NORTHWESTERN UNIVERSITY

Some Effective Problems in Algebraic Geometry and Their Applications

A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Mathematics

By

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EVANSTON, ILLINOIS

September 2019

ABSTRACT

Some Effective Problems in Algebraic Geometry and Their Applications

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In this thesis, we study pushforwards of canonical and log-pluricanonical bundles on projective log canonical pairs over the complex numbers. We partially answer a Fujita-type conjecture proposed by Popa and Schnell in the log canonical setting. Built on Kawamata's result for morphisms that are smooth outside a simple normal crossing divisor, we show a global generation result for morphisms that are log-smooth with respect to a reduced snc pair outside such divisors. Furthermore, we partially generalize this result to arbitrary log canonical pairs and obtain generic effective global generation.

In the pluricanonical setting, we show two different effective statements. First, when the morphism surjects onto a projective variety, we show a quadratic bound for generic generation for twists by big and nef line bundles. Second, when the morphism is fibred over a smooth projective variety, we give a linear bound for twists by ample line bundles. In each context we give descriptions of the loci over which these global generations hold.

These results in particular give effective nonvanishing statements. As an application, we prove an effective weak positivity statement for log-pluricanonical bundles in this

setting, with a description of the loci where this positivity is valid. We discuss its most remarkable application by presenting a proof of a well-known case of the Iitaka conjecture for subadditivity of log Kodaira dimensions. Finally, using the description of the positivity loci, we show an effective vanishing theorem for pushforwards of certain log-pluricanonical bundles under smooth morphisms.

Acknowledgements

I am indebted to a number of people whose presence helped me in many different ways during the preparation of this dissertation over the course of the past few years. First and foremost, this thesis might not have seen the light of the day without the invaluable wisdom, constant support, and countless hours of time and energy from my advisor Mihnea Popa. This thesis stemmed from a problem he suggested and my research greatly benefitted from numerous insightful discussions with him. I am fortunate to have had supervision from such a talented mathematician and a skilled advisor. He has played an instrumental role not only as a mathematical guide, but as a source of encouragement through ebb and flow of the PhD experience.

I am also very grateful to Lawrence Ein, Mircea Mustaţă, Christian Schnell and Valentino Tosatti for finding time to meet me and for sharing their wisdom and insights, from which I have benefitted immensely. I am thankful to several fellow graduate students and friends, especially Kenny Ascher, Rankeya Datta, Alyssa Kenney, Sudeshna Kolay, Nicole Looper, Sebastián Olano, Sean Pohorence, Akash Sengupta, Guillermo Tochi, Lei Wu, Kitty Yang and Mingyi Zhang for their support, friendship and for innumerable exciting mathematical discussions. A special thanks to Yajit Jain for some of the beautiful diagrams in this thesis and to Takumi Murayama for many helpful discussions, for teaching me how to use github and for being a fun and extremely diligent collaborator.

I am filled with gratitude to have been part of the Northwestern family, to have benefitted from the excellent faculty members especially Antonio Auffinger, Ursula Porod, Valentino Tosatti, Jared Wunsch and Eric Zaslow, and the enthusiastic administrative staff, especially Emily Green, Greg Jue, Jenima Lyon, Deavon Mitchell and Niki Koumoutsos. I was also fortunate enough to have spent a year at the University of Illinois at Chicago and am indebted to Ramin Takloo-Bighash for being constantly supportive, even after I left UIC.

It is a pleasure to thank my mother Sharmila Dutta, my sister Sayantanee Dutta Ghosh and my partner Emanuel Reinecke for their unwavering love and for simply being there for me. I am thankful to my college professors, especially Sundararaman Ramanan, Pramathnath Sastry and Vikraman Balaji for their excellent classes that got me interested in Algebraic Geometry and to my childhood math teachers Udayaditya Bhattacharyya and Kallol Mazumdar for spurring and nurturing my interest in the abstractness of pure Mathematics. Finally, I am grateful to my psychotherapist Dr. Christine Yoo for patiently taking time to help me understand the roots of many problems during difficult times in the past few years.

"All that I am or ever hope to be, I owe to my angel mother." – Abraham Lincoln.

Dedicated to my mother, Sharmila Dutta.

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CHAPTER 1

Introduction

The main goal of this thesis is to study complex projective varieties and their families. In complex geometry and differential topology, manifolds are often studied via holomorphic functions on them. However, globally defined holomorphic functions on compact complex manifolds are constants by Liouville's theorem. Moreover, it is known that connected complex projective varieties underlie connected compact complex analytic spaces and therefore holomorphic functions on them are not interesting. More information can be extracted from the study of local holomorphic functions with a gluing data, i.e. global sections of holomorphic line bundles. Sometimes, these global sections define morphisms to complex projective spaces, which are fairly well understood objects in complex geometry.

More precisely, given a line bundle $\mathcal{L}$ on a complex projective variety, assume that for every closed point $x \in X$, there is a global section $s \in \Gamma(X, \mathcal{L})$ such that $s(x) \neq 0$. Then define

$$\varphi_m \colon U \longrightarrow \mathbb{P}^m$$

by

$$x \longmapsto (s_0(x) : \cdots : s_m(x))$$

where s_i's are linearly independent global sections of $\mathcal{L}$ and $U \subseteq X$ is an open set around x. The assumptions on the sections of $\mathcal{L}$ ensures that the morphism φ_m is well defined around an open subset U of x. Such a line bundle $\mathcal{L}$ is said to be *globally generated* at x; see

Definition 2.0.1 for different characterizations of this property. Furthermore, this global generation property, more generally for coherent sheaves, encodes various information about the variety. This will be the central theme of this thesis. The methods involved dwell in the realm of Hodge theory and birational geometry.

Bombieri [**Bom73**], Kawamata [**Kaw84**], Shokurov [**Sho85**] et al., studied the space of sections of canonical bundles $\omega_X \coloneqq \bigwedge^{\dim X} \Omega_X^1$ and their higher tensor powers, i.e. pluricanonical bundles on smooth varieties of general type. They concluded that large enough powers of the canonical bundle admits a lot of sections. More precisely, it is known that, if $\omega_X^{\otimes m}$ is nef (see Definition 2.0.4) for some m and has "enough" sections to map a large open subset of X isomorphically on to its image, then there exists a large integer ℓ depending only on the dimension of X and m, so that $\omega_X^{\otimes \ell}$ is globally generated; see [**Kol93**].

Likewise, given an ample line bundle $\mathcal{L}$ (see Definition 2.0.3) on X, it is known [**AS95**] that there exists a large integer ℓ depending only on the dimension of X such that, $\omega_X \otimes \mathcal{L}^{\otimes \ell}$ is globally generated. These kind of bounds are said to be "effective". It is the latter kind of global generation that we will consider in this thesis.

In 1985, Takao Fujita [**Fuj85**] conjectured that $\omega_X \otimes \mathcal{L}^{\otimes \ell}$ is globally generated for all $\ell \geqslant \dim X + 1$. In particular, the bound on ℓ is independent of the choice of $\mathcal{L}$, and thus is an *effective* bound. For example, the canonical bundle $\omega_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$ of the projective space needs to be tensored by $\mathcal{O}_{\mathbb{P}^n}(1)$ at least $n+1$ times to obtain global generation. Indeed,

$$\omega_{\mathbb{P}^n} \otimes \mathcal{O}(1)^{\otimes n} = \mathcal{O}(-1)$$

which does not have any global sections. Even though the conjecture remains unsolved as of today, partial progress was made by Angehrn–Siu [**AS95**], Heier [**Hei02**] and Helmke [**Hel97, Hel99**] establishing the effectiveness with non-linear bounds. Moreover, Ein–Lazarsfeld [**EL93**], Kawamata [**Kaw97**], Reider [**Rei88**], Ye–Zhu [**YZ15**] et al. established the conjecture when $n \leqslant 5$. We survey some parts of their arguments in Chapter 2.

Taking this one step further, we study the canonical and pluri-canonical bundles on varieties varying in families. We consider a similar problem for pushforwards (pluri)-canonical bundles, i.e. $f_*\omega_X^{\otimes k}$ for $k \in \mathbb{Z}_{\geqslant 0}$, where $f\colon X \to Y$ is a surjective morphism of projective varieties X and Y. As mentioned earlier, the notion of *global generation* can be generalised to this setting. A result of Kawamata [**Kaw02**] shows the global generation of

$$f_*\omega_X \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes \dim Y+1}$$

when f is smooth outside a simple normal crossing divisor and $\dim Y \leqslant 4$. Kawamata's result plays an instrumental role in some of the main results of this dissertation. This dissertation also draws its inspiration from a conjecture of Popa and Schnell (see Conjecture 3.0.1) predicting an extension of Kawamata's result for pluricanonical bundles and for any surjective morphism of smooth projective varieties. When Y is a smooth projective curve or when $\mathcal{L}$ is ample and globally generated along with X and when Y is mildly singular, in [**PS14**] they confirm the conjecture. This kind of global generation statements are applied to study a plethora of properties, such as *weak positivity*, *nefness* of pushforwards, the *Iitaka conjecture* for subadditivity of *log-Kodaira dimensions* etc. We will discuss a few of these implications in Chapter 5.

1.1. Summary of Results

Most of the results in this thesis are part of [**Dut17**] and [**DM18**]. They are organized as follows.

In Chapter 2, we begin by establishing some of the vocabularies necessary to talk about Fujita-type conjectures. The rest of the Chapter is devoted to a survey of the known cases of the original conjecture and some of the strategies involved in their proofs.

Chapter 3 is concerned with various generalisations of Kawamata's global generation result. Building on his Hodge theoretic arguments, I generalise this result in Theorem 3.4.3 first to the case of a pair (X, D) with X smooth and D simple normal crossing so that f is log-smooth (see Definition A.3.1) with respect to (X, D) outside of a simple normal crossing divisor $\Sigma \subset Y$. Roughly speaking, this means that restricted to every component of D and every component of their intersections, f satisfies the hypotheses of Kawamata's theorem. More generally, such global generation holds when X itself has simple normal crossing singularities. Using this and Kawamata's cyclic covering arguments, in Theorem 3.4.8, I show a generic global generation result when (X, D) is a log-canonical pair (see Definition 3.4.6). In Theorem 3.1.3, I present a seemingly unrelated technique that gives a similar global generation when Y is possibly singular, however with a slightly weaker bound. This method uses a measure of positivity named after C. S. Seshadri and theorems concerning injectivity of cohomology groups.

In Chapter 4, I extend these global generation statements to the pluricanonical case. Since $f_*\omega_X^{\otimes k}$ is not a Hodge theoretic object, I follow the arguments of Popa and Schnell in [**PS14**] and tackle the pluricanonical case by reducing to the global generation problem for $k = 1$. Various results from Chapter 3 applies to this case. In Theorem 4.2.1, assuming Y

is non-singular, this reduction uses the weak positivity of the pushforwards of the twisted relative canonical bundles, i.e.

$$f_*\mathcal{O}_X(k(K_{X/Y} + D)).$$

Theorem 4.3.1 on the other hand, inspired by Popa–Schnell's original arguments, uses an argument involving minimality of the twists of the ample line bundle $\mathcal{L}$ in order to perform this reduction. Combining this with the bounds from Theorem 3.1.3 based on Seshadri constants, in Theorem 4.3.1, I present a generic global generation when Y is not necessarily smooth.

In Chapter 5, I discuss some applications of various global generation results from the previous chapters. First, in Theorem 5.1.4, I present the weak positivity statement used in the reduction argument for Theorem 4.2.1. Roughly speaking weak positivity is a generic global generation statement and if true globally, this notion is equivalent to nefness; see Definition 5.1.2 for a more precise definition. The weak positivity Theorem 5.1.4 for log-canonical pairs, is a consequence of Popa–Schnell's theorem [**PS14**, Theorem 1.7]. I also present an effective version in Theorem 5.1.5 for a special class of line bundles, as well as describe the locus on which the weak positivity holds when D has simple normal crossing support. Second, I discuss a standard application of such positivity to certain cases of the Iitaka-type conjecture for subadditivity of Kodaira and log-Kodaira dimensions in families. In Theorem 5.2.2, I present a proof for a case that was previously obtained by Campana [**Cam04**] and Nakayama [**Nak04**], – for statements of similar flavour see [**Vie83, Kol86, PS14, Fuj17**] et al. I present a slightly different proof of this statement. Third, I present a Kollár-type effective vanishing theorem for pushforwards of twisted pluricanonical bundles for certain morphisms. Due to the lack of Hodge theory, the proof is again based on a reduction to the case $k = 1$. However, due to the global nature of vanishing statements, we need to impose certain hypotheses on the morphism and on the twisted-pluricanonical bundle in order to ensure that the Theorems in Chapter 3 and 4 hold globally. See Theorem 5.3.1 for a precise statement.

CHAPTER 2

The Fujita Conjecture

The notion of global generation of a line bundle $\mathcal{L}$ on a variety X has been discussed in Chapter 1. Formally, we have the following equivalent definitions of global generation of a line bundle:

Definition 2.0.1. A line bundle $\mathcal{L}$ on a complex variety X is said to be *globally generated* at a point $x \in X$ if any of the following equivalent condition is true:

(1) There exists a subset $S \subseteq \Gamma(X, \mathcal{L})$, an open subset $U \ni x$ and a morphism $\phi_L \colon U \to \mathbb{P}^{|S|-1}$ such that $\phi_L^* \mathcal{O}_{\mathbb{P}^{|S|-1}}(1) \simeq \mathcal{L}|_U$.

(2) The space of global sections $\Gamma(X, \mathcal{L}) \neq \emptyset$ and there exists $s \in \Gamma(X, \mathcal{L})$ such that $s(x) \neq 0$.

(3) The morphism of vector spaces $\Gamma(X, \mathcal{L}) \to \mathcal{L} \otimes \kappa(x)$ is surjective, where $\kappa(x)$ denotes the residue field at x.

We say that a line bundle $\mathcal{L}$ is globally generated if it is globally generated for all $x \in X$.

Example 2.0.2. (1) Let $X \to \operatorname{Spec} k$ is a scheme over a ground field k. The structure sheaf $\mathcal{O}_X$ is globally generated and the corresponding map retrieves the structure map. In other words, $\phi_L \colon X \to \mathbb{P}^0 \simeq \operatorname{Spec} k$.

(2) On the projective space $\mathbb{P}^n$, $\mathcal{O}_{\mathbb{P}^n}(i)$ is globally generated for all $i \geq 0$. Moreover, $\phi_i \colon \mathbb{P}^n \to \mathbb{P}^N$, for $N = \binom{n+i}{n} - 1$ are in fact embeddings and are known as the *Veronese embeddings*. More generally, a line bundle whose global sections define

an embedding to some projective space is called a *very ample* line bundle. On the other hand $\mathcal{O}_{\mathbb{P}^n}(-i)$ for $i > 0$ do not admit any global sections and hence they are not globally generated.

(3) If C is a curve of genus $g > 0$, then ω_C is globally generated. Indeed, for any point $p \in C$ by Riemann-Roch, $h^0(C, \omega_C(-p)) - h^0(C, \mathcal{O}_C(p)) = 2g - 3 + 1 - g = g - 2$. By Clifford's theorem [**Har77**, Theorem IV.5.4] (or by the fact that $C \neq \mathbb{P}^1$) we have $h^0(C, \mathcal{O}_C(p)) = 1$. Therefore, $h^0(C, \omega_C(-p)) = g - 1$ and hence there exists $s \in H^0(C, \omega_C) - H^0(C, \omega_C(-p))$ for all $p \in C$. Therefore, ω_C is globally generated [**Har77**, Theorem IV.3.1].

(4) If $\mathcal{L}$ is a line bundle on a smooth projective curve C such that $\deg \mathcal{L} \geqslant 2$ then $\omega_C \otimes \mathcal{L}^{\otimes 2}$ is globally generated. Indeed, $\deg(\omega_C \otimes \mathcal{L}^{\otimes 2}) \geqslant 2g$, which is sufficient to ensure global generation. This can be checked by an argument analogous to (3) above; see also [**Har77**, Corollary IV.3.2]. The Fujita conjecture is a generalisation of this phenomenon.

(5) For Calabi-Yau manifolds the canonical bundle is trivial and therefore, is globally generated.

(6) Canonical bundle of a fano manifolds is anti-ample and hence it does not admit any global section. Therefore it is not globally generated.

To study projective varieties via maps to projective spaces, it is desirable to find maps that are defined by globally generated line bundles and are somewhat intrinsic to the variety. Canonical bundles are therefore natural candidates for this purpose. However, often (e.g. Example 6) neither canonical bundles, nor their higher tensor powers define maps to projective spaces. We define a class of line bundles that compensate for this lack of global generation.

Definition 2.0.3 (Ample Line Bundles)**.** A line bundle $\mathcal{L}$ on a variety X is said to be ample if it satisfies one of the following equivalent properties:

(1) There exists an integer $k \geqslant 1$ such that $\mathcal{L}^{\otimes k}$ defines an embedding to the projective space, i.e. very ample.

(2) For any coherent sheaf $\mathcal{F}$ on X, there exists an integer $k \geqslant 1$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes k}$ is globally generated.

(3) For any irreducible subvariety $Z \subset X$, $\mathcal{L}^{\dim Z} \cdot Z > 0$. Here when Z is smooth, the intersection product is defined by $\int_{[Z]} c_1(\mathcal{L})|_Z^{\dim Z}$ where $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$ is a closed 2-form and $[Z]$ is the fundamental class of Z. It can be defined similarly when Z is singular; see [**Har77**, Appendix A] for a brief survey on Intersection Theory.

(4) For every point $x \in X$, there exists a real number $\varepsilon > 0$ such that for all curves $C \subset X$, $\mathcal{L} \cdot C \coloneqq \deg_C \mathcal{L}|_C > \varepsilon \cdot \operatorname{mult}_x(C)$. The smallest of such real numbers is known as the *Seshadri constant of* L *at* x and is denoted by $\varepsilon(\mathcal{L}; x)$; see Definition 3.1.2.

The last characterisation of ample line bundles is numerical and has the advantage that under any morphism $f : X' \to X$, $f^*\mathcal{L}$ satisfies a weakened version of the above intersection properties. Such line bundles are called *nef*.

Definition 2.0.4 (Nef Line Bundles)**.** A line bundle $\mathcal{L}$ on a variety X is said to be nef if it satisfies one of the following equivalent (Kleiman's theorem) properties:

(1) For any subvariety $Z \subset X$, $\mathcal{L}^{\dim Z} \cdot Z \geqslant 0$.

(2) For any curve $C \subset X$, $\deg \mathcal{L}|_C \geqslant 0$.

While for a morphism $f : X' \to X$, $f^*\mathcal{L}$ still satisfies non-negative intesection properties, it may fail to satisfy Property 2.0.3(1) above, i.e. none of its tensor powers has enough sections to embed Y to a projective space. Indeed, a morphism defined by the sections of tensor poweres of $f^*\mathcal{L}$ factors through X and hence contracts all the fibres of f. In fact, if f is finite, then $\mathcal{L}$ is ample if an only if $f^*\mathcal{L}$ is so. However, if the morphism is only generically finite, the sections of tensor powers of $f^*\mathcal{L}$ has enough sections, i.e. $h^0(X, \mathcal{L}^{\otimes m})$ grows at the rate of $m^{\dim X}$, which is indeed true for ample line bundles as a consequence of Property 2.0.3(1). This gives rise to the notion of *bigness*.

Definition 2.0.5 (big line bundles)**.** A line bundle $\mathcal{L}$ on a projective variety X is said to be big if the following equivalent conditions are satisfied:

(1) There exists an integer $m \gg 0$ such that $\mathcal{L}^{\otimes m}$ admits enough sections so that it maps a non-empty open subset of X isomorphically onto its image.

(2) The ring $R(X; \mathcal{L}) \coloneqq \bigoplus_m H^0(X, \mathcal{L}^{\otimes m})$ has transcendence degree $\dim X + 1$. The ring structure on $R(X; \mathcal{L})$ comes from the multiplication of sections.

(3) The dimension of the vector space $H^0(X, \mathcal{L}^{\otimes m})$ grows at $\sim m^{\dim X}$.

2.1. The Statement

By the Definition 2.0.3 of ample line bundles, we know that there exists an integer $\ell > 0$ such that for an ample line bundle $\mathcal{L}$ on a projective variety X, $\omega_X \otimes \mathcal{L}^{\otimes \ell}$ is globally generated. A priori, there is no reason to expect that ℓ would depend only on the $\dim X$.

The following conjecture of Fujita predicts that when X is a projective variety, ℓ must depend linearly, only on the $\dim X$.

Conjecture 2.1.1 (The Fujita Conjecture)**.** *Let X be a smooth projective variety of $\dim X = n$ and let $\mathcal{L}$ be an ample line bundle on X, then*

$$\omega_X \otimes \mathcal{L}^{\otimes \ell}$$

is globally generated for all $\ell \geqslant n+1$.

The conjecture can be seen as a higher dimensional generalisation of Item 4 in the examples of globally generated line bundles on curves. The Conjecture remains unsolved till date. Nonetheless this, together with the study of linear system in general have given rise to number of interesting tools in algebraic geometry. For instance, the technique of producing a singular divisor D passing through a given point x, so that it is highly singular at x but not so singular at other points around x. There is an invariant that materialises this subtle difference in singularities. Coined first in the work of Demailly [**Dem93**] and Nadel [**Nad90**], this is known as the *multiplier ideal*.

Definition 2.1.2 (Multiplier Ideals)**.** Let D be an effective $\mathbb{Q}$-divisor on a smooth complex variety X. Let $\mu : X' \to X$ be a log resolution of D so that

$$K_{X'} = \mu^*(K_X + D) + F$$

for some $\mathbb{Q}$-divisor F. Then the multiplier ideal is defined as follows

$$\mathcal{J}(D) \coloneqq \mu_* \mathcal{O}_{X'}(\lceil F \rceil).$$

Remark 2.1.3.

(1) The definition is independent of the log resolution μ chosen.

(2) If D is integral $\mathcal{J}(D) \simeq \mathcal{O}_X(-D)$. This follows from

$$\lceil F \rceil = K_{X'} - \lfloor \mu^*(K_X + D) \rfloor = K_{X'/X} - \mu^* D,$$

together with the fact that $\mu_* \omega_{X'} \simeq \omega_X$ for proper birational morphism μ of smooth projective varities.

(3) If $F = \sum_i a_i F_i$ and $a_i \in (-1, 0]$ then $\mathcal{J}(D) = \mathcal{O}_X$. This by definition implies that D has *kawamata-log-terminal singularities* (klt).

(4) Similarly if $a_i \in [-1, 0]$, then $\mathcal{J}((1 - \epsilon)D) \simeq \mathcal{O}_X$ for some $\epsilon \ll 1$. This by definition implies that D is log-canonical (lc).

(5) If the conditions of the coefficients a_i are satisfied only for the components lying above a certain point, then the notions of the above singularities can be made local. For instance, if $a_i \in (-1, 0)$ only for i such that $x \in \mu(F_i)$, then $\mathcal{J}(D)_x \simeq \mathcal{O}_x$ and D is klt at x, and similarly for log-canonical.

The independence of log resolution (1) makes this notion very useful to work with. The following Lemma describes how this is used to show the known cases of the Fujita conjecture.

Lemma 2.1.4. *Let $\mathcal{L}$ be an ample line bundle on a smooth projective variety X, such that for a fixed point $x \in X$, there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} \lambda L$ for $\lambda < 1$, where L denotes a Cartier class of $\mathcal{L}$ and such that x is an isolated point of the cosupport of the multiplier ideal $\mathcal{J}(D)$. Then $\omega_X \otimes \mathcal{L}$ is globally generated at x.*

Proof. The proof follows from the Nadel vanishing theorem, for $i > 0$

$$H^i(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{J}(D)) = 0.$$

Indeed, since $x \in \mathrm{Zero}(\mathcal{J}(D))$ is an isolated point, it is enough to show that

$$H^0(X, \omega_X \otimes \mathcal{L}) \to H^0(X, \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_X/\mathcal{J}(D))$$

is surjective, which is a consequence of the Nadel vanishing. □

The global generation in Lemma 2.1.4 in contingent on getting hold of such a divisor D. The works of [**EL93, Kaw97, AS95, Hel97, Hel99**] show its existence with various bounds on the intersection numbers of $\mathcal{L}$, as in 2.0.3(3). We state here the ones imposed by Angehrn and Siu [**AS95**]. This was the first breakthrough towards getting an effective statement in all dimensions.

Lemma 2.1.5. *Let $\mathcal{L}$ be an ample line bundle on a smooth projective variety X, such that for a point $x \in X$, $\mathcal{L}^{\dim V} \cdot V > \left(\frac{n^2+n}{2}\right)^{\dim V}$ for all irreducible subvariety V passing through x and $\mathcal{L}^n \geqslant \left(\frac{n^2+n}{2}\right)^n$ then there exists an effective $\mathbb{Q}$-divisor D such that $D \sim_{\mathbb{Q}} \lambda L$ satisfying the hypotheses of Lemma 2.1.4. In particular,*

$$\omega_X \otimes \mathcal{L}$$

is globally generated at x.

Roughly, the idea is to use the lower bound on the top intersection product $L^n > N^n$ with $N = \frac{n^2+n}{2}$ to produce an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \frac{n}{N}L$ such that $\mathrm{mult}_x(D) > n$.

The multiplicity lower bound ensures that $\mathcal{J}(D)_x \subsetneq \mathcal{O}_X$. If x is not an isolated singular point of the cosupport of $\mathcal{J}(D)$, one proceeds to cut down the smallest component of the cosupport containing x by divisors in $\mathbb{Q}$-linearly equivalent to $\mathcal{L}$.

Notice that if $\mathcal{L}$ is any ample line bundle, it satisfies $\mathcal{L}^{\dim V} \cdot V \geqslant 1$ for any subvariety $V \subseteq X$. Therefore, for any ample line bundle $\mathcal{L}$, the above Lemma 2.1.5 implies $\omega_X \otimes \mathcal{L}^{\otimes N}$ is globally generated.

Furthermore, Angehrn and Siu, showed this result allowing X to be mildly (klt) singular as well. From the point of view of birational geometry it is fundamental to allow singularities. Various results stated in this thesis will allow singularities.

2.2. The Easy Case

When a line bundle $\mathcal{L}$ on a smooth projective variety X is ample and globally generated, the Conjecture 2.1.1 has a rather simpler proof. The key ingredients is the following Vanishing Theorem.

Theorem 2.2.1 (Kodaira Vanishing Theorem). *Let $\mathcal{L}$ be an ample line bundle on a smooth projective variety X. Then*

$$H^i(X, \omega_X \otimes \mathcal{L}) = 0 \text{ for all } i > 0.$$

Now since $\mathcal{L}$ itself is globally generated we have a surjective morphism

$$H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \longrightarrow\!\!\!\!\!\rightarrow \mathcal{L}$$

Denoting by $V = H^0(X, \mathcal{L})$ and $r = \dim V$, consider the Koszul complex $\mathcal{K}^\bullet$ associated to this surjection; see [**Laz04a**, §B.2]

$$\mathcal{K}^\bullet := [0 \to \bigwedge^r V \otimes \mathcal{L}^{\otimes -r} \to \bigwedge^{r-1} V \otimes \mathcal{L}^{\otimes -r+1} \to \cdots \to \bigwedge^2 V \otimes \mathcal{L}^{\otimes -2} \to V \otimes \mathcal{L}^{-1} \to \mathcal{O}_X]$$

Tensoring this by $\omega_X \otimes \mathcal{L}^{\otimes n+k+1}$ with $n = \dim X$ and $k \gg 0$, and using Kodaira vanishing Theorem 2.2.1 $H^i(X, \omega_X \otimes \mathcal{L}^{\otimes n+k-i}) = 0$ for $k \geqslant 0$ and for all $i > 0$, to chase through the cohomologies to obtain

$$H^0(X, \mathcal{L}) \otimes H^0(X, \omega_X \otimes \mathcal{L}^{\otimes n+k}) \simeq H^0(X, V \otimes \omega_X \otimes \mathcal{L}^{\otimes n+k}) \twoheadrightarrow H^0(X, \omega_X \otimes \mathcal{L}^{\otimes n+k+1})$$

is surjective. Therefore, the global generations of $\mathcal{L}$ and $\omega_X \otimes \mathcal{L}^{\otimes n+k+1}$ imply the global generation of $\omega_X \otimes \mathcal{L}^{\otimes n+k}$. Repeating this argument until $k = 0$, we therefore obtain the global generation of $\omega_X \otimes \mathcal{L}^{\otimes n+1}$.

For any coherent sheaf $\mathcal{F}$ on X and a line bundle $\mathcal{L}$, a vanishing of the form $H^i(X, \mathcal{F} \otimes \mathcal{L}^{m-i}) = 0$ for all $i > 0$ defines *m-CM-regular (or Castelnuovo–Mumford Regular)* sheaf $\mathcal{F}$ with respect to $\mathcal{L}$. The above discussion establishes Mumford's Theorem [**Laz04a**, Theorem 1.8.5]:

Theorem 2.2.2 (Mumford's Theorem). *Let $\mathcal{F}$ be a coherent sheaf on X, m-regular with respect to $\mathcal{L}$. Then for every $k \geqslant 0$*

(1) *$\mathcal{F} \otimes \mathcal{L}^{\otimes m+k}$ is generated by its global sections.*

(2) *The natural maps*

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \otimes H^0(X, \mathcal{L}^{\otimes k}) \to H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m+k})$$

are surjective.

This concludes the following easy case of the Fujita conjecture:

Proposition 2.2.3. *Let $\mathcal{L}$ be an ample and globally generated line bundle on a smooth projective variety X. Then*

$$\omega_X \otimes \mathcal{L}^{\otimes \ell}$$

is globally generated for all $\ell \geqslant \dim X + 1$.

CHAPTER 3

The Relative Fujita Conjecture

In [**PS14**], Popa and Schnell proposed the following relative version of Fujita's conjecture:

Conjecture 3.0.1 ([**PS14**, Conjecture 1.3])**.** *Let $f\colon X \to Y$ be a morphism of smooth projective varieties, with $\dim Y = n$, and let $\mathcal{L}$ be an ample line bundle on Y. For each $k \geq 1$, the sheaf*

$$f_*\omega_Y^{\otimes k} \otimes \mathcal{L}^{\otimes \ell}$$

is globally generated for all $\ell \geq k(n+1)$.

This Chapter is devoted to the canonical case, i.e. $k = 1$ of this conjecture. It is instructive to note that if $\mathcal{L}$ is ample and globally generated the case k=1 is a consequence of arguments very similar to the one in Proposition 2.2.3; see §3.2. However, when $k > 1$, due to the lack of vanishing theorems, it is not straightforward to see why the conjecture should hold with this additional assumption. This was shown by Popa and Schnell in [**PS14**]; see also §4.1.

The main results of this section are Theorem 3.1.3, 3.4.3 and 3.4.8. The last two are built on Kawamata's freeness result, where assuming that the morphism is smooth outside a simple normal crossing divisor, he showed the global generation when $\dim Y \leqslant 4$ or more generally with Angehrn–Siu-type or Helmke-type non-linear bounds. As a result, under similar assumptions, in Theorem 3.4.3 we prove a global generation result for pushforwards of twisted canonical bundles, i.e. $\omega_X(D)$ from a pair (X, D) with simple normal crossing singularities and the morphism is suitably transversal under with respect to this pair. More generally, in Theorem 3.4.8, we establish a generic effective generation statement for log-canonical pairs, proving in particular a generic version of Popa–Schnell's conjecture when $k = 1$ and $\dim Y \leqslant 4$. The case when $k > 1$ is dealt with in Chapter 4.

We begin the section with a discussion on the role Seshadri constants play in the original Fujita conjecture, as well as a generic result in the relative setting giving a coarser bound in terms of Seshadri constants. This is the content of Theorem 3.1.3.

3.1. Effectivity of the Relative Statement

Seshadri constant $\varepsilon(\mathcal{L}; x)$ associated to nef line bundles is a useful measure of positivity. They have been proven to be helpful in showing global generation. For instance,

Proposition 3.1.1. *Let X be a smooth projective variety of dimension n and $\mathcal{L}$ an ample line bundle on X with $\varepsilon(\mathcal{L}; x) > n$, then $\omega_X \otimes \mathcal{L}$ is globally generated.*

Recall that,

Definition 3.1.2 (Seshadri Constants)**.** If $\mu\colon X' \to X$ is the blow-up of a projective variety X at x with exceptional divisor E, i.e. $\mu(E) = x$, then the *Seshadri constant* of a nef Cartier divisor L at x is defined by

$$\varepsilon(L; x) \coloneqq \sup\big\{t \in \mathbb{R}_{\geq 0} \;\big|\; \mu^* L - tE \text{ is nef}\big\}.$$

Proof of Proposition 3.1.1. By Definition of the Seshadri constants, $\varepsilon(\mathcal{L}; x) > n$ implies that $\mu^*\mathcal{L}(-nE)$ is ample. Then by the Kodaira vanishing theorem, $H^1(X, \omega_{X'} \otimes$

$\mu^*\mathcal{L}(-nE)) = 0$. Therefore, we have a surjection

$$H^0(X', \omega_{X'} \otimes \mu^*\mathcal{L}(-(n-1)E) \to H^0(E, \omega_E \otimes \mu^* L(-nE))$$

Now, note that $\omega_{X'} \simeq \mu^*\omega_X((n-1)E)$, then by the projection formula we obtain

$$H^0(X', \mu^*\omega_X((n-1)E) \otimes \mu^*\mathcal{L}(-(n-1)E)) \simeq H^0(X, \omega_X \otimes \mathcal{L}).$$

Moreover, there is a vector space isomorphism $H^0(E, \omega_E \otimes \mu^* L(-nE)) \simeq \omega_X \otimes \mathcal{L} \otimes \kappa(x)$. Thus, we get the required surjection for global generation. $\square$

In a joint work with Takumi Murayama, [**DM18**, Theorem C, Corollary 3.2], we show that in the relative setting a similar statement hold, only generically however. Here, the arguments are a bit more involved. This is in part inspired by [**dC98**] and by the analytic arguments of Deng in [**Den17**], where he proved similar generic global generation statements in terms of Seshadri constants.

Theorem 3.1.3. *Let $f\colon X \to Y$ be a surjective morphism of projective varieties, where Y is of dimension n. Let (X, Δ) be a log-canonical $\mathbb{R}$-pair, and let $\mathcal{L}$ be a nef and big line bundle and H be a semiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y such that there exists a Cartier divisor P on X satisfying*

$$P - (K_X + \Delta) \sim_{\mathbb{R}} f^*H.$$

Let ℓ be a positive integer such that $\ell > \frac{n}{\varepsilon(\mathcal{L};y)}$ for general $y \in Y$, then the sheaf $f_*\mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes \ell}$ is generically globally generated.

Proof. After taking a log resolution we may assume that Δ has simple normal crossings support and coefficients in $(0,1]$. Indeed, letting $\mu : \widetilde{X} \to X$ denote the log resolution, note that since (X,Δ) is log-canonical one can write

$$K_{\widetilde{X}} = \mu^*(K_X + \Delta) + E - N$$

with $E-N$ is a divisor with simple normal crossing support with no common components, $\lceil N \rceil = 0$ and E exceptional. Then, we define $\widetilde{\Delta} \coloneqq N + \lceil E \rceil - E$. Then, we redefine $\widetilde{X}$ by X and $\widetilde{\Delta}$ by Δ.

Let $y \in U(f,\Delta)$, where $U(f,\Delta)$ is as in the notation 3.4.9. We claim that it is enough to show the following extension statement; the restriction map

$$H^0\big(X, \mathcal{O}_X(P) \otimes f^*\mathcal{L}^{\otimes \ell}\big) \longrightarrow H^0\big(X_y, \mathcal{O}_{X_y}(P) \otimes f^*\mathcal{L}^{\otimes \ell}\big) \tag{3.1}$$

is surjective. Indeed, consider the commutative diagram

$$\begin{array}{ccc} H^0(Y, f_*\mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes \ell}) & \longrightarrow & f_*\mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes \ell} \otimes_{\mathcal{O}_Y} \kappa(y) \\ \| & & \downarrow \beta \\ H^0(X, \mathcal{O}_X(P) \otimes f^*\mathcal{L}^{\otimes \ell}) & \twoheadrightarrow & H^0(X_y, \mathcal{O}_{X_y}(P) \otimes f^*\mathcal{L}^{\otimes \ell}) \end{array}$$

where the bottom arrow is surjective by assumption, and hence so is β. This implies that β is an isomorphism. Indeed, tensoring with $\kappa(y)$ is the same as restricting first to an open neighbourhood $U \subset Y$, such that $f|_{f^{-1}(U)}$ is flat over U and then tensoring with $\kappa(y)$. Since f is flat over U, by cohomology and base change [**Ill05**, (8.3.2.3)], we have

$$f_*\mathcal{O}_{f^{-1}(U)}(P) \otimes \mathcal{L}^{\otimes \ell} \otimes_{\mathcal{O}_U} \kappa(y) \overset{\beta}{\simeq} H^0(X_y, \mathcal{O}_{X_y}(P) \otimes f^*\mathcal{L}^{\otimes \ell}).$$

Therefore, the top horizontal arrow is also surjective.

To see the claim, we use Injectivity Theorem 3.1.4 to show that for $y \in U(f, \Delta)$,

$$H^1\big(X, \mathcal{O}_X(P) \otimes f^*\mathcal{L}^{\otimes \ell} \otimes \mathscr{I}_{X_y}\big) \hookrightarrow H^1\big(X, \mathcal{O}_X(P) \otimes f^*\mathcal{L}^{\otimes \ell}\big)$$

where $\mathscr{I}_{X_y}$ is the ideal sheaf of X_y. In order to apply Theorem 3.1.4, we blow up X_y and since f is flat over y, we observe that the following diagram is Cartesian and the pullback variety X' is smooth; see [**Stacks**, Tag 0805].

$$\begin{array}{ccc} X' \coloneqq \operatorname{Bl}_{X_y} X & \xrightarrow{B} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' \coloneqq \operatorname{Bl}_y Y & \xrightarrow{b} & Y \end{array}$$

Figure 3.1. Blowing-up a smooth fibre.

Denoting by E the exceptional divisor $b^{-1}(y)$, note that

$$B^*(K_X + \Delta) = K_{X'} + B^*\Delta - (n-1)f'^*E.$$

Indeed, f'^*E is the exceptional divisor of B. Moreover, since $y \in U(f, \Delta)$ i.e. the components of Δ intersects X_y transversely, $B^*\Delta$ is the strict transform of Δ.

Now, by hypothesis $\varepsilon(\mathcal{L}^{\otimes \ell}; y) > n$ and hence, one can choose $D \sim_{\mathbb{Q}} f'^*(b^*\mathcal{L}^{\otimes \ell}(-(n+\delta)E))$ for some small $\delta \in \mathbb{Q}_{>0}$ so that $D + B^*\Delta + f'^*E$ has simple normal crossing support with coefficients in $(0, 1]$. Let $D' \sim_{\mathbb{Q}} f'^*b^*H$ be a similar suitable choice of representative whose support intersect components of $B^*\Delta + D + f'^*E$ transversely. Denoting $\Delta' \coloneqq D' + D + B^*\Delta + \delta f'^*E$, we can therefore define a Cartier divisor

$$P' \coloneqq B^*f^*\mathcal{L}^{\otimes \ell}(B^*P) \quad \text{and hence } P' \sim_{\mathbb{R}} K_{X'} + \Delta' + f'^*E.$$

Since f'^*E is in the support of Δ', the injectivity Theorem implies the injectivity of

$$H^1(X', \mathcal{O}_{X'}(P' - f'^*E)) \hookrightarrow H^1(X', \mathcal{O}_{X'}(P')).$$

Now by our choice of P',

$$B_*\mathcal{O}_{X'}(P' - f'^*E) \simeq B_*\mathcal{O}_X(B^*P - f'^*E) \otimes f^*\mathcal{L}^{\otimes \ell} \simeq \mathcal{O}_X(P) \otimes f^*\mathcal{L}^{\otimes \ell} \otimes \mathscr{I}_{X_y}$$

and $B_*\mathcal{O}_{X'}(P') \simeq \mathcal{O}_X(P) \otimes f^*\mathcal{L}^{\otimes \ell}$. □

The injectivity theorem we use above is due to Fujino for divisors Cartier up to $\mathbb{R}$-linear equivalence. It is built on the previous works of Kollár [**Kol95**] and Esnault–Viehweg [**EV92**].

Theorem 3.1.4 ([**Fuj17**, Theorem 5.4.1])**.** *Let X be a smooth complete variety and let Δ be an $\mathbb{R}$-divisor on X with coefficients in $(0,1]$ and simple normal crossings support. Let P be a Cartier divisor on X and let D be an effective irreducible divisor on X whose support is contained in* Supp Δ*. Assume that $P \sim_{\mathbb{R}} K_X + \Delta$. Then, the natural homomorphism*

$$H^i\big(X, \mathcal{O}_X(P)\big) \longrightarrow H^i\big(X, \mathcal{O}_X(P + D)\big)$$

induced by the inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$ is injective for every i.

3.1.1. A Lower Bound For The Seshadri Constants.

Ein, Küchle and Lazarsfeld in [**EKL95**] showed that on an open dense set U, one can find an effective lower bound for $\varepsilon(\mathcal{L}; y)$ for all $y \in U$.

Theorem 3.1.5 ([**EKL95**, Theorem 1])**.** *Let Y be a projective variety of dimension n. Let L be a big and nef Cartier divisor on Y. Then, for every $\delta > 0$, the locus*

$$\left\{ y \in Y \;\middle|\; \varepsilon(L; y) > \frac{1}{n+\delta} \right\}$$

contains an open dense set.

Remark 3.1.6. If in the notation of Theorem 3.1.5, we also assume that X is smooth and L is ample, then better lower bounds are known if $n = 2, 3$. Under these additional assumptions, the locus

$$\left\{ y \in Y \;\middle|\; \varepsilon(L; y) > \frac{1}{(n-1)+\delta} \right\}$$

contains an open dense set if $n = 2$ [**EL93**, Theorem] or $n = 3$ [**CN14**, Theorem 1.2]. Here, we use [**EKL95**, Lemma 1.4] to obtain results for general points from the cited results, which are stated for very general points.

In general, it is conjectured that in the situation of Theorem 3.1.5, the locus

$$\left\{ y \in Y \;\middle|\; \varepsilon(L; y) > \frac{1}{1+\delta} \right\}$$

contains an open dense set [**Laz04a**, Conjecture 5.2.5]. Assuming this conjectured bound, one obtains generic generation with effective bound of $n + 1$ in Theorem 3.1.7 below.

Using Theorem 3.1.5, we obtain the following effective version of Theorem 3.1.3.

Theorem 3.1.7. *Let $f\colon X \to Y$ be a surjective morphism of projective varieties, where Y is of dimension n. Let (X, Δ) be a log-canonical $\mathbb{R}$-pair, and let $\mathcal{L}$ be a nef and big line bundle and H be a semiample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Let ℓ be a real number* *for which there exists a Cartier divisor P on Y such that $P - (K_X + \Delta) \sim_{\mathbb{R}} f^*H$. then there exists a nonempty open set U such that*

$$f_*\mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes \ell}$$

is globally generated at $y \in U$ for all $\ell \geqslant n^2 + 1$.

When Y is non-singular and $\mathcal{L}$ is ample, Theorem 3.4.8 below improves this bound $\frac{n^2+n}{2} + 1$, with a relatively better grasp on the open set U. Assuming X is smooth, Iwai [**Iwa18**], built on a previous weaker bound by Deng [**Den17**], showed this global generation at all regular values of f. The rest of this Chapter is devoted to the generic global generation result in Theorem 3.4.8 for log-canonical pairs.

3.2. The "Easy" Case

One of the first evidences towards the relative Fujita conjecture is the "easy" statement, i.e. when $\mathcal{L}$ is an ample and globally generated line bundle. In Proposition 2.2.3 we showed the Fujita conjecture under this assumption using Kodaira vanishing and Castelnuovo-Mumford regularity.

In the relative setting, one uses Kollár's vanishing [**Kol95**, Theorem 10.19]:

$$H^i(Y, f_*\omega_Y \otimes \mathcal{L}^{\otimes n+1-i}) = 0$$

for all $i > 0$ where $f\colon X \to Y$ is a morphism of smooth projective varieties. This implies that $f_*\omega_Y \otimes \mathcal{L}^{\otimes n+1}$ is CM-regular with respect to $\mathcal{L}$ and hence globally generated by Mumford's Theorem 2.2.2. In fact, a little more is true due to vanishing theorems by Ambro [**Amb03**, Theorem 3.2] and Fujino [**Fuj11**, Theorem 6.3] for more singular spaces.

Here we present a statement that is a consequence of Fujino's vanishing theorem [**Fuj15**, Theorem 3.1] for semi-log-canonical pairs, and CM-regularity.

Theorem 3.2.1. *Let (X, Δ) be a semi-log-canonical pair and let $f : X \to Y$ be a surjective morphism onto a projective variety Y with $\dim Y = n$. Moreover, suppose D is a Cartier divisor on Y such that $D - (K_X + \Delta) \sim_{\mathbb{R}} f^*H$ for some ample $\mathbb{R}$-divisor H on Y and let $\mathcal{L}$ be an ample and globally generated line bundle on Y. Then $f_*\mathcal{O}_X(D) \otimes \mathcal{L}^{\otimes n+1}$ is globally generated.*

Semi-log-canonical singularities appear naturally in the study of modular compactification of the moduli functor of smooth varieties [**KSB88**]. These can be seen an a higher dimensional analogues of nodal curves. Therefore it is natural to extend these effective generation questions for morphisms from slc pairs. For the purpose of this Theorem we give a definition of semi-log-canonical pairs here:

Definition 3.2.2. (semi-log-canoical pairs) A pair (X, Δ) is called a *semi-log-canonical pair* (or, slc pair) if X is an equidimensional variety satisfying Serre's S_2 condition, is double normal crossings in codimension one and Δ is an effective $\mathbb{R}$-divisor on X such that irreducible components of Δ intersects X_{reg} non-trivially. Moreover, the divisor $K_X + \Delta$ is $\mathbb{R}$-Cartier, and there is an effective $\mathbb{R}$-divisor Δ' on the normalisation X^ν such that (X^ν, Δ') is log-canonical and

$$K_{X^\nu} + \Delta' = \nu^*(K_X + \Delta) \tag{3.2}$$

under the normalisation morphism $\nu : X^\nu \to X$.

Example 3.2.3. A *simple normal crossing pair* (X, Δ) consists of a scheme X of pure dimension n, such that X can be embedded Zariski locally in to a smooth variety Y of pure dimension $n+1$, so that on Y, there is a simple normal crossing divisors B with the property that $X + B$ is also simple normal crossings and $\operatorname{Supp}\Delta = B \cap X$. Moreover $\Delta = \sum_i a_i \Delta_i$ with $a_i \leqslant 1$. A simple normal crossing pair is semi-log-canonical since X is Cohen-Macaulay i.e. satisfies S_k for all $k \leqslant n$ and is clearly normal crossing in codimension 1. Furthermore, $X^\nu = \bigsqcup_i X_i$ with X_i's being the irreducible components of X and $\Delta' = \nu^*\Delta + \mathfrak{C}_X$ is a divisor on X^ν with simple normal crossing support and coeffecients of its components are $\leqslant 1$ that satisfies (3.2). Indeed, $\mathfrak{C}_X$ is the *conductor* divisor defined by the ideal

$$\mathfrak{cond}_X \coloneqq \nu^{-1}\mathcal{H}om(\nu_*\mathcal{O}_{X^\nu}, \mathcal{O}_X) \subseteq \mathcal{O}_{X^\nu}. \tag{3.3}$$

By definition, $\nu(\mathfrak{C}_X)$ is supported along $X_i \cap X_j$ for $i \neq j$ and hence, it does not share any component with $\mu^*\Delta$. Consequently, (X^ν, Δ') is log-canonical.

In §3.4, we discuss the generation problem for morphisms from reduced snc pairs, i.e. when $a_i = 1 \quad \forall i$.

3.3. The Smooth Case

The first non trivial case of the relative generation problem, was done by Kawamata in [**Kaw02**, Theorem 1.7], where with an additional assumption that f is smooth outside a simple normal crossing divisor $\Sigma \subset Y$, he showed that $f_*\omega_X \otimes \mathcal{L}^{\otimes n+1}$ is globally generated when $n = \dim Y \leqslant 4$. As the Hodge theory of simple normal crossing varieties are relatively well understood, a posteriori, it is no surprise that the statement is true in this setting.

When $\Sigma = 0$, Kawamata's theorem for smooth morphisms $f : X \to Y$ between smooth projective varieties already requires a bit of an involved argument. However, it is possible to avoid Hodge theory in this case, yet demonstrate the essences of some of the main ideas from his original arguments. We outline a non-Hodge-theoretic interpretation of his proof.

Theorem 3.3.1. *Let $f\colon X \to Y$ be a smooth morphism of smooth projective varieties and let $\mathcal{L}$ be an ample line bundle on X satisfying the intersection properties as in Lemma 2.1.5. Then $R^q f_*\omega_X \otimes \mathcal{L}$ is globally generated for all $q \geqslant 0$.*

Proof. Fixing a point $y \in Y$, we know by Lemma 2.1.5 that there exists a divisor $D \subset Y$ such that $\mathcal{J}(D)_y = \mathcal{O}_{Y,y}$ and $\mathcal{J}(D)_{y'} \subsetneq \mathcal{O}_{Y,y'}$ for all y' close to y. Moreover, f is smooth, $h^q(X_y, \omega_{X_y})$ does not depend on y. Hence by Grauert's Theorem [**Har77**, Corollary III.12.9], $R^q f_*\omega_{X/Y}$ is locally free. Then, consider the short exact sequence as in the proof of 2.1.4

$$0 \to R^q f_*\omega_X \otimes \mathcal{J}(D) \otimes \mathcal{L} \to R^q f_*\omega_X \otimes \mathcal{L} \to f_*\omega_X \otimes \mathcal{L} \otimes \mathcal{O}_X/\mathcal{J}(D) \to 0$$

Claim 3.3.2. *The relative version of the Nadel vanishing theorem is true in this case, i.e. $H^i(Y, R^q f_*\omega_X \otimes \mathcal{L} \otimes \mathcal{J}(D)) = 0$ for all $i > 0$.*

Assuming the claim we obtain a surjection

$$H^0(Y, R^q f_*\omega_X \otimes \mathcal{L}) \to H^0(Y, R^q f_*\omega_X \otimes \mathcal{L} \otimes \mathcal{O}_X/\mathcal{J}(D))$$

Since y is an isolated point in the cosupport of $\mathcal{J}(D)$, we obtain a surjection

$$H^0(Y, f_*\omega_X \otimes \mathcal{L}) \to f_*\omega_X \otimes \mathcal{L} \otimes \kappa(y).$$

It now remains to show the claim 3.3.2. It follows from Lemma 3.3.3. □

The proof of Claim 3.3.2 is a prototype of Kawamata's original argument in [**Kaw02**]. One of the key points of both the arguments is the fact that $R^q f_*\omega_{X/Y}$ pulls-back nicely under log resolutions. When f is smooth outside Σ with certain restrictions on monodromy, this behaviour is again satisfied under smooth blow ups; see e.g. [**Kaw02**, Lemma 2.1]. When f is smooth we do this using usual arguments from algebraic geometry.

Lemma 3.3.3. *Let $f : X \to Y$ be a smooth morphism of smooth projective varieties. Let D be an effective $\mathbb{Q}$-divisor on Y and L an effective Cartier divisor such that $L - D$ is big and nef. Then*

$$H^i(Y, R^q f_*\omega_X \otimes \mathcal{O}_X(L) \otimes \mathcal{J}(D)) = 0$$

for all $i > 0$.

Proof. Let $\mu : Y' \to Y$ be a log resolution of D such that μ is an isomorphism outside $\mathrm{Supp}(D)$. Then the following commutative diagram is a Cartesian square:

$$\begin{array}{ccc} X' & \xrightarrow{\tau} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\mu} & Y \end{array}$$

Figure 3.2. Log resolution of (X, D)

Here f' is smooth by base change [**Har77**, Proposition III.10.1] and hence so is Y'. Further, the base change morphism

$$R^q f'_* \omega_{X'/Y'} \to \mu^* R^q f_* \omega_{X/Y}$$

is an isomorphism. Indeed, the sheaves are isomorphic over $\mu^*(X \setminus \operatorname{Supp} D)$ and $R^q f'_* \omega_{X'/Y'}$ has no torsion [**Kol86**, Theorem 2.1] and therefore the map is an injection everywhere. Let Q denote the quotient of this injection. Since f' is smooth, around any point $y' \in Y'$ by Cohomology and Base Change Theorem [**Har77**, Theorem III.12.11] we have

$$R^q f'_* \omega_{X'/Y'} \otimes \kappa(y') \simeq H^q(X'_{y'}, \omega_{X'_{y'}})$$

where $X'_{y'} = f'^{-1}(y')$. Since tensoring is right exact, the following vector space isomorphisms

$$\mu^* R^q f_* \omega_{X/Y} \otimes \kappa(y') \simeq R^q f_* \omega_{X/Y} \otimes \kappa(y) \simeq H^q(X_y, \omega_{X_y}) \simeq H^q(X'_{y'}, \omega_{X'_{y'}})$$

imply that $Q \otimes \kappa(y') = 0$ for all $y' \in Y'$. Hence $Q = 0$ [**Har77**, Exercise II.5.8]. Now, letting $\mu^* D = \widetilde{D} + E$, we write:

$$K_{Y'} + \mu^*(L - D) = \mu^*(K_Y + L) + F \tag{3.4}$$

where $F = K_{Y'/Y} - \mu^* D$. By the local vanishing for multiplier ideals [**Laz04b**, Theorem 9.4.1.], we know that $R^i\mu_*\mathcal{O}_{Y'}(\lceil F \rceil) = 0$ for all $i > 0$. Therefore, by the degeneration of the Leray spectral sequence and the projection formula we have,

$$H^i(Y', \mu^* R^q f_* \omega_X(\mu^* L) \otimes \mathcal{O}_{Y'}(\lceil F \rceil)) \simeq H^i(Y, R^q f_* \omega_X(L) \otimes \mathcal{J}(D))$$

for all i. Therefore it is enough to show that the former is 0 for all $i > 0$. To this end, note that by (3.4)

$$H^i(Y', \mu^* R^q f_* \omega_X(\mu^* L) \otimes \mathcal{O}_{Y'}(\lceil F \rceil)) \simeq H^i(Y', R^q f'_* \omega_{X'} \otimes \mathcal{O}_{Y'}(\lceil \mu^*(L - D) \rceil)) = 0$$

for all $i > 0$ and $q \geqslant 0$. The vanishing part of the above follows by an application of Kollár's vanishing theorem [**Kol95**, Theorem 10.19]. Indeed, since f is smooth, we can write $f^*(\mu^* D - \lfloor \mu^* D \rfloor) = \Sigma_i a_i \Delta_i$ for some simple normal crossing divisor $\Sigma_i \Delta_i$ and $0 < a_i < 1$. Then the assertion follows by letting P be the Cartier divisor satisfying

$$P - (K_{X'} + \Sigma_i a_i \Delta_i) \sim_{\mathbb{Q}} f^* H$$

with $H = \mu^*(L - D)$ a big and nef divisor. □

Remark 3.3.4. When f is smooth outside a simple normal crossing divisor Σ and the local system $R^q f_* \mathbb{C}_{X_0}$ with $X_0 = f^{-1}(Y \setminus \Sigma)$ has unipotent mordromies along the components of Σ, the isomorphism $R^q f'_* \omega_{X'/Y'} \simeq \mu^* R^q f_* \omega_{X/Y}$ is still true; see [**Kaw02**, Lemma 2.1].

3.4. Generic Generation for LC Pairs

This section is devoted to establish the global generation statement Theorem 3.4.3 for reduced snc pairs. Since the argument relies on Hodge theory, we detailed a few preliminary definitions and results required in our argument in Appendix A.

3.4.1. Generalisations to Reduced Simple Normal Crossing Pairs

The much discussed global generation statement of Kawamata [**Kaw02**] is as follows.

Theorem 3.4.1. *Let $f\colon X \to Y$ be a surjective morphism from of smooth projective varieties so that f is smooth outside a simple normal crossing divisor $\Sigma \subset Y$. Furthermore, let $\mathcal{L}$ be a big and nef line bundle on Y, satisfying intersection properties as in 2.1.5 around $y \in Y$, then $R^q f_* \omega_X \otimes \mathcal{L}$ is globally generated at y for all $q \geqslant 0$.*

Pictorially the situation is as follows.

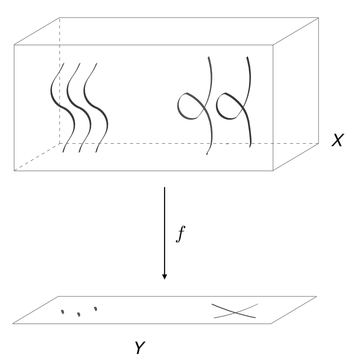

Figure 3.3. Kawamata's global generation result

The key technique of Kawamata's original proof was to exploit the fact that variations of Hodge structures outside simple normal crossing divisors has particularly nice extensions. A Hodge module theoretic formulation of Kawamata's constructions was done in the PhD thesis of Wu [**Wu17**], where he showed similar global generation properties for Hodge modules with strict support.

Theorem 3.4.2 ([**Wu18**, Theorem 4.2])**.** *Let $\mathcal{M}$ be a pure Hodge module on Y with strict support on Y, i.e. it has no submodule or quotient module supported on a proper subvariety of Y. Furthermore let $\mathcal{M}|_{Y\setminus\Sigma}$ be a polarised variation of Hodge structure and let $\mathcal{L}$ be a nef and big bundle on Y, satisfying intersection properties as in 2.1.5 around $y \in Y$, then*

$$\omega_Y \otimes F^{\text{low}}\mathcal{M} \otimes \mathcal{L}$$

is globally generated at y.

Using these two theorems we establish the following. An snc pair (X, D) with with coefficients of the components of D equal to 1is said to be *reduced simple normal crossing pair*; see Example 3.2.3. For the notion of log-smooth see Definition A.3.1.

Theorem 3.4.3. *Let $f : X \to Y$ be a surjective morphism from a reduced simple normal crossing pair (X, D) to a smooth projective variety Y of dimension n. Furthermore, assume that f is log-smooth outside a simple normal crossing divisor $\Sigma \subset Y$ and let $\mathcal{L}$ be a nef and big line bundle on Y, satisfying intersection properties as in 2.1.5 around $y \in Y$, then for all $q > 0$ $R^q f_*\omega_X(D) \otimes \mathcal{L}$ is globally generated at y.*

Proof. *A Simple Case.*

We first discuss the situation in Figure A.3; namely let $f:(X,D)\to Y$ be a morphism of smooth projective varieties with D a smooth divisor on X. Furthermore, for Σ a simple normal crossing divisor on Y and $Y'=Y\setminus\Sigma$, let $f':(X',D')\to Y'$ log-smooth where $X'=f^{-1}(Y')$ and $D'=X'\cap D$. Denoting by $j:X'\setminus D'\hookrightarrow X'$ the inclusion, there is a short exact sequence of Hodge modules

$$0\to W_0\mathcal{O}_X(*D)\to\mathcal{O}_X(*D)\to gr_1^W\mathcal{O}_X(*D)\to 0.$$

with weights described in Example A.2.2 (A.4), i.e.

$$W_0\mathcal{O}_X(*D)=\mathcal{O}_X,\quad gr_1^W\mathcal{O}_X(*D)=\mathcal{O}_D\quad\text{and}\quad gr_i^W\mathcal{O}_X(*D)=0\text{ for }i\neq 0,1 \tag{3.5}$$

The derived functor f_+ thus produces a long exact sequence of Hodge modules

$$0\to\mathcal{H}^0f_+\mathcal{O}_X\to\mathcal{H}^0f_+\mathcal{O}_X(*D)\to\mathcal{H}^0f_+\mathcal{O}_D\to\mathcal{H}^1f_+\mathcal{O}_X\to\cdots. \tag{3.6}$$

Just as in the case of mixed Hodge structure, by [**Sai88**, Proposition 5.3.5] the morphisms $\mathcal{H}^qf_*\mathcal{O}_D\to\mathcal{H}^{q+1}f_+\mathcal{O}_X$, being morphisms of E_1 page of the weight spectral sequence, are morphisms of Hodge modules. Hence, $gr_0^W\mathcal{H}^qf_+\mathcal{O}_X(*D)$ and $gr_1^W\mathcal{H}^qf_+\mathcal{O}_X(*D)$ are both mixed Hodge modules so that restricted to Y' they are mixed Hodge structures given by $R^qf_*\mathbb{C}_X$ and $R^qf_*\mathbb{C}_D$.

By the decomposition by strict support, (mixed) Hodge modules split into direct sums of (mixed) Hodge modules with support along closed subvarieties of Y so that no sub or quotient Hodge module has support along Y. Therefore we can restrict to the unique direct summand of $\mathcal{H}^qf_+\mathcal{O}_X(*D)$ supported all over Y. Denote by $ß_Y(-)$ taking the strict support along Y. Applying $ß_Y(-)$ we obtain Hodge modules

$$\mathcal{W}_0 := ß_Y(gr_0^W \mathcal{H}^q f_+ \mathcal{O}_X(*D)) \text{ and } gr_1^{\mathcal{W}} := ß_Y(gr_1^W \mathcal{H}^q f_+ \mathcal{O}_X(*D))$$

with strict support along Y, underlying variations of Hodge structures when restricted to Y'.

These pure Hodge modules with strict support fit into the weight short exact sequence of $\mathcal{H}^q f_+ \mathcal{O}_X(*D)$ breaking the strict support version of the long exact sequence (3.6) into smaller pieces.

$$0 \to \mathcal{W}_0 \to ß_Y(\mathcal{H}^q f_+ \mathcal{O}_X(*D)) \to gr_1^{\mathcal{W}} \to 0.$$

Taking F^{low}, tensoring with $\mathcal{L}$ and then taking the global sections of the above we obtain:

(3.7)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(Y, \omega_Y \otimes F^{\text{low}}\mathcal{W}_0 \otimes \mathcal{L}) & \longrightarrow & H^0(Y, R^q f_* \omega_X(D) \otimes \mathcal{L}) & \longrightarrow & H^0(Y, \omega_Y \otimes F^{\text{low}} gr_1^{\mathcal{W}} \otimes \mathcal{L}) & \longrightarrow & 0 \\ & & \twoheaddownarrow & & \downarrow & & \twoheaddownarrow & & \\ 0 & \longrightarrow & \omega_Y \otimes F^{\text{low}}\mathcal{W}_0 \otimes \mathcal{L} \otimes \kappa(y) & \longrightarrow & R^q f_* \omega_X(D) \otimes \mathcal{L} \otimes \kappa(y) & \longrightarrow & \omega_Y \otimes F^{\text{low}} gr_1^{\mathcal{W}} \otimes \mathcal{L} \otimes \kappa(y) & \longrightarrow & 0 \end{array}$$

Indeed, the top row is exact because of the Vanishing Theorem A.4.9

$$H^1(X, \omega_Y \otimes F^{\text{low}}\mathcal{W}_0 \otimes \mathcal{L}) = 0.$$

Moreover, by Theorem 3.4.2, we know that $\omega_X \otimes F^{\text{low}}\mathcal{W}_0 \otimes \mathcal{L}$ and $\omega_Y \otimes F^{\text{low}} gr_1^{\mathcal{W}} \otimes \mathcal{L}$ are globally generated at y. Hence the surjectivity of the middle column follow from the snake lemma.

The General Case.

The proof is very similar to the previous case, except in this case we have a filtration of $\mathcal{H}^q f_+\mathcal{O}_X(*D)$ by possibly more than two weights and therefore we need to argue by induction. With the Notation from Example A.2.5, let $\mathcal{M}$ denote the mixed Hodge module corresponding to the extension $Rj_*\mathbb{L}_{\mathbb{Q}}$ on X where $j : X' \to X$ is the open immersion and $\mathbb{L}_{\mathbb{Q}}$ is as in (A.8).

By [**Sai88**, Proposition 5.3.5] we have a spectral sequence

$$E_1^{-m,m+q} = \mathcal{H}^q f_+ gr_m^W \mathcal{M} \Rightarrow gr_m^W \mathcal{H}^q f_+\mathcal{M}$$

degenerating at E_2 with $E_2^{p,q}$ mixed Hodge modules; see [**FFS14**, Corollary 1]. Furthermore, restricting $\mathcal{H}^q f_+\mathcal{M}$ to Y' we obtain the mixed Hodge complex

$$((R^q f_*\mathbb{L}_{\mathbb{Q}}, W), (R^q f'_*\widetilde{\Omega}^\bullet_{X'/Y'}(\log D'), W, F))$$

underlying a mixed Hodge structure. Similarly, restricting the $E_1^{-m,q+m}|_{Y'}$ to Y', we get the variation of Hodge structures corresponding to the pushforwards of the graded weight filtration of $\mathbb{L}$, namely $R^q f_* gr_m^W R^q f_*\mathbb{L}_{\mathbb{Q}}$. Therefore restricting $E_2^{-m,q+m}|_{Y'}$ we obtain the variation of Hodge structures corresponding to the mixed Hodge complex

$$(gr_{-m}^W R^q f_*\mathbb{L}_{\mathbb{Q}}, gr_{-m}^W R^q f'_*\widetilde{\Omega}^\bullet_{X'/Y'}(\log D'), F).$$

Taking strict supports we obtain $\text{ß}_Y(E_2^{-m,q+m}(f_+\mathcal{M}))$ are pure Hodge modules underlying variation of Hodge structures on Y' for all m and q.

We now apply Theorem 3.4.2 on the lowest graded piece of the Hodge module with strict support $\text{ß}_Y(E_2^{-m,q+m}) = \text{ß}_Y(gr_m^W \mathcal{H}^q f_+\mathcal{M})$ to obtain

$$F^{\text{low}}\text{ß}_Y(gr_m^W \mathcal{H}^q f_+\mathcal{M}) \otimes \mathcal{L}$$

is globally generated at y.

By induction we obtain $F^{\text{low}}\text{ß}_Y(W_m \mathcal{H}^q f_+\mathcal{M}) \otimes \mathcal{L}$ is globally generated at y. Indeed, consider

$$0 \to \text{ß}_Y(W_{m-1}\mathcal{H}^q f_+\mathcal{M}) \to \text{ß}_Y(W_m \mathcal{H}^q f_+\mathcal{M}) \to \text{ß}_Y(gr_m^W \mathcal{H}^q f_+\mathcal{M}) \to 0.$$

Using the Vanishing Theorem A.4.9 and taking $H^0(Y, F^{\text{low}}(-) \otimes \mathcal{L})$ we obtain:

(3.8)
$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0(Y, F^{\text{low}}\mathcal{W}_{m-1} \otimes \mathcal{L}) & \longrightarrow & H^0(Y, F^{\text{low}}\mathcal{W}_m \otimes \mathcal{L}) & \longrightarrow & H^0(Y, F^{\text{low}} gr_m^{\mathcal{W}} \otimes \mathcal{L})) & \longrightarrow 0 \\
& \downarrow\!\!\downarrow & & \downarrow & & \downarrow\!\!\downarrow & \\
0 \longrightarrow & F^{\text{low}}\mathcal{W}_{m-1} \otimes \mathcal{L} \otimes \kappa(y) & \longrightarrow & F^{\text{low}}\mathcal{W}_m \otimes \mathcal{L} \otimes \kappa(y) & \longrightarrow & F^{\text{low}} gr_m^{\mathcal{W}} \otimes \mathcal{L} \otimes \kappa(y) & \longrightarrow 0
\end{array}$$

where for brevity we are using the notation $\mathcal{W}_{m-1} \coloneqq \text{ß}_Y(W_{m-1}\mathcal{H}^q f_+\mathcal{M})$, $\mathcal{W}_m \coloneqq \text{ß}_Y(W_m \mathcal{H}^q f_+\mathcal{M})$ and $gr_m^{\mathcal{W}} \coloneqq \text{ß}_Y(gr_m^W \mathcal{H}^q f_+\mathcal{M})$. The left vertical arrow is surjective by induction and the right vertical arrow is surjective by Theorem 3.4.2. Therefore $R^q f_*\omega_X(D)$ is globally generated at y, since $R^q f_*\omega_X(D) = F^{\text{low}}\text{ß}_Y W_m \mathcal{H}^q f_+\mathcal{M}$ for some $m \geqslant 0$. □

3.4.2. Generic Generation for LC Pairs: Kawamata Cover Technique

A key ingredient of the main Theorem 3.4.8 for the effective generic generation for log-canoincal pairs is Kawamata covering technique. Given a divisor D, a Kawamata cover of D is way to take roots of D. This is a two step process: A) Bloch–Gieseker Cover and B) Cyclic Cover. We first briefly recall these constructions. The main reference for this part is [**Laz04a**, §4.1].

Lemma 3.4.4 (Bloch–Gieseker Construction)**.** *Let X be a smooth projective variety and let D be a smooth irreducible effective divisor on X. For any positive integer b, there exists a smooth projective variety Z a finite flat surjective morphism $p : Z \to X$, together with a line bundle M on Z, such that $p^*\mathcal{O}_X(D) \simeq M^{\otimes b}$. Furthermore, if $\dim X \geqslant 2$, letting V be any smooth irreducible subvariety of X intersecting D transversely, we can arrange that $V' = p^*V$ is again smooth irreducible and intersects p^*D transversely.*

Proof Sketch. For a proof and a more generalised statement, we refer the readers to [**Laz04a**, Theorem 4.1.10]. Here we highlight a few important steps that are essential for our purpose. Write $D = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$, where $\mathcal{L}_i$'s are very ample line bundles for $i = 1, 2$. Therefore it is enough to construct p and Z for a very ample line bundle $\mathcal{L}$. This is done by embedding $X \hookrightarrow \mathbb{P}^N$ via sections of $\mathcal{L}$ and then taking the Frobenius fibration

$$\pi : \mathbb{P}^N \to \mathbb{P}^N$$

defined by

$$(z_0 : \cdots : z_N) \mapsto (z_0^b : \cdots : z_N^b).$$

Precomposing this with an action of $g \in \mathrm{GL}_{N+1}(\mathbb{C})$, we obtain $\pi_g \colon \mathbb{P}^N \to \mathbb{P}^N$. For g general enough, the fibre product over π_g given by $Z \coloneqq X \times_{\mathbb{P}^N} \mathbb{P}^N$ is smooth and irreducible. Letting $p : Z \to X$ denote the resulting morphism, observe that

$$p^*\mathcal{L} \simeq \pi_g^*\mathcal{O}(1)|_Z \simeq \mathcal{O}(b)|_Z.$$

Furthermore $V' = p^*V$, $D' = p^*D$ and $D' \cap V' = p^*(D \cap V)$ are all smooth and irreducible.

□

Lemma 3.4.5 (Cyclic Cover)**.** *Let X be a smooth variety and $D \in |M^{\otimes b}|$ a smooth effective divisor in the b^{th}-tensor power of a line bundle M on X. Then there exists a smooth variety X' and a finite flat morphism $p : X' \to X$, called the b^{th}-cyclic cover along D, satisfying the reduced scheme structure on $D' = (p^*D)_{\mathrm{red}}$ is smooth and $bD' \sim p^*D$. Furthermore, if $Z \subseteq X$ is a smooth subvariety of X intersecting D transversely, i.e. $D \cap Z$ is smooth then, $p^{-1}(Z)$ and $p^{-1}(D \cap Z)$ are both smooth. In other words, $D' = p^{-1}(D)$ with reduced scheme structure intersects Z transversely.*

Proof. Let D be given by $s \in \Gamma(X, M^{\otimes b})$. Denote by $\mathbb{M}$ (resp. $\mathbb{M}_Z$) the total space of the line bundle M (resp. M). Then we have cartesian squares

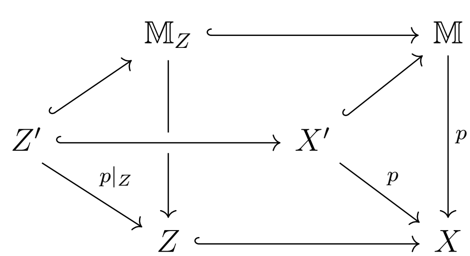

Figure 3.4. Cyclic Covering

where $X' \coloneqq \mathrm{Zeroes}(T^b - p^*s)$ and $T \in \Gamma(\mathbb{M}, p^*M)$ is the tautological section. Then by commutativity $Z' \coloneqq \mathrm{Zeroes}(T^b - p^*s)|_{\mathbb{M}_Z}$. Since Z and $D \cap Z$ are both smooth, so is Z'; see [**Laz04a**, Proposition 4.1.6.]. By a similar logic D' and $D' \cap Z'$ are also smooth. Indeed, by a local choice of parameters

$$(t, z_2, \cdots, z_r, x_{r+1}, \cdots, x_n)$$

around a point $x \in D \cap Z$, so that $D = (t = 0)$ and $Z = \mathrm{Zeroes}(x_{r+1}, \cdots, x_n)$, by construction we have $X' \overset{\text{loc}}{\simeq} \mathrm{Spec}\dfrac{\mathcal{O}_X[T]}{(T^b - t)}$. Hence, p is locally defined by

$$p\colon (T, z_2, \cdots, z_r, x_{r+1}, \cdots, x_n) \mapsto (T^b, z_2, \cdots, z_r, x_{r+1}, \cdots, x_n).$$

Therefore, $D' = (T = 0)$.

Restricted to Z, with coordinates $(t, z_1, \cdots, z_{r-1})$ and $Z' = p^{-1}(Z)$, $p|_{Z'} : Z' \to Z$ is defined by

$$p|_{Z'}\colon (T, z_1, \cdots, z_r) \mapsto (T^b, z_2, \cdots, z_r)$$

and hence Z' and $Z' \cap D'$ are both regular. Indeed, in terms of local parameters $Z' = \mathrm{Zeroes}(x_{r+1}, \cdots, x_n)$ and $Z' \cap D' = \mathrm{Zeroes}(T, x_{r+1}, \cdots, x_n)$. □

Having established a global generation statement in Theorem 3.4.3 for reduced snc-pair for morphisms log-smooth outside a simple normal crossing divisor, it is natural to try to extend such global generations for more general pairs (X, D). Recall that

Definition 3.4.6 (Log-Canonical and Kawamata-Log-Terminal Pairs)**.** Let X be a normal projective variety and Δ an effective $\mathbb{Q}$-Cartier divisor on X such that $\lfloor \Delta \rfloor = 0$. We say that (X, Δ) is a *Kawamata-log-terminal*, or klt pair (resp. log-canonical pair) if for any log resolution $\mu : X' \to X$, we can write

$$K_{X'} = \mu^*(K_X + \Delta) + F$$

where $F = \Sigma_i a_i F_i$ and $a_i > -1$. (resp, $a_i \geqslant -1$). In the language of multiplier ideals, this is equivalent to $\mathcal{J}(D) \simeq \mathcal{O}_X$ (resp. $\mathcal{J}((1-\epsilon)D) \simeq \mathcal{O}_X$) by Remark 2.1.3 (3) (resp. (4))

We now extend the global generation to log-canonical pairs. After a birational modification, we can always assume that a log-canonical pair consists of a smooth projective variety X and a divisor D on it with simple normal crossing support such that coeffecients of its components are in $(0, 1]$. We will use this reduction repeatedly and therefore we include its proof in the following

Lemma 3.4.7 (snc-modification of pairs)**.** *Let (X, D) be a log-canonical (resp. klt) $\mathbb{R}$-pair, and suppose there exists a Cartier divisor P on X such that*

$$P - (K_X + D) \sim_{\mathbb{R}} H$$

for some some $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H. Then, for every proper birational morphism $\mu\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and $\mu^{-1}(D) + Exc(\mu)$ has simple normal crossings support, then there exists a divisor $\widetilde{P}$ on $\widetilde{X}$ and an $\mathbb{R}$-divisor $\widetilde{D}$ such that

i. *$\widetilde{D}$ has coefficients in $(0, 1]$ (resp. $(0, 1)$) and simple normal crossings support;*
ii. *The divisor $\widetilde{P} - \mu^*P$ is an effective divisor with support in $\operatorname{Supp}\big(Exc.(\mu)\big)$;*
iii. *The divisor $\widetilde{P}$ satisfies $\widetilde{P} - (K_{\widetilde{X}} + \widetilde{D}) \sim_{\mathbb{R}} \mu^*H$; and*
iv. *There is an isomorphism $\mu_*\mathcal{O}_{\widetilde{X}}(\widetilde{P}) \simeq \mathcal{O}_X(P)$.*

Proof. On $\widetilde{X}$, we can write

$$K_{\widetilde{X}} - \mu^*(K_X + D) = Q - N$$

where Q and N are effective $\mathbb{R}$-divisors without common components, such that $Q - N$ has simple normal crossings support and Q is μ-exceptional. Note that since (X, D) is log-canonical (resp. klt), all coefficients in N are less than or equal to 1 (resp. less than 1). Let

$$\widetilde{D} \coloneqq N + \lceil Q \rceil - Q,$$

so that by definition, $\widetilde{D}$ has simple normal crossings support and coefficients in $(0, 1]$ (resp. $(0, 1)$). Now setting $\widetilde{P} \coloneqq \mu^* P + \lceil Q \rceil$, we have

$$\begin{aligned} \widetilde{P} - (K_{\widetilde{X}} + \widetilde{D}) &\sim_{\mathbb{R}} \mu^* P + \lceil Q \rceil - (\mu^*(K_X + D) + Q - N + N + \lceil Q \rceil - Q) \\ &\sim_{\mathbb{R}} \mu^* H. \end{aligned}$$

Since $\lceil Q \rceil$ is μ-exceptional, we get $\mu_* \mathcal{O}_{\widetilde{X}}(\widetilde{P}) \simeq \mathcal{O}_X(P)$ by using the projection formula. □

We prove a generic version of Theorem 3.4.3 for log-canonical pairs without any additional assumption on f. A version of the following for klt pairs with simple normal crossing support in [**Dut17**, Proposition 1.3] is a consequence of the following

Theorem 3.4.8. *Let $f\colon (X, D) \to Y$ be a surjective morphism from a log-canonical pair (X, D) to a smooth projective variety Y of dimension n, such that f is smooth outside of a closed subvariety B in Y. Assume that D is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor and that there is* *a Cartier divisor P satisfying*

$$P - (K_X + D) \sim_{\mathbb{Q}} f^*H$$

for some semiample $\mathbb{Q}$-divisor H on Y. Furthermore, let $\mathcal{L}$ be a nef and big line bundle on Y satisfying Angehrn–Siu type intersection properties as in Lemma 2.1.5. Then there exists a Zariski open set $U \subseteq Y$ such that for all $y \in U$

$$R^q f_* \mathcal{O}_X(P) \otimes \mathcal{L}$$

is globally generated at y.

A Discussion on the Proof. By taking a log resolution it is possible to reduce the statement to X smooth and D with simple normal crossing support with coefficient in $(0, 1]$. Then we reduce to Theorem 3.4.3, i.e. when D is reduced and all its components are log-smooth over an appropriate open set. To this end, we use cyclic covering techniques to remove the components of D with coeffecients strictly smaller than 1 and then prove a global generation statement on the locus where (X, D) is log-smooth. This locus automatically avoids the components of D that maps onto proper subvarieties of Y and therefore we can make a precise statement about the locus of global generation; namely $U = U(f, D)$ as in the following

Notation 3.4.9. We denote by $U(f, D)$ the largest open subset of Y such that

- $U(f, D)$ is contained in the smooth locus Y_{reg} of Y;
- $f\colon f^{-1}(U(f, D)) \to U(f, D)$ is smooth; and

- The fibers $X_y \coloneqq f^{-1}(y)$ intersect each component of D transversely for all closed points $y \in U(f, D)$.

This open set $U(f, D)$ is non-empty by generic smoothness; see [**Har77**, Corollary III.10.7] and [**Laz04a**, Lemma 4.1.11].

Proof. By Lemma 3.4.7 we assume that X is smooth and D has simple normal crossing support with coefficients in $(0, 1]$.

Since H is semiample, so is f^*H and therefore by Bertini's theorem (see Remark III.10.9.2 [**Har77**]), we can pick a fractional $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} f^*H$ with smooth support such that $D' + D$ still has simple normal crossings support, $\mathrm{Supp}(D')$ is not contained in the support of the D and intersects the fibre over y transversely or not at all and $D' + D$ has coefficient in $(0, 1)$. We rename $D' + D$ by D and denote the fractional part of D by Δ, i.e. $\Delta \coloneqq D - \lfloor D \rfloor$.

Step 1. Kawamata Covering of Δ. If $\Delta = 0$ we move to Step 3.4.2.

Otherwise let $\Delta = \frac{l}{k} D_1 + D_2$ with $l, k \in \mathbb{Z}_{>0}$, $l < k$ and D_1 smooth irreducible. We choose a Bloch-Gieseker cover $p \colon Z \to X$ along D_1, so that $p^*D_1 \sim kM$ for some Cartier divisor (possibly non-effective) M on Z and so that the components of $p^*\Delta$ and the fibre $(f \circ p)^{-1}(y)$ are smooth and intersect each other transversely or not at all; see Lemma 3.4.4.

Moreover since p is flat and f is smooth over a neighbourhood around y, we can conclude that there is a open neighbourhood U around y such that $f \circ p$ is still smooth over U [**Har77**, Ex. III.10.2].

Set $g = f \circ p$ and denote by $B \subset Y$, the non-smooth locus of g and note that $y \notin B$.

Now, ω_X is a direct summand of $p_*\omega_Z$ via the trace map and $R^q p_*\omega_Z = 0$ for $q > 0$. Define by $P_Z \coloneqq p^*P + K_{Z/X}$, i.e. P_Z is a Cartier divisor $\mathbb{Q}$-linearly equivalent to $K_Z + p^*D_2$. Hence

$$R^q f_* \mathcal{O}_X(P) \otimes \mathcal{L}$$

is a direct summand of

$$R^q g_* \mathcal{O}_Z(P_Z + lM) \otimes \mathcal{L}.$$

Hence it is enough to show that the latter is globally generated at y.

To this end, we take the k^{th} cyclic cover $p_1 : X_1 \to Z$ of p^*D_1. By Lemma 3.4.5 above (see also [**Laz04a**, Remark 4.1.8]), the smoothness of the components of $\mathrm{Supp}(p^*\Delta)$ and of $g^{-1}(y)$, and the intersection properties carry over to X_1, i.e. $(g \circ p_1)^{-1}(y)$ and $p_1^* p^* D_i$ are smooth and intersect each other transversely or not at all. Furthermore, $g \circ p_1$ is still smooth over y, and hence over an open subset U around y. In other words y is not in the branch locus (denoted B again) of $g \circ p_1$. Set $f_1 := g \circ p_1$ and note that, (see for instance, [**EV92**, §1])

$$p_{1_*}\omega_{X_1} \simeq \bigoplus_{i=0}^{k-1} \omega_Z(p^*D_1 - iM) \simeq \bigoplus_{i=0}^{k-1} \omega_Z((k-i)M).$$

The last isomorphism is due to the fact that $p^*D_1 \sim kM$. Further, since $k > l$, the direct sum on the right hand side contains the term $\omega_Z(lM)$ when $i = k - l$.

Let $P_1 = p_1^* P_Z + K_{X_1/Z}$, i.e. P_1 is a Cartier divisor such that $P_1 \sim_{\mathbb{Q}} K_{X_1} + p_1^* p^* D_2$. Since $\mathbb{R}p_{1_*}\omega_{X_1} = \omega_Z$ for $q > 0$, it is enough to show that,

$$R^q f_{1_*} \mathcal{O}_{X_1}(P_1) \otimes \mathcal{L}$$

is generated by global sections at y.

Proceeding inductively this way, it is enough to show that

$$f_{s*}\omega_{X_s}(\lfloor p_s^* D \rfloor) \otimes \mathcal{L}$$

is globally generated at y, where $p_s : X_s \to X$ is the composition of Kawamata covers along the components of Δ (here s is the number of components of Δ) and $f_s = f \circ p_s$. Our argument in Step 1 ensures that $f_s^* D_s$ is a reduced simple normal crossing divisor. We rename f_s by f, X_s by X and $f_s^* D_s$ by D. We again call the non-smooth locus of f_s by B and note that $y \notin B$.

Step 2. Base Case of the Induction. Let U be an open set around y, such that $f\colon (X, D) \to U$ is log-smooth. Redefine $B := Y \setminus U$. Take a birational modification Y' of Y such that $\mu^{-1}(B)_{\mathrm{red}} =: \Sigma$ in Y', as in the diagram below, is a simple normal crossing divisor and such that $Y' \setminus \Sigma \simeq Y \setminus \mathrm{Supp}(B)$. We can find such μ by Sźabo's theorem. In particular, μ is an isomorphism around y. Let $X' \to X$ be a resolution of the largest irreducible component of the fibre product $Y' \times_Y X$. The situation is described in the following commutative diagram and a pictorial illustration.

$$\begin{array}{ccc} X' & \xrightarrow{\tau} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\mu} & Y \end{array}$$

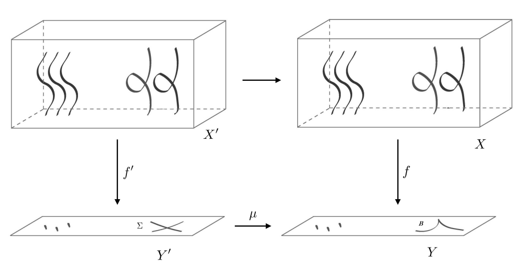

Figure 3.5. Birational Modification of Y.

Let Δ denote the union of the components of τ^*D that satisfy $f'(\Delta) = Y'$. Further note that f' is log-smooth over $Y' \setminus \Sigma$. Moreover, μ is an isomorphism over a neighbourhood U around y and hence $\mu^*\mathcal{L}$ satisfies the intersection properties, as in the hypothesis, at the point $\mu^{-1}(y)$. Therefore, we can apply Theorem 3.4.3 to conclude that $R^q f'_*\omega_{X'}(\Delta) \otimes \mu^*\mathcal{L}$ is globally generated at $\mu^{-1}(y)$ for all q. Therefore, so is $R^q f'_*\omega_{X'}(\tau^*D) \otimes \mu^*\mathcal{L}$ at $\mu^{-1}(y)$. Indeed, we have the following diagram

$$\begin{array}{ccc} H^0(Y', R^q f'_*\omega_{X'}(\Delta) \otimes \mathcal{L}) & \longrightarrow & H^0(Y', R^q f'_*\omega_{X'}(\tau^*D) \otimes \mathcal{L}) \\ \downarrow\!\!\!\!\downarrow & & \downarrow \\ R^q f_*\omega_{X'}(\Delta) \otimes \mathcal{L} \otimes \kappa(\mu^{-1}(y)) & \xrightarrow{\simeq} & R^q f_*\omega_{X'}(\tau^*D) \otimes \mathcal{L} \otimes \kappa(\mu^{-1}(y)) \end{array} \tag{3.9}$$

and hence the right vertical arrow is surjective. The isomorphism in the lower row can be justified using the formal function theorem [**Har77**, Theorem 11.1].

Additionally we have,

$$\mu_*(R^q f'_*\omega_{X'}(\tau^*D) \otimes \mu^*\mathcal{L}) \simeq R^q f_*\omega_X(D) \otimes \mathcal{L}.$$

Therefore the sheaf $R^q f_*\omega_X(D) \otimes \mathcal{L}$ is generated by global sections at y for all $q \geqslant 0$.

$\square$

As an upshot of the arguments above we obtain the following local version of Kawamata's theorem 3.4.1 as well as the of the statement for reduced snc pairs in Theorem 3.4.3.

Corollary 3.4.10. *Let $f : X \to Y$ be a surjective morphism from a reduced simple normal crossing pair (X, D) to a smooth projective variety Y of dimension n. Furthermore, around $y \in Y$, assume that there exists an open set $U \subseteq Y$ such that $f|_{f^{-1}(U)}$ is log-smooth outside a simple normal crossing divisor $\Sigma \subset U$ and let $\mathcal{L}$ be a nef and big line bundle on Y, satisfying intersection properties as in 2.1.5 around $y \in Y$, then for all $q > 0$, $R^q f_* \omega_X(D) \otimes \mathcal{L}$ is globally generated at y.*

Proof. Define $B := Y \setminus U$. Take a birational modification Y' of Y such that $\mu^{-1}(B + \overline{\Sigma})_{\text{red}} =: \Sigma'$ in Y', as in the diagram below, is a simple normal crossing divisor and such that $Y' \setminus \operatorname{Supp}(\mu^{-1}(B)) \simeq Y \setminus \operatorname{Supp}(B)$. We can find such μ by Sźabo's theorem. In particular, μ is an isomorphism around y. Let $X' \to X$ be a resolution of the largest irreducible component of the fibre product $Y' \times_Y X$ such that they satisfy the outcome of Sźabo's theorem. The situation is described in the following commutative diagram and a pictorial illustration.

$$\begin{array}{ccc} X' & \xrightarrow{\tau} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\mu} & Y \end{array}$$

Figure 3.6. Birational Modification of Y.

Then the global generation of $R^q f_* \omega_X(D) \otimes \mathcal{L}$ at y follow similarly as in the proof above by the global generation of $R^q f'_* \omega_X(\tau^* D) \otimes \mu^* \mathcal{L}$ at $\mu^{-1}(y)$. $\square$

CHAPTER 4

Global Generation for Pushforwards of Pluricanonical Sheaves

One of the main results of this section Theorem 4.2.1 shows a generic version of the $k > 1$ case of the Conjecture 3.0.1, establishing the bound predicted by Popa and Schnell in dimension of $Y \leq 4$. Theorem 4.3.1, on the other hand shows a generic global generation result allowing Y to be singular. Below we provide a list of various effective bounds known in the pluricanonical case and the methods that they use. The last two entries hold globally and are due to Popa and Schnell. The locus where the respective global generation holds will be discussed later in more details.

Theorem	$\dim Y$	Y	$\mathcal{L}$	**Technique**	**Twist on** $\mathcal{L}$
Remark 4.3.6	$\leqslant 4$	smooth	ample	minimality and Theorem 3.4.8	k(n+1)
Theorem 4.2.1	any	smooth	ample	weak positivity & Theorem 3.4.8	$k(n+1) + \frac{n^2-n}{2} + 1$
Theorem 4.3.1	any	singular	big & nef	minimality and Theorem 3.1.3	$k(n^2+1)$
[**PS14**]	any	singular	ample and g.g.	minimality and vanishing theorems	k(n+1)
[**PS14**]	1	smooth	ample	semipositivity	2k

Table 4.1. Known Bounds for PS conjecture

4.1. The Theorem of Popa and Schnell

Recall that, given a surjective morphism $f\colon X \to Y$ of smooth projective varieties, Conjecture 3.0.1 predicts that $f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes k(n+1)}$ is globally generated. Additionally assuming that $\mathcal{L}$ is globally generated itself, Popa and Schnell proved the conjecture more generally for log-canonical pairs (X, Δ). Their idea was to reduce the $k > 1$ to $k = 1$ for a different log-canonical pair using [**PS14**] an argument via minimality. We first present a somewhat unnecessary improvement of Popa and Schnell's statement based on their original theorem. Nonetheless, this version will be crucial for establishing the weak positivity statement in §5.1.

Theorem 4.1.1. *Let $f\colon X \to Y$ be a morphism of projective varieties where X is normal and Y is of dimension n. Let Δ be an $\mathbb{R}$-divisor on X and H a semiample $\mathbb{Q}$-divisor on Y such that for some integer $k \geq 1$, there is a Cartier divisor P on X satisfying*

$$P - k(K_X + \Delta) \sim_{\mathbb{R}} f^*H.$$

Suppose, moreover, that Δ can be written as $\Delta = \Delta' + \Delta^v$ where (X, Δ') is log-canonical, Δ^v is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is vertical over Y, i.e. $f(\mathrm{Supp}(\Delta^v)) \subsetneq Y$ and Δ' and Δ^v do not share any component. Let $\mathcal{L}$ be an ample and globally generated line bundle on X. Then, the sheaf

$$f_*\mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes \ell}$$

is generated by global sections on some open set U for all $\ell \geqslant k(n+1)$. Moreover, when Δ has simple normal crossings support, we have $U = Y \smallsetminus f(\mathrm{Supp}(\Delta^v))$.

Proof. Possibly after a log resolution of (X, Δ), by Lemma 3.4.7 we assume that X is smooth and Δ has simple normal crossing support satisfying all the hypotheses in the statement. Furthermore, denote by

$$\Delta' \coloneqq \Delta + \Delta^v - \lfloor \Delta^v \rfloor \quad \text{and rename} \quad \Delta^v \coloneqq \lfloor \Delta^v \rfloor.$$

With this notation the coefficients of the components of Δ' are in $(0,1]$ and Δ^v is an effective Cartier divisor with $f(\operatorname{Supp}\Delta^v) \subsetneq Y$.

Now since f^*H is semiample, by Bertini's theorem we can pick a $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \frac{1}{k} f^*H$ with smooth support and satisfying the conditions that $D+\Delta$ has simple normal crossing support and D does not share any components with Δ'. Letting $\Delta'' \coloneqq \Delta' + D$, we note that (X, Δ'') is log-canonical. Furthermore the Cartier divisor P' satisfies

$$P' \sim_{\mathbb{R}} k(K_X + \Delta'' + \Delta^v).$$

Since $\mathcal{L}$ is ample and globally generated, by [**PS14**, Variant 1.6] we obtain

$$f_*\mathcal{O}_X(P' - k\Delta^v) \otimes \mathcal{L}^{\otimes \ell}$$

is generated by global sections for all $\ell \geqslant k(n+1)$. But

$$f_*\mathcal{O}_X(P' - k\Delta^v) \otimes \mathcal{L}^{\otimes \ell} \hookrightarrow f_*\mathcal{O}_X(P') \otimes \mathcal{L}^{\otimes \ell},$$

and they have the same stalks at every point $y \in U = Y \setminus f(\operatorname{Supp}\Delta^v)$. Thus, the sheaf on the right hand side is globally generated at y for all $y \in U$ and for all $\ell \geqslant k(n+1)$. □

4.2. Generic Generation for LC Pairs: Pluricanonical Case

In this section we proceed without the global generation assumption on $\mathcal{L}$. We begin by following the same idea which was used in Popa–Schnell's original arguments in [**PS14**], i.e. reduce the global generation problem for $k > 1$ to the global generation problem of $k = 1$ for a different log-canonical pair. Roughly speaking, there exists, a log canonical pair (X', Δ'), a proper birational morphism $\tau : X' \to X$ and a constant C such that,

$$K_{X'} + \Delta' + (\ell - C)\tau^* f^* L \sim_{\mathbb{R}} \tau^*(k(K_X + \Delta) + \ell f^* L)$$

and therefore ℓ can be estimated solely from the left hand side which we have studied in Chapter 3. This reduction is done using the weak positivity Theorem 5.1.4.

From here on we assume that the morphism $f\colon X \to Y$ is a *fibration*, i.e. the generic fibre of f is connected and irreducible.

Theorem 4.2.1. *Let $f\colon X \to Y$ be a fibration of projective varieties where X is smooth of dimension n. Let (X, Δ) be a log-canonical $\mathbb{R}$-pair and let $\mathcal{L}$ be an ample line bundle on Y. Consider a Cartier divisor P on Y such that $P \sim_{\mathbb{R}} k(K_X + \Delta)$ for some integer $k \geq 1$. Then, there exists a nonempty open set $U \subseteq Y$, so that the sheaf*

$$f_* \mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes \ell}$$

is globally generated at $y \in U$ for $\ell \geqslant k(n+1) + \frac{n^2 - n}{2} + 1$.

Proof. We start with some preliminary reductions. Since the adjunction morphism

$$f^* f_* \mathcal{O}_X(P) \xrightarrow{a} \mathcal{O}_X(P) \tag{4.1}$$

is induced by the identity morphism $f_* \mathcal{O}_X(P) \to f_* \mathcal{O}_X(P)$, we may assume that the image of a is nonzero.

Step 1 (see [**PS14**, Theorem 1.7, Step 1])**.** *We can reduce to the case where X is smooth, Δ has simple normal crossings support with coefficients in $(0, 1]$, the image of*

(4.1) *is of the form* $\mathcal{O}_X(P - E)$ *for a divisor* E *such that* $\Delta + E$ *has simple normal crossings support.*

A priori, the image of the adjunction (4.1) is of the form $\mathscr{I} \cdot \mathcal{O}_X(P)$, where $\mathscr{I} \subseteq \mathcal{O}_X$ is the *relative base ideal* of $\mathcal{O}_X(P)$. Consider a log resolution $\mu \colon \widetilde{X} \to X$ of $\mathscr{I}$ and (X, Δ). Since (X, Δ) is log-canonical, by Lemma 3.4.7 one can find $\widetilde{\Delta}$ such that $(\widetilde{X}, \widetilde{\Delta})$ is log-canonical and there exists a Cartier divisor $\widetilde{P}$ so that

$$\widetilde{P} \sim_{\mathbb{R}} k(K_{\widetilde{X}} + \widetilde{\Delta}) \sim_{\mathbb{R}} \mu^* P + \widetilde{N}$$

where $\widetilde{N}$ is an effective μ-exceptional Cartier divisor. Furthermore, assuming $\mu^{-1}\mathscr{I} \simeq \mathcal{O}_{\widetilde{X}}(-E')$, we obtain that the image of the adjunction morphism

$$\mu^* f^* f_* \mu_* \mathcal{O}_{\widetilde{X}}(\widetilde{P}) \simeq \mu^* f^* f_* \mathcal{O}_X(P) \twoheadrightarrow \mathcal{O}_{\widetilde{X}}(\mu^* P(-E')) \simeq \mathcal{O}_{\widetilde{X}}(\widetilde{P} - \widetilde{N} - E'). \tag{4.2}$$

Denote by $E \coloneqq \widetilde{N} + E'$. We rename, $\widetilde{X}$ by X, and $\widetilde{\Delta}$ by Δ.

Step 2. *Reducing to* $k = 1$ *and a log-canonical pair.*

With Notation 3.4.9, throughout this proof we fix U to denote the intersection of $U(f, \Delta + E)$ with the open set over which $f_* \mathcal{O}_X(P)$ is locally free.

Abusing notation to identify $P \otimes \mu^* f^* \omega_Y^{-1}$ and $k(K_{X/Y} + \Delta)$ we note that

$$\begin{array}{ccc} f^*\Big(\big(f_* \mathcal{O}_X(k(K_{X/Y} + \Delta))\big)^{\otimes b}\Big) & \twoheadrightarrow & \mathcal{O}_X\big(bk(K_{X/Y} + \Delta) - bE\big) \\ \downarrow & & \| \\ f^*\Big(\big(f_* \mathcal{O}_X(k(K_{X/Y} + \Delta))\big)^{[b]}\Big) & \dashrightarrow & \mathcal{O}_X\big(bk(K_{X/Y} + \Delta) - bE\big) \end{array}$$

the dashed map exists making the diagram commute. Indeed, the map exists over the locus where $f_*\mathcal{O}_Y(k(K_{X/Y}+\Delta))$ is locally free. Since the sheaf is torsion free, the locally free locus has a complement of codimension ≥ 2. Furthermore, the bottom right sheaf is locally free and hence by Corollary B.1.4 we can extend the dashed map to all of X.

Now the top arrow is the surjective map obtained by taking the b^{th}-tensor power of (4.2). Then the commutativity of the diagram implies that the bottom arrow is also surjective. By Theorem 5.1.5 we know that over U, there exists $b \in \mathbb{Z}_{\geqslant 1}$ such that

$$f_*\mathcal{O}_X\big(k(K_{X/Y}+\Delta)\big)^{[b]} \otimes \mathcal{L}^{\otimes b}$$

is generated by global sections. Therefore so is $\mathcal{O}_X\big(bk(K_{X/Y}+\Delta)-bE\big) \otimes f^*\mathcal{L}^{\otimes b}$ over $f^{-1}(U)$.

We now fix a point $y \in U$. We can apply Bertini's theorem to choose a divisor

$$D \in \left|\mathcal{O}_X\big(bk(K_{X/Y}+\Delta)-bE\big) \otimes f^*\mathcal{L}^{\otimes b}\right|$$

such that on $f^{-1}(U)$, D is smooth, $D+\Delta+E$ has simple normal crossing support, D is not contained in the support of $\Delta+E$, and D intersects the fibre over y transversely. Letting L denote a Cartier divisor class of $\mathcal{L}$, write

$$\frac{1}{b}D \sim_{\mathbb{R}} k(K_{X/Y}+\Delta)-E+f^*L.$$

Multiplying both sides by $\frac{k-1}{k}$, and then adding $\frac{k-1}{k}E - \frac{k-1}{k}f^*L$, we have

$$(4.3) \qquad \frac{k-1}{k}\left(\frac{1}{b}D+E\right)-\frac{k-1}{k}f^*L \sim_{\mathbb{Q}} (k-1)(K_{X/Y}+\Delta).$$

Let us now define the line bundle $H \coloneqq \omega_Y \otimes \mathcal{L}^{\otimes n+1}$ and denote a divisor class in it by H at the same time. Note that by Mori's cone theorem [**KM98**, Theorem 1.24] H is nef and hence semiample by the base point free theorem [**KM98**, Theorem 3.3]. Therefore, we can assume that H is $\mathbb{Q}$-effective and $\mathrm{Supp}(H)$ satisfies Bertini-type intersection properties with Δ. For a positive integer ℓ, we add $K_X+\Delta+(k-1)f^*H+(\ell-(k-1)(n+1))f^*L$ to both sides of (4.3) to obtain

(4.4)

$$K_X+\frac{k-1}{k}\left(\frac{1}{b}D+E\right)+\Delta+(k-1)f^*H+\left(\ell-\frac{k-1}{k}-(k-1)(n+1)\right)f^*L \sim_{\mathbb{R}} P+\ell f^*L.$$

Step 3. *Applying Theorem 3.4.8 to obtain global generation.*

In order to apply Theorem 3.4.8 to the left hand side of (4.4), we now adjust coeffecients of $\frac{k-1}{k}\left(\frac{1}{b}D+E\right)+\Delta+(k-1)f^*H$ and make suitable modifications. To this end, write

$$E=\sum_i s_i\Delta_i+\widetilde{E} \quad \text{and} \quad \Delta=\sum_i a_i\Delta_i$$

where $\widetilde{E}$ and Δ do not have any common component. Note that, by hypothesis, $0<a_i\leqslant 1$ and $s_i\in\mathbb{Z}_{\geq 0}$. We want to pick non-negative integers b_i, such that

$$0\leq a_i+\frac{k-1}{k}s_i-b_i\leqslant 1 \quad \text{and} \quad b_i\leq s_i.$$

Denote by

$$\gamma_i := a_i + \frac{k-1}{k} s_i$$

and note that $\gamma_i < 1 + s_i$. We pick b_i as follows. For some integer j with $0 \le j \le s_i$, we can write $s_i - j + 1 > \gamma_i \geqslant s_i - j$. Then we pick

$$b_i = s_i - j.$$

Now let

$$E' := \sum_i b_i \Delta_i + \left\lfloor \frac{k-1}{k} \widetilde{E} \right\rfloor \preceq E.$$

and assign

$$\widetilde{\Delta} := \Delta + \frac{k-1}{k} E - E' = \sum_i \alpha_i \widetilde{\Delta}_i.$$

Note that $\widetilde{\Delta}$ has coeffecients in $(0,1]$. Subtracting E' from both sides in (4.4), we write

$$K_X + \Delta' + \frac{k-1}{kb} D + (\ell - (k-1)(n+1) - 1) f^* L + f^* H' \sim_{\mathbb{R}} P - E' + \ell f^* L$$

where $H' := (k-1)H + \frac{1}{k} L$ is a semi-ample $\mathbb{Q}$-divisor.

With this choice of E', we have $f_* \mathcal{O}_Y(P - E') \simeq f_* \mathcal{O}_Y(P)$. Indeed, $E' \preceq E$ is an effective Cartier divisor in the relative base locus E of P, and

$$f_* \mathcal{O}_Y(P) \to f_* f^* f_* \mathcal{O}_Y(P) \to f_* \mathcal{O}_Y(P - E)$$

is the identity, and hence $f_* \mathcal{O}_Y(P) \twoheadrightarrow f_* \mathcal{O}_Y(P - E)$ implying it is in fact an isomorphism.

Recall that, D is a smooth divisor that is transversal to Δ' over $f^{-1}(U)$. To obtain such transversal intersection outside $f^{-1}(U)$, we take a log resolution D which is an isomorphism over $f^{-1}(U)$. We can do so by Sźabo's Theorem [**KK13**, Theorem 10.45(1)]. Let $\mu\colon X' \to X$ be a log resolution of $\frac{k-1}{kb}D + \Delta'$, which is an isomorphism over $f^{-1}(U)$. Write

$$\mu^* D = \widetilde{D} + F, \qquad \mu^*\Delta' = \widetilde{\Delta}' + F_1$$

where $\widetilde{D}$ is the strict transform of the components of D that lie above U and $\widetilde{\Delta}'$ is the strict transform of Δ'. Note that both F and F_1 has support outside of $f^{-1}(U)$.

Denote,

$$F' := \left\lfloor \frac{k-1}{kb} F + F_1 \right\rfloor, \qquad \widetilde{\Delta} := \mu^* D + \mu^*\Delta' - F', \qquad \widetilde{P} := \mu^* P + K_{X'/X}$$

By definition $\widetilde{\Delta}$ has coefficients in $(0,1]$. Now adding $K_{X'/X} - F'$ we rewrite (4.4) as:

$$K_{X'} + \widetilde{\Delta} + (\ell - (k-1)(n+1) - 1)\,\mu^* f^* L + \mu^* f^* H \sim_{\mathbb{R}} \widetilde{P} - \mu^* E' - F' + \ell \mu^* f^* L. \tag{4.5}$$

In order to fit in to the framework of Theorem 3.4.8 denote the Cartier divisor on the right hand side of (4.5) P' and write

$$P' - (K_{X'} + \widetilde{\Delta}) \sim_{\mathbb{R}} \mu^* f^* H + (\ell - (k-1)(n+1) - 1)\,\mu^* f^* L.$$

Therefore, by Theorem 3.4.3, $f_*\mu_*\mathcal{O}_{X'}(P') \otimes \mathcal{L}^{\otimes(\ell-(k-1)(n+1)-1)}$ is globally generated at y for all $\ell \geqslant k(n+1) + \frac{n^2-n}{2} + 1$. In other words,

$$f_*\mu_*\mathcal{O}_{X'}(\widetilde{P} - \mu^* E' - F') \otimes \mathcal{L}^{\otimes \ell}$$

is globally generated at y for all $y \in U$ and for all $\ell \geqslant k(n+1) + \frac{n^2-n}{2} + 1$. This implies the desired generic global generation for

$$f_* \mu_* \mathcal{O}_{X'}(\widetilde{P}) \otimes \mathcal{L}^{\otimes \ell}$$

for the same choice of ℓ. Indeed, by construction we have

$$f_* \mu_* \mathcal{O}_{X'}(\widetilde{P} - \mu^* E' - F') \hookrightarrow f_* \mu_* \mathcal{O}_{X'}(\widetilde{P} - \mu^* E') \simeq f_* \mu_* \mathcal{O}_{X'}(\widetilde{P})$$

with isomorphic stalks on U. □

4.3. Seshadri Constant Technique in the Pluricanonical Setting

For an arbitrary morphism $f\colon (X, \Delta) \to Y$ from a log-canonical pair to a projective variety Y, using the coarser bound in Theorem 3.1.7, we obtain an effective generic generation statement.

Theorem 4.3.1. *Let $f\colon (X, \Delta) \to Y$ be a surjective morphism of projective varieties where Y is of dimension n. Let (X, Δ) be a log-canonical $\mathbb{R}$-pair and let $\mathcal{L}$ be a big and nef line bundle on X. Consider a Cartier divisor P on Y such that $P \sim_{\mathbb{R}} k(K_X + \Delta)$ for some integer $k \geq 1$. Then, the sheaf*

$$f_* \mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes \ell}$$

is generated by global sections on an open set U for every integer $\ell \geq k(n^2+1)$.

The argument follows the path of Theorem 4.2.1, namely we reduce to the case $k = 1$. This time, we resort to Popa and Schnell's minimality arguments in order to perform this reduction. Many of the techniques like weak positivity and theorems from the minimal model program do not work without assuming mild enough singularities on Y and therefore unlike before we do not use weak positivity. Furthermore, the global generation for $k = 1$ case uses Theorem 3.1.7.

The reason why the theorem works for nef and big line bundle $\mathcal{L}$ is that whenever $\varepsilon(\mathcal{L}; y) > c$, for some $c > 0$, $\mathcal{L}$ "behave" like an ample line bundle. We present here a few points towards this claim. These results will also be used in the proof of Theorem 4.3.1. An invariant that measures lack of bigness of L is the Augmented base locus.

Definition 4.3.2 (Augmented Base Locus)**.** Let X be a projective variety. If L is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then the *stable base locus* of L is the closed set

$$\mathbf{B}(L) \coloneqq \bigcap_m \mathbf{B}s|mL|_{\mathrm{red}},$$

where m runs over all integers such that mL is Cartier and the subscript "red" denotes the reduced scheme structure. If L is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X, the *augmented base locus* of L is the closed set

$$\mathbf{B}_+(L) \coloneqq \bigcap_A \mathbf{B}(L - A)$$

where A runs over all ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisors A such that $L - A$ is $\mathbb{Q}$-Cartier. By definition, if L is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then

$$\mathbf{B}(L) \subseteq \mathbf{B}_+(L).$$

The following is an example Theorem that shows that outside the augmented base loci line bundles behave somewhat like ample line bundles

Theorem 4.3.3 ([**Kür13**, Proposition 2.7])**.** *Let X be a projective variety and $\mathcal{L}$ a line bundle on X, then for any coherent sheaf $\mathcal{F}$, there exists m large and divisible enough so that outside $\mathbf{B}_+(\mathcal{L})$ the morphism*

$$\mathcal{O}_X \longrightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes m}$$

is surjective.

Note that, $L^{\dim V} \cdot V = L|_V^{\dim V}$. If L is nef, $L|_V^{\dim V} = 0$ if and only if $L|_V$ is not big. With this formulation we define

Definition 4.3.4 (Null Locus)**.** Let X be a projective variety and L be a nef $\mathbb{R}$-Cartier divisor on X, then

$$\text{Null}(L) \quad := \bigcup_{\{V | L^{\dim V} \cdot V = 0\}} V \quad = \bigcup_{\{V | L|_V^{\dim V} = 0\}} V$$

Note that $\text{Null}(L) = X$ if an only if $L^{\dim X} = 0$ if and only if L is not big and hence is another invariant that measure the bigness of L.

A result of Birkar [**Bir17**, Theorem 1.4], built on a result of Nakamaye for smooth projective varieties implies that when L is a nef $\mathbb{Q}$-Cartier divisor

$$\text{Null}(L) = \mathbf{B}_+(L).$$

A consequence of this is the following

Lemma 4.3.5. *Let X be a projective variety, and let $x \in X$ be a closed point. Suppose L is a big and nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. If $\varepsilon(L; x) > 0$, then $x \notin \mathbf{B}_+(L)$.*

Proof. If $x \in \mathbf{B}_+(L) = \mathrm{Null}(L)$, then there exists a closed subvariety $V \subseteq X$ containing x such that $L^{\dim V} \cdot V = 0$. Then $\varepsilon(L; x) = 0$. Indeed, let $\mu\colon \mathrm{Bl}_x X \to X$ denote the blow-up and $\widetilde{C}$ denote the strict transform of a curve $C \subset V$ passing through x with multiplicity 1. Since $C \subset V$, $L \cdot C = 0$, we have

$$(\mu^* L - \delta E) \cdot \widetilde{C} = L \cdot C - \delta = -\delta > 0 \text{ if and only if } \delta < 0$$

and hence maximal such δ is 0.

□

Proof of Theorem 4.3.1. Using Lemma 3.4.7, we assume that X is smooth and Δ has snc support. Furthermore, as in the first few steps of the proof of Theorem 4.2.1, we assume that E is an effective Cartier divisor with simple normal support such that $\Delta + E$ has snc support as well and that E is the relative base locus of P, i.e.

$$f^* f_* \mathcal{O}_X(P) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{O}_X(P - E) \tag{4.6}$$

Let L denote the divisor class of $\mathcal{L}$. and U be the subset of $U(f, \Delta + E)$ where

$$\varepsilon(L; y) > \frac{1}{n + \frac{1}{kn}}$$

for every $y \in U$, which is nonempty by Notation 3.4.9 and Theorem 3.1.5.

We set m to be the smallest positive integer such that $f_* \mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes m}$ is globally generated on U. Since by Corollary 4.3.5, $U \cap \mathbf{B}_+(L) = \emptyset$, this integer m exists; see Theorem 4.3.3.

Step 1. *Reducing the problem to $k = 1$ and a suitable pair.*

This step is a bit different from the one in the proof of Theorem 4.2.1, namely we use the minimality of m above.

From now on, fix a closed point $y \in U$. The surjection in (4.6) tensored with $f^*\mathcal{L}^{\otimes m}$, namely

$$f^*f_*\mathcal{O}_Y(P) \otimes_{\mathcal{O}_X} f^*\mathcal{L}^{\otimes m} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{O}_Y(P-E) \otimes_{\mathcal{O}_X} f^*\mathcal{L}^{\otimes m}$$

implies that $\mathcal{O}_X(P-E) \otimes_{\mathcal{O}_X} f^*\mathcal{L}^{\otimes m}$ is globally generated on $f^{-1}(U)$. Using [**Har77**, Corollary III.10.9 and Remark III.10.9.3], choose a general member

$$D \in |P - E + mf^*L|.$$

so that D is smooth away over $f^{-1}(U)$ and intersects the fibre X_y transversely or not at all. Furthermore the support of $\Delta + E$ has simple normal crossing on $f^{-1}(U)$; see [**Laz04a**, Lemma 4.1.11]. We then have

$$k(K_X + \Delta) \sim_{\mathbb{R}} K_X + \Delta + \frac{k-1}{k}D + \frac{k-1}{k}E - \frac{k-1}{k}mf^*L,$$

hence for every integer ℓ,

$$k(K_X + \Delta) + \ell f^*L \sim_{\mathbb{R}} K_X + \Delta + \frac{k-1}{k}D + \frac{k-1}{k}E + \left(\ell - \frac{k-1}{k}m\right)f^*L.$$

We now adjust the coefficients of Δ and E so they do not share any components. This is done exactly as in the proof of Theorem 4.2.1,

Write

$$E = \sum_i s_i\Delta_i + \widetilde{E} \quad \text{and} \quad \Delta = \sum_i a_i\Delta_i$$

where $\widetilde{E}$ and Δ do not have any common component. Note that, by hypothesis, $0 < a_i \leqslant 1$ and $s_i \in \mathbb{Z}_{\geq 0}$. Then we pick non-negative integers b_i, such that

$$0 \leq a_i + \frac{k-1}{k} s_i - b_i \leqslant 1 \quad \text{and} \quad b_i \leq s_i.$$

Now let

$$E' := \sum_i b_i \Delta_i + \left\lfloor \frac{k-1}{k} \widetilde{E} \right\rfloor \preceq E.$$

and assign

$$\widetilde{\Delta} := \Delta + \frac{k-1}{k} E - E' = \sum_i \alpha_i \widetilde{\Delta}_i.$$

Note that $\widetilde{\Delta}$ has coeffecients in $(0,1]$. Subtracting E' from both sides we write

$$P - E' + \ell f^* L \sim_{\mathbb{R}} K_X + \widetilde{\Delta} + \frac{k-1}{k} D + \left(\ell - \frac{k-1}{k} m \right) f^* L. \tag{4.7}$$

Since E' is contained in the relative basae locus of P, we have $f_* \mathcal{O}_X(P - E') \simeq f_* \mathcal{O}_X(P)$. It therefore suffices to show that

$$f_* \mathcal{O}_X(P - E') \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes \ell} \tag{4.8}$$

is globally generated at y.

Step 2. *Applying Theorem 3.1.7 to obtain global generation.*

We first modify D to allow us to apply Theorem 3.1.7. By Sźabo's Theorem (see e.g. [**KK13**, Theorem 10.45]), there exists a common log resolution $\mu \colon \widetilde{X} \to X$ for D and

(Y,Δ) that is an isomorphism over $f^{-1}(U)$. We write

$$\mu^*D = D' + F, \qquad \mu^*\widetilde{\Delta} = \mu_*^{-1}\widetilde{\Delta} + F_1$$

where D' is a smooth divisor intersecting the fibre over y transversely and F, F_1 are supported on $\widetilde{X} \setminus \mu^{-1}(f^{-1}(U))$. Define

$$F' \coloneqq \left\lfloor \frac{k-1}{k}F + F_1 \right\rfloor, \qquad \Delta' \coloneqq \mu^*\widetilde{\Delta} + \frac{k-1}{k}\mu^*D - F', \qquad \widetilde{P} \coloneqq \mu^*P + K_{\widetilde{Y}/Y}.$$

Note that Δ' has simple normal crossings support and coefficients in $(0,1]$ by assumption on the log resolution and by definition of F'. Moreover, the support of Δ' intersects the fibre over y transversely. Pulling back the equation in (4.7) and adding $K_{\widetilde{Y}/Y} - F'$ yields

$$\widetilde{P} - \mu^*E' - F' + \ell(f\circ\mu)^*L \sim_{\mathbb{R}} K_{\widetilde{Y}} + \mu^*\Delta' + \frac{k-1}{k}\mu^*\mathfrak{D}_x - F' + \left(\ell - \frac{k-1}{k}m\right)(f\circ\mu)^*L$$
$$\sim_{\mathbb{R}} K_{\widetilde{Y}} + \widetilde{\Delta} + \left(\ell - \frac{k-1}{k}m\right)(f\circ\mu)^*L. \tag{4.9}$$

Now note that

(4.10)
$$(f\circ\mu)_*\mathcal{O}_{\widetilde{X}}(\widetilde{P} - \mu^*E' - F') \otimes \mathcal{L}^{\otimes\ell} \hookrightarrow (f\circ\mu)_*\mathcal{O}_{\widetilde{X}}(\widetilde{P} - \mu^*E') \otimes \mathcal{L}^{\otimes\ell} \simeq f_*\mathcal{O}_X(P - E') \otimes \mathcal{L}^{\otimes\ell}$$

with isomorphic stalks at y. Therefore it is enough to show that the right side of the linea equivalence is globally generated.

To this end, we apply Theorem 3.1.7 to Equation (4.9) defining

$$H \coloneqq \frac{k-1}{k}m - \left\lfloor \frac{k-1}{k}m \right\rfloor$$

to conclude that the sheaf in (4.10) is globally generated at y for all $\ell - \left\lfloor \frac{k-1}{k} m \right\rfloor \geqslant n^2+1$.

By the minimality of m, we must have

$$\frac{k-1}{k} m + n^2 + 1 \geqslant m.$$

i.e. $m \leqslant k(n^2+1)$. Therefore $f_*\mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes \ell}$ is globally generated on U for $\ell \geq k(n^2+1)$. □

Remark 4.3.6. If Y is smooth, $\mathcal{L}$ ample and $n = \dim Y \leqslant 4$, in Step 2 above we could apply Theorem 3.4.8 with the linear bound $n+1$, to obtain a linear bound of $k(n+1)$ in the pluricanonical case as well.

4.3.1.1. Further Remarks. It is worth mentioning that when $f\colon X \to Y$ is a morphism of smooth projective varities, using analytic techniques, Iwai [**Iwa18**] and Deng [**Den17**] could show similar effective global generation at all regular values of f. In particular, the open set they obtain in this case do not depend on the relative base locus of $\omega_X^{\otimes k}$.

In a different direction, more recently, Fulgar and Murayama [**FM19**] showed a generic separation of higher order jets for such pushforwards.

Theorem 4.3.7 ([**FM19**, Theorem 8.1])**.** *Let $f\colon (X, D) \to Y$ be a surjective morphism of from a projective log-canonical $\mathbb{R}$-pair (X, D) to a complex projective variety Y with $\dim Y = n$. Let $\mathcal{L}$ be a big and nef line bundle on Y. Assume that there is a Cartier divisor P on X such that*

$$P \sim_{\mathbb{R}} k(K_X + D)$$

for some integer $k \geq 1$, and suppose for a general point $y \in Y \setminus \mathbf{B}_+(\mathcal{L})$ we have

$$\varepsilon(\mathcal{L}; y) > k(n+s)$$

then the sheaf $f_*\mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes k}$ separates s-jets at y for some $s \in \mathbb{Z}_{\geq 0}$ i.e.

$$H^0(Y, f_*\mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes k}) \otimes \mathcal{O}_Y \twoheadrightarrow f_*\mathcal{O}_X(P) \otimes \mathcal{L}^{\otimes k} \otimes \mathcal{O}_Y/\mathfrak{m}_y^{s-1}$$

Remark 4.3.8 (Global Generation for Higher Direct Images of the Pluricanonical Bundles)**.** By an argument of Shibata [**Shi16**, Lemma 4.4.] we know that, when X is a smooth projective irregular variety of dimension n with big anti-canonical bundle, there cannot be a number $N(k, q, n)$ dependent only k, q and the dimension of Y such that $R^q f_*\omega_X^{\otimes k} \otimes \mathcal{L}^{\otimes N}$ is globally generated for any ample line bundle $\mathcal{L}$ on X. Such an example of a smooth projective variety is given by the projective bundle $X = \mathbb{P}_A(E)$ over an Abelian variety A with $E = \mathcal{L}^{-1} \oplus \mathcal{O}_A$ and $\mathcal{L}$ ample. Indeed,

$$\omega_X \simeq \pi^*(\omega_A \otimes \det E) \otimes \mathcal{O}_X(-\mathrm{rk}E) \simeq \pi^*\mathcal{L}^{-1} \otimes \mathcal{O}_X(-2)$$

where $\pi : X \to A$ is the bundle morphism. Now, $H^0(X, \mathcal{O}_X(1)) \simeq H^0(A, \mathcal{L}^{-1}) \oplus H^0(A, \mathcal{O}_A) \neq 0$. Letting D denote this effective divisor class, and L denote a Cartier class of $\mathcal{L}$, we write $-K_X = L + 2D$ which is big by Kodaira's lemma. It is therefore natural to ask whether $R^q f_*\omega_X^{\otimes k}$ satisfies Fujita-type generation properties for X when the ω_X^{-1} is not big.

CHAPTER 5

Applications

In this Chapter we demonstrate how global generation results can be applied to study varieties in families. As observed before sections of the (pluri)-canonical bundles often do not define a morphism on X, but they do so on the complement of the common zero loci of their global sections. This gives rise to the so-called *Kodaira dimension*.

Definition 5.0.1 (Kodaira Dimension)**.** Let X be a smooth projective variety. Then

$$\kappa(X) = \sup_m \dim \overline{\phi_m(X)}$$

where $\phi_m : X \dashrightarrow \mathbb{P}^{P_m}$ is the morphism defined on an open subset of X by the global sections of $\omega_X^{\otimes m}$ and $P_m = |\Gamma(X, \omega_X^{\otimes m})|$ is called the m^{th}-*plurigenus* of X. Another way to interpret this is

$$\kappa(X) \coloneqq \operatorname{trdeg}_k(\bigoplus_m H^0(X, \omega_X^{\otimes m})) - 1$$

where the ring structure on the right side is given by the multiplication map.

The Kodaira dimension is an important object of study. For instance, smooth projective surfaces can be classified up to finitely many isomorphism classes according to its Kodaira dimensions. A celebrated conjecture of Iitaka predicts how the Kodaira dimension of a general fibre is somewhat dictated by the Kodaira dimension of the base and of the total space; namely

$$\kappa(X) \geqslant \kappa(Y) + \kappa(F)$$

where F is the generic fibre of a fibration $f : X \to Y$ of smooth projective varieties. An ad-hoc argument for this conjecture shows how global generations of pushforwards of pluricanonical bundles lead to such subaddivity statements.

5.0.1.1. An ad-hoc Argument. Assume that $f_*\omega_{X/Y}^{\otimes m}$ is generically globally generated for all m. Let r_m denote the generic rank of $f_*\omega_{X/Y}^{\otimes m}$. Then by cohomology and base change, $r_m = h^0(F, \omega_F^{\otimes m})$. By the global generation, we have an injection

$$\mathcal{O}_X^{\oplus r_m} \hookrightarrow f_*\omega_{X/Y}^{\otimes m}.$$

Indeed the morphism is generically an isomorphism and hence injective. Tensoring by $\omega_Y^{\otimes m}$ we observe that

$$h^0(Y, \omega_Y^{\otimes m}) \cdot h^0(F, \omega_F^{\otimes m}) \leqslant h^0(X, \omega_X^{\otimes m})$$

and for m large and divisible enough, $h^0(Y, \omega_Y^{\otimes m}) \sim O(m^{\kappa(Y)})$ and similarly for F and X. Taking logarithm we obtain Iitaka's formula.

In general such global generation is a rather strong assumption. Nonetheless, one has a similar global generation up to a twist by an ample line bundle $\mathcal{L}$. The formula is roughly as follows: there exists a positive integer b such that the reflexive hull of $(f_*\omega_{X/Y}^{\otimes m})^{\otimes b} \otimes \mathcal{L}^{\otimes b}$ is generically globally generated; see Theorem 5.1.4 for a more precise statement. This is known as the *weak positivity* property and is intertwined with the Fujita-type relative generation questions. As a consequence of this positivity one obtains a case of the Iitaka conjecture obtained in the log-canonical setting by Campana [**Cam04**] and Nakayama [**Nak04**]. Theorem 5.2.2 surveys a proof in this case.

In a different direction, we present a vanishing Theorem 5.3.1 for pushforwards of pluricanonical bundles under certain assumptions on the morphism. This can be seen as a partial extension of Kodaira–Kollár-type vanishing statements. Nonetheless due to the lack of Hodge theory for pushforwards of pluricanonical bundles when $k > 1$, the result arises out of the reduction arguments involved in the proof of Theorem 4.2.1.

5.1. Twisted Weak Positivity

The results in this section were obtained in a joint work with Takumi Murayama in [**DM18**]. See Appendix B for some relevant results on reflexive sheaves.

Notation 5.1.1 ([**Hör10**, Notation 3.3])**.** Let $\mathcal{F}$ be a torsion-free coherent sheaf on a normal variety X. Let $j\colon X^* \hookrightarrow X$ be the largest open set such that $\mathcal{F}|_{X^*}$ is locally free. We define

$$\operatorname{Sym}^{[b]}\mathcal{F} := j_*\operatorname{Sym}^b(\mathcal{F}\big|_{X^*}) \quad \text{and} \quad \mathcal{F}^{[b]} := j_*\big((\mathcal{F}\big|_{X^*})^{\otimes b}\big).$$

We can also describe these sheaves as follows:

$$\operatorname{Sym}^{[b]}\mathcal{F} \simeq \big(\operatorname{Sym}^b(\mathcal{F})\big)^{\vee\vee} \qquad \text{and} \quad \mathcal{F}^{[b]} \simeq (\mathcal{F}^{\otimes b})^{\vee\vee}.$$

Indeed, these pairs of reflexive sheaves coincide in codimension 1 and hence are isomorphic by [**Har94**, Theorem 1.12]. We now recall the following notion defined by Viehweg [**Vie83**, Definition 1.2].

Definition 5.1.2 (Weak positivity)**.** Let X be a normal variety, and let $U \subseteq X$ be a non-empty open set. A torsion-free coherent sheaf $\mathcal{F}$ on X is said to be *weakly positive on U* if for every positive integer a and every ample line bundle $\mathcal{L}$ on X, there exists an integer $b \geq 1$ such that $\operatorname{Sym}^{[ab]} \mathcal{F} \otimes \mathcal{L}^{\otimes b}$ is globally generated on U. We say $\mathcal{F}$ is *weakly positive* if $\mathcal{F}$ is weakly positive on some open set U.

Remark 5.1.3. Let $\mathcal{M}$ be a vector bundle on a smooth projective variety X. Then $\mathcal{M}$ is weakly positive all over X if and only if $\mathcal{M}$ is nef. Indeed, if $\mathcal{M}$ is nef, for any ample line bundle $\mathcal{L}$, $\mathcal{M} \otimes \mathcal{L}$ is ample [**Laz04a**, Proposition 6.2.11.] and therefore there exists $b \in \mathbb{Z}_{\geqslant 1}$ such that $\operatorname{Sym}^b \mathcal{M} \otimes \mathcal{L}^{\otimes b}$ is global generated; see [**Laz04a**, Theorem 6.1.10.]. Conversely, if $\mathcal{M}$ is weakly positive everywhere, for all $a \in \mathbb{Z}_{\geqslant 1}$ there exists a $b \in \mathbb{Z}_{\geqslant 1}$ such that $\operatorname{Sym}^{ab} \mathcal{M} \otimes \mathcal{L}^{\otimes b}$ is globally generated. Let $\pi : \mathbb{P}(\mathcal{M}) \to X$ denote the bundle morphism, then via the natural surjective morphism [**Har66**, Proposition 7.11(b)]

$$\pi^* \mathcal{M} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)$$

we obtain that, $\mathcal{O}_{\mathbb{P}(\mathcal{M})}(ab) \otimes \pi^* \mathcal{L}^{\otimes b}$ is globally generated and hence nef. As $\mathbb{Q}$ divisors, dividing by b and letting $a \to \infty$, we observe that $\mathcal{O}_{\mathbb{P}(\mathcal{M})}(1)$ is nef. Hence, weak positivity is in a way "local" nefness. See also [**B+15**, Theorem 1.1 (1), (4)].

In this section we show that the generalisation of Popa–Schnell's result as in Theorem 4.1.1 paired with Viehweg's fibre product trick imply the weak positivity of the push-forwards of log-pluricanonical sheaves on an open set U. If (X, Δ) is a simple normal crossing pair with X smooth variety, the weak positivity locus is in fact the intersection of $U(f, \Delta)$ with the open set where the pushforward is locally free.

Theorem 5.1.4 (Weak Positivity). *Let $f\colon X \to Y$ be a fibration of normal projective varieties such that Y is normal Gorenstein of dimension n. Let Δ be an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that (X, Δ) is log-canonical and for some integer $k \geq 1$ there is a Cartier divisor P such that $P \sim_{\mathbb{R}} k(K_{X/Y} + \Delta)$. Then, $f_*\mathcal{O}_X(P)$ is weakly positive.*

The theorem is a direct consequence of the following effective statement

Theorem 5.1.5 (Effective Weak Positivity). *Let $f\colon X \to Y$ be a fibration of normal projective varieties such that Y is normal and Gorenstein of dimension n. Let Δ be an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X such that (X, Δ) is log-canonical and for some integer $k \geq 1$ there is a Cartier divisor P such that $P \sim_{\mathbb{R}} k(K_{X/Y} + \Delta)$. Then there exists a non-empty open set $U \subseteq Y$ such that for $H \coloneqq \omega_Y \otimes \mathcal{L}^{\otimes n+1}$ with $\mathcal{L}$ an ample and globally generated line bundle on X, the sheaf*

$$\big(f_*\mathcal{O}_X(P)\big)^{[s]} \otimes H^{\otimes \ell}$$

is generated by global sections on U for all integers $\ell \geqslant k$ and $s \geq 1$.

Proof of Theorem 5.1.4. We first show that Theorem 5.1.5 implies Theorem 5.1.4. This is fairly straightforward. Indeed, let $\mathcal{A}$ be any ample line bundle. Let b be an integer such that $H^{-\otimes k} \otimes \mathcal{A}^{\otimes b}$ is globally generated. Then

$$\big(f_*\mathcal{O}_X(P)\big)^{[s]} \otimes H^{\otimes k} \otimes H^{-\otimes k} \otimes \mathcal{A}^{\otimes b}$$

is globally generated on U. For an integer a, taking $s = ab$, we obtain, for any a there is an integer b such that

$$\big(f_*\mathcal{O}_X(P))\big)^{[ab]} \otimes \mathcal{A}^{\otimes b}$$

is globally generated on U. $\square$

We prove Theorem 5.1.5 using Viehweg's fibre product trick. This trick enables us to reduce the global generation of the reflexivised s-fold tensor product $\big(f_*\mathcal{O}_Y(P)\big)^{[s]}$ to $s = 1$ with X replaced by a suitable modification of the s-fold product of X over Y. The reduction is done by producing a morphism of sheaves between the pushforward from the s-fold product to the reflexivised s-fold tensor of the pushforward. However, the s-fold product may have pretty bad singularities, therefore we first need to prove the existence of trace morphisms for pushforwards of pluricanonical bundles for certain schemes. We believe that this construction is known to the experts and therefore we include it in Appendix C.

5.1.1. Proof of the Effective Weak Positivity Statement

We are now ready to prove Theorem 5.1.5. Readers are encouraged to consult [**PS14**, §4], [**Vie83**, §3], or [**Hör10**, §3].

Throughout the proof we use $\mathcal{O}_X(K_Y)$ and ω_Y interchangeably whenever Y is a normal variety, since on normal varieties $\omega_Y \simeq j_*\omega_{Y_{\mathrm{reg}}} \simeq j_*\mathcal{O}_{Y_{\mathrm{reg}}}(K_{Y_{\mathrm{reg}}})$, where Y_{reg} is the regular locus of Y.

Proof of Theorem 5.1.5. After a log resolution we may assume that X is smooth and Δ has simple normal crossing support; see Lemma 3.4.7. Following Notation 3.4.9, denote by U the intersection of $U(f, \Delta)$ with the open set where the pushforward is locally free.

For every positive integer s, let X^s denote the unique irreducible component of

$$\underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{s \text{ times}}$$

that surjects onto Y; note that it is unique since f has irreducible generic fibre. Define $V \coloneqq f^{-1}(U)$ and $V^s \coloneqq f^{s^{-1}}(U)$.

Let $d\colon X^{(s)} \to X^s$ be a desingularisation of X^s so that d is an isomorphism over V^s. Again, we can do so by Sźabo's lemma. We denote by V^s the pre-image of V^s under any birational modification of X^s which is an isomorphism over V^s. Define $d_i \coloneqq \pi_i \circ d$ for $i \in \{1, 2, \ldots, s\}$, where $\pi_i\colon X^s \to X$ is the i^{th} projection. Since d_i is a surjective morphism between integral varieties, the pullback $d_i^* \Delta_j$ of the Cartier divisor Δ_j is well defined for every component Δ_j of Δ (see [**Stacks**, Tag 02OO(1)]).

Let $\mu\colon \widetilde{X}^s \to X^{(s)}$ be a log resolution of the pair $\left(X^{(s)}, \sum_i d_i^* \Delta\right)$ so that μ is an isomorphism over V^s. Indeed, $\sum_i d_i^* \Delta$ remain a simple normal crossing over V^s and thus we can choose such a resolution by Sźabo's Theorem; see [**KK13**, Theorem 10.45]. Denote

$$\widetilde{\Delta} \coloneqq \mu^* \sum_i d_i^* \Delta.$$

and hence denoting the Cartier divisor $(P - kK_{X/Y})$ by M, one has $\widetilde{\Delta} \sim_{\mathbb{R}} \mu^* \sum_i d_i^* M$. Therefore, we can define a Cartier divisor $\widetilde{P}^s$ on $\widetilde{X}^s$ and

$$\widetilde{P}^s \coloneqq K_{\widetilde{X}^s/Y} + \mu^* \sum_i d_i^* M \sim_{\mathbb{R}} K_{\widetilde{X}^s/Y} + \widetilde{\Delta}$$

Then we

Claim 5.1.6. *There exists a map*

$$\widetilde{f}^s_* \mathcal{O}_{\widetilde{X}^s}\big(\widetilde{P}^s\big) \longrightarrow \big(f_*\mathcal{O}_X\big(P\big)\big)^{[s]} \tag{5.1}$$

is an isomorphism over U.

To this end, define $Y_0 \subseteq Y$ to be the non-empty open subset

- The map f is flat over Y_0;
- The regular locus of Y contains Y_0; and
- The sheaf $f_*\mathcal{O}_X(P)$ is locally free over Y_0.

Note that $\operatorname{codim}(Y \setminus Y_0) \geqslant 2$, as Y is normal and both $f_*\mathcal{O}_X$ and $f_*\mathcal{O}_X(P)$ are torsion-free. Define further $X_0 := f^{-1}(Y_0)$ and similarly X_0^s and $\widetilde{X}_0^s$. The situation is best described by the following commutative diagram

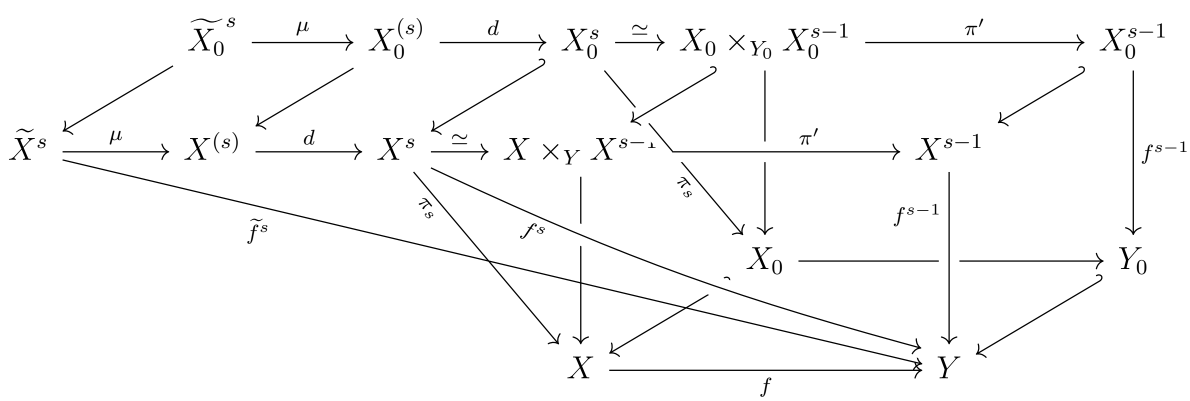

Figure 5.1. Viehweg's Fibre Product Trick

In this situation, by [**Hör10**, Corollary 5.24] we know that

$$X_0^s = \underbrace{X_0 \times_Y X_0 \times_Y \cdots \times_Y X_0}_{s \text{ times}} \simeq \underbrace{X_0 \times_{Y_0} X_0 \times_{Y_0} \cdots \times_{Y_0} X_0}_{s \text{ times}}$$

and that X_0^s is Gorenstein and hence $\omega_{X_0^s}$ is a line bundle. We can therefore apply Lemma C.1.4 to $d \circ \mu$, to obtain a morphism

$$(d \circ \mu)_* \omega^{\otimes k}_{\widetilde{X}_0^s/Y_0} \longrightarrow \omega^{\otimes k}_{X_0^s/Y_0}$$

which is an isomorphism over V^s. Here $\omega_{X_0^s/Y_0} \coloneqq \omega_{X_0} \otimes f^{s*}\omega_{Y_0}^{-1}$ and similarly for $\omega_{\widetilde{X}_0^s/Y_0}$. This induces a map

$$\widetilde{f}^s_* \mathcal{O}_{\widetilde{X}_0^s}(P_0^s) \longrightarrow f^s_* \Big(\omega^{\otimes k}_{X_0^s/Y_0} \otimes \bigotimes_i \pi_i^* \mathcal{M}\big|_{X_0^s} \Big) \tag{5.2}$$

which is an isomorphism over U. Here $\mathcal{M} \coloneqq \mathcal{O}_X(M)$ is the line bundle associated to the Cartier divisor $P - kK_X \sim_{\mathbb{R}} k\Delta$.

We will now show that the sheaf on the right-hand side of (5.2) admits an isomorphism to

$$\big(f_* \mathcal{O}_{X_0}(P_0)\big)^{\otimes s}$$

where $P_0 \coloneqq P|_{X_0}$. Note that this would show Claim 5.1.6, since (5.2) is an isomorphism over U. We proceed by induction, adapting the argument in [**Hör10**, Lemma 3.15] to our twisted setting. Note that the case $s = 1$ is clear, since in this case $X^s = X$ and the sheaves in question are equal.

By [**Hör10**, Corollary 5.24] we have that

$$\omega^{\otimes k}_{X_0^s/Y_0} \otimes \bigotimes_i \pi_i^* \big(\mathcal{M}\big|_{X_0}\big) \simeq \pi_s^* \big(\omega^{\otimes k}_{X_0/Y_0} \otimes \mathcal{M}\big|_{X_0}\big) \otimes \pi'^* \big(\omega^{\otimes k}_{X_0^{s-1}/Y_0} \otimes \mathcal{M}^{s-1}\big|_{X_0^{s-1}}\big)$$

where $\pi' \colon X^s \to X^{s-1}$ and $\mathcal{M}^{s-1} := \bigotimes_{i=1}^{s-1} \pi_i^* \mathcal{M}$. Since $\omega^{\otimes k}_{X_0^{s-1}/Y_0} \otimes \mathcal{M}^{s-1}\big|_{X_0^{s-1}}$ is locally free, by the projection formula we obtain

$$f^s_*\Big(\omega^{\otimes k}_{X_0^s/Y_0} \otimes \bigotimes_{i=1}^{s} \pi_i^* \mathcal{M}\big|_{X_0}\Big) \simeq f_*\Big(\big(\omega^{\otimes k}_{X_0/Y_0} \otimes \mathcal{M}\big|_{X_0}\big) \otimes \pi_{s_*}\pi'^*\big(\omega^{\otimes k}_{X_0^{s-1}/Y_0} \otimes \mathcal{M}^{s-1}\big|_{X_0^{s-1}}\big)\Big).$$

Now by flat base change [**Har77**, Proposition III.9.3],

$$\pi_{s_*}\pi'^*\big(\omega^{\otimes k}_{X_0^{s-1}/Y_0} \otimes \mathcal{M}^{s-1}\big|_{X_0^{s-1}}\big) \simeq f^* f^{s-1}_*\big(\omega^{\otimes k}_{X_0^{s-1}/Y_0} \otimes \mathcal{M}^{s-1}\big|_{X_0^{s-1}}\big).$$

By induction the latter is isomorphic to

$$f^*\big(f_*\mathcal{O}_{X_0}(P_0)^{\otimes s-1}\big).$$

Therefore

$$f^s_*\Big(\omega^{\otimes k}_{X_0^s/Y_0} \otimes \bigotimes_i \pi_i^* \mathcal{M}\big|_{X_0}\Big) \simeq f_*\Big(\omega^{\otimes k}_{X_0/Y_0} \otimes \mathcal{M}\big|_{X_0} \otimes f^*\big(f_*\mathcal{O}_{X_0}(P_0)^{\otimes s-1}\big)\Big).$$

Since $f_*\mathcal{O}_X(k(K_{X/Y}+\Delta))$ is locally free over Y_0, we can apply the projection formula to obtain

$$f^s_*\Big(\omega^{\otimes k}_{X_0^s/Y_0} \otimes \bigotimes_i \pi_i^* \mathcal{M}\big|_{X_0}\Big) \simeq \big(f_*\mathcal{O}_{X_0}(P_0)\big)^{\otimes s}.$$

Now by construction, we have $U \subseteq Y_0$. Since $(f_*\mathcal{O}_X(P))^{[s]}$ is reflexive and is isomorphic to $(f_*\mathcal{O}_X(P))^{\otimes s}$ on Y_0, a map

$$\widetilde{f}^s_*\mathcal{O}_{\widetilde{X}^s}(\widetilde{P}^s) \longrightarrow \big(f_*\mathcal{O}_X(P)\big)^{\otimes s}$$

over Y_0 will extend to a map of the form in (5.1) on Y by Corollary B.1.4. This morphism is an isomorphism over U.

We now use the global generation in Theorem 4.1.1 to finish the proof of Theorem 5.1.5. First, note that $\widetilde{\Delta}$ satisfies the hypothesis of Theorem 4.1.1. Indeed, π_i is flat over X_0, and therefore by flat pullback of cycles we have

$$\pi_i^*(\Delta_j)\big|_{X_0^s} = \pi_i^{-1}(\Delta_j\big|_{X_0}) = X_0 \times_{Y_0} \cdots \times_{Y_0} \underbrace{\Delta_j}_{i^{\text{th}} \text{ position}} \times_{Y_0} \cdots \times_{Y_0} X_0.$$

Since $X_0 \supseteq V$ and both d and μ are isomorphisms over V^s, the pullback $\mu^*(\pi_i \circ d)^*\Delta_j^h\big|_{V^s}$ of the horizontal components of Δ are smooth above U for all $i \in \{1, 2, \ldots, s\}$. In other words, the components of $\widetilde{\Delta}$ either do not intersect V^s, or intersect the fibre over x transversely for all $x \in U$. Thus,

$$\widetilde{\Delta}\big|_{V^s} = \mu^{-1}d^{-1}\sum_i \pi_i^{-1}\big(\Delta^h\big|_V\big).$$

In particular, the horizontal part $\widetilde{\Delta}^h$ of $\widetilde{\delta}$ equals the closure $\overline{\widetilde{\Delta}\big|_{V^s}}$ of $\widetilde{\Delta}\big|_{V^s}$ in $\widetilde{X}^s$. We can therefore write

$$\widetilde{\Delta} = \widetilde{\Delta}^h + \widetilde{\Delta}^v,$$

where by construction, $\widetilde{f}^s$-images of the components of $\widetilde{\Delta}^h$ is Y, the coefficients of $\widetilde{\Delta}^h$ are in $(0, 1]$ and $\widetilde{f}^s\big(\widetilde{\Delta}^v\big) \cap U = \emptyset$.

Finally, we note from Mori's cone theorem [**KM98**, Theorem 1.24] that $H = \omega_X \otimes \mathcal{L}^{\otimes n+1}$ is nef and hence semiample by the base point free theorem [**KM98**, Theorem 3.3].

Using H again to denote a divisor class of H, we note that

$$\widetilde{P}^s + \ell H \sim_{\mathbb{R}} k(K_{\widetilde{X}^s} + \widetilde{\Delta}') + k(n+1)L \tag{5.3}$$

where $\widetilde{\Delta}' = \widetilde{\Delta} + D'$ for some effective $\mathbb{Q}$-divisor $D' \sim_{\mathbb{Q}} (\ell - k)f^*H$, whose support intersects the components of $\widetilde{\Delta}$ transversely.

Since $\mathcal{L} = \mathcal{O}_X(L)$ is ample and globally generated, we can apply Theorem 4.1.1 to conclude that

$$\widetilde{f}^s_* \mathcal{O}_{\widetilde{X}^s}(P) \otimes H^\ell$$

is generated by global sections over U for all $\ell \geqslant k$. Now fix a closed point $y \in U$. We have the commutative diagram

$$\begin{array}{ccc} H^0\big(X, \widetilde{f}^s_* \mathcal{O}_{\widetilde{X}^s}(\widetilde{P}^s) \otimes H^{\otimes \ell}\big) & \twoheadrightarrow & \big(\widetilde{f}^s_* \mathcal{O}_{\widetilde{X}^s}(\widetilde{P}^s) \otimes H^{\otimes \ell}\big) \otimes \kappa(y) \\ \downarrow & & \downarrow \wr \\ H^0\big(X, \big(f_* \mathcal{O}_X(P)\big)^{[s]} \otimes H^{\otimes \ell}\big) & \longrightarrow & \big(\big(f_* \mathcal{O}_X(P)\big)^{[s]} \otimes H^{\otimes \ell}\big) \otimes \kappa(y) \end{array}$$

where the vertical arrows are induced by the map (5.1) from Claim 5.1.6, and the top horizontal arrow is surjective by the global generation of the sheaves in (5.3) over U. Since (5.1) is an isomorphism over U, the right vertical arrow is an isomorphism, hence by the commutativity of the diagram, the bottom horizontal arrow is surjective. We therefore conclude that

$$\big(f_* \mathcal{O}_X(P)\big)^{[s]} \otimes H^{\otimes \ell}$$

is generated by global sections over U for all $\ell \geqslant k$. □

Remark 5.1.7. When X is smooth, Δ has simple normal crossing support and $\lfloor\Delta\rfloor = 0$, if we moreover take $U(f,\Delta)$ to be an open set over f is log-smooth, then the invariance of log plurigenera [**HMX18**, Theorem 4.2] implies that $f_*\mathcal{O}_X(P)\big|_{U(f,\Delta)}$ is locally free. In this case we can take Y_0 to be simply the locus inside Y_{reg} over which f is flat. Moreover, the isomorphism

$$\big(f_*\mathcal{O}_X(P)\big)^{\otimes s} \simeq \big(f_*\mathcal{O}_X(P)\big)^{[s]}$$

automatically holds over $U(f,\Delta)$. Thus, Theorem 5.1.5 holds more generally over $U(f,\Delta)$.

5.2. Subdditivity of Log-Kodaira Dimensions

As a consequence of the weak positivity we establish a special case of the Iitaka type conjeture for log Kodaira dimensions. In this special case, $\Delta = 0$ version was done by Viehweg; see [**Vie83**, Corollary IV]. Just like in the Definition 5.0.1 of Kodaira dimension, one defines the Iitaka dimension associated to a $\mathbb{Q}$-Cartier divisor.

Definition 5.2.1. Let X be a projective variety defined over a field k and let L be a $\mathbb{Q}$-linealy equivalent to a Cartier divisor on X. Let k be the smallest integer such that $kL \sim P$ where P is Cartier. The *Iitaka dimension* of L is defined as

$$\kappa(X;L) \coloneqq \mathrm{trdeg}_k(\bigoplus_m H^0(X,\mathcal{O}_X(mP)).$$

For a pair (X,D) so that $K_X + D$ is $\mathbb{Q}$-Cartier, $\kappa(X, K_X + D)$ is also called as the *log-Kodaira dimension* of (X,D).

The following case of the Iitaka-type conjecture for log-Kodaira dimension appears around the same time in the works of Campana [**Cam04**] and Nakayama [**Nak04**, Chapter

V. Theorem 4.1(2)]. Based on their idea, here we present a re-interpretation of the methods involved.

Theorem 5.2.2. *Let $f : (X, D) \to Y$ be a fibration from a log-canonical pair to a normal Gorenstein projective variety Y so that $K_X + \Delta$ is $\mathbb{Q}$-linearly equivalent to a $\mathbb{Q}$-Cartier divisor and $\kappa(X, K_X + D) > 0$. Furthermore assume that Y is of general type, i.e. $\kappa(Y) = \dim Y$. Then,*

$$\kappa(X, K_X + D) = \kappa(X_y, K_{X_y} + D_y) + \dim Y$$

for a general fibre X_y of f.

Proof. The $\leqslant$ direction is relatively easy and hence is known in the literature as the "*Easy Addition Formula*". Indeed, from the restriction morphism for sufficiently large and divisible b and for a general enough fibre X_y

$$H^0(X, \mathcal{O}_X(k(K_X + \Delta))) \to H^0(X_y, \mathcal{O}_{X_y}(k(K_{X_y} + \Delta|_{X_y})))$$

it follows that we have the following diagram with the top row $\phi_k \colon X \to \mathbb{P}^N$ denoting the Iitaka fibration of X.

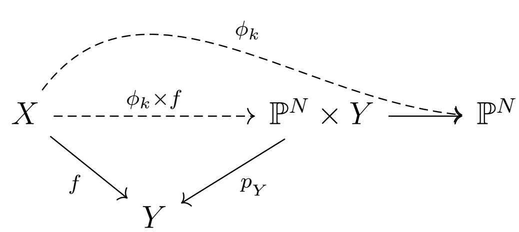

Figure 5.2. Iitaka Fibration

Denote by $X' \coloneqq \overline{\operatorname{Im}((\phi_k \times f)(X))}$ Note that, $\dim(p_Y\big|_{X'})^{-1}(y) = \dim \phi_k(X_y)$ for a general $y \in Y$. Therefore,

$$\kappa(X, K_X + D) = \dim \phi_k(X) \leqslant \dim X' \leqslant \dim X'_y + \dim Y \leqslant \kappa(X_y, K_{X_y} + D) + \dim Y.$$

Here $X'_y = (p_Y\big|_{X'})^{-1}(y)$ and p_Y denotes the projection onto Y.

For the opposite direction, we use Lemma 5.2.3 below. Note that for large and divisible k, there exists a Cartier divisor P such that $P \sim_{\mathbb{R}} k(K_{X/Y} + D)$ and hence by Theorem 5.1.4, $f_*\mathcal{O}_X(P)$ is weakly positive. We follow the argument in the beginning of the proof of Theorem 4.2.1 and consider the following diagram

$$\begin{array}{ccc} f^*\Big(\big(f_*\mathcal{O}_X(P)\big)^{\otimes b}\Big) & \xrightarrow{\ \phi\ } & \mathcal{O}_X\big(bP\big) \\ \downarrow & & \big\| \\ f^*\Big(\big(f_*\mathcal{O}_X(P)\big)^{[b]}\Big) & \overset{\phi'}{\dashrightarrow} & \mathcal{O}_X\big(bP\big). \end{array}$$

Then the dashed map exists making the diagram commute. Indeed, the map exists over the locus where $f_*\mathcal{O}_Y(P)$ is locally free. Since the sheaf is torsion free, the locally free locus has a complement of codimension ≥ 2, and the bottom right sheaf is locally free, we can extend the dashed map to all of Y by Corollary B.1.4.

Next by the weak positivity property of $f_*\mathcal{O}_X(P)$ there exists a positive integer b, for which we have a global section of $f^*\Big(\big(f_*\mathcal{O}_X(P)\big)^{[2b]}\Big) \otimes f^*\mathcal{L}^{\otimes b}$. This in turn gives a section

$$\mathcal{O}_X \longrightarrow \mathcal{O}_X\big(2bP\big) \otimes f^*\mathcal{L}^{\otimes b}.$$

Since $\mathcal{O}_X$ is torsion free, twisting by $f^*\mathcal{L}^{\otimes b}$ we obtain an injection,

$$f^*\mathcal{L}^{\otimes b} \hookrightarrow \mathcal{O}_X\big(2bP\big) \otimes f^*\mathcal{L}^{\otimes 2b}.$$

Denoting by L a Cartier class of $\mathcal{L}$, observe that Fujita's Lemma 5.2.3 implies

$$\kappa(X, k(K_{X/Y} + \Delta) + f^*L) \geqslant \kappa(F, K_F + \Delta|_F\big) + \dim Y.$$

Now, since Y is of general type (i.e. K_Y is big) and $\mathcal{L}$ is ample, by Kodaira's lemma, there exist a large integer k and an effective divisor E on Y such that

$$kK_Y \sim L + E.$$

Therefore for all large and divisible k, one obtains

$$H^0(Y, f_*\mathcal{O}_X\big(2bP\big) \otimes \omega_Y^{\otimes 2bk}\big) \hookrightarrow H^0(Y, f_*\mathcal{O}_X\big(2bP\big) \otimes \mathcal{L}^{\otimes 2b}) \otimes \omega_Y^{\otimes kb}).$$

In conclusion,

$$\kappa(X, k(K_{X/Y} + D) + f^*L) \geqslant \kappa(X, k(K_{X/Y} + D) + f^*kK_Y) = \kappa(X, K_X + D).$$

□

The proof crucially hinges on the following observation by Fujita; see [**Fuj77**, Proposition 1].

Lemma 5.2.3. *Let $f : X \to Y$ be a fibration between normal projective varieties. Let H be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Assume that, for sufficiently large number k there* *exists a big line bundle $\mathcal{L}$ on Y so that $f^*\mathcal{L} \hookrightarrow \mathcal{O}_X(kH)$, then*

$$\kappa(X, H) \geqslant \kappa(X_y, H|_{X_y}) + \dim Y$$

for a general fibre X_y of f.

Proof. Denote by $\mathcal{H} = \mathcal{O}_X(kH)$. For sufficiently large and divisible b, we have a rational map

$$X \dashrightarrow \mathbb{P}^M$$

where $M = |\Gamma(X, \mathcal{H}^{\otimes b})|$. From the inclusion of line bundles, we obtain that

$$\Gamma(X, \mathcal{H}^{\otimes b}) \supseteq \Gamma(X, f^*\mathcal{L}^{\otimes b}) \simeq \Gamma(Y, \mathcal{L}^{\otimes b})$$

where the last isomorphism follows from the projection formula using the fact that $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$. Thus, $M \geqslant M_\ell := |\Gamma(Y, \mathcal{L}^{\otimes b})|$. Taking a projection $\phi\colon \mathbb{P}^M \twoheadrightarrow \mathbb{P}^{M_\ell}$ and replacing X and Y by their suitable birational modifications we obtain the following commutative diagram of morphisms:

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^M \\ \downarrow & & \downarrow{\scriptstyle \phi} \\ Y & \longrightarrow & \mathbb{P}^{M_\ell} \end{array}$$

Figure 5.3. Iitaka Fibration of H and L

By construction the bottom row is defined by the section of $\mathcal{L}^{\otimes b}$ and hence is a birational morphism. Therefore a general point, $y \in \mathbb{P}^{M_\ell}$ can be identified with a general point $y \in Y$. One has,

$$\dim X_y = \phi^{-1}(y) = \kappa(X, H) - \dim Y.$$

By the definition of Iitaka dimension, $\kappa(X_y, H|_{X_y}) \leqslant \dim X_y$, hence the lemma. $\square$

5.3. An Effective Vanishing Theorem

With the help of the effective twisted weak positivity, in [**DM18**] we also prove a vanishing statement for pushforwards of twisted pluricanonical bundles; see also [**PS14**, Theorem 3.2], [**Dut17**, Theorem 3.1]:

Theorem 5.3.1. *Let $f\colon (X, D) \to Y$ be a fibration of smooth projective varieties with $\dim Y = n$. Let D be a $\mathbb{Q}$-divisor with simple normal crossing support with coefficients in $(0, 1)$, such that f is log-smooth with respect to (X, D) over Y, and let $\mathcal{L}$ be an ample line bundle on Y. Assume also that for some fixed integer $k \geqslant 1$, there exists a Cartier divisor P such that*

$$P \sim_{\mathbb{R}} k(K_X + D)$$

and $\mathcal{O}_X(P)$ is relatively base point free. Then, for every $i > 0$ and all $\ell \geqslant k(n+1) - n$, we have

$$H^i\big(Y, f_*\mathcal{O}_X\big(P\big) \otimes \mathcal{L}^{\otimes \ell}\big) = 0.$$

Moreover, if K_Y is semiample, for every $i > 0$ and every ample line bundle $\mathcal{L}$ we have

$$H^i\big(Y, f_*\mathcal{O}_X\big(P\big) \otimes \mathcal{L}\big) = 0.$$

Proof. The hypothesis on f and D ensures that $f_*\mathcal{O}_X(k(K_{X/Y}+D))$ is locally free by the invariance of log-plurigenera when $\lfloor D \rfloor = 0$, as noted in Remark 5.1.7, hence $f_*\mathcal{O}_X(k(K_{X/Y}+D))$ is locally free. Furthermore, as the morphism is log-smooth and the relative base locus of P is empty, we deduce that the locus where weak positivity holds is in fact Y. Then by Theorem 5.1.4, we have a positive integer b such that

$$\big(f_*\mathcal{O}_X\big(P\big)\big)^{[b]} \otimes \mathcal{L}^{\otimes b}$$

is globally generated everywhere on Y. Now since $\mathcal{O}_X(P)$ is relatively base point free, we can choose a divisor $\frac{1}{b}D' \sim_{\mathbb{R}} k(K_{X/Y}+D) + f^*L$, satisfying the Bertini-type properties as in Step 2 of Theorem 4.2.1. Denote $H \coloneqq K_Y + (n+1)L$, which is semiample by Mori's cone theorem and the base point free theorem. Following the divisor arithmetic in the proof of Theorem 4.2.1 or 4.3.1, we write

$$K_X + D + \frac{k-1}{kb}D' + (k-1)f^*H + \left(\ell - \frac{k-1}{k} - (k-1)(n+1)\right)f^*L \sim_{\mathbb{R}} k(K_X+D) + \ell f^*L.$$

Now, the vanishing of the right hand side follows from that of the left by an application of Kollár's vanishing theorem [**Kol95**, Theorem 10.19] or Ambro–Fujino type vanishing theorems [**Amb03, Fuj11**]. Indeed, observe that the divisor $D + \frac{k-1}{kb}D'$ is log-canonical and $(k-1)H + \big(\ell - \frac{k-1}{k} - (k-1)(n+1)\big)L$ is ample for all $\ell \geqslant k(n+1) - n$. Therefore we obtain

$$H^i\big(Y, f_*\mathcal{O}_X\big(P\big) \otimes \mathcal{L}^{\otimes \ell}\big) = 0$$

for all $\ell \geqslant k(n+1) - n$ and for all $i > 0$.

For the last part of the statement, if K_Y is already semiample, we take $H = K_Y$. In this case, the linear equivalence above looks as follows:

$$K_X + D + \frac{k-1}{kb} D' + (k-1) f^* H + \big(\ell - \frac{k-1}{k}\big) f^* L \sim_{\mathbb{R}} k(K_X + D) + \ell f^* L.$$

Then, we obtain the desired vanishing for all $\ell \geqslant 1$ and $i > 0$. $\square$

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APPENDIX A

Hodge Theory and Hodge Modules

The cohomology groups of smooth projective varieties X have more structures, namely a decomposition of its singular cohomologies with complex coefficients in to subspaces

$$H^n(X,\mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}(X) \tag{A.1}$$

satisfying the symmetry $H^{p,q}(X) \simeq \overline{H^{q,p}(X)}$ where the overline denotes the complex conjugation. One can define a decreasing filtration $F^pH^n(X,\mathbb{C}) \subset H^n(X,\mathbb{C})$ by

$$F^pH^n(X,\mathbb{C}) \coloneqq \bigoplus_{q\geqslant p} H^{q,n-q}(X).$$

This is known as the *Hodge filtration* of the Hodge structure $H^n(X,\mathbb{C})$. Furthermore, $H^{p,q}(X) \simeq H^q(X,\Omega^p_X)$. One of the key ingredient in the proof of this decomposition is the exactness of the filtered deRham complex of $\mathbb{C}$ given by

$$\mathrm{DR}(\mathbb{C}) \coloneqq [\mathcal{O}_X \to \Omega^1_X \to \cdots \to \omega_X]$$

with the filtration $F^k\,\mathrm{DR}(\mathbb{C}) = \Omega_X^{\bullet\geqslant p}$. Since $\mathrm{DR}(\mathbb{C})$ is exact, it follows that, $H^{p,n-p} = gr^F_p\,\mathrm{DR}(\mathbb{C}) = H^{n-p}(X,\Omega^p_X)$. When X is singular or not projective, even though the decomposition fails to hold, there is a E_1-degeneration of the Hodge filtration F on the cohomology groups. We will discuss this in more detail a bit later.

A.1. Variations of Hodge structures.

Now, let $f : X \to Y$ be a smooth surjective morphism of smooth projective varieties. Ehresmann's Lemma [**Ehr51**] states that f is a locally trivial fibration. In other words, for every point $y \in Y$, there exists a small analytic open set $U \ni y$, $f^{-1}(U) \simeq U \times F$ where $F = f^{-1}(y)$ and $\simeq$ denotes diffeomorphism. This, in particular implies that $H^i(F, \mathbb{Z})$ is invariant on the fibres. In fact, a little more is true for submersions of projective manifolds; namely the Hodge numbers $h^{p,q}$ of the fibres are also constant. Indeed, the coherent sheaves $\Omega^p_{X/Y} \coloneqq \bigwedge^p \Omega^1_{X/Y} \coloneqq \bigwedge^p \operatorname{Coker}\{f^*\Omega^1_Y \to \Omega^1_X\}$ satisfy

$$\Omega^p_{X/Y} \otimes \mathcal{O}_{X_y} \simeq \bigwedge \Omega^1_{X/Y}|_{X_y} \simeq \Omega^p_{X_y}.$$

Therefore the functions

$$h^{p,q}(y) \coloneqq h^q(X_y, \Omega^p_{X_y})$$

are upper-semicontinuous [**Har77**, Theorem III.12.8]. Since $h^n(X_y, \mathbb{C})$ does not depend on y, and by the Hodge decomposition, $h^n(y) = \Sigma_{p+q=n} h^{p,q}(y)$, the functions $h^{p,q}(y)$ are also constant. Therefore by Grauert's theorem [**Har77**, Corollary III.12.9] we obtain, $R^q f_* \Omega^p_{X/Y}$ are locally free for all p, q. This situation is called the *geometric variation of Hodge structure* and is an example of the following

Definition A.1.1 (Abstract Variation of Hodge Structures)**.** Let X be a complex manifold. A variation of Hodge structure of weight n on X consists of the following triple $(\mathbb{L}_{\mathbb{Q}}, \mathscr{E}, F)$ with

(1) $\mathbb{L}_{\mathbb{Q}}$, a $\mathbb{Q}$-local system of finitely generated abelian groups

(2) $\mathscr{E} \simeq \mathbb{L}_{\mathbb{C}} \otimes_{\mathbb{C}} \mathcal{O}_X$, a vector bundle with a finite decreasing filtration $\mathcal{F}^p$ given by $\mathcal{F}^p := F^p\mathbb{L}_{\mathbb{C}} \otimes \mathcal{O}_X \subseteq \mathscr{E}$, where $\{F^p\mathbb{L}_{\mathbb{C}}\}_p$ is a finite decreasing filtration of the local system

such that $F^p|_x$ makes the fibres $\mathbb{L}_x \simeq L$ Hodge structures of weight n, i.e.

$$F^pL \bigcap \overline{F^{n-p+1}L} \simeq L \tag{A.2}$$

and the connection

$$\nabla : \mathscr{E} \to \Omega^1_X \otimes \mathscr{E}$$

defined by $v \otimes f \mapsto df \otimes v$ with $v \in \mathbb{L}$ $f \in \mathcal{O}_X$ satisfies the *Griffiths' transversality* condition with respect to $\mathcal{F}^p$, i.e. $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_X$.

Note that by definition the connection ∇ is integrable, i.e. $\nabla \circ \nabla = 0$

Definition A.1.2 (deRham Complex)**.** Given a Hodge structure $(\mathbb{L}_{\mathbb{Q}}, \mathscr{E}, F)$ of weight n, there is an associated $\mathbb{C}$-linear complex

$$\mathrm{DR}(\mathscr{E}) := [0 \to \mathscr{E} \to \Omega^1_X \otimes \mathscr{E} \to \cdots \to \omega_X \otimes \mathscr{E} \to 0]$$

and its filtered version

$$F^k\,\mathrm{DR}(\mathscr{E}) := [0 \to F^k\mathscr{E} \to \Omega^1_X \otimes F^{k-1}\mathscr{E} \to \cdots \to \omega_X \otimes F^{k-n}\mathscr{E} \to 0].$$

Note that $F^k\,\mathrm{DR}(\mathscr{E})$ is well-defined by the Griffiths transversality condition in Definiton A.1.1 and the integrability of the connection implies that $\mathrm{DR}(\mathscr{E})$ is indeed a complex. Furthermore in the derived category of complexes of abelian sheaves, $\mathrm{DR}(\mathscr{E}) \otimes \mathbb{C} \simeq \mathbb{L}_{\mathbb{C}}$.

Example A.1.3. Let $f\colon X \to Y$ be a smooth surjective morphism of smooth projective varieties with f. Then $R^q f_* \mathbb{Q}_X$ is a local system since by proper base change $R^q f_*\mathbb{Q}_X \otimes \kappa(y) \simeq H^q(X, \mathbb{Q})$. Then, as already discussed above, $(R^q f_*\mathbb{Q}_X, \mathscr{E}^q, F^\bullet)$ is a variation of Hodge structures [**PS08**, Corollary 10.32]. Moreover, by [*loc. cit.*, Theorem 10.26] $\mathrm{DR}(\mathscr{E}^q) \simeq R^q f_* \Omega^\bullet_{X/Y}$ where,

$$\Omega^\bullet_{X/Y} \simeq \frac{\Omega^\bullet_X}{f^*\Omega^1_Y \wedge \Omega^{\bullet-1}_X}.$$

A.2. Mixed Hodge Structures.

When X is singular and (or) not projective, $H^n(X, \mathbb{C})$ loses the nice decomposition as in (A.1). However, it still keeps the decreasing Hodge filtration $F^p H^n(X, \mathbb{C})$. Moreover, while it does not satisfy the direct summand property in (A.2), there is a finite increasing weight filtration $\{W_m H^n(X, \mathbb{Q})\}_{m \in \mathbb{Q}}$ so that

$$F^p gr^W_m H^n(X, \mathbb{C}) \coloneqq gr^W_m H^n(X, \mathbb{Q}) \otimes \mathbb{C} \cap F^p H^n(X, \mathbb{C})$$

induces a Hodge structure of weight $n + m$ on $gr^W_m H^n(X, \mathbb{C})$.

Strictness. A morphism between mixed Hodge structures,

$$\phi : (V_1, W_{1,m}, F^p_1) \to (V_2, W_{2,m}, F^p_2)$$

is *strict* with respect to both W and F filtration, i.e. $F^p_2 V_2 \simeq \phi(V_1) \cap F^p_2 V_2$ and similarly for W; see e.g [**PS08**, Corollary 3.6.].

In general, it is really hard to understand the weight filtration of $H^n(X, \mathbb{C})$. We work it out in the following situations

Example A.2.1 (Punctured Torus)**.** Let $X = T \setminus \{\mathrm{pt}_1, \mathrm{pt}_2\}$ where T is a torus:

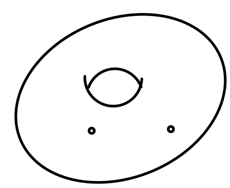

Figure A.1. 2-Puntured Torus

Note that, X has a deformation retraction onto the bouquet of three circles, i.e. $S^1 \vee S^1 \vee S^1$. Therefore, $h^1(X, \mathbb{C}) = 3$, i.e. odd dimensional and hence $H^1(X, \mathbb{C})$ cannot admit a Hodge structure. Indeed, if it did, we would have $h^1 = h^{1,0} + h^{0,1}$ with $h^{1,0} = h^{0,1}$ and hence h^1 must be an even number. We have the following long exact sequence of cohomologies of pairs of spaces

$$0 \to H^0(T, X) \to H^0(T) \to H^0(X) \to H^1(T, X) \to \cdots$$

By Thom isomorphism we have $H^i(T, X) \simeq H^{i-2}(\{\mathrm{pt}_1, \mathrm{pt}_2\})$ and hence the above long exact sequence results in

$$0 \to H^1(T) \to H^1(X) \to H^0(\{\mathrm{pt}_1, \mathrm{pt}_2\}) \to H^2(T) \to 0.$$

i.e.

$$0 \to \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \oplus \mathbb{Q} \to \mathbb{Q} \to 0.$$

Since $H^1(T,\mathbb{C})$ (resp. $H^0(\{\mathrm{pt}_1,\mathrm{pt}_2\})$) is a pure Hodge structure of weight 1 (resp. 0), we obtain that

$$W_0 H^1(X,\mathbb{C}) \simeq H^0(\{\mathrm{pt}\},\mathbb{C}) \text{ and } gr_1^W H^1(X,\mathbb{C}) \simeq H^1(T,\mathbb{C}).$$

Example A.2.2 (Complement of an snc divisor)**.** A simple normal crossing divisor $D = \sum_{i=1}^{r} D_i$ on a smooth variety X is roughly speaking, a union of hypersurfaces D_i intersecting each other transversely. In other words, D_i's are smooth and around a point $x \in X$, $D_i = (x_i = 0)$ where $x_1, \cdots, x_n$ are the local parameters of the local ring $\mathcal{O}_{X,x}$. In the above example of a 2-punctured torus $\{\mathrm{pt}_1,\mathrm{pt}_2\}$ is a simple normal crossing divisor.

For a simple normal crossing divisor D on a smooth projective variety X, we now discuss the mixed Hodge structure on $X \setminus D$. The underlying idea is similar to Example A.2.1 i.e. the weight filtration on $X \setminus D$ comes from various intersections of the components of D. To this end, we need to understand the mixed Hodge structure on D. Needless to say, this is technically more challenging than Example A.2.1. We first set some notations

Notation A.2.3 (Stratification of D)**.** Letting $D = \Sigma_i D_i$ denote

$$D(0) \coloneqq X$$

and

$$D(q) \coloneqq \bigsqcup_{|I|=q} D_I \text{ with } D_I \coloneqq \cap_{i\in I} D_i$$

Consider the log-deRham complex associated to D

$$\Omega_X^\bullet(\log D) \coloneqq [\Omega_X^1(\log D) \xrightarrow{d} \Omega_X^2(\log D) \longrightarrow \cdots \omega_X(D)]. \tag{A.3}$$

In local coordinates on U the maps are given by

$$d(f\eta) \longmapsto \sum_i \partial_i f dz_i \wedge \eta$$

where $f \in \mathcal{O}_X(U)$ and $\eta \in \Omega_X^p(\log D)(U)$. It is known [**Del74**, Scholie 8.1.8] that $Rj_*\mathbb{Q}_{X\setminus D} \otimes \mathbb{C} \simeq \Omega_X^\bullet(\log D)$. Therefore,

$$H^i(X \setminus D, \mathbb{C}) \simeq \mathbb{H}^i(X, \Omega_X^\bullet(\log D)).$$

The mixed Hodge structure on $H^i(X \setminus D, \mathbb{C})$ is thus defined via first defining a weight filtration on the complex $\Omega_X^\bullet(\log D)$. This is done as follows. Define

$$W_m\Omega_X^p(\log D) \coloneqq \begin{cases} 0 & \text{for } m < 0 \\ \Omega_X^{p-m} \wedge \Omega_X^m(\log D) & \text{for } 0 \leqslant m \leqslant p \\ \Omega_X^p(\log D) & \text{for } m \geqslant p \end{cases}$$

Then for $|I| = m$ the residue map:

$$\mathrm{res}_I : \Omega_X^\bullet(\log D) \to \Omega_{D_I}^\bullet(\Sigma_{j\notin I} D_I \cap D_j)[-m]$$

restricts to

$$\mathrm{res}_m \colon W_m\Omega_X^\bullet(\log D) \to i_{m_*}\Omega_{D(m)}^\bullet[-m]$$

where $\bigsqcup_{|I|=m} i_I =: i_m \colon D(m) \to X$ is the inclusion map. With this notation by [**PS08**, Lemma 4.6] we obtain

$$gr_m^W \Omega_X^\bullet(\log D) \simeq i_{m_*}\Omega_{D(m)}^\bullet[-m] \simeq i_{m_*}\mathbb{Q}_{D(m)}[-m] \otimes \mathbb{C}.$$

The last isomorphism can be justified by noting that $D(m)$ is a smooth projective variety of dimension $n-m$. The above data can thus be encoded by

$$((Rj_*\mathbb{Q}_{X\setminus D}, W), (\Omega_X^\bullet(\log D), W, F))$$

with

$$gr_m^W \Omega_X^\bullet(\log D) = \bigoplus_I (i_{I_*}\mathbb{Q}_{D_I}[-m], \Omega_{D_I}, F(-m)[-m]) \tag{A.4}$$

where the shifted filtration is defined by

$$F(-m)^k(gr_m^W \Omega_X^\bullet(\log(D)) = \bigoplus_I [\Omega_{D_I}^{k-m} \to \Omega_{D_I}^{k-m+1} \to \cdots].$$

Note that with the appropriate shifts, $(i_{I_*}\mathbb{Q}_{D_I}[-m], \Omega_{D_I}, F(-m)[-m])$ induces a Hodge structure of weight $m+i$ on $\mathbb{H}^i(D_I, \mathbb{Q}_{D_I}[-m])$.

Now we have the spectral sequence (see e.g. [**PS08**, §A.3.3])

$$E_1^{-m,k+m} := \mathbb{H}^k(X, gr_m^W \Omega_X^\bullet(\log D)) \Rightarrow H^k(X \setminus D, \mathbb{C}). \tag{A.5}$$

By [**Del74**, Scholie 8.1.9(iv)] this degenerates at E_2. Furthermore, by [*loc. cit.*, Scholie 8.1.9(iv)] the maps in the E_1 page are maps of Hodge structures. Thus, we obtain a weight filtration on $H^k(X \setminus D, \mathbb{C})$. The F-filtration can be defined by a similar spectral sequence with E_1 degeneration. [*loc. cit.*, Scholie 8.1.9(iii)].

Remark A.2.4. In general, it is more convenient to talk about the complexes itself as opposed to the computing the singular cohomology groups that realise the complex as a (mixed) Hodge structure. Such complexes, are said to be the *cohomological mixed Hodge complexes* and were introduced by Delgne in [**Del71**]. They are given by the data

$$((\mathbb{L}, W), (C^\bullet, W, F))$$

where

- $\mathbb{L}$ is a $\mathbb{Q}$-local system with a filtration by $\mathbb{Q}$-local systems,
- $C^\bullet$ is $\mathbb{C}$-linear complex of $\mathcal{O}_X$-modules with a decreasing F-filtration and an increasing W-filtration with a filtred isomorphism $(C^\bullet, W) \simeq (\mathbb{L}, W) \otimes \mathbb{C}$ in the derived category of constructible sheaves. The filtrations W (resp. F) is called the *weight filtration* (resp. *Hodge filtration*).
- This data via the degenerations of appropriate spectral sequences induces a mixed Hodge structure on the cohomology groups $H^k(X, \mathbb{L})$ for all k.
- Moreover, the filtrations should satisfy certain compatibility conditions such as

  $$\left(gr_\ell^W \mathbb{L}, gr_\ell^W C^\bullet, F\right)$$

  is a pure Hodge complex of weight ℓ, i.e. it induces pure Hodge structure of weight $k + \ell$ on $H^k(X, gr_\ell^W(\mathbb{L}))$.

Instead of going into further technicalities and subtleties of the definition, we give an example. The readers are advised to consult the excellent survery by Zein and Tráng [C+14, Chapter 3] for a detailed treatment of mixed Hodge structures and further discusions.

Example A.2.5 (MHS Asscociated to Reduced SNC Pairs)**.** In the previous example we noticed that the description of the mixed Hodge structures associated to a simple normal crossing divisor is technical yet explicit. We now discuss a similar scenario where the ambient scheme X has simple normal crossing singularities, i.e. an snc pair with coeffecients of the components equal to 1; see Example 3.2.3 for the definition.

We describe the filtrations W and F only at the level of mixed Hodge complexes. For this we first need to construct the so called *log deRham complex* in this setting. Let (X, D) be a reduced simple normal crossing pair as in the Definition above. Let X_i (resp. D_i) be the irreducible components of X (resp. D). Following Notation A.2.3, define

$$X(q) \coloneqq \bigsqcup_{|I|=q+1} \bigcap_{i\in I} X_i \qquad (\text{resp. } D(q) \coloneqq D \cap X(q))$$

. The *logarithmic deRham complex* can be defined in this setting via

$$\widetilde{\Omega}^{\bullet}_X(\log D) = [\Omega^{\bullet}_{X(0)}(\log D(0)) \xrightarrow{d_0} \Omega^{\bullet}_{X(1)}(\log D(1))[-1] \xrightarrow{d_1} \cdots] \tag{A.6}$$

where

$$\Omega^{\bullet}_{X(q)}(\log D(q)) = \frac{\Omega^{\bullet}_{X(q)}(\log D(q))}{f^*\Omega^1_Y \wedge \Omega^{\bullet-1}_{X(q)}(\log D(q))}.$$

In this complex the morphisms d_q are given by the "residue morphisms" constructed as follows. Let X_I be a component of $X(q)$ and $X_{I+1} \subseteq X_I$ is a closed component of $X(q+1)$. Let $D_I = D \cap X_I$ and $D_{I+1} = D \cap X_{I+1}$. Then we have a closed immersion

$i_I : X_{I+1} \setminus D_{I+1} \hookrightarrow X_I \setminus D_I$ with the complement denoted by $U_I = X_I \setminus (D_I \cup X_{I+1}) \overset{j}{\hookrightarrow} X_I \setminus D_I$. A rough sketch of the situation is as follows:

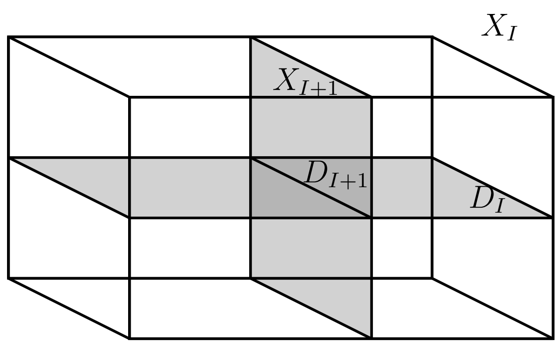

Figure A.2. Construction of the Residue Morphism

We then obtain an exact triangle:

$$Rj_*\mathbb{Q}_{U_I} \longrightarrow \mathbb{Q}_{X_I \setminus D_I} \longrightarrow i_{I*} i_I^! \mathbb{Q}_{X_I \setminus D_I}[1] \overset{+1}{\longrightarrow} . \tag{A.7}$$

Note that the last sheaf computes the local cohomologies of U_I supported along $X_{I+1} \setminus D_{I+1}$ and therefore we have (see also [**Kaw11**, p. 1435])

$$i_{I*} i_I^! \mathbb{C}_{X_I \setminus D_I}[1] \simeq i_{I*} \mathbb{C}_{X_{I+1} \setminus D_{I+1}}[-1]$$

where the isomorphism is given by the Poincaré–Lefschetz duality [**PS08**, Theorem B.28]. Then the last two terms of (A.7) defines the desired "residue maps" d_q is given via the identifications [**Del74**, Scholie 8.1.8.]

$$\Omega^\bullet_{X(q)}(\log D(q)) \simeq \bigoplus_{|I|=q} \mathbb{Q}_{X_I\setminus D_I}\otimes\mathbb{C} \quad \text{and} \quad \Omega^\bullet_{X(q+1)}(\log D(q+1)) \simeq \bigoplus_{|I|=q+1} i_{I_*}\mathbb{Q}_{X_{I+1}\setminus D_{I+1}}\otimes\mathbb{C}$$

Kawamata showed [**Kaw11**, Lemma 3.1] that there exists a $\mathbb{Q}$-local system $\mathbb{L}_\mathbb{Q}$ so that

$$((\mathbb{L}_\mathbb{Q}, W), (\widetilde{\Omega}^\bullet_X(\log D), W, F)) \tag{A.8}$$

defines a mixed Hodge complex with the filtrations defined by

$$F^p\widetilde{\Omega}^\bullet_X(\log D) := \widetilde{\Omega}^{\bullet\geqslant p}_X(\log D)$$

and the weight filtration given by

$$gr^W_m\widetilde{\Omega}^\bullet_X(\log D) := \bigoplus_I(\mathbb{Q}_{D'_I}[-m], \Omega^\bullet_{D'_I}, F(-m)[-m]).$$

A.3. Variations of Mixed Hodge Structures via an Example.

Given a family $f\colon X \to Y$, mixed Hodge structures on the fibres of $X_y := f^{-1}(y)$ for some $y \in Y$ set the premise for a discussion on variation of mixed Hodge structures. However, the compatibilities of the various filtrations get far more intricate. For instance, the situation when f is a morphism of smooth varieties with non-proper fibres, f lacks Ehresmann-type local trivialisations upto diffeomorphism. Here, we will only discuss one example where the variation is particularly nice, namely under a morphism that is *log smooth* like defined below.

Definition A.3.1 (Log-smooth Morphisms)**.** A morphism $f : (X, D) \to Y$ from a projective simple normal crossing pair to a smooth projective variety is said to be *log-smooth* if using Notation A.2.3, we have $f|_{X(q)} : X(q) \to Y$ and $f|_{D(q)} : D(q) \to Y$ are smooth surjective morphisms for all q.

In this case, the mixed Hodge complex is defined by the pushforward of $(\mathbb{L}, W)$ as constructed in Example A.2.5, with the filtrations coming from the degenerations of spectral sequences of the pushforwards of the respective filtrations on $\mathbb{L}_{\mathbb{C}}$. The main reference for this discussion is again [**Kaw11**]. He showed

Theorem A.3.2 ([**Kaw11**, Theorem 4.1, Corollary 4.2])**.** *Let $f : (X, D) \to Y$ be a log-smooth surjective morphism from a projective reduced simple normal crossing pair (X, D) to a smooth projective variety Y. In this situation, the relative log deRham , defined via the total complex*

$$\widetilde{\Omega}^{\bullet}_{X/Y}(\log D) \coloneqq [\Omega^{\bullet}_{X(0)/Y}(\log D(0)) \to \Omega^{\bullet}_{X(1)/Y}(\log D(1))[-1] \to \cdots] \tag{A.9}$$

satisfy $R^q f_* \mathbb{L}_{\mathbb{Q}} \otimes \mathcal{O}_Y \simeq R^q f_* \widetilde{\Omega}^{\bullet}_{X/Y}(\log D)$, where $\mathbb{L}_{\mathbb{Q}}$ is as in (A.8).

Furthermore, $W_m(\widetilde{\Omega}^{\bullet}_{X/Y}(\log D)) \coloneqq W_m(\widetilde{\Omega}^{\bullet}_X(\log D)) \otimes f^{-1}\mathcal{O}_Y$ and $F^p\widetilde{\Omega}^{\bullet}_{X/Y}(\log D) \simeq \widetilde{\Omega}^{\bullet \geqslant p}_{X/Y}(\log D)$ which defines the weight (W) and the Hodge (F) filtrations on $R^q f_* \widetilde{\Omega}^{\bullet}_{X/Y}(\log D)$ via the degeneration of

$${}^{W}E_1^{-m,q+m} \coloneqq R^q f_* gr^W_m(\widetilde{\Omega}^{\bullet}_{X/Y}(\log D)) \Rightarrow R^q f_* \mathbb{L}_{\mathbb{Q}} \tag{A.10}$$

at level E_2 and the degeneration of

$$ {}^{F}E_1^{q-p,p} \coloneqq R^q f_* \widetilde{\Omega}^p_{X/Y}(\log D) \Rightarrow R^q f_* \mathbb{L}_{\mathbb{C}} $$

at level E_1.

In conclusion, $R^q f_* \mathbb{L}_{\mathbb{Q}}$ is a local system that along with

$$ ((R^q f_* \mathbb{L}_{\mathbb{Q}}, W), (R^q f_* \widetilde{\Omega}^{\bullet}_{X/Y}(\log D), W, F)) $$

gives a cohomological mixed Hodge complex.

A.3.1. When X and D both are smooth

An easy case of the scenario above is when X and D are both smooth projective varieties. For instance:

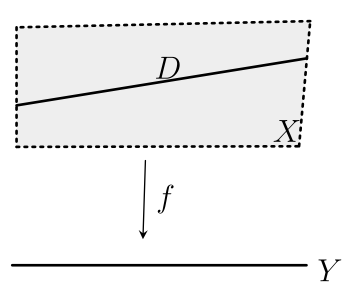

Figure A.3. A Simple Case

In this case, $\widetilde{\Omega}^\bullet_{X/Y}(\log D) = \Omega^\bullet_{X/Y}(\log D) = \dfrac{\Omega^\bullet_X(\log D)}{f^{-1}\Omega^1_Y \wedge \Omega^{\bullet-1}_X(\log D)}$ and

$$R^q f_* gr^W_m(\Omega^\bullet_{X/Y}(\log D)) = \begin{cases} R^q f_* \Omega^\bullet_{X/Y} & \text{when } m = 0 \\ R^q f_* \Omega^\bullet_{D/Y} & \text{when } m = 1 \end{cases} \tag{A.11}$$

Now by the degeneration of (A.5) at E_2, we obtain,

$$gr^W_0(R^q f_* \Omega^\bullet_{X/Y}(\log D)) \simeq \mathrm{Coker}(E_1^{-1,q} \to E_1^{0,q}) = \mathrm{Coker}(R^{q-1} f_* \Omega^\bullet_{D/Y} \to R^q f_* \Omega^\bullet_{X/Y})$$

$$gr^W_1(R^q f_* \Omega^\bullet_{X/Y}(\log D)) \simeq \ker(E_1^{-1,q+1} \to E_1^{0,q+1}) = \ker(R^q f_* \Omega_{D/Y} \to R^{q+1} f_* \Omega_{X/Y})$$

$$\text{and} \qquad gr^W_m(R^q f_* \Omega^\bullet_{X/Y}(\log D)) = 0 \text{ for all } m \neq 0, 1$$

The lowest graded pieces of the respective Hodge filtrations fit nicely in the long exact sequence

(A.12)

$$0 \to f_*\omega_X \to f_*\omega_X(D) \to f_*\omega_D \to R^1 f_*\omega_X \to \cdots \to R^d f_*\omega_X \to R^d f_*\omega_X(D) \to 0.$$

A.4. A Brief Discussion on Hodge modules.

The theory of (mixed) Hodge modules is in a way a vast generalisation of the theory of variations of mixed Hodge structures. For instance, This theory provides a systematic way to study the Hodge theory of pushforwards of certain Hodge structures when the morphism is smooth or log-smooth only over an open set. A Hodge module, an $\mathcal{O}_X$-module in particular, admits a flat connection akin to its vHs-counterpart. This endows such a module with a left (or, right) action of the holomorphic tangent bundle $\mathcal{T}_X$. By iterating this action, one obtains an action of the non-commutative ring $\mathcal{D}_X$, generated by $\mathcal{O}_X$ and the holomorphic tangent sheaf $\mathcal{T}_X$. We make this precise in the following definition; see [**HTT95**] for a thorough treatment of $\mathcal{D}$-modules.

Definition A.4.1 ($\mathcal{D}$-modules)**.** Let X be a smooth algebraic variety. The sheaf of differential operators $\mathcal{D}_X$-modules on X, is the $\mathbb{C}$ algebra generated by $\mathcal{O}_X$ and the holomorphic tangent sheaf $\mathcal{T}_X$. Locally, with local system of parameteres $t_1, \cdots, t_n$,

$$\mathcal{D}_X \overset{\text{loc}}{\simeq} \mathbb{C}[t_1, \cdots, t_n, \partial_1, \cdots, \partial_n]/\sim$$

where $\{\partial_i\}_i$ are the local sections of the sheaf Θ_X and $\sim$ denotes the relations

$$[t_i, t_j] = 0 = [\partial_i, \partial_j] \quad \text{and } [t_i, \partial_j] = \delta_{ij}.$$

An $\mathcal{O}_X$-modules $\mathcal{M}$ is said to be a left (or, right) $\mathcal{D}_X$-module, if it has a left $\mathcal{D}_X$-action. This is equivalent to a $\mathbb{C}$-linear morphism $\nabla\colon \mathcal{T}_X \to \mathscr{E}nd_{\mathbb{C}}(M)$ locally satisfying

$$(1) \quad \nabla_{f\theta} = f\nabla_\theta \qquad (2) \quad \nabla_\theta(fm) = \theta(f)m + f\nabla(m) \quad \text{and } (3) \quad \nabla_{[\theta_1,\theta_2]} = [\nabla_{\theta_1}, \nabla_{\theta_2}]$$

where $\theta_i \in \Theta_X, m \in M, f \in \mathcal{O}_X$. Furthermore, the above are equivalent to having a $\mathbb{C}$-linear morphism $\nabla' : M \to \Omega^1_X \otimes_{\mathcal{O}_X} M$ locally satisfying

$$\nabla'(fm) = \sum_i dt_i \otimes \partial_i fm + f\nabla'(m) \quad \text{and } \nabla'(Pm) = \sum_i dt_i \otimes \partial_i P(m) + P\nabla'(m)$$

and is a flat connection, i.e. $\nabla' \circ \nabla' = 0$.

To a $\mathcal{D}_X$-module $\mathcal{M}$ one can thus associate a complex, the deRham complex much like in the Definition A.1.2.

$$\mathrm{DR}(\mathcal{M}) \coloneqq [\mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \cdots \to \omega_X \otimes \mathcal{M}].$$

Example A.4.2. With the notation in Definition A.1.1, it follows that a variation of Hodge structure underlies the $\mathcal{D}_X$-module $(\mathscr{E}, \nabla)$.

A Hodge module, roughly speaking is a (filtered left) $\mathcal{D}_X$-module with respect to the filtration on $\mathcal{D}_X$ is defined by

$$F_k\mathcal{D}_X \stackrel{\text{loc}}{\coloneqq} \bigoplus_{\sum_{i\in I} a_i \leqslant k} \mathbb{C} \cdot \partial_I^{a_I},$$

such that restricted to a Zariski open set $U \subseteq X$, $\mathrm{DR}\,\mathcal{M}|_U$ gives a local system underlying a variation of Hodge structures and such that outside U the filtrations satisfy appropriate compatibilities with monodromy weight filtration coming from taking limits. Again instead of going into the technicalities of the definition, we focus on the pushforwards under log-smooth morphisms. For this purpose, we can take *(mixed) Hodge modules* to be simply an extension of mixed Hodge structures via the following structure theorem.

Theorem A.4.3 ([**Sai90**, Theorem 3.27])**.** *Let X be a smooth projective variety. A graded-polarisable variation of mixed Hodge structure on a Zariski-open subset $U \subseteq X$; $((\mathbb{L}, W), (C^\bullet, W, F))$ can be extended to a mixed Hodge module $(\mathcal{M}, W, F)$ on X.*

Conversely, for any mixed Hodge $\mathcal{D}_X$-module $\mathcal{M}$, there exists a Zariski open set $U \subseteq X$ and a polarized variation of mixed Hodge structure $((\mathbb{L}, W), (C^\bullet, W, F))$ such that $\mathcal{M}|_U \simeq \mathbb{L} \otimes \mathcal{O}_U$.

Associated to $(\mathcal{M}, W, F)$, there is a filtered deRham complex

$$F^k \operatorname{DR}(\mathcal{M}) \coloneqq [F^{k+n}\mathcal{M} \to \Omega^1_X \otimes F^{k+n-1}\mathcal{M} \to \cdots \to \omega_X \otimes F^k\mathcal{M}]$$

Caveat A.4.4. Note that in Saito's theory the Hodge filtration F is considered to be increasing. The dictionary between the classical Hodge theoretic increasing filtration and Saito's filtration is simply given by $F_k \coloneqq F^{-k}$.

Although we will not get into the definition of polarisations in this thesis, we remark that one of the most basic examples of a polarisable Hodge module is $(\mathcal{O}_X, F)$ with $F^{-k}\mathcal{O}_X = \mathcal{O}_X$ and $F^{k+1}\mathcal{O}_X = 0$ for $k \geqslant 0$. This corresponds to the constant variation of Hodge structures and the corresponding Hodge complex is given by $(\mathbb{C}, \Omega^\bullet_X, F)$ where $F^p\Omega^\bullet_X \simeq \Omega^{\bullet \geqslant p}_X$.

Saito's theory is compatible with Grothendieck's six functor formalism, namely under the deRham functor DR commutes with the derived operations

$$\mathcal{H}om, \quad \otimes, \quad f_*, \quad f_!, \quad f^!, \quad f^*$$

appropriately defined for the derived category of (mixed) Hodge modules, for any morphism $f : X \to Y$ of algebraic varieties. For instance, given a (mixed) Hodge module $\mathcal{M}$ on X, $\mathcal{H}^i f_+\mathcal{M}$ are (mixed) Hodge modules[1] on Y for all i and $f_* \operatorname{DR}(\mathcal{M}) \simeq \operatorname{DR}(f_+\mathcal{M})$. Saito's theory also prescribes a way to assign a Hodge filtration on $\mathcal{H}^i f_+\mathcal{M}$; see e.g. [**Pop16**, *Strictness* p. 55–56] for a discussion on this. For our purposes, we only need

[1]To distinguish Hodge module pushforward from $\mathcal{O}_X$-module pushforward, following the standard norm in the literature, we use f_+ for (derived) pushforwards of Hodge modules.

Saito's formula [**Sai88**, §2.3.7] for graded filtered $\mathcal{D}_X$-modules:

$$R^q f_* gr^k_F \operatorname{DR}(\mathcal{M}) \simeq gr^k_F \operatorname{DR} \mathcal{H}^q f_* \mathcal{M}$$

We use this formula to compute the lowest graded pieces of Hodge modules associated to the variations discussed in Example A.2.2 and A.2.5

Example A.4.5. The above formula applied to $\mathcal{O}_X$ gives

$$R^q f_* gr^0_F \operatorname{DR} \mathcal{O}_X \simeq R^q f_* \omega_X \simeq gr^0_F \mathcal{H}^q f_+ \mathcal{M} \simeq \omega_Y \otimes F^0 \mathcal{H}^q f_* \mathcal{M}$$

Hence

$$F^{\text{low}} \mathcal{H}^q f_* \mathcal{M} \simeq R^q f_* \omega_{X/Y} \tag{A.13}$$

where $\text{low}(\mathcal{M}) \coloneqq \max\{q | F^q \mathcal{M} \neq 0\}$[2]

Example A.4.6 (The SNC Case)**.** More generally let $\mathcal{M}$ be the Hodge module corresponding to the mixed Hodge structure in Example A.2.2, i.e. $\mathcal{M} \simeq \mathcal{O}_X(*D)$, the sheaf of holomorphic functions with arbitrary poles along D. The Hodge filtration of this Hodge module is an interesting object and has been studied extensively in [**MP16, MP18**]. Here we only need that $F^{\text{low}} \operatorname{DR} \mathcal{O}_X(*D) \simeq \omega_X(D)$; see e.g. [**MP16**, Proposition 8.2]. Therefore, $F^{\text{low}} \operatorname{DR} \mathcal{H}^q f_*(\mathcal{O}_X(*D)) \simeq R^q f_* \omega_{X/Y}(D)$.

Example A.4.7 (The Reduced SNC-Pair Case)**.** The mixed Hodge module corresponding to the variation of mixed Hodge structures in Example A.2.5 works similarly.

[2] The discrepancy between the nomenclature and the definition is again due to sticking to the convention of increasing Hodge filtration due to classical Hodge theory. In Saito's theory, this is truly the lowest piece of the Hodge filtration.

We should remark that because of the lack of regularity of X, notationally it is a bit more complicated. This situation is discussed in great detail in [**FFS14**]. By [*loc. cit.*, Theorem 1], $\omega_X(D)$ is the lowest graded piece of a Hodge $\mathcal{D}$-module on X. Similarly by [*loc. cit.*, Theorem 2] $R^q f_* \omega_{X/Y}(D)$ is the lowest graded piece of the Hodge filtration of the Hodge module associated to the variation of Hodge structure $R^q f_* \mathbb{L}_{\mathbb{C}}$.

Example A.4.8 (Reduced Effective Divisor Case)**.** When D is any reduced effective divisor on a smooth projective variety X, $F^{\text{low}} \operatorname{DR} \mathcal{O}_X(*D) \simeq \omega_X(D) \otimes \mathcal{J}((1-\epsilon)D)$; see [**MP16**, §10]. Now let $f : X \to Y$ is a surjective morphism of smooth projective varieties, then Saito's formula gives

$$F^{\text{low}} \operatorname{DR} \mathcal{H}^q f_+ \mathcal{O}_X(*D) \simeq \omega_Y \otimes F^{\text{low}} \mathcal{H}^q f_+ \mathcal{O}_X(*D) \overset{\text{Saito's Formula}}{\simeq} R^q f_*(\omega_X(D) \otimes \mathcal{J}((1-\epsilon)D).$$

The last isomorphism can be re-written as $F^{\text{low}} \mathcal{H}^q f_+ \mathcal{O}_X(*D) \simeq R^q f_*(\omega_{X/Y}(D) \otimes \mathcal{J}((1-\epsilon)D)$

A.4.1. Vanishing Theorem

Akin to Kodaira vanishing or Ambro–Kollár–Fujino type vanishing, there exists vanishing for the lowest graded pieces of Hodge modules by the work of Suh [**Suh15**, Theorem 3.2] and Wu [**Wu15**] building on Saito's vanishing [**Sai88**, §2.g].

Theorem A.4.9. *Let $\mathcal{M}$ be a pure Hodge module with strict support on X and let $\mathcal{L}$ be a big and nef line bundle on X, then for all $i > 0$*

$$H^i(X, \omega_X \otimes F^{\text{low}} \mathcal{M} \otimes \mathcal{L}) = 0.$$

Remark A.4.10. When $\mathcal{L}$ is ample, it is Saito's Vanishing Theorem and holds more generally for all the F-graded pieces of $\mathrm{DR}(\mathcal{M})$ where $\mathcal{M}$ is more generally a mixed Hodge module.

We write down what the Vanishing theorem imply in each of the above Examples.

Example A.4.11. Example A.4.5: A case of Kollár's Vanishing Theorem [**Kol95**, Theorem 10.19]: $H^i(Y, R^q f_* \omega_X \otimes \mathcal{L}) = 0$ for all $i > 0$.

Example A.4.12. Example A.4.6 and A.4.7: A case of the Ambro–Fujino-type vanishing theorems [**Amb03, Fuj11**]: $H^i(Y, R^q f_* \omega_X(D) \otimes \mathcal{L}) = 0$ for all $i > 0$.

Example A.4.13. Example A.4.8: A case of the Ein–Popa Vanishing Theorem [**EP08**, Theorem 3.2(3)] when $\mathcal{L}$ is ample: $H^i(Y, R^q f_*(\mathcal{O}_X(K_X+D) \otimes \mathcal{J}((1-\epsilon)D)) \otimes \mathcal{L}) = 0$ for all $i > 0$ and $1 \gg \epsilon \in \mathbb{Q}$.

APPENDIX B

Reflexive Sheaves

Fix an integral noetherian scheme X.

Definition B.1.1. A coherent sheaf $\mathcal{F}$ on X is *reflexive* if the natural morphism $\mathcal{F} \to \mathcal{F}^{\vee\vee}$ is an isomorphism, where $\mathcal{G}^{\vee} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{O}_X)$. In particular, locally free sheaves are reflexive.

A coherent sheaf $\mathcal{F}$ on X is *normal* if the restriction map

$$\Gamma(U, \mathcal{F}) \longrightarrow \Gamma(U \smallsetminus Z, \mathcal{F})$$

is bijective for every open set $U \subseteq X$ and every closed subset Z of U of codimension at least 2.

Proposition B.1.2 (see [**Har94**, Proposition 1.11])**.** *If X is normal, i.e. $\mathcal{O}_X$ is normal, then every reflexive coherent sheaf $\mathcal{F}$ is normal.*

Lemma B.1.3 ([**Stacks**, Tag 0AY4])**.** *Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X, and assume that $\mathcal{F}$ is reflexive. Then, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is also reflexive.*

We will often use these facts to extend morphisms from the complement of codimension at least 2, as recorded in the following:

Corollary B.1.4. *Suppose X is normal, and let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on X such that $\mathcal{F}$ is reflexive. If $U \subseteq X$ is an open subset such that $\operatorname{codim}(X \smallsetminus U) \geq 2$, then every morphism $\varphi\colon \mathcal{G}|_U \to \mathcal{F}|_U$ extends uniquely to a morphism $\widetilde{\varphi}\colon \mathcal{G} \to \mathcal{F}$.*

Proof. The morphism φ corresponds to a section of the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ over U. The sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ is reflexive by Lemma B.1.3, hence the section φ extends uniquely to a section $\widetilde{\varphi}$ of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{F})$ over X by Proposition B.1.2. □

APPENDIX C

Dualising Complexes and Canonical Sheaves

The main reference for this section is Hartshorne's book [**Har66**] on dualising sheaves. Recall that when X is a smooth projective variety of dimension n, the canonical sheaf of X is given by n-fold exterior power of the cotangent bundle, namely $\omega_X \simeq \bigwedge \Omega_X$. Therefore it is a line bundle. Furthermore, they satisfy the Serre duality, namely, for any line bundle $\mathcal{L}$ on X (see [**Har77**, Theorem III.7.6])

$$H^i(X, \mathcal{L}^{-1}) \simeq H^{n-i}(X, \mathcal{L} \otimes \omega_X)$$

or more generally for any coherent sheaf $\mathcal{F}$

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{O}_X) \simeq H^{n-i}(X, \mathcal{F} \otimes \omega_X).$$

Unfortunately for singular varieties one does not have such duality for free. Nonetheless there is a complex, known as *the dualising complex* (defined below) that satisfy similar properties in the derived categories of coherent sheaves denoted $D^b_{\mathrm{coh}}(\mathrm{Mod}(\mathcal{O}_X))$.

Recall the following

Definition C.1.1 (Dualising Complex)**.** Let X be a locally noetherian scheme. An object

$$\omega_X^\bullet \in D^b_{\mathrm{coh}}(\mathrm{Mod}(\mathcal{O}_X))$$

of finite injective dimension is a *dualising complex* for X if the natural map

$$\mathcal{F}^\bullet \longrightarrow R\mathcal{H}om_{\mathcal{O}_X}\big(R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \omega_X^\bullet), \omega_X^\bullet\big)$$

is an isomorphism for every $\mathcal{F}^\bullet \in D^b_{\mathrm{coh}}(\mathrm{Mod}(\mathcal{O}_X))$.

Example C.1.2.

(1) When X is a Cohen-Macaulay scheme $\omega_X^\bullet$ is concentrated in one degree, in other words we can treat it as a coherent sheaf.
(2) When X is Gorenstein this sheaf is in fact invertible [**Har66**, Proposition V.9.3].
(3) Needless to say, when X is smooth over a field, then $\omega_X^\bullet$ is again a coherent sheaf and moreover it coincides with $\omega_X \coloneqq \Omega_X^{\dim X}$ [**Har66**, III.2].
(4) When $X = \operatorname{Spec} k$ for a field k, $\omega_X^\bullet \simeq k$.

This leads to the notion of the canonical sheaf, denoted ω_X. We first need

C.1.1. Grothendieck Duality.

Given an equidimensional proper scheme of finite type over a field k with structure map $h : X \to k$, for any complex $\mathcal{F} \in D^b_{\mathrm{coh}}(X)$,

$$R\mathcal{H}om(Rh_*\mathcal{F}, k) \simeq Rh_*\mathcal{H}om(\mathcal{F}, h^! k)$$

where $h^!$ is the exceptional pullback of Grothendieck duality [**Har66**, Corollary VII.3.4].

Definition C.1.3 (Canonical Sheaves)**.** Let $h\colon X \to \operatorname{Spec} k$ be as above. Then the *normalized dualising complex* for X is $\omega_X^\bullet \coloneqq h^! k$ as an object in $D^b_{\mathrm{coh}}(X)$.

One defines the *canonical sheaf* on X to be the coherent sheaf

$$\omega_X \coloneqq \mathbf{H}^{-\dim X}\omega_X^\bullet.$$

Any noetherian scheme of finite type over a field admits a dualising complex unique up to quasi-isomorphism [**Har66**, V.10], hence has a canonical sheaf ω_X.

C.1.2. Explicit description of $\nu^!$ for finite morphisms.

For our purposes, we need an explicit description of the exceptional pullback functor for finite morphisms. Let $\nu\colon Y \to X$ be a finite morphism of equidimensional schemes of finite type over a field. Consider the functor

$$\overline{\nu}^*\colon \operatorname{Mod}(\nu_*\mathcal{O}_Y) \longrightarrow \operatorname{Mod}(\mathcal{O}_Y)$$

obtained from the morphism $\overline{\nu}\colon (Y,\mathcal{O}_Y) \to (X,\nu_*\mathcal{O}_Y)$ of ringed spaces. This functor $\overline{\nu}^*$ satisfies the following properties (see [**Har66**, III.6]):

(1) The functor $\overline{\nu}^*$ is exact since the morphism $\overline{\nu}$ of ringed spaces is flat.

(2) For every $\mathcal{O}_X$-module $\mathcal{G}$, we have $\nu^*\mathcal{G} \simeq \overline{\nu}^*(\mathcal{G} \otimes_{\mathcal{O}_X} \nu_*\mathcal{O}_Y)$.

(3) We define the functor

$$\begin{aligned}\nu^!\colon D^+(\operatorname{Mod}(\mathcal{O}_X)) &\longrightarrow D^+(\operatorname{Mod}(\mathcal{O}_Y))\\ \mathcal{F} &\longmapsto \overline{\nu}^* R\mathcal{H}om_{\mathcal{O}_X}(\nu_*\mathcal{O}_Y,\mathcal{F})\end{aligned}$$

If $\omega_X^\bullet$ is the normalized dualising complex for X, then $\nu^!\omega_Y^\bullet$ is the normalized dualising complex for Y.

Using the above description, we construct the following *pluri-trace map* for integral schemes over fields, which we used in the proof of Theorem 5.1.5. We believe that this construction is already known to the experts. We include the details for future references.

Lemma C.1.4 (Pluri-trace Map)**.** *Let $d\colon Y' \to Y$ be a dominant proper birational morphism of integral schemes of finite type over a field, where Y' is normal and Y is Gorenstein. Then, there is a map of pluricanonical sheaves*

$$d_*\omega_{Y'}^{\otimes k} \longrightarrow \omega_Y^{\otimes k}$$

which is an isomorphism where d is an isomorphism.

Proof. By the universal property of normalisation [**Stacks**, Tag 035Q], we can factor d as

$$Y' \xrightarrow{d'} \overline{Y} \xrightarrow{\nu} Y, \qquad d = \nu \circ d'$$

where ν is the normalisation. Note that d' is proper and birational since d is.

We first construct a similar morphism for ν. Let $n = \dim Y$. Since Y is Gorenstein, the canonical sheaf ω_Y is invertible and in $D^b_{\mathrm{coh}}(\mathrm{Mod}(Y))$, we have $\omega_Y^\bullet \simeq \omega_Y[n]$. Using property (3) above we have

$$\begin{aligned}\omega_{\overline{Y}} = \mathbf{H}^{-n}(\nu^!\omega_Y^\bullet) &\simeq \overline{\nu}^*\big(\mathbf{R}^{-n}\mathcal{H}om_{\mathcal{O}_Y}(\nu_*\mathcal{O}_{\overline{Y}}, \mathcal{O}_Y[n]) \otimes_{\mathcal{O}_Y} \omega_Y\big)\\ &\simeq \overline{\nu}^*\big(\mathcal{H}om_{\mathcal{O}_Y}(\nu_*\mathcal{O}_{\overline{Y}}, \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \omega_Y\big)\end{aligned}$$

where we get the first isomorphism since $\overline{\nu}^*$ is exact by (1) and since ω_Y is invertible.

Now $\mathcal{H}om_{\mathcal{O}_Y}(\nu_*\mathcal{O}_{\overline{Y}}, \mathcal{O}_Y)$ admits a morphism to $\nu_*\mathcal{O}_{\overline{Y}}$ which makes it the largest ideal in $\nu_*\mathcal{O}_{\overline{Y}}$ that is also an ideal in $\mathcal{O}_Y$. It is the so-called *conductor ideal* $\mathfrak{cond}_X$ of the normalisation map [**KK13**, §(5.2)]. Thus, we get a morphism

$$\omega_{\overline{Y}} \hookrightarrow \overline{\nu}^*(\nu_*\mathcal{O}_{\overline{Y}} \otimes \omega_Y) \simeq \nu^*\omega_Y.$$

The last isomorphism follows from (2) above. By taking the $(k-1)$-fold tensor product of the above morphism we have

$$\omega_{\overline{Y}}^{\otimes(k-1)} \hookrightarrow \nu^*\omega_Y^{\otimes(k-1)}. \tag{C.1}$$

Finally, we use (C.1) to construct a map

$$d_*\omega_{Y'}^{\otimes k} \longrightarrow \nu^*\omega_Y^{\otimes(k-1)} \otimes_{\mathcal{O}_{\overline{Y}}} \omega_{\overline{Y}}.$$

First, we construct the above morphism over U where d' is an isomorphism. Denote $V := d'^{-1}(U)$. The identity map

$$\mathrm{id}\colon d'_*\omega_V^{\otimes k} \longrightarrow \omega_U^{\otimes k}$$

composed with map obtained from (C.1) gives the following map

$$\tau\colon \omega_U^{\otimes k} \hookrightarrow \nu^*\omega_Y^{\otimes(k-1)}\big|_U \otimes_{\mathcal{O}_U} \omega_U.$$

Since $\nu^*\omega_Y^{\otimes(k-1)}$ is invertible and $\omega_{\overline{Y}}$ is reflexive, the sheaf $\nu^*\omega_Y^{\otimes(k-1)} \otimes \omega_{\overline{Y}}$ is also reflexive. Now $\mathrm{codim}(Y \setminus U) \geqslant 2$ by Zariski's Main Theorem (see [**Har77**, Theorem V.5.2]).

Therefore by Corollary B.1.4 we obtain

$$\widetilde{\tau}\colon d'_*\omega_{Y'}^{\otimes k} \longrightarrow \nu^*\omega_Y^{\otimes(k-1)} \otimes_{\mathcal{O}_{\overline{Y}}} \omega_{\overline{Y}}.$$

Composing $\nu_*\widetilde{\tau}$ with one copy of the trace morphism $\nu_*\omega_{\overline{Y}} \to \omega_Y$ [**Har66**, Proposition III.6.5], we get

$$d_*\omega_{Y'}^{\otimes k} \xrightarrow{\nu_*\widetilde{\tau}} \nu_*(\nu^*\omega_Y^{\otimes(k-1)} \otimes_{\mathcal{O}_{\overline{Y}}} \omega_{\overline{Y}}) \simeq \omega_Y^{\otimes(k-1)} \otimes_{\mathcal{O}_Y} \nu_*\omega_{\overline{Y}} \xrightarrow{\mathrm{id}\otimes\mathrm{Tr}} \omega_Y^{\otimes k}. \tag{C.2}$$

The last part of the statement holds by construction of the maps above. Indeed, in (C.2) the trace morphism is compatible with flat base change [**Har66**, Proposition III.6.6(2)], hence compatible with restriction to the open set where d is an isomorphism.

□

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