioned types of operators coincide with the positive disjointness preserving operators on the same subspaces. It is clear that this will not hold generally, as the space of polynomials on for example $[0, 1]$ does not have a pair of non trivial disjoint elements, hence all its operators are disjointness preserving. More research in the lines of [8] needs to be done to give a full answer to this question.

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