COHOMOLOGY OF $\mathcal{H}_L$

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Abstract. This note is a continuation of the computation of a certain cohomology ring from my talk on Kuznetsov's theory of homological projective duality. This is an attempt to consolidate various discussions I had with Pieter, Daniel and Mirko and to incorporate some of their comments, ideas and references. That said, any mistake in these notes is definitely the author's fault.

1.1. **Decomposition of the cohomology of $\mathcal{H}_L$.** The object of interest is the universal hyperplane section associated to a linear system L of rank ℓ on a smooth projective variety X. Consider the following diagram (see [Tho18, §3] or slides from my talk for more details about the set-up).

$$\begin{array}{ccccc} X_L \times \mathbb{P}(L) & \xhookrightarrow{j} & \mathcal{H}_L & \xrightarrow{\iota} & X \times \mathbb{P}(L) \\ \downarrow p & \square & \downarrow \pi & \swarrow \rho & \\ X_L & \xrightarrow{i} & X & & \end{array}$$

Here we recall that X_L is the base locus of the linear system L and $\mathcal{H}_L$ is the projectivisation of the coherent sheaf $K := \ker(\mathcal{O}_X \otimes L \twoheadrightarrow \mathscr{I}_{X_L}(1))$ where $\mathscr{I}_{X_L}$ is the ideal sheaf of X_L. Since L is base point free on $X \setminus X_L$, K is a rank $\ell - 1$ locally free sheaf on $X \setminus X_L$.

1.1.1. *Decomposition over* $\mathbb{Q}$. We first show a decomposition theorem over $\mathbb{Q}$. Note that, over the smooth locus of π, namely $X \setminus X_L$ we have

$$R\pi_*\mathbb{Q}_{\mathcal{H}_L}|_{X\setminus X_L} \simeq \oplus_{i=0}^{2\ell-4} R^i\pi_*\mathbb{Q}_{\mathcal{H}_L}|_{X\setminus X_L}[-i].$$

Since the fibres are $\mathbb{P}^{\ell-2}$, by proper base change we note that when i is even the fibre

$$R^i\pi_*\mathbb{Q}_{\mathcal{H}_L}|_{X\setminus X_L} \otimes \kappa(x) \simeq \mathbb{Q}.$$

Since X is smooth, $IC_X(\mathbb{Q}) \simeq \mathbb{Q}$. Therefore, on X, the decomposition is given by

$$R\pi_*\mathbb{Q}_{\mathcal{H}_L}|_{X\setminus X_L} \simeq \bigoplus_{i=0}^{\ell-2} \mathbb{Q}_X[-2i] \oplus \mathcal{B}_{X_L}$$

where $\mathcal{B}_{X_L}$ is supported on X_L. Taking cohomology of both sides we note that $\mathcal{H}^i(\mathcal{B}_{X_L}) = 0$ for $i \leq 2\ell - 4$ and for $i = 2\ell - 2$, again by taking fibres we obtain

$$R^{2\ell-2}\pi_*\mathbb{Q}_{\mathcal{H}_L} \otimes \kappa(x) \simeq \mathbb{Q}$$

for all $x \in X_L$ and 0 otherwise. Since $p\colon X_L \times \mathbb{P}^{\ell-1} \to X_L$ is a trivial fibration, we infact have more, namely $R^{2\ell-2}\pi_*\mathbb{Q}_{\mathcal{H}_L} \simeq \mathcal{H}^{2\ell-2}(\mathcal{B}) \simeq \mathbb{Q}_{X_L}$.

1.1.2. *Decomposition over $\mathbb{Z}$.* In our particularly nice situation, this decomposition in fact carries over to $\mathbb{Z}$. Moreover, when X_L is singular, $\mathbb{Q}_{\mathcal{H}_L}[\dim \mathcal{H}_L]$ may not be a simple perverse and hence the general theory does not give a decomposition even with $\mathbb{Q}$-coefficient. However in this situation, we still obtain a decomposition of cohomology. Let us first fix notations for the open immersions $\alpha\colon U_L := \mathcal{H}_L \setminus X_L \times \mathbb{P}(L) \hookrightarrow \mathcal{H}_L$ and $\beta\colon U := X \setminus X_L \hookrightarrow X$. Consider the following triangle of functors

$$\beta_! \beta^* \to \mathrm{id} \to i_* i^*.$$

This applied to $R\pi_*\mathbb{Z}[\dim \mathcal{H}_L]$ and proper base change we obtain

$$\beta_! R\pi_* \mathbb{Z}_{U_L} \to R\pi_*\mathbb{Z} \to i_* Rp_* \mathbb{Z}_{X_L \times \mathbb{P}(L)}.$$

By derived Leray-Hirsch [dC11, 2.4.3] we get decompositions $R\pi_*\mathbb{Z}_{U_L} \simeq \bigoplus_{i=0}^{\ell-2} \mathbb{Z}_U[-2i]$ and $Rp_*\mathbb{Z}_{X_L\times\mathbb{P}(L)} \simeq \bigoplus_{i=0}^{\ell-1} \mathbb{Z}_{X_L}[-2i]$. Then the updated triangle

$$\bigoplus_{i=0}^{\ell-2} \beta_! \mathbb{Z}_U[-2i] \to R\pi_*\mathbb{Z}_{\mathcal{H}_L} \to \bigoplus_{i=0}^{\ell-1} i_*\mathbb{Z}_{X_L}[-2i]$$

glues piecewise via the triangle $\beta_!\mathbb{Z}_U \to \mathbb{Z}_X \to i_*\mathbb{Z}_{X_L}$ to give

$$R\pi_*\mathbb{Z}_{\mathcal{H}_L} \simeq \bigoplus_{i=0}^{\ell-2} \mathbb{Z}_X[-2i] \oplus \mathbb{Z}_{X_L}[-2\ell+2].$$

Taking cohomologies it reads as follows

$$H^k(\mathcal{H}_L, \mathbb{K}) \simeq \bigoplus_{i=0}^{\ell-2} H^{k-2i}(X, \mathbb{K}) \oplus H^{k-2\ell+2}(X_L, \mathbb{K})$$

for $\mathbb{K} = \mathbb{Z}, \mathbb{Q}$.

Remark 1.1. When X_L is smooth the "other" triangle, namely

$$i_* i^! \to \mathrm{id} \to \beta_* \beta^* \tag{1}$$

gives a similar story. However when X_L is singular, $\mathbb{Q}_{\mathcal{H}_L}[\dim \mathcal{H}_L]$ may not even be perverse (in our complete intersection situation it in fact is perverse; see [Dim04, 5.1.20]). Instead, one can apply the triangle to $R\pi_*\mathscr{K}_{\mathcal{H}_L}$ where $\mathscr{K}_{\mathcal{H}_L}$ is the dualising complex and is isomorphic to $\pi^!\mathbb{Z}_X[2\dim X] \simeq D\pi^* D\mathbb{Z}_X[2\dim X] \simeq D\mathbb{Z}_{\mathcal{H}_L}$. This gives us a similar story in the homology setting, using the definition $H_k^{\mathrm{BM}} := \mathrm{Hom}(\mathbb{Z}, \mathscr{K}[-k])$.

Question 1.2. Is $\mathscr{K} = \mathbb{Q}[2\dim]$ for Cohen-Macaulay varieties? Recall that in coherent setting the dualising complex is $\mathcal{O}[2\dim]$.

Remark 1.3. For the smooth case, an argument using the topological translation of the triangle (1) namely the long exact sequence of pairs $(\mathcal{H}_L, \mathcal{H}_L \setminus X_L \times \mathbb{P}(L))$ can be found in [BEM19]. A similar argument was also communicated to me by Mirko Mauri. The Chow theoretic incarnation is described in [Jia19].

1.2. **Ring structure.** An immediate multiplication structure in the above decomposition can be specified by mulplying the summands with $h = c_1(\mathcal{O}_{\mathcal{H}_L}(1))$ and using the Gysin morphism j_*. More precisely, we have

$$H^k(\mathcal{H}_L, \mathbb{K}) \simeq \bigoplus_{i=0}^{\ell-2} h^i \smile \pi^* H^{k-2i}(X, \mathbb{K}) \oplus j_* p^* H^{k-2\ell+2}(X_L, \mathbb{K}).$$

We have the following multiplications,

$$\begin{aligned} \pi^*\alpha \smile \pi^*\beta &= \pi^*(\alpha \smile \beta) \\ \pi^*\alpha \smile j_*p^*\gamma &= j_*(p^*i^*\alpha \smile \gamma) \\ j_*\gamma \smile j_*\delta &= j_*(\gamma \smile \delta \smile c_{\ell-1}(N_{X_L \times \mathbb{P}(L)/\mathcal{H}_L}) \end{aligned} \tag{2}$$

Since $H^*(\mathcal{H}_L)$ is generated by h and the classes in $\pi^*H^*(X)$ and $j_*p^*H^*(X_L \times \mathbb{P}(L))$, we are only left to determine $c_{\ell-1}(N_{X\times\mathbb{P}(L)/\mathcal{H}_L})$ in terms of the generators. To do this we use the following short exact sequence of normal bundles

$$0 \to N_{X\times\mathbb{P}(L)/\mathcal{H}_L} \to \pi^*\mathcal{O}_X(1) \otimes L^* \to \mathcal{O}_{X\times\mathbb{P}(L)}(1,1)|_{\mathcal{H}_L} \to 0$$

Now $h = c_1(\mathcal{O}_{X\times\mathbb{P}(L)}(0,1)|_{\mathcal{H}_L})$. Letting $\alpha := c_1(\mathcal{O}_X(1))$, $c_1(\mathcal{O}_{X\times\mathbb{P}(L)}(1,1)|_{\mathcal{H}_L}) = \pi^*\alpha + h$. Similarly, $c_t(\pi^*\mathcal{O}_X(1) \otimes L^*) = (1 + \pi^*\alpha t)^\ell$. Therefore,

$$\begin{aligned} c_{\ell-1}(N_{X\times\mathbb{P}(L)/\mathcal{H}_L}) &= \text{ coefficient of } t^{\ell-1} \text{ in } (1+\pi^*\alpha t)^\ell (1 + (\pi^*\alpha + h)t)^{-1} \\ &= \sum_{i=0}^{\ell-1} (-1)^i \binom{\ell}{i+1} h^{\ell-i-1} \smile (\pi^*\alpha + h)^i \end{aligned} \tag{3}$$

References

[BEM19] Marcello Bernardara, Fatighenti Enrico, and Laurent Manivel. Nested varieties of k3 type. *arXiv.1912.03144 [math.AG]*, Dec, 2019. 2

[dC11] Mark Andrea A. de Cataldo. Lectures on perverse sheaves and decomposition theorem. *url: https://www.math.stonybrook.edu/mde/AnnArborLectures.pdf*, Ann Arbor Lectures, 2011. 2

[Dim04] Alexandru Dimca. *Sheaves in topology*. Universitext. Springer-Verlag, Berlin, 2004. 2

[Jia19] Qingyuan Jiang. On the chow theory of projectivization. *arXiv.1910.06730 [math.AG]*, Oct, 2019. 2

[Tho18] Richard P. Thomas. Notes on homological projective duality. In *Algebraic geometry: Salt Lake City 2015*, volume 97 of *Proc. Sympos. Pure Math.*, pages 585–609. Amer. Math. Soc., Providence, RI, 2018. 1

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