S.L. van Lieshout

$\triangle$-normality versus functional-$\triangle$-normality

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Scriptiebegeleider: dr. K.P. Hart

Mathematisch Instituut, Universiteit Leiden

1 Abstract

We will investigate the position of the diagonal in the square of a topological space. Suppose a closed set disjoint from the diagonal can be separated from the diagonal by a continuous function. Then certainly this closed set and the diagonal can be separated by open sets. On the other hand, if a closed set and the diagonal can be separated by open sets then it does not always hold that the closed set and the diagonal can be separated by a continuous function. Examples of spaces that show this are not Tychonov however and this prompted R.Z. Buzyakova to ask whether separation by open sets implies separation by continuous functions in the class of Tychonov spaces. Since we expect it also does not hold for Tychonov spaces our main focus will be constructing a counterexample for this problem.

2 Introduction

From now on, when we write space we will mean topological space. We will start with some definitions.

Definition 2.1. *If X is a space then we let*

$$\triangle := \{(x, x) \mid x \in X\} \subset X^2$$

*and call $\triangle$ the **diagonal** of X^2.*

Definition 2.2. *A space X is called $\boldsymbol{T}_1$ if for each pair of different points $x, y \in X$ there exists an open set $O \subset X$ so that $x \in O$ and $y \notin O$.*

Definition 2.3. *A space X is called **completely regular** if for every nonempty closed set $F \subset X$ and point $x \in F^c$ there exists a continuous map $f : X \to [0, 1]$ so that $f[F] = \{0\}$ and $f[\{x\}] = \{1\}$*

Definition 2.4. *A space X is called **Tychonov** if it is T_1 and completely regular.*

Definition 2.5. *A space X is called $\triangle$**-normal** if for every open set $U \subset X^2$ for which $\triangle \subset U$ there exists an open set $V \subset X^2$ for which $\triangle \subset V$ and $\overline{V} \subset U$.*

Definition 2.6. *A space X is called **functionally** $\triangle$**-normal** if for every open set $U \subset X^2$ for which $\triangle \subset U$ there exists a continuous map $f : X^2 \to [0, 1]$ so that $f[\triangle] \subset \{0\}$ and $f[U^c] \subset \{1\}$.*

The problem we will try to solve, which is also stated in the two articles [1] and [2], is the following:

Problem 2.7. *Is it true that for any Tychonov space X it holds that*

$$X \text{ is } \triangle\text{-normal} \to X \text{ is functionally } \triangle\text{-normal.}$$

In article [3] we see that T_1 $\triangle$-normal spaces are regular.

Definition 2.8. *A space X is called **regular** if for every closed set $F \subset X$ and point $x \in F^c$ there exists an open set O containing F and an open set P containing x so that $O \cap P = \emptyset$.*

Anyhow, we have to note that the assumption of complete regularity in problem 2.7 is necessary because T_1 functionally $\triangle$-normal spaces are completely regular and in [3] one finds an example of a T_1 $\triangle$-normal space that is not completely regular.
Now we will start to look for a counterexample, that is, we try to find a $\triangle$-normal Tychonov space which is not functionally $\triangle$-normal. Let us first look at the metrizable spaces.

3 Metrizable Spaces

Definition 3.1. *A **metrizable space** is a space for which the topology can be induced by a metric.*

We have the following definition and lemma.

Definition 3.2. *A space X is called **normal** if for every pair of closed sets $A, B \subset X$ for which $A \cap B = \emptyset$ there exists an open set O containing A and an open set V containing B so that $O \cap V = \emptyset$.*

Lemma 3.3. *If X is a metrizable space then X is normal.*

Proof. Suppose d is a metric on X and $A, B \subset X$ is a pair of closed sets for which $A \cap B = \emptyset$. For each $x \in A$ we let r_x be so that $B_{r_x}(x) \subset B^c$, where $B_r(x)$ denotes the open ball around x with radius r. Now define

$$C := \bigcup \{B_{\frac{1}{2}r_x}(x) \mid x \in A\} \supset A.$$

Note that we are done when we show that $\overline{C} \subset B^c$. So suppose $y \in \overline{C}$ and $y \notin B^c$, then in particular $y \notin A$ so let $s := \frac{1}{2} \inf\{d(x,y) \mid x \in A\}$. Note that $s = 0$ would imply $y \in \overline{A} = A$ which is not the case, so $s > 0$. So we can consider $B_{\frac{1}{2}s}(y)$ and because $y \in \overline{C}$ we have $B_{\frac{1}{2}s}(y) \cap C \neq \emptyset$ and thus there must be an $x \in A$ so that $B_{\frac{1}{2}s}(y) \cap B_{\frac{1}{2}r_x}(x) \neq \emptyset$. By the triangle inequality we then have $d(x,y) \leq \frac{1}{2}s + \frac{1}{2}r_x$. While on the other hand by construction we have $s < d(x,y)$. So it follows that $r_x > s$ and thus $d(x,y) < r_x$ and thus $y \in B_{r_x}(x) \subset B^c$, contradiction. $\square$

It is not to difficult to check the following fact:

Lemma 3.4. *A space X is Hausdorff if and only if X^2 has a closed diagonal.*

Proof. Suppose X is Hausdorff, then take an element $(x,y) \notin \triangle$. Then we have $x \neq y$ so there are open sets U_x and U_y so that $x \in U_x$, $y \in U_y$ and $U_x \cap U_y = \emptyset$. Now we have $U_x \times U_y$ is open in X^2, contains (x,y) and $U_x \times U_y \cap \triangle = \emptyset$. So then X^2 has a closed diagonal.

Suppose X^2 has a closed diagonal and consider two different points $x, y \in X$ then clearly $(x, y) \notin \triangle$ so we can find a basic open set $U \times V$ of X^2 so that $U \times V \cap \triangle = \emptyset$ and $(x, y) \in U \times V$. So then $x \in U$ and $y \in V$ but $U \cap V = \emptyset$. $\square$

Now note that if X is metrizable then X^2 is also metrizable. Since if d defines a metric on X then $d_{X^2}((x_1, y_1), (x_2, y_2)) = \sqrt{d^2(x_1, x_2) + d^2(y_1, y_2)}$ defines a metric on X^2 that induces the product topology. We now get

Corollary 3.5. *Metrizable spaces are $\triangle$-normal.*

Proof. Consider a metrizable space X and an open set $U \subset X^2$ for which $\triangle \subset U$. Then X^2 is metrizable and thus normal by lemma 3.3. Also X^2 has a closed diagonal $\triangle$ since X is hausdorff, see lemma 3.4. So we have two disjoint closed sets U^c and $\triangle$ in X^2 for which we can find disjoint open sets W and V so that $U^c \subset W$ and $\triangle \subset V$. Then we have that W^c is closed and $V \subset W^c$ so $\triangle \subset V \subset \overline{V} \subset W^c \subset U$. So X is $\triangle$-normal. $\square$

But regretfully all metrizable spaces are also functionally $\triangle$-normal. To see this we take a look at the lemma of Urysohn which is well known from general topology [5]. We state it as a fact:

Fact 3.6. *(Urysohn's Lemma) If a space X is normal then for each pair of disjoint closed sets $A, B \subset X$ there exists a continuous map $f : X \to [0, 1]$ so that $f[A] \subset \{0\}$ and $f[B] \subset \{1\}$.*

Corollary 3.7. *Metrizable spaces are functionally $\triangle$-normal.*

So we learned that all metrizable spaces are not suited for a counterexample of problem 2.7. So it is useful to check the following important result about metrizable spaces.

Fact 3.8. *(Urysohn metrization Theorem) A space X is separable and metrizable if and only if X is regular and has a countable basis.*

Definition 3.9. *A space X is called **separable** if it contains a countable dense subset.*

We already noted that a T_1 $\triangle$-normal space is regular so by fact 3.8 we conclude that the counterexample we are looking for may not have a countable basis. An example of a space which does not need to have a countable basis is the topological space induced by an almost disjoint family (a.d.f) on ω. These spaces are often used to construct counterexamples.

4 a.d.f. spaces

First we define the natural numbers in a formal way.

Definition 4.1. *We define $0 := \emptyset$, $n := \{0, 1, \ldots, n-1\}$ and $\omega := \{0, 1, 2, \ldots\}$.*

Definition 4.2. *A set $\mathcal{A} \subset \mathcal{P}(\omega)$ is called an* ***almost disjoint family*** *on ω or an* ***a.d.f.*** *on ω if*

1. *$\forall(A, B \in \mathcal{A})(A \cap B$ is finite) and*
2. *$\forall(A \in \mathcal{A})(A$ is infinite).*

To make the right space we are going to deal with uncountable a.d.f.'s. At first glance it might be counterintuitive that uncountable a.d.f.'s on ω even exist. But they do, and a nice example is the following one:

Example 4.3. For each $a \in \mathbb{R}$ we have a pair of lines in $\mathbb{R}^2$

$$l_{a,1} := \{(x, ax - 1/2) \mid x \in \mathbb{R}\} \text{ and } l_{a,2} := \{(x, ax + 1/2) \mid x \in \mathbb{R}\}.$$

Each pair determines a subset of $\mathbb{Z}^2$, namely the set of lattice points between these lines:

$$N_a := \{(x, y) \in \mathbb{Z}^2 \mid l_{a,1}(x) \leq y \leq l_{a,2}(x)\}.$$

The vertical distance between the lines is 1 so for every $a \in \mathbb{R}$ the disjoint sets $S_n = \{(n, m) \mid m \in \mathbb{Z}\}$ with $n \in \mathbb{Z}$ each intersect N_a in at least 1 point. So the N_a are infinite. If $a, b \in \mathbb{R}$ and $a \neq b$ we note that we can find an $r \in \mathbb{Z}$ so that $N_a \cap N_b \subset B_r(0)$ and thus $N_a \cap N_b$ is finite. So we can use any bijection $f : \mathbb{Z}^2 \to \omega$ and conclude that $\{f[N_a] \mid a \in \mathbb{R}\}$ is an a.d.f. on ω.

Remark 4.4. By example 4.3 we also give a way to represent the elements $a \in \mathbb{R}$ in a unique way, namely as subsets of ω via the injection $a \mapsto f[N_a]$.

Definition 4.5. *The* ***space*** *X* ***induced by an a.d.f.*** *$\mathcal{A}$ on ω is defined by $X := \mathcal{A} \cup \omega$, with its topology induced by the basis:*

$$\mathcal{B} := \{\{x\} \mid x \in \omega\} \cup \{U_n(A) \mid A \in \mathcal{A}, n \in \omega\}$$

where for each pair $A \in \mathcal{A}$, $n \in \omega$ we define

$$U_n(A) := \{A\} \cup A \backslash n$$

Remark 4.6. The $\mathcal{B}$ defined above indeed forms a basis. We have $X = \cup\mathcal{B}$, and if $x \in A \cap B$ with $A, B \in \mathcal{B}$ then either $x \in \omega$ so that $\{x\} \in \mathcal{B}$ has $x \in \{x\} \subset A \cap B$, or $x \in \mathcal{A}$ and then A, B must be equal to $U_{n_A}(x)$ respectively $U_{n_B}(x)$ for certain $n_A, n_B \in \omega$ so then we have $U_{\max(n_A,n_B)}(x) \in \mathcal{B}$ with $x \in U_{\max(n_A,n_B)}(x) = A \cap B$.

To clarify this space a little more intuitively, we can think of this space as follows. We have a set of points $x \in \omega$ next to a set of points $A \in \mathcal{A}$. The points of ω are connected components and the n-th basic open sets around a point $A \in \mathcal{A}$ are of the form $U_n(A)$ consisting of the point A itself plus points of ω which are also in the set A and greater or equal than n.

Remark 4.7. An a.d.f. which is countable induces a space which has a countable basis and is regular and thus is separable and metrizable. So we only need to consider the uncountable a.d.f.'s.

That this kind of topological spaces forms a good candidate for a counterexample follows from the lemma below.

Lemma 4.8. *If X is a space induced by an uncountable a.d.f. $\mathcal{A}$ on ω then X is T_1, completely regular and has no countable basis.*

Proof. If $x \in \mathcal{A}$ then we have $X \backslash \{x\} = \omega \cup \bigcup \{U_0(y) \mid y \in \mathcal{A} \backslash \{x\}\}$ is open, and if $x \in \omega$ we have that $X \backslash \{x\} = \omega \backslash \{x\} \cup \bigcup \{U_{x+1}(y) \mid y \in \mathcal{A}\}$ is open. So $\{x\}$ is closed for any $x \in X$, so X is T_1.
To prove complete regularity we show that whenever $x \in O$ and $O \subset X$ is open we can take a $D \subset O$ clopen so that $x \in D$. This is enough since then

$$\begin{array}{llll} f: & X & \to & [0,1] \\ & x & \mapsto & \begin{cases} 0 & \text{if } x \in D \\ 1 & \text{if } x \in D^c \end{cases} \end{array}$$

defines a continuous function separating x from the closed set O^c. First suppose $x \in \mathcal{A}$, then there exists a n_x for which $x \in U_{n_x}(x) \subset O$. $U_{n_x}(x)$ is (also) closed since if $y \in \omega \backslash U_{n_x}(x)$ then $\{y\} \cap U_{n_x}(x) = \emptyset$ and if $y \in \mathcal{A} \backslash U_{n_x}(x)$ then $U_{\max\{y \cap x\}+1}(y) \cap U_{n_x}(x) = \emptyset$. Finally if $x \in \omega$ then we simply take $\{x\}$ as clopen set.
Let $\mathcal{C}$ be a basis of X and consider the map:

$$\begin{array}{lllll} m: & \mathcal{A} & \to & \mathcal{C} & \\ & A & \mapsto & m(A) & \text{so that } A \in m(A) \subset U_0(A) \end{array}$$

This map is injective, for if $A, B \in \mathcal{A}$ are different then $A \in m(A) \backslash m(B)$ so $m(A) \neq m(B)$. And since $\mathcal{A}$ is uncountable also $\mathcal{C}$ is uncountable. $\square$

We want to investigate whether these a.d.f. spaces are $\triangle$-normal, but we find it rather difficult to say something about these spaces in general. So we decide to look at a specific kind of a.d.f. spaces, the Lusin spaces.

4.1 Lusin spaces

A Lusin space is a space induced by a Lusin a.d.f.. The construction of a Lusin a.d.f makes use of ordinal numbers. Therefore we will begin defining them and introducing some facts about them. More explanation about these facts can be found in [6].

Definition 4.9. *A set α is called an **ordinal number** if it is a transitive set that is well-ordered by $\in$.*

When we have two ordinal numbers α and β we denote $\alpha < \beta$ when $\alpha \in \beta$ and $\alpha \leq \beta$ when $\alpha \in \beta$ or $\alpha = \beta$.

Definition 4.10. *A set α is called **transitive** if it holds that $\beta \in \alpha \to \beta \subset \alpha$.*

Example 4.11. ω is an ordinal number.

Remark 4.12. One can check that if α is an ordinal number and $\beta \in \alpha$ then β is also an ordinal number.

And to every ordinal number belongs a unique cardinal number.

Definition 4.13. *If α is an ordinal number we define the **cardinal number** of α by $|\alpha| := \min\{\beta \leq \alpha \mid \text{there is a bijection between } \beta \text{ and } \alpha\}$.*

Also we have the following.

Fact 4.14. *(Transfinite induction over **ON**)*

$$\text{The class } \boldsymbol{ON} := \{\alpha : \alpha \text{ is an ordinal}\} \text{ is well-ordered by} \in .$$

So there is a unique smallest uncountable ordinal number ω_1.
Now we are ready to define the family:

Definition 4.15. *A **Lusin family** (or a **Lusin a.d.f.**) is an a.d.f.*

$$\mathcal{A} = \{A_\eta \mid \eta \in \omega_1\}$$

with the property that for all $\eta \in \omega_1$ and for all $k \in \omega$ we have that

$$\{\xi \in \eta \mid \max(A_\eta \cap A_\xi) < k\} \text{ is finite.}$$

This property we will refer to as the Lusin property.

Theorem 4.16. *Lusin families exist.*

Proof. We start by defining sets A_n for each $n \in \omega$. We take $A_0 \subset \omega$ so that A_0 is infinite and $\omega \backslash A_0$ is infinite. We now continue inductively taking for each $n > 0$ an infinite set $A_n \subset \omega \backslash \bigcup\{A_d \mid d \in n\}$ so that A_n is infinite and $\omega \backslash \bigcup\{A_d \mid d \in n+1\}$ is infinite. Note that these A_n with $n \in \omega$ are pairwise disjoint and clearly satisfy the Lusin property. Now we still need to define A_ζ for $\zeta \in \omega_1 \backslash \omega$ in the right manner. We do this with transfinite induction, suppose $\zeta \in \omega_1 \backslash \omega$ and suppose A_η is well defined for all $\eta \in \zeta$. We then have $\zeta \notin \omega$ so $\omega \subset \zeta$ and $\zeta \in \omega_1$ so $|\zeta| \subset \omega$ so there is a bijection $b : \omega \to \zeta$. If $n \in \omega$ and $k \in n$ then $b(k) \neq b(n)$, so either $b(k) \in b(n) \in \zeta$ or $b(n) \in b(k) \in \zeta$. Then by the induction hypothesis $A_{b(n)}$ and $A_{b(k)}$ are both infinite subsets of ω and $A_{b(n)} \cap A_{b(k)}$ is finite, so the set $S(n) := A_{b(n)} \backslash \bigcup\{A_{b(k)} \mid k \in n\} \subset \omega$ is infinite for all $n \in \omega$. Now we make an unbounded function $a : \omega \to \omega$ by picking $a(n) \in S(n)$ so that $a(n) > n$ for each $n \in \omega$. We finally define $A_\zeta := \operatorname{ran}(a) := \{y \in \omega \mid \exists x : a(x) = y\}$. We now only need to check that A_ζ satisfies the desired properties.

Suppose $n \in \omega$ then if $a(l) \in A_{b(n)}$ for a certain $l \in \omega$ we must have $l \leq n$ because $a(l) \in S(l)$, from this follows $|\operatorname{ran}(a) \cap A_{b(n)}| \leq n + 1$. So $A_\zeta \cap A_{b(n)} = \operatorname{ran}(a) \cap A_{b(n)}$ is finite for all $n \in \omega$. So $A_\zeta \cap A_\xi$ is finite whenever $\xi \in \zeta$. Also A_ζ is infinite since a is unbounded.

Now we prove the property for ζ, suppose $k \in \omega$ then for all $n \in \omega$ for which $n \geq k$ we have $A_\zeta \cap A_{b(n)} = \operatorname{ran}(a) \cap A_{b(n)} \ni a(n) > n \geq k$ so

$$\{n \in \omega \mid \max\{A_\zeta \cap A_{b(n)}\} < k\}$$

is finite. And so $\{\xi \in \zeta \mid \max\{A_\zeta \cap A_\xi\} < k\}$ is finite for each $k \in \omega$.

Now we may conclude that A_ζ is well defined for all $\zeta \in \omega_1$, since otherwise $\{\zeta \in \omega_1 \mid A_\zeta \text{ is not well defined}\} \subset \omega_1$ would be nonempty and its minimum would yield a contradiction. $\square$

One can find more about Lusin families in [4]. The Lusin property makes a Lusin space more easy to describe then a general a.d.f. space. We found the following result.

Theorem 4.17. *A Lusin space is not $\triangle$-normal.*

Proof. Let X be a space induced by a Lusin a.d.f. $\mathcal{A} = \{A_\eta \mid \eta \in \omega_1\}$. If we have an open set $U \subset X^2$ with $\triangle \subset U$ then for every $(x,x) \in \triangle$ we can pick a basic open set around it which is contained in U. The union of these basic open sets gives us a 'typical' open set $O \subset U$ which is of the form:

$$O := \bigcup\{(x,x) \mid x \in \omega\} \cup \bigcup\{U^2_{n_\eta}(A_\eta) \mid \eta \in \omega_1\} = \triangle \cup \bigcup\{U^2_{n_\eta}(A_\eta) \mid \eta \in \omega_1\}.$$

These kind of typical open sets in fact form a neighborhood basis of the diagonal. Now we prove the following.

Claim 4.18. *For any typical open $O \subset X^2$ we have $(\mathcal{A}^2 \backslash \triangle) \cap \overline{O} \neq \emptyset$.*

Proof. Let O be as above and $\mathcal{A} = \{A_\eta \mid \eta \in \omega_1\}$ be a Lusin a.d.f.. We have the map

$$\begin{array}{cccc} n: & \omega_1 & \to & \omega \\ & \eta & \mapsto & n_\eta. \end{array}$$

So there is a certain $l \in \omega$ so that $|n^{-1}(l)| = \omega_1$. Note that we then can take a $\delta \in n^{-1}(l)$ so that $|\delta \cap n^{-1}(l)| = \omega$. Now let $\alpha, \beta \in \omega_1 \backslash \delta$ be a given pair and let $U_k(A_\alpha) \times U_k(A_\beta)$ be any basic neighbourhood of (A_α, A_β), where without loss of generality we assume $k \geq l$. It follows from the Lusin property that the sets

$$\{\xi \in \alpha \mid \max(A_\alpha \cap A_\xi) \leq k\} \text{ and } \{\xi \in \beta \mid \max(A_\beta \cap A_\xi) \leq k\}$$

are both finite. So there exists a $\xi \in n^{-1}(l) \cap \delta$ so that both:

$$i := \max(A_\alpha \cap A_\xi) > k \text{ and } j := \max(A_\beta \cap A_\xi) > k,$$

then we have $(i,j) \in U^2_l(A_\xi) \subset O$ as well as $(i,j) \in U_k(A_\alpha) \times U_k(A_\beta)$. So we must have $(A_\alpha, A_\beta) \in \overline{O}$. $\square$

To see that the space induced by a Lusin a.d.f. is not delta normal we assume without loss of generality that U is a typical open set around the diagonal. And thus U has the property that $(\mathcal{A}^2 \backslash \triangle) \cap U = \emptyset$. If X is $\triangle$-normal we could find another typical open set V so that $\triangle \subset V$ and $\overline{V} \subset U$. But by the claim we have $(\mathcal{A}^2 \backslash \triangle) \cap \overline{V} \neq \emptyset$, which is in contradiction with $\overline{V} \subset U$. $\square$

So the Lusin spaces will not provide us with a counterexample.

Remark 4.19. We showed that if $\mathcal{A}$ is a Lusin set and O is an open set around the diagonal in the induced product space then $(\mathcal{A}^2 \backslash \triangle) \cap \overline{O} \neq \emptyset$. This while typical open sets around the diagonal have the property $(\mathcal{A}^2 \backslash \triangle) \cap O = \emptyset$.

A key role in the proof above is played by the Lusin property. We couldn't say something about about the $\triangle$-normality of a.d.f. spaces in general but thanks to this property we found a special class of a.d.f spaces which are not $\triangle$-normal; the Lusin spaces. Anyhow, to find a counterexample for problem 2.7 we want to find an a.d.f. space which is $\triangle$-normal. So in the next section we will discuss a new class of a.d.f. spaces which are not Lusin spaces.

4.2 f.f. spaces

We start with some notations. For $n \leq \omega$ we let 2^n denote the set of functions from n to 2. We also let $2^{<\omega} = \bigcup\{2^n \mid n < \omega\}$. Note that $|2^\omega| = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$ and $|2^{<\omega}| = |\mathbb{N}| = \omega$. Also the restriction of a map $f \in 2^\omega$ to $n = \{0, 1, \ldots, n-1\}$ is denoted as $f \upharpoonright n$.

Definition 4.20. *The* ***space*** X ***induced by a family of functions (f.f.)*** $\mathcal{F} \subset 2^\omega$ *is defined by* $X := 2^{<\omega} \cup \mathcal{F}$*, with its topology induced by the basis:*

$$\mathcal{B} := \{\{f \upharpoonright n\} \mid f \in 2^\omega, n \in \omega\} \cup \{U_n(f) \mid f \in \mathcal{F}, n \in \omega\}$$

where for $f \in \mathcal{F}$ *and* $n \in \omega$ *we define*

$$U_n(f) := \{f\} \cup \{f \upharpoonright k \mid k \in \omega \backslash n\}$$

Remark 4.21. Note again, like in Remark 4.7, we only need to consider uncountable f.f.'s.

Remark 4.22. Note that when A and B are two given f.f. sets with $A \subset B$ and the space induced by B is $\triangle$-normal then also the space induced by A is $\triangle$-normal. Indeed the smaller space is closed in the larger one.

The first thing we are going to prove now, is that these spaces are indeed a.d.f. spaces:

Lemma 4.23. *A space induced by an f.f. is homeomorphic to a certain space induced by an a.d.f..*

Proof. Suppose the space $\mathcal{F} \cup 2^{<\omega}$ is induced by the f.f. $\mathcal{F}$. We have a bijection:

$$\begin{array}{llll} b: & 2^{<\omega} & \to & \omega \\ & (f : n \to 2) & \mapsto & \begin{cases} f(0) \cdot 2^0 + \ldots + f(n-1) \cdot 2^{n-1} + 2^n - 1 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \end{array}$$

Let $b^* : \mathcal{P}(2^{<\omega}) \to \mathcal{P}(\omega)$ denote the bijection induced by b. Also we have an injection:

$$\begin{array}{llll} r: & 2^\omega & \to & \mathcal{P}(2^{<\omega}) \\ & f & \mapsto & \{f \upharpoonright n \mid n \in \omega\} \end{array}$$

We now can make the injection:

$$\begin{array}{llll} b^* \circ r: & 2^\omega & \to & \mathcal{P}(\omega) \\ & f & \mapsto & \{b(f\restriction 0), b(f\restriction 1), b(f\restriction 2), \ldots\} \end{array}$$

Now let $s := b^* \circ r$, $A_f := s(f)$ for all $f \in \mathcal{F}$ and let $\mathcal{A}_\mathcal{F} := \{A_f \mid f \in \mathcal{F}\}$. If $f, g \in \mathcal{F}$ then $(b^*)^{-1}(A_f \cap A_g) = r(f) \cap r(g)$ which is finite if $f \neq g$ so then $A_f \cap A_g$ is also finite. So $\mathcal{A}_\mathcal{F}$ is an a.d.f.. Let $\mathcal{A}_\mathcal{F} \cup \omega$ be the space induced by $\mathcal{A}_\mathcal{F}$ and denote $\tilde{s} := s \restriction \mathcal{F}$ then

Claim 4.24. *The map $\tilde{s} \cup b : \mathcal{F} \cup 2^{<\omega} \to \mathcal{A}_\mathcal{F} \cup \omega$ is a homeomorphism.*

Proof. $\tilde{s} \cup b$ is a well defined bijection, so it is enough to prove that $\tilde{s} \cup b$ and $(\tilde{s} \cup b)^{-1}$ send basic open sets to basic open sets. We first check whether the isolated sets behave well. We have that $(\tilde{s} \cup b)[\{f \restriction n\}] = \{b(f \restriction n)\}$ is open in $\mathcal{A}_\mathcal{F} \cup \omega$ and also we have that $(\tilde{s} \cup b)^{-1}(\{x\}) = \{b^{-1}(x)\}$ is open in $\mathcal{F} \cup 2^{<\omega}$. Now we check whether the other basic open sets behave well. Suppose $n \in \omega$ and $f \in \mathcal{F}$, let $l(n) := \min\{i \in \omega \mid b(f \restriction i) \geq n\}$. Then

$$\begin{array}{rcl} (\tilde{s} \cup b)^{-1}[U_n(A_f)] & = & \tilde{s}^{-1}[\{A_f\}] \cup b^{-1}[\{x \in A_f \mid x \in \omega \backslash n\}] \\ & = & \{f\} \cup \{f \restriction i \mid i \in \omega \backslash l(n)\} \\ & = & U_{l(n)}(f), \end{array}$$

and $U_{l(n)}(f)$ is open in $\mathcal{F} \cup 2^{<\omega}$. Also we have

$$\begin{array}{rcl} (\tilde{s} \cup b)[U_n(f)] & = & \tilde{s}[\{f\}] \cup b[\{f \restriction x \mid x \in \omega \backslash n\}] \\ & = & \{A_f\} \cup \{x \in A_f \mid x \in \omega \backslash b(f \restriction n)\} \\ & = & U_{b(f\restriction n)}(A_f), \end{array}$$

and $U_{b(f\restriction n)}(A_f)$ is open in $\mathcal{A}_\mathcal{F} \cup \omega$. □

□

One could expect that Lusin spaces and f.f. spaces are two different classes of spaces. This is indeed true and it can been seen when we compare the result below with claim 4.24 and remark 4.19 of the previous section.

Lemma 4.25. *For any typical open $O \subset X^2$ containing the diagonal we have $(\mathcal{F}^2 \backslash \triangle) \cap \overline{O} = \emptyset$.*

Proof. Let a typical open set $O = \triangle \cup \bigcup\{U^2_{n_f}(f) \mid f \in \mathcal{F}\}$ and a point $(f, g) \in \mathcal{F}^2 \backslash \triangle$ be given. Let $k := \min\{n \mid f(n) \neq g(n)\}| + 1$, then for all $h \in \mathcal{F}$ at least one of the following two holds:

1. $\forall(n \geq \max(k, n_h))(f \restriction n \neq h \restriction n)$
2. $\forall(n \geq \max(k, n_h))(g \restriction n \neq h \restriction n)$.

Because if not, then there would be a pair $n_1, n_2 \geq \max(k, n_h)$ so that

$$f \restriction n_1 = h \restriction n_1 \text{ and } g \restriction n_2 = h \restriction n_2,$$

which implies $f \restriction \min\{n_1, n_2\} = h \restriction \min\{n_1, n_2\} = g \restriction \min\{n_1, n_2\}$, so in particular $f \restriction k = g \restriction k$, which is impossible by the choice of k. So we must have for all $h \in \mathcal{F}$ that $U_k(f) \times U_k(g) \cap U^2_{n_h}(h) = \emptyset$. Note that also

$$U_k(f) \times U_k(g) \cap \triangle = \emptyset,$$

so $U_k(f) \times U_k(g) \cap O = \emptyset$. □

So we have the following:

$$\begin{gathered}\{\text{Lusin spaces}\}, \{\text{f.f. spaces}\} \subset \{\text{a.d.f. spaces}\} \\ \{\text{Lusin spaces}\} \cap \{\text{f.f. spaces}\} = \emptyset,\end{gathered}$$

and thus it makes sense to try to find a counterexample among the f.f. spaces. So we have to consider the right sets $\mathcal{F} \subset 2^\omega$. A priori it is difficult to say what these right sets are but they can be found when we put a certain topology on 2^ω. The topology we will work with is the discrete topology on 2 and the product topology on 2^ω. We can check that the space 2^ω then becomes a complete metric space with the metric

$$\begin{array}{llll} d: & 2^\omega \times 2^\omega & \to & [0,1] \\ & (f,g) & \mapsto & \begin{cases} 2^{-\min\{i: f(i) \neq g(i)\}} & \text{if } f \neq g \\ 0 & \text{if } f = g \end{cases} \end{array}.$$

Remark 4.26. Note that the basic open sets are of the form

$$\begin{array}{lll} B_{2^{-n}}(f) & := & \{g \in 2^\omega \mid d(f,g) < 2^{-n}\} \\ & = & \{g \in 2^\omega \mid 2^{-\min\{i: f(i) \neq g(i)\}} < 2^{-n}\} \\ & = & \{g \in 2^\omega \mid \min\{i : f(i) \neq g(i)\} > n\} \\ & = & \{g \in 2^\omega \mid f \restriction (n+1) = g \restriction (n+1)\} \end{array}$$

with $n \in \omega$.

The subsets of 2^ω we will first consider are the perfect sets.

4.2.1 The Perfect sets

Definition 4.27. *A* ***perfect set*** *is a non-empty closed set without isolated points.*

Because a perfect set of 2^ω is closed it becomes a non-empty complete metric space. Also we can prove that this set must be uncountable.

Theorem 4.28. *A perfect set of a complete metric space is uncountable.*

Proof. Suppose F is a perfect set of the complete metric space X. Because F is non-empty we can take an element $x_{(0)} \in F$ and consider the open ball with radius 1 around $x_{(0)}$; $B_1(x_{(0)})$. Recursively we now construct for each element $(a_i)_{i=0}^n \in 2^{<\omega}$ an element $x_{(a_i)_{i=0}^n} \in F$ and a ball $B_{r_{(a_i)_{i=0}^n}}(x_{(a_i)_{i=1}^n})$ around it. Suppose for $(a_i)_{i=0}^n \in 2^{<\omega}$ an element $x_{(a_i)_{i=0}^n}$ and a ball $B_{r_{(a_i)_{i=0}^n}}(x_{(a_i)_{i=0}^n})$ are constructed. Then because $x_{(a_i)_{i=0}^n}$ is not isolated we can find a pair of distinct points $x_{(a_0,\ldots,a_n,0)}, x_{(a_0,\ldots,a_n,1)} \in B_{r_{(a_i)_{i=0}^n}}(x_{(a_i)_{i=0}^n})$. Because X is a metric space and thus Hausdorff and regular we can find two open balls

$$B_{r_{(a_0,\ldots,a_n,0)}}(x_{(a_0,\ldots,a_n,0)}) \text{ and } B_{r_{(a_0,\ldots,a_n,1)}}(x_{(a_0,\ldots,a_n,1)})$$

with radii smaller then 2^{-n-1} both contained in $B_{r_{(a_i)_{i=0}^n}}(x_{(a_i)_{i=0}^n})$ so that

$$\emptyset = \overline{B_{r_{(a_0,\ldots,a_n,0)}}(x_{(a_0,\ldots,a_n,0)})} \cap \overline{B_{r_{(a_0,\ldots,a_n,1)}}(x_{(a_0,\ldots,a_n,1)})}$$

Now we can consider the map

$$\begin{array}{ccc} 2^\omega & \to & F \\ (a_i)_{i=0}^\infty & \mapsto & \lim_{n\to\infty} x_{(a_i)_{i=0}^n} \end{array}$$

It is enough to show that this map is well defined and injective. Suppose $\epsilon > 0$ is given then we can take $N \in \mathbb{Z}_{>0}$ so that $2^{-N} < \epsilon$ so that for all $n > N$ we have:

$$x_{(a_i)_{i=0}^n} \in B_{r_{(a_i)_{i=0}^n}}(x_{(a_i)_{i=1}^n}) \subset B_{r_{(a_i)_{i=0}^{n-1}}}(x_{(a_i)_{i=0}^{n-1}}) \subset \ldots \subset B_{r_{(a_i)_{i=0}^N}}(x_{(a_i)_{i=0}^N})$$

So if $n, m > N$ then $d(x_{(a_i)_{i=0}^n}, x_{(a_i)_{i=0}^m}) < 2^{-N} < \epsilon$ so $(x_{(a_i)_{i=0}^n})_{n=0}^\infty$ is a Cauchy sequence in F and this sequence must converge to a point in F because F is a closed subset of the complete metric space X. So the map is well defined.

Now suppose we have two distinct elements $(a_i)_{i=0}^\infty, (b_i)_{i=0}^\infty \in 2^\omega$. Then we can take $N := \min\{i : a_i \neq b_i\}$ so that for all $n > N$ we have

$$x_{(a_i)_{i=0}^n} \in \overline{B_{r_{(a_i)_{i=0}^N}}(x_{(a_i)_{i=0}^N})}$$
$$x_{(b_i)_{i=0}^n} \in \overline{B_{r_{(b_i)_{i=0}^N}}(x_{(b_i)_{i=0}^N})}$$

and both sets on the right hand side are disjoint so

$$\lim_{n\to\infty} x_{(a_i)_{i=0}^n} \neq \lim_{n\to\infty} x_{(b_i)_{i=0}^n}.$$

So the map is also injective. □

And even faster we can prove this is via the following well known theorem from general topology which we can find in [5].

Fact 4.29. *(Baire category theorem) A nonempty complete metric space cannot be represented as the union of a countable family of nowhere dense subsets.*

Corollary 4.30. *A perfect set of a non empty complete metric space is uncountable.*

Proof. Suppose F is a perfect set of a nonempty complete metric space, then F itself is a nonempty complete metric space. Suppose F is countable then F can be written as the countable union of its elements; $F = \bigcup\{\{x\} \mid x \in F\}$, which means that one of these singletons $\{x\}$ must be somewhere dense on an open set O. But this is only possible if $O = \{x\}$ which means that $\{x\}$ is isolated, contradiction. □

The Baire category theorem is also nice since it gives us another way to formalize small and big from the cardinalities.

Remark 4.31. A technique that is often used in proofs of set theory is using that there could occur a situation where sets are small or big. What we do here is analysing a typical open set V around the diagonal in the f.f. space induced by a perfect set $\mathcal{P} \subset 2^\omega$. We can write $V := \triangle \cup \bigcup\{U^2_{k_f}(f) \mid f \in \mathcal{P}\}$, so from V we get an induced map:

$$\begin{array}{cccc} k: & \mathcal{P} & \rightarrow & \omega \\ & f & \mapsto & k_f \end{array}$$

So we can write $\mathcal{P} := \bigcup\{k^{-1}(n) \mid n \in \omega\}$. Looking at the cardinalities we see that there must be an $n \in \omega$ so that $k^{-1}(n)$ is uncountable. This is one way of saying $k^{-1}(n)$ is big for a certain $n \in \omega$. But we can also apply the Baire Category theorem, since perfect sets of 2^ω are complete metric spaces themselves. So there has to be a certain $n \in \omega$ so that $k^{-1}(n)$ is somewhere dense, which is is another way of saying $k^{-1}(n)$ is big for a certain $n \in \omega$.

Also one can read in [1] how the Baire category theorem is used to prove that the Niemytski plane is not $\triangle$-normal. We now will use it in the same way to prove that the space induced by perfect sets of 2^ω are not $\triangle$-normal. But first we need to consider a lemma which we will use.

Lemma 4.32. *If X is a separable and metrizable space without isolated points then X can be written as a countable disjoint union of dense sets D_n:*

$$X = \bigcup\{D_n \mid n \in \omega\}$$

Proof. We will construct the D_n recursively for all $n \in \omega$. Since X is separable and metrizable it has a countable base $\mathcal{B} = \{B_i \mid i \in \omega\}$ by fact 3.8. By an analogous argument as in theorem 4.28 we see that the B_i must be uncountable. Use the axiom of choice to take $d_{i,0} \in B_i$ for each $i \in \omega$ and let

$$D_0 := \{d_{i,0} \mid i \in \omega\},$$

this set is dense by construction. Suppose D_k is defined for all $k \in n$ then we construct D_n by choosing $d_{n,i} \in B_i \backslash \bigcup\{D_k \mid k \in n\}$. Then $D_n = \{d_{n,i} \mid i \in \omega\}$ is again dense by construction and clearly disjoint from the D_k if $k \in n$. We can define D_k for all $k \in \omega$ in this way. The set we are left with is

$$D_\omega = X \backslash \bigcup\{D_n \mid n \in \omega\},$$

we may assume it is also dense, otherwise we add it to D_0. □

Now we give the proof.

Theorem 4.33. *The f.f. space induced by a perfect set $\mathcal{P}$ of 2^ω is not $\triangle$-normal.*

Proof. Let X be the f.f. space induced by $\mathcal{P}$. Using lemma 4.32 we may write $\mathcal{P} = \bigcup\{D_i \mid i \in \omega\}$, with the D_i pairwise disjoint and dense in $\mathcal{P}$. For each $f \in \mathcal{P}$ let n_f be so that $f \in D_{n_f}$. Now let

$$O := \triangle \cup \bigcup\{U^2_{n_f}(f) \mid f \in \mathcal{P}\}$$

be the basic open set around $\triangle$ determined by the function

$$\begin{array}{rccc} n: & \mathcal{P} & \to & \omega \\ & f & \mapsto & n_f. \end{array}$$

Suppose that $V := \triangle \cup \bigcup\{U^2_{k_f}(f) \mid f \in \mathcal{P}\}$ is another basic open set around $\triangle$, it suffices to prove $\overline{V} \subsetneq O$. Therefore consider the map induced by the open set V:

$$\begin{array}{rccc} k: & \mathcal{P} & \to & \omega \\ & f & \mapsto & k_f. \end{array}$$

$\mathcal{P}$ is a complete metric space, and $\mathcal{P} := \bigcup\{k^{-1}(n) \mid n \in \omega\}$. Hence the Baire Category theorem tells us that there exists a $m \in \omega$ so that $k^{-1}(m)$ is somewhere dense, say on the open set $J \subset \mathcal{P}$. Pick a fixed $g \in D_{m+1} \cap J$. We have $g \restriction m \notin U_{m+1}(g) = U_{n_g}(g)$ so $(g, g \restriction m) \notin O$. We will show that $(g, g \restriction m) \in \overline{V}$. Suppose $W \subset X^2$ is an open neighborhood of $(g, g \restriction m)$ then we can find an $l \in \omega \backslash m$ so that

$$(g \restriction l, g \restriction m) \in (U_l(g) \times \{g \restriction m\}) \subset W.$$

Now since $k^{-1}(m)$ is dense in J we can let $(h_i)_{i\in\omega}$ be a sequence in $k^{-1}(m)$ so that $\lim_{i\to\infty} h_i = g$ and thus we can fix an $i \in \omega$ so that we have $h_i \restriction l = g \restriction l$. Hence $(g \restriction l, g \restriction m) \in U^2_m(h_i)$. And because $h_i \in k^{-1}(m)$ we have

$$(g \restriction l, g \restriction m) \in U^2_m(h_i) = U^2_{k_{h_i}}(h_i) \subset V,$$

so $W \cap V \neq \emptyset$. □

So also the perfect sets of 2^ω will not give us a counterexample. And we can generalize this proof even more.

Fact 4.34. *(Suslin perfect set theorem) If X is a complete separable metric space then every uncountable Borel set of X contains a perfect set of X.*

Definition 4.35. *Let X be a space, then a set $B \subset X$ is called a **Borel set** if it can be formed from open sets through the operations of countable union, countable intersection and relative complement.*

Example 4.36. Perfect sets are Borel sets.

We can formulate the more general statement as:

Theorem 4.37. *The f.f. space induced by an uncountable Borel set of 2^ω is not $\triangle$-normal.*

Proof. Follows directly from Theorem 4.33, remark 4.22 and fact 4.34. □

So the f.f. space induced by the Borel sets of 2^ω won't give us a counterexample either. It seems a good idea to look for a subset of 2^ω that behaves different from a perfect set. The sets we are going to analyse next will behave different and are a bit more complicated.

4.2.2 The Q-sets

We will consider the $\mathcal{Q}$-sets of cardinality ω_1.

Definition 4.38. *Let Z be a space. A set $Q \subset Z$ is called a **$\mathcal{Q}$-set** if all subsets $A \subset Q$ can be written as the union of countably many relatively closed sets of Q.*

The existence of $\mathcal{Q}$-sets of 2^ω is not provable from the axioms of **ZFC**. But its existence can be proved after adding two axioms to **ZFC**, namely the negation of the continuum hypothesis ($\neg$**CH**) and Martins axiom (**MA**). (**MA** + $\neg$**CH**) can't be proven from **ZFC** but is independent and consistent with it. One can read more about these axioms in [7] and [8]. To prove the existance of the $\mathcal{Q}$-sets we will need a combinatorial fact which one can also find in [8]. This fact follows from **ZFC+MA**.

Fact 4.39. *Suppose we have two families $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(\omega)$, with cardinalities $|\mathcal{A}|, |\mathcal{B}| < |2^\omega|$ and with the property that for all finite $\mathcal{C} \subset \mathcal{A}$ and $B \in \mathcal{B}$ we have that $B \backslash \bigcup \mathcal{C}$ is infinite. Then there is an $M \subset \omega$ such that $B \backslash M$ is infinite for all $B \in \mathcal{B}$ but $A \backslash M$ is finite for all $A \in \mathcal{A}$.*

We now can prove the following. The topology on 2^ω is again the one given before where we put the discrete topology on 2 and the product topology on 2^ω.

Theorem 4.40. *If we have a set $X \subset 2^\omega$ of cardinality $|X| < |2^\omega|$, then X is a $\mathcal{Q}$-set.*

Proof. Suppose we have a set $Y \subset X$, we will prove that Y can be written as a union of countably many relatively closed sets of X. Let $\{U_n\}_{n\in\omega}$ be a countable basis for 2^ω such that every intersection of infinitely many $U_n's$ does not contain more than one element. Let $\lambda = \max\{|Y|, |X \backslash Y|\}$ so we can index $Y = \{y_\alpha\}_{\alpha\in\lambda}$ and $X \backslash Y = \{x_\alpha\}_{\alpha\in\lambda}$, we assume without loss of generality that neither is empty. For $\alpha \in \lambda$ let

$$A_\alpha := \{n \in \omega \mid y_\alpha \in U_n\} \text{ and } B_\alpha := \{n \in \omega \mid x_\alpha \in U_n\}$$

so that we can define two families $\mathcal{A} := \{A_\alpha\}_{\alpha\in\lambda}$ and $\mathcal{B} := \{B_\alpha\}_{\alpha\in\lambda}$. By fact 4.39 there is a set $M \subset \omega$ such that $B \backslash M$ is infinite for all $B \in \mathcal{B}$ but $A \backslash M$ is finite for all $A \in \mathcal{A}$. We now can define a set $L := \omega \backslash M$ so that for all $B_\alpha \in \mathcal{B}$ we have $B_\alpha \backslash M = B_\alpha \backslash (\omega \backslash L) = L \cap B_\alpha$ is infinite. Similarly $L \cap A_\alpha$ is finite for all $A_\alpha \in \mathcal{A}$. For $n \in \omega$ we now define $L_n := \bigcup\{U_m \mid m \geq n, m \in L\}$. Since each $B_\alpha \cap L$ is infinite we have $x_\alpha \in \bigcap\{L_n \mid n \in \omega\}$. But each $A_\alpha \cap L$ is finite so $y_\alpha \notin \bigcap\{L_n \mid n \in \omega\}$ so $Y = \bigcup\{(2^\omega \backslash L_n) \cap X | n \in \omega\}$. □

Now we can use the negation of the continuum hypothesis.

Axiom 4.41. *(continuum hypothesis) There is no set whose cardinality is strictly between that of the integers and that of the real numbers.*

So **ZFC+MA+¬CH** implies that there is a subset of 2^ω with cardinality ω_1 which is a $\mathcal{Q}$-set. We now can show that this set induces an f.f. space which is $\triangle$-normal. But before we give the proof we will first give a fact which we will use, which can be learned in [6].

Fact 4.42. *If κ is an infinite ordinal number then there is a bijection $\kappa \to \kappa \times \kappa$*

Theorem 4.43. *The f.f. space induced by a $\mathcal{Q}$-set $Q \subset 2^\omega$ of cardinality ω_1 is $\triangle$-normal.*

Proof. Let $O := \triangle \cup \bigcup\{U^2_{n_f}(f) \mid f \in Q\}$ be a typical open set around the diagonal in X^2. Let $n : Q \to \omega$ be the map that O induces. Now we shrink O by increasing the values of n in two steps. First let k be such that $|n^{-1}(k)| = \omega_1$ and define

$$\begin{array}{cccc} n_2 : & Q & \to & \omega \\ & q & \mapsto & \left\{ \begin{array}{rl} n(q) & \text{if } n(q) > k \\ k & \text{if } n(q) \le k \end{array} \right. . \end{array}$$

Then we increase n_2 further by choosing any bijection

$$b : n_2^{-1}(k) \to \omega \backslash k \times n_2^{-1}(k),$$

denoting $b(q) = (b_1(q), b_2(q))$ for each $q \in n_2^{-1}(k)$ and defining

$$\begin{array}{cccc} n_3 : & Q & \to & \omega \\ & q & \mapsto & \left\{ \begin{array}{rl} n_2(q) & \text{if } q \notin n_2^{-1}(k) \\ b_1(q) & \text{if } q \in n_2^{-1}(k) \end{array} \right. . \end{array}$$

Now we are done shrinking O. That is, we have

- $|n^{-1}(q)| = \omega_1$ if $q \in w\backslash k$
- $n^{-1}(q) = \emptyset$ if $q \in k$.

Now we want to find a typical open set $V := \triangle \cup \bigcup\{U^2_m(f) \mid f \in Q\}$ so that $\overline{V} \subset O$. This comes down to constructing a map

$$\begin{array}{cccc} m : & Q & \to & \omega \\ & f & \mapsto & m_f \end{array}$$

so that it induces the open set V in the right way. Therefore we first prove the following property of n.

Claim 4.44. *There exist a map $m : Q \to \omega$ and a map $s : Q \to \omega$ so that for every pair $f, h \in Q$ for which $m(h) < n(f)$ the following three conditions hold:*

1. $f \restriction s(f) \neq h \restriction s(f)$
2. $s(f) \geq n(f)$
3. $m(f) > n(f)$

Proof. Suppose $l \in \omega \backslash k$, then $|n^{-1}(l)| = \omega_1$ and $n^{-1}(l) \subset Q$. So then $n^{-1}(l)$ can be written as a countable union of closed sets. So for each $j \in \omega$ let $n^{-1}(k+j) = \bigcup\{G_{j,i} \mid i \in \omega\}$ with the $G_{j,i}$ closed in Q. We may assume without loss of generality $G_{j,i} \subset G_{j,i+1}$ for all $i \in \omega$. Suppose $f \in Q$ then there exists a unique $i \in \omega$ so that $f \in n^{-1}(k+i)$. Also there exists a unique $l \in \omega$ so that $f \in G_{i,l} \backslash G_{i,l-1}$ (we set $G_{i,-1} = \emptyset$ for all $i \in \omega$). In this way we set $m(f) := k+1+i+l$ for all $f \in Q$. Clearly m then satisfies 3. by construction.

Now we are going to define closed sets M_p with $p \in \omega$ as follows:

$$
\begin{array}{lllll}
M_0 & := & G_{0,0} & = & m^{-1}(k+1) \\
M_1 & := & G_{0,1} \cup G_{1,0} & = & m^{-1}(k+1) \cup m^{-1}(k+2) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
M_p & := & \bigcup\{G_{i,p-i} \mid 0 \leq i \leq p\} & = & \bigcup\{m^{-1}(i+1) \mid k \leq i \leq k+p\}.
\end{array}
$$

Now suppose $h, f \in Q$ are so that $m(h) < n(f)$. Then we have

$$h \in M_{m(h)-k-1} \subset M_{n(f)-k-2}.$$

On the other hand if $f \in M_{n(f)-k-2} = \bigcup\{m^{-1}(i+1) \mid k \leq i \leq n(f)-2\}$ then we have $f \in m^{-1}(i+1)$ for a certain $i \in \omega$ for which also $k \leq i \leq n(f)-2$. But then it holds that $i = m(f)-1 > n(f)-2 \geq i$, contradiction. So $f \notin M_{n(f)-k-2}$.

So we have that $(M_{n(f)-k-2})^c$ is open and contains f but not h. So we can find an $s(f) \geq n(f)$ so that $B_{2^{-s(f)+1}}(f) \subset (M_{n(f)-k-2})^c$. And because $B_{2^{-s(f)+1}}(f) = \{g \in 2^\omega \mid g \restriction s(f) = f \restriction s(f)\}$ we must have $f \restriction s(f) \neq h \restriction s(f)$. So also 1. and 2. are satisfied. $\square$

We now continue to prove $\overline{V} \subset O$ by proving $O^c \subset \overline{V}^c$. We consider an $x \in O^c$ and separate three cases.

First suppose that $x \in 2^{<\omega} \times 2^{<\omega} \cap O^c$ then x is an isolated point. So if $x \in \overline{V}$ then $x \in V \subset O$, contradiction.

Secondly suppose $x \in Q \times Q \cap O^c$. Then clearly we have $x \in \triangle^c$ so x can be written as $x = (f, g)$ with $f \neq g$. If we set $p := \min\{n : f(n) \neq g(n)\} + 1$ we can show

$$U_p(f) \times U_p(g) \cap \triangle = \emptyset,$$

and that for all $h \in Q$ we have

$$U_p(f) \times U_p(g) \cap U^2_{m_h}(h) = \emptyset.$$

The first is true because $f \restriction p = g \restriction p$ would be in contradiction with the choice of p. The second is true because $f \restriction p = h \restriction p = g \restriction p$ would also be in contradiction with te choice of p. So what we now know is:

$$U_p(f) \times U_p(g) \cap V = \emptyset.$$

which actually shows that $(f,g) \in \overline{V}^c$.

Finally suppose $x \in Q \times 2^{<\omega} \cap O^c$ (if $x \in 2^{<\omega} \times Q \cap O^c$ we can make an analogous proof). We can write $x = (f, g \restriction p)$. To prove $x \in \overline{V}^c$ it is enough to show that $U_{s(f)}(f) \times \{g \restriction p\} \cap U^2_{m(h)}(h) = \emptyset$ for all $h \in Q$. We are going to assume the contrary, then there exists a $h \in Q$ so that all the following three statements hold.

1. $f \restriction \max(s(f), m(h)) = h \restriction \max(s(f), m(h))$
2. $g \restriction p = h \restriction p$
3. $p \geq m(h)$

We also have $x \in O^c$ so $(f, g \restriction p) \notin U^2_{n_f}(f)$, i.e., we have $g \restriction p \neq f \restriction p$ or $p < n(f)$. If $g \restriction p \neq f \restriction p$ then the 1. and 2. right above tell us that $\max(s(f), m(h)) < p$ which is in contradiction with 3.. On the other hand if $p < n(f)$ then the 3. right above gives us $m(h) < n(f)$ in which case we can use the 1. and 2. of the claim combined with the 1. above to give us a contradiction again. We conclude $x \in \overline{V}^c$. □

So the f.f. spaces induced by $\mathcal{Q}$-sets form a good candidate for a counterexample on problem 2.7. We did not have enough time to find out whether these spaces are indeed not functionally $\triangle$-normal.

5 Conclusion

We were not able to find a counterexample to problem 2.7. But maybe someone can try to prove that the f.f. spaces induced by the $\mathcal{Q}$-sets are not functionally $\triangle$-normal. The results we got is that there are different kinds of a.d.f. spaces. Among them we have the Lusin spaces which are not $\triangle$-normal. But also we have the f.f. spaces which behave topologically totally different. If we put a topology on 2^ω we can find among them spaces which are not $\triangle$-normal, namely the ones coming from the Borel sets. On the other hand, if we assume $\mathbf{ZFC} + \mathbf{MA} + \neg\mathbf{CH}$ we can find f.f. spaces which are $\triangle$-normal, namely the ones coming from the $\mathcal{Q}$-sets.

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