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The Morita Equivalence and (Graded) Brauer Groups

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Contents

1 Introduction

The main focus of this thesis is introducing the reader to the Morita equivalence and (graded) Brauer groups. The content of Chapter 3, which covers the Morita equivalence, is primarily based on [Sch04]. We supplement this work by addressing and filling in any gaps that may have been left out. We start by defining small, projective, and generating modules. Following this, we introduce the concept of Morita equivalence between two rings, which occurs when their categories of right modules are equivalent. This equivalence gives us a new way to characterize rings as being essentially the same, other than them being isomorphic.

Chapter 4 introduces the Brauer group of a field k, and is based on [Lam73]. Brauer groups essentially classify central simple algebras (CSAs) over k; see Definition 4.1 for the formal definition of CSA. An example of a CSA is the quaternion algebra, which will be defined formally in Chapter 4.2. One property of the Brauer group is that it can detect if the equation $ax^2 + by^2 = 1$ has a k-point solution for $a, b \in k$ non-zero. Namely, this equation has a solution precisely if the quaternion algebra associated to a and b is trivial in the Brauer group of k; see Theorem 2.7 of Chapter 3 in [Lam73]. More applications of Brauer groups are found in algebraic K-theory, topological K-theory, and for finding solutions of more general equations. However, these applications are beyond the scope of this thesis and will not be discussed here.

We end this chapter by showing that the Morita equivalence and Brauer equivalence are almost equivalent. Two Brauer equivalent CSAs are Morita equivalent, but two Morita equivalent CSAs are not immediately Brauer equivalent in general. We need to define a new k-algebra structure on one of the CSAs and only then they are Brauer equivalent. This result can be seen as a special case of the more general result found in [Ant16].

The final chapter of this thesis addresses the notion of graded Brauer groups. The graded Brauer group classifies central simple Z_2-graded algebras (CSGAs) over k, which differ slightly from CSAs. The formal definition is given in Definition 5.11. We conclude Chapter 5 by a short section discussing the possible relation between the Morita equivalence and graded Brauer equivalence.

2 Preliminaries

In this section, we briefly review some fundamentals of algebra and category theory. The material for the algebra part will primarily be taken from Prof. Dr. P. Stevenhagen's "*Algebra 1, 2, and 3*" notes, while the material for the category theory part will mainly draw from Dr. R.S. de Jong's "*Topics in Algebraic Topology*" notes. This is not an exhaustive list of definitions, theorems, and lemmas, but it should cover most of the essential concepts. The reader is expected to be familiar with most of this material.

2.1 Preliminaries

Definition 2.1. Let R be a ring. A *left R-module* is an abelian group M provided a map $R \times M \to M$, where $(r, m) \mapsto rm$, such that for all $r, s \in R$ and $m, n \in M$:

$$\begin{aligned} r(m+n) &= rm + rn, \\ (r+s)m &= rm + sm, \\ (rs)m &= r(sm), \\ 1m &= m. \end{aligned}$$

Definition 2.2. A *homomorphism of left R-modules M* and N is a group homomorphism $f\colon M \to N$, such that for all $r \in R$ we have $f(rm) = rf(m)$.

Remark 2.3. Similarly, one can define *right R-modules* and *homomorphisms of right R-modules*.

Definition 2.4. An *R-S-bimodule M* is a left R-module and a right S-module, such that for all $r \in R$, $s \in S$ and $m \in M$ we have $r(ms) = (rm)s$.

Definition 2.5. An R-module M is called *simple* if it is not 0 and has no proper submodules.

Definition 2.6. An R-module F is called *free* if there exists an index set I, such that $\bigoplus_{i \in I} R \simeq F$ as modules.

Definition 2.7. A *category* $\mathcal{C}$ consists of the following.

- A collection of objects.
- For every pair of objects A and B, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ consisting of morphisms (or maps) from A to B.
- For all objects $A, B, C \in \mathcal{C}$ a map $\circ\colon \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ called composition.
- For every object A of $\mathcal{C}$ an element $1_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ call the identity element.

Furthermore, the composition is associative and for all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ we have $1_B \circ f = f = f \circ 1_A$.

Example 2.8. The category of sets (often denoted by **Set**) together with maps as we learn in high school. The category of groups (often denoted by **Grp**) and group homomorphisms as you learn in Algebra 1, but also abelian groups (**Ab**) and group homomorphisms. Rings (**Rng**) together with ring homomorphisms. Vector spaces (**Vec**) with linear maps. Topological spaces (**Top**) or pointed topological spaces ($\mathbf{Top}_*$) together with continuous maps. There are plenty of more categories, but we shall be focusing primarily on the category of right R-modules, denoted by Mod-R, where R is a ring. We denote the category of left R-modules by R-Mod.

Definition 2.9. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *(covariant) functor* $F\colon \mathcal{C} \to \mathcal{D}$ consists of the following.

- For every object A in $\mathcal{C}$ an object $F(A)$ in $\mathcal{D}$.

- For all objects A and B of $\mathcal{C}$ a map $F(-)\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(A),F(B))$.

Furthermore, for every object $A \in \mathcal{C}$ we have that $F(1_A) = 1_{F(A)}$, and for all $A,B,C \in \mathcal{C}$, $f \in \mathrm{Hom}_{\mathcal{C}}(B,C)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ we have $F(f \circ g) = F(f) \circ F(g)$.

Remark 2.10. Similarly, we define a *contravariant functor*, but we change the last property to: for all A,B,C in $\mathcal{C}$ a map $F(-)\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(F(B),F(A))$, such that $F(1_A) = 1_{F(A)}$ and for all $f \in \mathrm{Hom}_{\mathcal{C}}(B,C)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ we have $F(f \circ g) = F(g) \circ F(f)$.

Example 2.11. For those who are new to category theory, without realising, you are already familiar with plenty of functors. For example, taking the fundamental group of a pointed topological space $\pi_1\colon \mathbf{Top}_* \to \mathbf{Grp}$, or abelianizing a group $(-)_{\mathrm{ab}}\colon \mathbf{Grp} \to \mathbf{Ab}$, where $G \mapsto G/[G,G]$. Taking the dual of a vector space is a contravariant functor $(-)^*\colon \mathbf{Vec} \to \mathbf{Vec}$. However, there is one functor I will introduce briefly, which plays a big part in this thesis. For an object X of a category $\mathcal{C}$ we can define the well known $\mathrm{Hom}_{\mathcal{C}}(X,-)\colon \mathcal{C} \to \mathbf{Set}$, by mapping an object A of $\mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(X,A)$ and mapping a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ to the morphism $f \circ -\colon \mathrm{Hom}_{\mathbf{Set}}(X,A) \to \mathrm{Hom}_{\mathbf{Set}}(X,B)$, where $f \circ -(\varphi) = f \circ \varphi$. The reader should verify that this is indeed a functor.

Definition 2.12. Let $F,G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *morphism of functors*, sometimes called a *natural transformation*, written as $\alpha\colon F \to G$, consists of the following. For all objects $A,B \in \mathcal{C}$ an element $\alpha_A \in \mathrm{Hom}_{\mathcal{D}}(F(A),G(A))$ such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ the following diagram in $\mathcal{D}$ commutes.

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

If each α_A is an isomorphism, then we call α an isomorphism of functors.

A *morphism between contravariant functors* is defined analogously, only we need the following diagram to commute instead.

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ \uparrow{\scriptstyle F(f)} & & \uparrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

Theorem 2.13 ***(Yoneda's Lemma).*** *Let $F\colon \mathcal{C} \to \mathbf{Set}$ be a functor, and let A and B objects of $\mathcal{C}$. If there exist isomorphisms of functors $F \xrightarrow{\sim} Hom_{\mathcal{C}}(A,-)$ and $F \xrightarrow{\sim} Hom_{\mathcal{C}}(B,-)$, then A is isomorphic to B as objects of $\mathcal{C}$.*

Definition 2.14. We say that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *equivalence of categories*, if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ and isomorphisms of functors $\alpha\colon GF \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$ and $\beta\colon FG \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$.

Remark 2.15. From now on, all categories will be assumed to be Abelian categories. We will not go into the exact definition, as we will only be working with the category of right and left modules, both of which are examples of Abelian categories. An Abelian category is essentially a category that has a zero object, kernels, cokernels, and some additional properties. Furthermore, it is a fact that equivalence of Abelian categories commute with (possibly infinite) direct sums.

Definition 2.16. Let R and S be rings. A functor $F\colon \text{Mod-}R \to \text{Mod-}S$ is *left exact* if it preserves the exactness of every exact sequence $0 \to A \to B \to C$ in Mod-R. The notion of a *right exact* functor is defined similarly. We say that F is *exact* if it is both left and right exact.

Lemma 2.17. *Let R and S be rings and P an R-S-bimodule. If $N \in \text{Mod-}S$, then $\text{Hom}_S(P, N)$ is a right R-module, by $[fr](p) := f(rp)$. In addition, the functor $\text{Hom}_S(P, -)\colon \text{Mod-}S \to \text{Mod-}R$ is left exact.*

Lemma 2.18. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories, then the following statements hold.*

1. *If $0_\mathcal{C}$ and $0_\mathcal{D}$ are the zero objects of $\mathcal{C}$, respectively $\mathcal{D}$, then $F(0_\mathcal{C}) = 0_\mathcal{D}$.*
2. *If $0_\mathcal{C}$ and $0_\mathcal{D}$ are the zero maps of $\mathcal{C}$, respectively $\mathcal{D}$, then $F(0_\mathcal{C}) = 0_\mathcal{D}$.*
3. *If $f \in \text{Hom}_\mathcal{C}(A, B)$ is surjective, then so is $F(f) \in \text{Hom}_\mathcal{D}(FA, FB)$.*

Definition 2.19. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that F is *fully faithful*, if for all $A, B \in \mathcal{C}$ we have $F\colon \text{Hom}_\mathcal{C}(A, B) \xrightarrow{\sim} \text{Hom}_\mathcal{D}(FA, FB)$. Furthermore, F is said to be *essentially surjective*, if for all $N \in \mathcal{D}$ there exists $M \in \mathcal{C}$ such that $FM \simeq N$.

Theorem 2.20. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

Proof. Assume that $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories with quasi-inverse $G\colon \mathcal{D} \to \mathcal{C}$ and isomorphisms of functors $\alpha\colon GF \xrightarrow{\sim} \text{id}_\mathcal{C}$ and $\beta\colon FG \xrightarrow{\sim} \text{id}_\mathcal{D}$. Let $Y \in \mathcal{D}$, then $G(Y) \in \mathcal{C}$ such that $FG(Y) \simeq \text{id}_\mathcal{C}(Y) = Y$. Hence F is essentially surjective. Let $A, B \in \mathcal{C}$, then let $f, f' \in \text{Hom}_\mathcal{C}(A, B)$ such that $F(f) = F(f')$ in $\text{Hom}_\mathcal{D}(FA, FB)$. Then $GF(f) = GF(f')$, hence by β we have that $f = f'$. Thus $\text{Hom}_\mathcal{C}(A, B)$ is embedded into $\text{Hom}_\mathcal{D}(FA, FB)$. To show that it is fully faithful we let $g \in \text{Hom}_\mathcal{D}(FA, FB)$ and show that we can find an original f in $\text{Hom}_\mathcal{C}(A, B)$. Let $f = G(g)$, then $F(f) = FG(g) \simeq \text{id}_\mathcal{D}(g) = g$. Hence F is fully faithful. Likewise, the same can be shown for G.

The following implication will be based on Proposition XI.1.5 of [Kas95]. Assume that $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective. We must construct the quasi-inverse $G\colon \mathcal{D} \to \mathcal{C}$ and then find the two desired natural isomorphisms. For each object Y of $\mathcal{D}$, fix $X \in \mathcal{C}$, such that $FX \simeq Y$, and denote this isomorphism by α_Y. Note that we can do this because F is essentially surjective. So now we can define $GY := X$, where X is the fixed object for Y as above. This is well-defined, since if we have that $FX_1 = FX_2 = Y$, then for any object X in $\mathcal{C}$ we have $\text{Hom}_\mathcal{C}(X, X_1) \simeq \text{Hom}_\mathcal{D}(FX, FX_1) \simeq \text{Hom}_\mathcal{D}(FX, FX_2) \simeq \text{Hom}_\mathcal{C}(X, X_2)$, so by Yoneda's lemma we have that $X_1 \simeq X_2$. Also, for two objects X_1 and X_2 in $\mathcal{C}$, we shall write $F_{X_1X_2}\colon \mathrm{Hom}_{\mathcal{C}}(X_1,X_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(FX_1,FX_2)$, for the isomorphism in fully faithfulness. Now define for $\varphi \in \mathrm{Hom}_{\mathcal{D}}(Y_1,Y_2)$, $G(\varphi) := F^{-1}_{X_1X_2}(\alpha^{-1}_{Y_2} \circ \varphi \circ \alpha_{Y_1})$. Now for our natural transformation, $\alpha\colon FG \to \mathrm{id}_{\mathcal{D}}$, for an element $Y \in \mathcal{D}$ we have $FX \simeq Y$ fixed, so now use the isomorphism $\alpha_Y\colon FG(Y) \xrightarrow{\sim} Y$ from earlier. Now to show that the desired diagram commutes, take any $\varphi \in \mathrm{Hom}_{\mathcal{D}}(Y_1,Y_2)$, then we need $\alpha_{Y_2}FG(\varphi) = \varphi\alpha_{Y_1}$. We can rewrite the equality to $FG(\varphi) = \alpha^{-1}_{Y_2}\varphi\alpha_{Y_1}$. Now clearly the equality holds by writing out $FG(\varphi) = F_{X_1X_2}F^{-1}_{X_1X_2}(\alpha^{-1}_{Y_2}\varphi\alpha_{Y_1}) = \alpha^{-1}_{Y_2}\varphi\alpha_{Y_1}$, and simply note that each α_Y is an isomorphism. To construct a natural transformation $\beta\colon GF \to \mathrm{id}_{\mathcal{C}}$, we can find a map $F(\beta_X)\colon FG(F(X)) \to F(X)$, by letting the $F(\beta_X) := \alpha^{-1}_X(\mathrm{id}_{F(X)})$, which we know exists and is unique. Likewise, we can find a map $\beta^{-1}_X\colon X \to GF(X)$ that is the inverse of β_X, to show that it is an isomorphism. As before, one can check themselves, that the desired diagram commutes. □

3 Morita Equivalence

As mentioned in the introduction, the content of this chapter supplements [Sch04], by filling in the gaps that have been left out. Unless specified, a module will always be a right module. The primary focus will be the Morita Equivalence Theorem (3.7), a powerful theorem that tacks together category theory and ring theory. This theorem indicates when two rings are Morita equivalent, based on the existence of some small, projective, generating bimodules over these rings. Proving this theorem takes some work, but the results are rewarding. For example, we use it to show that two rings are Morita equivalent if and only if their opposite rings are Morita equivalent. This is not evident and shows how useful this really is. Additionally, we will extend the concept of Morita equivalence to the category of left R-modules.

3.1 Morita Equivalence

Definition 3.1. Let R be a ring. We say that an R-module M is *small* if the $\mathrm{Hom}_R(M,-)$ functor preserves (possibly infinite) sums.

Notice that R is small as an R-module over itself. Since for any family of R-modules $\{M_i\}_{i\in I}$, we can consider $\bigoplus_{i\in I} M_i$ as an R-module by taking the action component wise, which gives us the following module isomorphisms.

$$\mathrm{Hom}_R(R, \bigoplus_{i\in I} M_i) \simeq \bigoplus_{i\in I} M_i \simeq \bigoplus_{i\in I} \mathrm{Hom}_R(R, M_i).$$

Definition 3.2. An R-module P is called *projective* if it is a summand of a free module.

Theorem 3.3. *An R-module P is a projective if and only if $\mathrm{Hom}_R(P,-)$ is exact.*

Definition 3.4. We say that an R-module M *generates* Mod-R, if for every module $N \in$ Mod-R there exists an index set I and a surjective module morphism

$$f\colon \bigoplus_{i\in I} M \to N.$$

Remark 3.5. For any R-module M, we can find a surjection $f\colon \bigoplus_{i\in I} R \to M$ by just letting I be indexed by the cardinality of M. Hence R is a generator of Mod-R. Furthermore, we shall call a projective generating R-module an *R-progenerator* or just progenerator if the context is clear.

Definition 3.6 (Morita equivalence). We say that two rings R and S are *Morita equivalent* if there exists an equivalence of categories $F\colon$ Mod-$R \to$ Mod-S.

The following theorem is the main focus for this section, as mentioned in the introduction of this chapter.

Theorem 3.7 ***(Morita Equivalence Theorem).*** *Let R and S be two rings, then following are equivalent.*

1. *R and S are Morita equivalent.*
2. *There exists a projective S-module P such that it is small, it generates Mod-S and $End_S(P) \simeq R$.*
3. *There exists an R-S-bimodule P, such that $- \otimes_R P\colon$ Mod-$R \to$ Mod-S is an equivalence of categories.*

Example 3.8. A classic non-trivial example of two Morita equivalent rings is if R is a ring and we let $S := M_n(R)$ the ring of $n \times n$ matrices with coefficients in R, where $n > 0$. You can check this by taking $P := R^n$ and then applying the second part of theorem 3.7. Notice that the centers are isomorphic by

$$\psi\colon Z(R) \to Z(S) = \{\lambda I_n : \lambda \in Z(R)\},$$

where we map $r \mapsto rI_n$.

Let us prove Theorem 3.7. We begin by observing that 3 is merely a special case of 1. Hence, it suffices to demonstrate that the first statement implies the second, and the second statement implies the third.

Proof. (1. $\implies$ 2.) Assume R and S are Morita equivalent rings and denote the equivalence of categories by $F\colon$ Mod-$R \to$ Mod-S. We seek an S-module P that is projective, small, generates Mod-S and whose endomorphism ring is isomorphic to $\mathrm{End}_S(P) \simeq R$. To find this module, remember that R is projective because it is free, small, and generates Mod-R. Furthermore, notice that we have a ring isomorphism $\mathrm{End}_R(R) \simeq R$. The proof of this is straightforward, the map $r \mapsto (s \mapsto rs)$ is clearly an injective ring homomorphism. If $f \in \mathrm{End}_R(R)$, then $f(1) \mapsto (s \mapsto f(1)s = f(s))$ shows surjectivity. This prompts one to define $P := FR \in$ Mod-S. We must now check all properties.

We show that P is projective by showing that $\mathrm{Hom}_S(P, -)$ is exact. Left exactness is given by Lemma 2.17, so all we need to show is that for any surjective map $f\colon N \to N'$ of S-modules, the induced map $\mathrm{Hom}_S(P, N) \to \mathrm{Hom}_S(P, N')$ is also surjective. F is essentially surjective, so we can find R-modules M and M', such that $FM \simeq N$ and $FM' \simeq N'$. Then $\mathrm{Hom}_S(P, N) \simeq \mathrm{Hom}_S(P, FM)$ and $\mathrm{Hom}_S(P, N') \simeq \mathrm{Hom}_S(P, FM')$. Since F is fully faithful, we have the following isomorphisms:

$$\mathrm{Hom}_S(P, FM) = \mathrm{Hom}_S(FR, FM) \simeq \mathrm{Hom}_R(R, M) \simeq M,$$

where the last isomorphism is a fact for general R-modules. Similarly, we have an isomorphism $\mathrm{Hom}_S(P, FM') \simeq M'$. Now Lemma 2.18(3) gives us that the induced map $M \to M'$ is a surjection and therefore $\mathrm{Hom}_S(P, N) \to \mathrm{Hom}_S(P, N')$ is also a surjection.

To show that P is small, we let $\{N_i\}_{i \in I}$ be a family of S-modules and to construct an isomorphism

$$\mathrm{Hom}_S(P, \bigoplus_{i \in I} N_i) \to \bigoplus_{i \in I} \mathrm{Hom}_S(P, N_i).$$

Since F is essentially surjective, we can find for each $i \in I$, an R-module M_i such that $FM_i \simeq N_i$. Furthermore, F commutes with direct sums because it is an equivalence of categories. This gives us the following isomorphisms.

$$\mathrm{Hom}_S(P, \bigoplus_{i \in I} N_i) \simeq \mathrm{Hom}_S(FR, \bigoplus_{i \in I} FM_i) \simeq \mathrm{Hom}_S(FR, F \bigoplus_{i \in I} M_i).$$

We use the fully faithfulness of F to find that

$$\mathrm{Hom}_S(FR, F(\bigoplus_{i \in I} M_i)) \simeq \mathrm{Hom}_R(R, \bigoplus_{i \in I} M_i) \simeq \bigoplus_{i \in I} M_i.$$

Similarly, we find that

$$\bigoplus_{i \in I} \mathrm{Hom}_S(P, N_i) \simeq \bigoplus_{i \in I} \mathrm{Hom}_S(FR, FM_i) \simeq \bigoplus_{i \in I} \mathrm{Hom}_R(R, M_i) \simeq \bigoplus_{i \in I} M_i.$$

We conclude that P is small.

To show that P generates Mod-S, let N be an S-module. Let M be an R-module such that $FM \simeq N$. R generates Mod-R, therefore there exists a surjection $\bigoplus_{i \in I} R \twoheadrightarrow M$, for some index set I. Given that F is an equivalence of categories, it preserves surjections. Additionally, since F commutes with direct sum, we obtain a surjection

$$\bigoplus_{i \in I} P = \bigoplus_{i \in I} FR \simeq F(\bigoplus_{i \in I} R) \twoheadrightarrow FM \simeq N.$$

We conclude that P generates Mod-S.

To show that $\mathrm{End}_S(FR) \simeq R$ as rings, simply note that $\varphi \mapsto F(\varphi)$ is a well-defined ring homomorphism from $\mathrm{End}_R(R)$ to $\mathrm{End}_S(FR)$, and is bijective as F is an equivalence of categories. This gives the second isomorphism in the sequence:

$$R \xrightarrow{\sim} \mathrm{End}_R(R) \xrightarrow{\sim} \mathrm{End}_S(FR).$$

We conclude that $P = FR$ was indeed the S-module we were looking for.

(2. $\implies$ 3.) Assume that $P \in$ Mod-S is a small progenerator as in the second statement, where the isomorphism is given by $\psi \colon R \to \mathrm{End}_S(P)$. P can be viewed as an R-S-bimodule by defining $R \times P \to P$, where $(r, x) \mapsto rx := \psi(r)x$. We will now show that $- \otimes_R P \colon \text{Mod-}R \to \text{Mod-}S$ is an equivalence of categories by showing that $\mathrm{Hom}_S(P, -) \colon \text{Mod-}S \to \text{Mod-}R$ is an inverse functor.

The first natural transformation is given by $\alpha\colon \mathrm{id}_{\text{Mod-}R} \to \mathrm{Hom}_S(P, -\otimes_R P)$, where for any R-module X we define the R-linear map $\alpha_X\colon X \to \mathrm{Hom}_S(P, X\otimes_R P)$, by $x \mapsto (y \mapsto x\otimes y)$. One can verify that this transformation is natural. To show that α_X is bijective, we will construct a commutative diagram and then apply the well known Snake Lemma (Proposition 2.10 [AM16]).

We first show that α_X is bijective for free modules. So let $X \simeq \bigoplus_{i\in I} R$, for some index set I and remember that $-\otimes_R P$ is additive, therefore

$$\mathrm{Hom}_S(P, \bigoplus_{i\in I} R\otimes_R P) \simeq \mathrm{Hom}_S(P, \bigoplus_{i\in I}(R\otimes_R P)) \simeq \mathrm{Hom}_S(P, \bigoplus_{i\in I} P).$$

By assumption P is small and $\mathrm{End}_S(P) \simeq R$, thus we indeed find that α_X is a bijection by

$$\mathrm{Hom}_S(P, \bigoplus_{i\in I} P) \simeq \bigoplus_{i\in I} \mathrm{Hom}_S(P,P) \simeq \bigoplus_{i\in I} R = X.$$

Now for the general case, let X be an R-module. First we are going to construct the exact sequence

$$\bigoplus_{i\in I} R \to \bigoplus_{j\in J} R \to X \to 0,$$

where I and J are some index sets. R generates Mod-R, so we can find index sets I and J and surjections $f\colon \bigoplus_{j\in J} R \to X$ and $g\colon \bigoplus_{i\in I} R \to \ker(f)$. Define the embedding $\iota\colon \ker(f) \hookrightarrow \bigoplus_{i\in I} R$, then $\mathrm{im}(\iota\circ g) = \ker(f)$. So $\iota\circ g$ and f give us the desired exact sequence. By exactness, we have that

$$\mathrm{coker}(\iota\circ g) = \frac{\bigoplus_{j\in J} R}{\mathrm{im}(\iota\circ g)} = \frac{\bigoplus_{j\in J} R}{\ker(f)} \simeq X.$$

Now remember that P is projective, so $\mathrm{Hom}_S(P,-)$ is exact and we know that tensoring is right exact. Hence, applying the functor $\mathrm{Hom}_S(P, -\otimes_R P)$ on the above sequence will preserve it's exactness. For ease of notation we shall write $\overline{f}$ for $\mathrm{Hom}_S(P, f\otimes_R P)$, and $\overline{\iota\circ g}$ for $\mathrm{Hom}_S(P, (\iota\circ g)\otimes_R P)$, then we obtain the following exact sequence.

$$\mathrm{Hom}_S(P, \bigoplus_{i\in I} R\otimes_R P) \xrightarrow{\overline{\iota\circ g}} \mathrm{Hom}_S(P, \bigoplus_{j\in J} R\otimes_R P) \xrightarrow{\overline{f}} \mathrm{Hom}_S(P, X\otimes_R P) \longrightarrow 0$$

Remember that α_X is bijective for free modules, so this sequence simplifies to

$$\bigoplus_{i\in I} R \xrightarrow{\overline{\iota\circ g}} \bigoplus_{j\in J} R \xrightarrow{\overline{f}} \mathrm{Hom}_S(P, X\otimes_R P) \longrightarrow 0.$$

By exactness, $\ker(\overline{f}) = \mathrm{im}(\overline{\iota\circ g})$ and using this we can compute:

$$\mathrm{coker}(\overline{\iota\circ g}) = \frac{\mathrm{Hom}_S(P, \bigoplus_{j\in J} R\otimes_R P)}{\mathrm{im}(\overline{\iota\circ g})} \simeq \mathrm{Hom}_S(P, X\otimes_R P).$$

Using the above results, we obtain the following commutative diagram.

$$\begin{array}{ccc} \bigoplus_{i\in I} R & \xrightarrow{\sim} & \mathrm{Hom}_S(P, \bigoplus_{i\in I} R \otimes_R P) \\ {\scriptstyle \iota\circ g}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\iota\circ g}} \\ \bigoplus_{j\in J} R & \xrightarrow{\sim} & \mathrm{Hom}_S(P, \bigoplus_{j\in J} R \otimes_R P) \end{array}$$

Above we found that $\mathrm{coker}(\iota \circ g) \simeq X$ and $\mathrm{coker}(\overline{\iota \circ g}) \simeq \mathrm{Hom}_S(P, X \otimes_R P)$, so now we can apply Snake Lemma to obtain the exact sequence

$$0 \to X \to \mathrm{Hom}_S(P, X \otimes_R P) \to 0.$$

We conclude that $X \simeq \mathrm{Hom}_S(P, X \otimes_R P)$ and thus α_X is bijective for all $X \in \text{Mod-}R$.

Now consider the natural transformation $\beta\colon \mathrm{Hom}_S(P, -) \otimes_R P \to \mathrm{id}_{\text{Mod-}S}$, where $\beta_Y\colon \mathrm{Hom}_S(P, Y) \otimes_R P \to Y$ is defined as $(\phi, y) \mapsto \phi(y)$. Again, the naturality is clear. To show bijectivity we construct a very similar exact sequence as before, compute the desired cokernels and then apply Snake Lemma on the proper diagram. Let $Y \in \text{Mod-}S$ and remember that P generates Mod-S by assumption. Hence, there exist index sets I and J and surjections $f\colon \bigoplus_{j\in J} P \twoheadrightarrow Y$ and $g\colon \bigoplus_{i\in I} P \twoheadrightarrow \ker(f)$. Via the in the embedding $\iota\colon \ker(f) \hookrightarrow \bigoplus_{j\in J} P$ we obtain the desired exact sequence:

$$\bigoplus_{i\in I} P \xrightarrow{\iota\circ g} \bigoplus_{j\in J} P \xrightarrow{f} Y \longrightarrow 0.^{1}$$

By the exactness of the above sequence we have that

$$\mathrm{coker}(\iota \circ g) = \frac{\bigoplus_{j\in J} P}{\mathrm{im}(\iota \circ g)} = \frac{\bigoplus_{j\in J} P}{\ker(f)} \simeq Y.$$

Now remember that P is projective, so $\mathrm{Hom}_S(P, -)$ is exact and we know that tensoring is right exact, hence by applying $\mathrm{Hom}_S(P, -) \otimes_R P$ we obtain the exact sequence

$$\mathrm{Hom}_S(P, \bigoplus_{i\in I} P) \otimes_R P \xrightarrow{\overline{\iota\circ g}} \mathrm{Hom}_S(P, \bigoplus_{j\in J} P) \otimes_R P \xrightarrow{\overline{f}} \mathrm{Hom}_S(P, Y) \otimes_R P \to 0.$$

Again, we would like to apply Snake Lemma. Therefore, we first find that

$$\begin{aligned} \mathrm{coker}(\overline{\iota \circ g}) &= \frac{\mathrm{Hom}_S(P, \bigoplus_{j\in J} P) \otimes_R P}{\mathrm{im}(\overline{\iota \circ g})} \\ &\simeq \frac{\mathrm{Hom}_S(P, \bigoplus_{j\in J} P) \otimes_R P}{\ker(\overline{f})} \\ &\simeq \mathrm{Hom}_S(P, Y) \otimes_R P. \end{aligned}$$

[1]These index sets I and J, and morphisms f, ι and g, are not necessarily the same as before, but we reuse them to indicate the same process as before, and for ease of notation.

By assumption, P is small and $\operatorname{End}_S(P) \simeq R$, thus we have that

$$\operatorname{Hom}_S(P, \bigoplus_{j\in J} P) \otimes_R P \simeq \bigoplus_{j\in J} \operatorname{Hom}_S(P,P) \otimes_R P \simeq \bigoplus_{j\in J} R \otimes_R P \simeq \bigoplus_{j\in J} P.$$

This gives us the following commutative diagram.

$$\begin{array}{ccc} \bigoplus_{i\in I} P & \xrightarrow{\sim} & \operatorname{Hom}_S(P, \bigoplus_{i\in I} P) \otimes_R P \\ \downarrow{\scriptstyle \iota\circ g} & & \downarrow{\scriptstyle \overline{\iota\circ g}} \\ \bigoplus_{j\in J} P & \xrightarrow{\sim} & \operatorname{Hom}_S(P, \bigoplus_{j\in J} P) \otimes_R P \end{array}$$

By Snake Lemma, we find that $\operatorname{Hom}_S(P,Y) \otimes_R P \simeq Y$ for all $Y \in \text{Mod-}S$ and thus β_Y is bijective for all $Y \in \text{Mod-}S$. Hence, there exists indeed an R-S-bimodule P, such that $- \otimes_R P \colon \text{Mod-}R \to \text{Mod-}S$ is an equivalence of categories.

As mentioned before, the third statement is a special case of the first statement. Hence, this concludes the proof. ☐

Remark 3.9. Here are some observations.

1. The Morita equivalence is symmetric in R and S, therefore if an R-S-bimodule P establishes an equivalence as in the second statement, then the inverse equivalence is also established by an S-R-bimodule Q.
2. If R is Morita equivalent to S and R' is Morita equivalent to S', then $R \otimes_{\mathbb{Z}} R'$ is Morita equivalent to $S \otimes_{\mathbb{Z}} S'$.
3. The center of a ring $Z(R)$ is Morita invariant, meaning that if R and S are Morita equivalent, then $Z(R)$ is isomorphic to $Z(S)$ as rings. Example 3.8 is a clear illustration of this. In particular, if R and S are both commutative and Morita equivalent, then they are isomorphic as rings. Therefore, the Morita equivalence is only interesting for non-commutative rings.

Proof. Remark 3.9(3). Assume that R and S are Morita equivalent. Consider the identity functor

$$I \colon \text{Mod-}R \to \text{Mod-}R,$$

and let $C = \operatorname{End}(I_{\text{Mod-}R})$ be the set of all natural transformations from I to itself. These natural transformations in C can be added component wise, and composed. The one and the zero elements are straightforward and it is a small exercise to check that this addition and multiplication provide a ring structure on C. Now we want to show that $Z(R) \simeq C$ as rings. We construct the following map $\lambda \colon Z(R) \to C$, where for any $r \in Z(R)$ and $M \in \text{Mod-}R$ we define $\lambda(r)_M \colon M \to M$ as the right multiplication by r. Now for any module morphism $f \colon M \to N$, we clearly have $f \circ \lambda_M(r) = \lambda_N(r) \circ f$, therefore $\lambda(r) \in C$ is a natural transformation. Since the elements in $Z(R)$ commute will all elements in R, λ is a well-defined ring homomorphism, which is clearly injective. For surjectivity, let $c \in C$ be a natural transformation. Then $c_R \colon R \to R$ commutes with all endomorphisms of R, hence c_R is given by left multiplication by a fixed $r \in Z(R)$. Now for an arbitrary $M \in \text{Mod-}R$, any module morphism from R to M is given by $f_m\colon R \to M$, where $f_m(x) = mx$ for some $m \in M$. For all $m \in M$ the following diagram commutes by naturality of c.

$$\begin{array}{ccc} R & \xrightarrow{f_m} & M \\ {\scriptstyle c_R}\downarrow & & \downarrow{\scriptstyle c_M} \\ R & \xrightarrow{f_m} & M \end{array}$$

Hence, this gives us that $c_M(m) = c_M \circ f_m(1) = f_m(c_R(1)) = f_m(r) = mr$. Hence, the natural transformation c is equal to $\lambda(r)$, and therefore $Z(R) \simeq C$. We conclude that $Z(R) \simeq \text{End}(I_{\text{Mod-}R}) \simeq \text{End}(I_{\text{Mod-}S}) \simeq Z(S)$.

□

We conclude this section with the following observation, which will be useful in the next section.

Lemma 3.10. *Let R and S be Morita equivalent, with P respectively Q the R-S-bimodule and S-R-bimodule as in the second statement in Theorem 3.7. Then we have module isomorphisms $P \otimes_S Q \simeq R$ and $Q \otimes_R P \simeq S$.*

Proof. By Theorem 3.7(3) and Remark 3.9(1) we have two equivalence of categories $- \otimes_R P\colon \text{Mod-}R \xrightarrow{\sim} \text{Mod-}S$ and $- \otimes_R Q\colon \text{Mod-}S \xrightarrow{\sim} \text{Mod-}R$. Hence, they must be inverses of each other. Remember that in the proof of $(2 \implies 3)$ from Morita's theorem, it is shown that $\text{Hom}_S(P, -)$ is the inverse functor of $- \otimes_R P$, which tells us that $\text{Hom}_S(P, -)$ and $- \otimes_R Q$ are isomorphic as functors. Which means that we have the following isomorphisms of modules

$$P \otimes_S Q \simeq \text{Hom}_S(P, P) \simeq R,$$

where the last isomorphism comes from the second statement of Morita's theorem. Similarly one can show that $Q \otimes_R P \simeq S$. □

3.2 Morita Equivalence via Left Modules

The previous theory is all done for the category of right modules, but it can in fact also be done for left modules. However, keep in mind that Morita equivalence refers specifically to right Morita equivalence.

Definition 3.11. We say that two rings R and S are *left Morita equivalent*, if there exists an equivalence of categories $F\colon R\text{-Mod} \to S\text{-Mod}$.

Likewise we obtain a very similar theorem as the Morita theorem above for right modules, however the proof will be omitted.

Theorem 3.12 ***(Left Morita Equivalence Theorem).*** *Let R and S be two rings, then following are equivalent.*

1. *R and S are left Morita equivalent.*

2. *There exists a projective S-module P such that it is small, it generates S-Mod and $End_S(P) \simeq R^{op}$ as rings.*
3. *There exists an S-R-bimodule Q, such that $Q \otimes_R -\colon$ Mod-$R \to$ Mod-S is an equivalence of categories.*

Morita theory works essentially the same for left modules as it does for right modules. Naturally, Remark 3.9 applies for left Morita as well. Moreover, the following lemma gives a connection between right and left Morita.

Lemma 3.13. *Two rings R and S are (right) Morita equivalent if and only if they are left Morita equivalent.*

Proof. Assume that R and S are Morita equivalent by the R-S-bimodule P and S-R-bimodule Q as in the second statement of Theorem 3.7. It would be most natural to show that $P \otimes_S -\colon S\text{-Mod} \to R\text{-Mod}$ and $Q \otimes_R -\colon R\text{-Mod} \to S\text{-Mod}$ are inverse equivalences of each other. Indeed, by Lemma 3.10 we have that

$$Q \otimes_R P \otimes_S - \xrightarrow{\sim} S \otimes_S - \xrightarrow{\sim} \mathrm{id}_{S\text{-Mod}}.$$

Likewise it also gives us that

$$P \otimes_S Q \otimes_R - \xrightarrow{\sim} R \otimes_R - \xrightarrow{\sim} \mathrm{id}_{R\text{-Mod}},$$

hence proving that R-Mod and S-Mod are equivalent as categories. A similar argument can be used to show that if two rings are left Morita equivalent, then they are also Morita equivalent. However, we will now be showing that. □

Corollary 3.14. *R is Morita equivalent to S if and only if R^{op} is Morita equivalent to S^{op}.*

Proof. Suppose that R and S are Morita equivalent, then Lemma 3.13 gives us an equivalence of categories R-Mod $\to$ S-Mod. But notice that in general, the categories R-Mod and Mod-R^{op} are equivalent. Hence, R^{op} and S^{op} are Morita equivalent. For the converse, simply note that the opposite of the opposite ring is isomorphic to the original ring. □

Remark 3.15. In general, R is not Morita equivalent to R^{op}. In Remark 4.24, we shall give an example where this is the case, by using the main result of the upcoming chapter.

4 The Brauer Group of a Field

From now on, k is a field and all k-algebras will be finite dimensional. This chapter starts by defining a central simple algebra (CSA), as given in Definition 4.1. We will provide an example of a CSA and briefly examine their properties before diving into quaternion algebras over a field of characteristic not two. Quaternion algebras serve as another example of a CSA. This will help our understanding of the Brauer group of a given field, as we now know more CSAs over that field.

We then give a construction of the Brauer group of a field, for which we will be following [Lam73]. This is done by defining the Brauer equivalence, an equivalence relation on CSAs that will give us a way to classify them. We then consider the set of equivalence classes of the CSAs over k and proceed by showing that the tensor product over k is a well-defined operator that makes this set a group. Following this, we shall show that the Brauer groups of finite fields and algebraically closed fields are trivial, and without proof give the Brauer group of $\mathbb{R}$. As mentioned before, we end this chapter by showing that the Morita equivalence and Brauer equivalence are almost equivalent. This will be our main theorem of this chapter.

4.1 Central Simple Algebras

Definition 4.1. Let k be a field and A be an algebra over k.

1. A is *central* over k if the center of A is isomorphic to $Z(A) \simeq k$.
2. A is called *simple* if it has no proper (two-sided) ideals.
3. A is said to be a *central simple algebra* (*CSA*) over k, if A is central over k and simple.

Remark 4.2. Henceforth, all ideals will be two-sided, unless stated otherwise.

Example 4.3. Clearly, k is a central simple k-algebra, and in fact, any central division k-algebra is CSA. To give a non trivial example, let V be a k-vector space of finite dimension n. Then the k-algebra $\mathrm{End}(V) \simeq M_n(k)$ is CSA over the field k by Theorem 4.4 and Lemma 4.6. Later, in Theorem 4.11, we will see that all CSAs are matrix algebras over division algebras.

Theorem 4.4. *The matrix algebra $M_n(k)$ is simple.*

We will prove this theorem using the following proposition.

Proposition 4.5. *For any ideal $J \subset M_n(R)$, there exists an ideal $I \subset R$, such that $J = M_n(I)$.*

Proof. Let E_{ij} denote the $n \times n$ matrix with all zeroes, except for the entry e_{ij}, which is equal to 1. Now let $A \in M_n(R)$, then the multiplication from the right AE_{ij} would simply be taking the i-th column of A as the resulting column j. So we have the matrix with zeroes everywhere, but for the j-th column, which is equal to the i-th column of A. Whilst the multiplication from the left $E_{ij}A$ would simply be taking the j-th row of A as the resulting column i. So we have the matrix with zeroes everywhere, but for the i-th row, which is equal to the j-th row of A. It now follows that $E_{ij}AE_{kl} = a_{jk}E_{il}$, where a_{jk} is the element of A in row j and column k.

Now let $J \subset M_n(R)$ be an ideal. Define $I := \{a_{11} \in A \mid A \in J\}$ as the set of all top left elements of the matrices in J. We shall show that I is the desired ideal. First we argue that I is an ideal. Let $A \in J$, then $E_{1j}AE_{k1} = a_{jk}E_{11} \in J$, for all $1 \leq j, k \leq n$. So in particular, we have that $a_{11}E_{11} \in J$, and if $r \in R$ then since J is an ideal it follows that $rI_n \cdot a_{11}E_{11} = ra_{11}E_{11} \in J$ and $a_{11}E_{11} \cdot rI_n = a_{11}rE_{11} \in J$. Hence, if $x \in I$, then rx and xr are also in I, and clearly I is closed under addition. So I is indeed an ideal. To show that $J = M_n(I)$, remember that $E_{1j}AE_{k1} = a_{jk}E_{11} \in J$, so $a_{jk} \in I$ by definition of I, for all $1 \leq j,k \leq n$. Since we can write $A = \sum_{j,k} a_{jk}E_{jk}$, we have $A \in M_n(I)$ and therefore $J \subset M_n(I)$.

Now if $B \in M_n(I)$, then we can write $B = \sum_{i,j} b_{ij}E_{ij}$, so it is enough to show that each $b_{ij}E_{ij} \in J$. By definition of I, we have that for each $b_{ij} \in I$ there exists $A \in J$ such that $a_{11} = b_{ij}$, namely let $A = a_{11}E_{11}$. Therefore, we conclude that $E_{i1}AE_{1j} = a_{11}E_{ij} = b_{ij}E_{ij}$ is contained in J, for each $1 \leq i,j \leq n$. Hence, $J \supset M_n(I)$ and so we have shown that $J = M_n(I)$. □

Proof. Theorem 4.4. All fields are simple, hence all ideals of $M_n(k)$ are trivial by Proposition 4.5. □

Lemma 4.6. *The k-algebra $M_n(k)$ is central.*

Proof. Let R be as above, then it is not hard to see that $k \cdot I_n$ is contained in $Z(R)$. Let $A \in Z(R)$, then using the E_{ij} matrix as above, we have that $AE_{ij} = E_{ij}A$, for all $1 \leq i,j \leq n$. Therefore, A is diagonal as we can see by picking $i \neq j$. Furthermore, $a_{ki} = a_{jk}$ for all $1 \leq k \leq n$, hence the elements on the diagonal of A are the same by picking $i = j$. Hence $A = aI_n$, where $a \in k$, so $A \in k \cdot I_n$ and therefore $Z(R) \simeq k$. □

In the next section we will give more examples of central simple k-algebras. But first we will dive deeper in some properties of k-algebras, which will be useful later on.

Theorem 4.7. *Let D be a division algebra and $R = M_n(D)$, then D^n is the only simple R-module up to isomorphism.*

Proof. First notice that D^n is a right R-module by $(v, A) \mapsto A^\top v$. To show that D^n is simple, take any non zero element $x = (x_1, \dots, x_n) \in D^n$ and assume that $x_i \neq 0$. Let $M_i := x_i^{-1}E_{ji} \in R$, where E_{ji} is the matrix from before. Then we have $M_i^\top x = x_i^{-1}E_{ij}x = e_j \in \langle x \rangle$ for all $1 \leq j \leq n$. So any element of D^n can found in $\langle x \rangle$. Hence, we have that $D = \langle x \rangle$ is cyclic and thus simple.

For uniqueness, we provide a proof for $n = 1$ and refer to Theorem 3.3(2) of [Lam01] for the general case. So now $R = D$ is a simple R-module. If V is another simple R-module, then any non-zero element $v \in V$ generates V, as otherwise $\langle v \rangle \subset V$ is a non trivial submodule. Therefore, there exists a surjective D-module homomorphism

$$f\colon D \twoheadrightarrow V,$$

which maps 1 to v. Clearly, f is not trivial and D is simple, thus $\ker(f) \simeq 0$ and therefore $D \simeq V$. □

The following lemma will be useful in proving Theorem 4.9 and Theorem 4.12.

Lemma 4.8. *Let A and B be k-algebras and $z \in A \otimes_k B$ a non-zero element. Then there exists a minimal n, such that $z = \sum_{i=1}^n a_i \otimes b_i$. Furthermore, if n is minimal, then the a_i are linearly independent, and the b_i are linearly independent.*

Proof. The fact that there exists a minimal n is clear. For the second part of the statement, assume n is minimal and that the a_i are linearly dependent. Then there exist elements $e_1, \dots, e_{n-1}$ of k, such that $a_n = \sum_{i=1}^{n-1} e_i a_i$. Now we find that

$$\begin{aligned}\sum_{i=1}^{n} a_i \otimes b_i &= \sum_{i=1}^{n-1} a_i \otimes b_i + a_n \otimes b_n \\ &= \sum_{i=1}^{n-1} a_i \otimes b_i + \sum_{i=1}^{n-1} e_i a_i \otimes b_n \\ &= \sum_{i=1}^{n-1} a_i \otimes b_i + \sum_{i=1}^{n-1} a_i \otimes e_i b_n \\ &= \sum_{i=1}^{n-1} a_i \otimes (b_i + e_i b_n).\end{aligned}$$

However, we assumed n was minimal. Thus, we have a contradiction, hence the a_i are indeed linearly independent. The same argument shows that the b_i are linearly independent. □

Theorem 4.9. *Let A and B be k-algebras, then we have $Z(A \otimes_k B) = Z(A) \otimes_k Z(B)$.*

Proof. Let A and B be k-algebras and $\sum_{i=1}^{n} a_i \otimes b_i \in Z(A \otimes_k B)$, such that n is minimal. By Lemma 4.8, the a_i are linearly independent and so are the b_i. Elements of the center commute with all elements, hence in particular, we have that

$$(\sum_{i=1}^{n} a_i \otimes b_i)(a \otimes 1) = \sum_{i=1}^{n} a_i a \otimes b_i = \sum_{i=1}^{n} a a_i \otimes b_i.$$

The a_i are linearly independent, thus we must have that $a_i a = a a_i$ for all $1 \leq i \leq n$. Since a was chosen arbitrarily, we conclude that each a_i lies in $Z(A)$. Similarly, one can show that each b_i lies in $Z(B)$, hence $Z(A \otimes_k B) \subset Z(A) \otimes_k Z(B)$. Clearly, the inclusion $Z(A \otimes_k B) \supset Z(A) \otimes_k Z(B)$ holds, so this concludes the proof. □

Theorem 4.10. *Let A be a k-algebra, then $A \otimes_k M_n(k) \simeq M_n(A)$, for some $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Consider the map $\psi\colon A \otimes_k M_n(k) \to M_n(A)$, where $a \otimes M \mapsto aM$, and then we extend this map linearly so that it is k-linear. We know that tensoring two k-algebras is again a k-algebra by defining the multiplication $(a \otimes b)(a' \otimes b') := (aa' \otimes bb')$. So ψ is a k-algebra morphism by the fact that

$$\begin{aligned}\psi((a \otimes M)(a' \otimes M')) &= \psi((aa' \otimes MM')) \\ &= aa'MM' \\ &= (aM)(a'M') \\ &= \psi(a \otimes M)\psi(a' \otimes M').\end{aligned}$$

To show injectivity assume that the image of $a \otimes M + a' \otimes M' \in A \otimes M_n(k)$ is zero. Note that we can rewrite $a \otimes M = a \otimes \sum_{kl} m_{kl} E_{kl} = \sum_{kl} a \otimes m_{kl} E_{kl} = \sum_{kl} am_{kl} \otimes E_{kl}$. Likewise, write $a' \otimes M' = \sum_{kl} a' m'_{kl} \otimes E_{kl}$, so then

$$a \otimes M + a' \otimes M' = \sum_{kl} am_{kl} \otimes E_{kl} + \sum_{kl} a' m'_{kl} \otimes E_{kl} = \sum_{kl} am_{kl} + a' m'_{kl} \otimes E_{kl}$$

Since this element is assumed to be mapped to zero, we must have $am_{kl} + a'm'_{kl} = 0$ for all $1 \leq k, l \leq n$, because E_{kl} is not the zero matrix. Thus, $a \otimes M + a' \otimes M'$ must be zero. We have shown it for the sum of two tensors, it follows that any element $\sum_i a_i \otimes M_i \in A \otimes k$ that gets mapped to zero, is equal to $\sum_i a_i \otimes M_i = 0$. Furthermore, by multiplicativity of ψ, we conclude that only the zero element gets mapped to zero, and thus ψ is injective.

For surjectivity, let $M \in M_n(A)$, then again we can rewrite $M = \sum_{ij} m_{ij} E_{ij}$, where $m_{ij} \in A$. Observe that $\sum_{ij} m_{ij} \otimes E_{ij} \in A \otimes M_n(k)$ is the original of M. □

Theorem 4.11. *[Artin–Wedderburn] Let A be a simple k-algebra. Then there exists a division k-algebra D, and an integer $n \geq 0$, such that $A \simeq M_n(D)$ as k-algebras.*

Theorem 4.12. *Let A be a CSA over k and B a simple k-algebra, then $A \otimes_k B$ is a simple k-algebra. In particular, if A and B are both CSAs, then so is $A \otimes_k B$.*

Proof. The proof presented here is an adaptation of Theorem 2.3 of Chapter 4 in [Lam73], which is the graded case of this statement. The fact that if A and B are CSAs, then $A \otimes_k B$ is central follows by Theorem 4.9. Now let A be a CSA and B a simple k-algebra. Let $I \subset A \otimes_k B$ be a non-zero ideal. Our goal is to show that $1 \in I$. Let $z = \sum_{i=1}^r a_i \otimes b_i \in I$ be non-zero, assume that r is minimal and that z is an element in I, such that for any other non-zero element $\sum_{i=1}^s \alpha_i \otimes \beta_i \in I$, we have $r \leq s$. In this proof we will refer to this last part as the *global minimality* of r. By Lemma 4.8, the a_i are linearly independent, and the b_i are linearly independent. Because r is minimal, the a_i and b_i are all non-zero. Hence, simplicity of A gives us that the ideal (a_1) is equal to A. Therefore, there exist $c_j, d_j \in A$, such that $\sum_j c_j a_1 d_j = 1$. Thus, we find that

$$\begin{aligned} z' &:= \sum_j (c_j \otimes 1) z (d_j \otimes 1) \\ &= \sum_j \sum_{i=1}^r c_j a_i d_j \otimes b_i \\ &= \sum_j c_j a_1 d_j \otimes b_1 + \sum_j \sum_{i=2}^r c_j a_i d_j \otimes b_i \\ &= 1 \otimes b_1 + \sum_j \sum_{i=2}^r c_j a_i d_j \otimes b_i \in I. \end{aligned}$$

Notice that this element z', is non-zero by the linear independence of the b_i. Now we do the same for b_1, namely, the ideal $(b_1) = B$ by simplicity of B. So there exist

$e_l, f_l \in B$ such that $\sum_l e_l b_1 f_l = 1$. Let $S := \sum_j \sum_{i=2}^r c_j a_i d_j \otimes b_i$, then we find that

$$z'' := \sum_l (1 \otimes e_l) z' (1 \otimes f_l) = 1 \otimes \sum_l e_l b_1 f_l + \sum_l e_l S f_l = 1 \otimes 1 + \sum_l e_l S f_l \in I.$$

Rewriting $\sum_l e_l S f_l = \sum_{i=2}^r a'_i \otimes b'_i$, where a'_i are linearly independent and b'_i are also linearly independent, and picking any $a \in A$ non-zero, gives us that the element $(a \otimes 1)z'' - z''(a \otimes 1) = \sum_{i=2}^r aa'_i \otimes b'_i - \sum_{i=2}^r a'_i a \otimes b'_i$ is contained in I. By global minimality of r, we must have that this element is zero, hence $a_i a'_i - a'_i a_i = 0$ for all $2 \leq i \leq r$. Hence, each a'_i lies in $Z(A) \simeq k$, thus we must have $r = 1$, as we assumed a'_i to be linearly independent. Therefore, $z'' = 1 \otimes 1$ is contained in I, hence $I = A \otimes_k B$ and thus $A \otimes_k B$ is simple. □

Remark 4.13. Note that in general, the centrality of either A or B is necessary for the first statement of Theorem 4.12. For example, take the simple $\mathbb{R}$-algebra $\mathbb{C}$, then $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not simple. If it were simple, it would be a field as it is a commutative algebra. But $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not even a domain as $(1 \otimes i + i \otimes 1)(1 \otimes i - i \otimes 1) = 0$.

4.2 Quaternion Algebra

In this section we shall construct the quaternion algebra over a field k of characteristic not two. This will give us another nice example of a CSA.

Definition 4.14. Let k be a field of characteristic not two, and $a, b \in k$ non zero, then we define the *quaternion algebra* $A = (\frac{a,b}{k})$ to be the k-algebra on two generators i and j, with the defining relations $i^2 := a$, $j^2 := b$ and $ij := -ji$.

Notice that A has k-basis $\{1, i, j, ij\}$, since $(ij)^2 = i(-ij)j = -i^2 j^2 = -ab \in k^*$ and is therefore four dimensional over k. Also, observe that each pair of i, j and ij anticommute. Lastly, observe that the construction of quaternion algebra is symmetric in a and b, that is $(\frac{a,b}{k})$ and $(\frac{b,a}{k})$ are k-algebra isomorphic. In literature, the quaternion algebra $(\frac{-1,-1}{\mathbb{R}})$ is often denoted by $\mathbb{H}$ and is usually referred to as the "real quaternions".

Lemma 4.15. *If K is a field extension of k then $K \otimes_k (\frac{a,b}{k}) \simeq (\frac{a,b}{K})$ as k-algebras.*

The following proposition tells us that all quaternion algebras are CSA.

Proposition 4.16. *Let $a, b, x, y \in k^*$, then the following statements hold.*

1. *We have a k-algebra isomorphism $(\frac{a,b}{k}) \simeq (\frac{ax^2, by^2}{k})$.*
2. *The quaternion $(\frac{a,b}{k})$ is a central algebra.*
3. *The quaternion $(\frac{a,b}{k})$ is a simple algebra.*
4. *The quaternion $(\frac{-1,1}{k})$ is isomorphic to the matrix algebra $M_2(k)$.*

Proof. For the first statement, we give an explicit k-algebra isomorphism. Suppose $A = (\frac{a,b}{k})$ and $A' = (\frac{ax^2, by^2}{k})$ with basis $\{1, i, j, ij\}$ respectively $\{1, i', j', i'j'\}$. Define

$\varphi\colon A' \to A$ by letting $\varphi(i') := xi$, $\varphi(j') := yj$ and $\varphi(i'j') := \varphi(i')\varphi(j')$, and extend this map linearly. Clearly this is now a k-linear map, and a well-defined ring homomorphism due to the following observations:

$$\varphi(i')^2 = (xi)^2 = x^2 i^2 = x^2 a = \varphi(i'^2);$$
$$\varphi(j')^2 = (yj)^2 = y^2 j^2 = y^2 b = \varphi(j'^2);$$
$$\varphi(i'j') = (xi)(yj) = xy(ij) = xy(-ji) = -(yj)(xi) = -\varphi(j')\varphi(i').$$

For all elements $\alpha+\beta i+\gamma j+\delta ij \in A$, the element $\alpha+\beta x^{-1}i+\gamma y^{-1}j+\delta(xy)^{-1}ij \in A'$ is an original. Thus, φ is surjective. Lastly, if $\varphi(\alpha+\beta i+\gamma j+\delta z) = 0$, then clearly all coefficients must be zero, thus φ is injective.

For the fourth statement, we also give an explicit isomorphism. Define the map

$$\psi\colon (\frac{-1,1}{k}) \to M_2(k),$$

by $\psi(i) = i_0 := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $\psi(j) = j_0 := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and $\psi(ij) = \psi(i)\psi(j)$. Extending ψ linearly makes it into a well-defined k-algebra morphism by the fact that $i_0^2 = -I_2$, $j_0^2 = I_2$ and $i_0 j_0 = -j_0 i_0$. The set $\{I_2, i_0, j_0, i_0 j_0\}$ forms a basis for $M_2(k)$ as k-vector space, so ψ is indeed an algebra isomorphism.

Lastly, for two and three, let $\overline{k}$ be an algebraic closure of k. By Lemma 4.15 we have that $\overline{k} \otimes_k (\frac{a,b}{k}) \simeq (\frac{a,b}{\overline{k}})$. Part one of the proposition gives us that $(\frac{a,b}{\overline{k}}) \simeq (\frac{ax^2,by^2}{\overline{k}})$, for all $x, y \in k$ non-zero and because $\overline{k}$ is algebraically closed we can find sufficient x and y such that $ax^2 = -1$ and $by^2 = 1$. By that fact, together with part four of the proposition we deduce that $(\frac{a,b}{\overline{k}}) \simeq M_2(\overline{k})$. Therefore, $(\frac{a,b}{\overline{k}})$ is CSA over $\overline{k}$ by Lemma 4.6 and Theorem 4.4. We leave it as an exercise for the reader to deduce from this that $(\frac{a,b}{k})$ is central and simple over k as a subalgebra of $M_2(\overline{k})$. □

Corollary 4.17. *Let $k = \mathbb{R}$, then there are only two quaternion algebras over $\mathbb{R}$, up to isomorphism. Namely,*

$$(\frac{a,b}{\mathbb{R}}) \simeq \begin{cases} \mathbb{H}, & \text{if } a,b < 0 \\ M_2(\mathbb{R}), & \text{else} \end{cases}$$

4.3 Construction of the Brauer Group

The idea behind forming the Brauer group of a field k, revolves around classifying all CSAs over k using a suitable similarity relation, which we will call Brauer equivalence. Subsequently, a group structure is imposed on the set of similarity classes through tensor product operations.

Definition 4.18 (Brauer equivalence). Let A and A' be CSAs over k, then we say that A is *Brauer equivalent* to A', if there exist finite dimensional vector spaces V and V' over k such that $A \otimes_k \mathrm{End}_k(V) \simeq A' \otimes_k \mathrm{End}_k(V')$ as k-algebras.

We will denote the Brauer equivalence by $\sim_{Br}$ and the class of A by $[A]$. Notice that this relation is clearly reflexive and symmetric. Not to mention, it is also transitive. To show this assume that $A \sim_{Br} B$ and $B \sim_{Br} C$, then there exists k-vector spaces V_i with finite dimension n_i, for each $1 \leq i \leq 4$, such that we have k-algebra isomorphisms $A \otimes_k \mathrm{End}(V_1) \simeq B \otimes_k \mathrm{End}(V_2)$ and $B \otimes_k \mathrm{End}(V_3) \simeq C \otimes_k \mathrm{End}(V_4)$. Rewrite $\mathrm{End}(V_i) \simeq M_{n_i}(k)$, then we have $A \otimes_k M_{n_1 n_3}(k) \simeq C \otimes_k M_{n_2 n_4}(k)$.

We define the product of two elements $[A_1]$ and $[A_2]$ as $[A_1] \cdot [A_2] := [A_1 \otimes_k A_2]$, but often we shall leave out the subscript k and the product symbol to avoid clutter. Because tensor products are associative, this binary operator is also associative. If $A \sim_{Br} A'$ then $A \otimes M_n(k) \simeq A' \otimes M_m(k)$, for some $m, n > 0$. So for another CSA B, we have $(A \otimes B) \otimes M_n(k) \simeq (A' \otimes B) \otimes M_m(k)$. Therefore, $[A \otimes B] = [A' \otimes B]$, which means that the product is independent of its representative. We conclude by Theorem 4.12 that the product is well-defined. We now have a non-empty set with a binary associative operator, or a semigroup, which we shall denote by $\mathrm{Br}(k)$. To see that $\mathrm{Br}(k)$ is a group, we note that the identity element is $[k]$, since in general $A \otimes_k k \simeq A \simeq k \otimes_k A$ and that the inverse of an element is given by $[A]^{-1} = [A^{op}]$, which is well-defined by the following proposition.

Proposition 4.19. *Let A be an k-algebra. If A is CSA, then so is A^{op} and we have an k-algebra isomorphism $A \otimes A^{op} \simeq End(A)$.*

Proof. Clearly, $Z(A^{op}) = Z(A)$, thus if A is central then so is A^{op}. Let $I \subset A^{op}$ be an ideal, then $I' = \{a \in A : a^{op} \in I\}$ is an ideal of A. So if A is simple then I' is trivial and therefore I is trivial as well. This concludes the first statement. Now define $\psi \colon A \otimes A^{op} \to \mathrm{End}(A)$, by $\psi(x \otimes y^{op}) := (\lambda_{x \otimes y^{op}} \colon z \mapsto xzy)$ for each $x, y, z \in A$, where the product is in A. This is a well-defined k-algebra homomorphism, so all we need to show is that it is bijective. For injectivity, notice that ψ is a non trivial map and that by Theorem 4.12 $A \otimes A^{op}$ is simple, so $\ker(\psi) \simeq 0$. Secondly, ψ is surjective by argument of dimension, as we have

$$\dim(A \otimes A^{op}) = \dim(A)\dim(A^{op}) = \dim(A)^2 = \dim(\mathrm{End}(A)).$$

□

All of this tells us that $\mathrm{Br}(k)$ is indeed a well-defined group. Furthermore, $\mathrm{Br}(k)$ is abelian for any field, as tensor products over commutative rings are commutative.

Remark 4.20. Let A be a CSA over k, then Artin–Wedderburn's Theorem tells us that there exists some division k-algebra D and $n > 0$ such that $A \simeq M_n(D)$. By Theorem 4.10, we have $A \simeq M_n(D) \simeq D \otimes M_n(k)$. Therefore, $A \sim_{Br} D$ and thus it is sufficient to determine all central division k-algebras to find $\mathrm{Br}(k)$.

Example 4.21. Here are some examples of Brauer groups:

1. $\mathrm{Br}(k) \simeq 1$, if k is a finite.
2. $\mathrm{Br}(\mathbb{R}) = \{[\mathbb{R}], [\mathbb{H}]\}$. For this, see Frobenius' Theorem (Theorem 3.20) of [FD93].
3. $\mathrm{Br}(k) \simeq 1$, if k is algebraically closed.

Proof. Example 4.21(1). Let k be a finite field. Then any central division algebra over k is finite. Wedderburn's Little Theorem of Chapter 6 in [Aig18] states that all finite division rings are fields. Hence, the only central simple k-algebra is k itself. □

Proof. Example 4.21(3). Assume that k is algebraically closed. Let D be a central division k-algebra. Our goal is to show that $D = k$. Clearly, k is contained in D, so for the other inclusion, let $\alpha \in D$ and let $m > \dim(D)$. Then there exist $c_0, \ldots, c_m \in k$ not all zero, such that

$$c_0 + c_1\alpha + c_2\alpha^2 + \ldots + c_m\alpha^m = 0,$$

as the α^i must be linearly dependent by the dimension of D. Hence, α is algebraic and thus $\alpha \in k$, which means that we have an equality $D = k$. □

4.4 Brauer Equivalence Versus Morita Equivalence

Now we come to the main result of this chapter, where we compare the Brauer equivalence and the Morita equivalence.

Theorem 4.22. *Let A and B be CSAs over a field k. If A is Brauer equivalent to B, then A is Morita equivalent to B.*

Proof. Suppose A and B are Brauer equivalent. Then there exists positive integers n and m such that $A \otimes M_n(k) \simeq B \otimes M_m(k)$. By Theorem 4.10 we have isomorphisms

$$M_n(A) \simeq A \otimes M_n(k) \simeq B \otimes M_m(k) \simeq M_m(B).$$

By Example 3.8 we find that

$$A \sim_m M_n(A) \simeq M_m(B) \sim_m B,$$

hence A is Morita equivalent to B. □

A more general version of the following statement can be found in [Ant16].

Theorem 4.23. *Let A and B be CSAs, and assume that they are Morita equivalent. Then there exists a k-algebra structure[2] on A, such that A and B are Brauer equivalent.*

Proof. Assume A and B are Morita equivalent CSAs. By Artin–Wedderburn there exist division algebras D_1 and D_2 such that $A \simeq M_n(D_1)$ and $B \simeq M_m(D_2)$, where n and m are positive integers. We find that D_1 and D_2 are Morita equivalent by

$$D_1 \sim_m M_n(D_1) \simeq A \sim_m B \simeq M_m(D_2) \sim_m D_2.$$

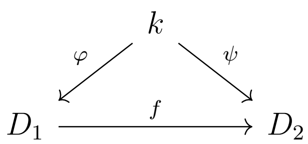
Denote the equivalence of categories by $F\colon \text{Mod-}D_1 \to \text{Mod-}D_2$, with inverse functor $G\colon \text{Mod-}D_2 \to \text{Mod-}D_1$, then it is easy to see that $F(D_1)$ is simple. Indeed let

[2]I thank Prof. Dr. H.W. Lenstra for pointing out an inaccuracy in the statement of Theorem 4.23 in an earlier state of the thesis, regarding the possible change of k-algebra structure on A.

$M \subset F(D_1)$ be a submodule, then $G(M) \subset GF(D_1) \simeq D_1$ is either 0 or D_1 itself, as D_1 is simple. Hence we conclude that $F(D_1)$ is simple as $M \simeq FG(M)$, which is either $F(0) \simeq 0$ or $F(D_1)$. Clearly, D_2 is a simple D_2-module, so Theorem 4.7 gives us that $F(D_1) \simeq D_2$. The first implication in the proof of Theorem 3.7 shows us that $F(D_1)$ is a small progenerator such that $\mathrm{End}_{D_2}(F(D_1)) \simeq D_1$ as rings. This gives us the following ring isomorphism

$$D_1 \simeq \mathrm{End}_{D_2}(F(D_1)) \simeq \mathrm{End}_{D_2}(D_2) \simeq D_2.$$

Denote this ring isomorphism by $f\colon D_1 \to D_2$. Say that the k-algebra structure of D_2 is given by $\psi\colon k \to D_2$. Defining the map $\varphi\colon k \to D_1$ by $\varphi := f^{-1} \circ \psi$ gives us a well-defined k-algebra structure on D_1, such that the following diagram commutes.



Hence, for this algebra structure, D_1 and D_2 are isomorphic as k-algebras. By Theorem 4.10 and the work above, we find that $M_n(D_1) \otimes M_m(k) \simeq M_m(D_2) \otimes M_n(k)$ as k-algebras. In other words, $A \sim_{Br} B$ for the k-algebra structure on A induced by D_1. □

Remark 4.24. The result of Example 4.21(2) tells us that $\mathbb{H} \otimes \mathbb{H} \sim_{Br} \mathbb{R}$, therefore $\mathbb{H} \sim_{Br} \mathbb{H}^{op}$. By Theorem 4.22, we find that $\mathbb{H} \sim_m \mathbb{H}^{op}$.

5 The Graded Brauer Group of a Field

The final chapter of this thesis addresses the notion of graded Brauer groups introduced by C.T.C Wall in [WAL64], in order to classify a higher dimensional version of the quaternion algebras, known as the Clifford algebras. This chapter will have a similar structure to that of Chapter 4, as the construction of graded Brauer groups is almost the same to that of Brauer groups. Due to this similarity, and the margin and time limit for this thesis, most proofs will be omitted, but can be found in Chapter 4 of [Lam73]. In this chapter, "graded" will mean $\mathbb{Z}_2$-graded, but for simplicity we shall just write graded. We will define what it means for a graded algebra to be graded central and graded simple. These graded algebras will be called central simple graded algebras (CSGAs). These central simple graded algebras (CSGAs) will play the same role as CSAs did for the Brauer groups. On the CSGAs, we will define the graded Brauer equivalence. Similar to before, we will construct the graded Brauer group to consist of all CSGAs, up to graded Brauer equivalence.

5.1 Central Simple Graded Algebras

Definition 5.1. A *graded* k-algebra A, is an algebra that is decomposed as the sum of two subspaces $A = A_0 \oplus A_1$, where $A_i A_j \subset A_{i+j}$ with subscript taken modulo two. We shall call elements of $\mathrm{h}(A) := A_0 \cup A_1$ the *homogeneous elements* of A. The *degree* of a homogeneous element a is defined as $\partial a := i$, if $a \in A_i$.

Remark 5.2. The "degree function" is not well-defined for $a = 0$, but in practise, this does not cause any difficulties.

Definition 5.3. A subspace $S \subset A$ is a *graded subspace* if it can be written as a direct sum

$$S = S \cap A_0 \oplus S \cap A_1.$$

Definition 5.4. A *graded subalgebra* $S \subset A$ is a subalgebra that is also a graded subspace.

Example 5.5. Let $V = V_0 \oplus V_1$ be a graded k-vector space. Then we can grade $\mathrm{End}(V)$ by defining $\mathrm{End}(V)_i = \{f \in \mathrm{End}(V) \mid f(V_j) \subset V_{i+j}\}$. Similarly, we can grade the matrix algebra $M_n(k)$, by defining $M_n(k)_0$ and $M_n(k)_1$, as the subspaces generated by all matrices $M \in M_n(k)$ such that the entry m_{ij} is zero if $i + j$ is even and odd, respectively. We will refer to this grading on $M_n(k)$ as the *graded matrix algebra*. If V has basis $e_1, \dots, e_n$, and we define the grading on V by letting $V_0 = \langle e_1, e_3, e_5, \dots \rangle$ and $V_1 = \langle e_2, e_4, e_6, \dots \rangle$. Then the grading on $\mathrm{End}(V)$ is consistent with the grading on $M_n(k)$. These two graded algebras will henceforth be used interchangeably and we denote them by $\hat{\mathrm{End}}(V)$ and $\hat{M}_n(k)$. Another important example of a CSGA is the following. Let k be a field of characteristic not 2, and $A = k \oplus kx$, with defining relations $x^2 = a \in k$ and $\partial x = 1$. Then A can be graded by $A_0 = k$ and $A_1 = kx$. We shall use the notation $A = k\langle\sqrt{a}\rangle$, to indicate this particular grading on A.

Another notable example of a graded algebra is the *Clifford algebra*, mentioned at the beginning of this chapter, which is constructed as follows. Let $\mathrm{char}(k) \neq 2$, and V be an m-dimensional k-vector space and $q\colon V \to k$ a quadratic form on V. Define the *n-th tensor power*, $T^n(V)$, of V as the n-fold tensor product of V:

$$T^n(V) = V \otimes V \otimes \cdots \otimes V \quad \text{(n times)},$$

where $T^0(V) = k$. Now we define the *tensor algebra* generated by V, as

$$T(V) := \bigoplus_{n=0}^{\infty} T^n(V).$$

Let I be the ideal of $T(V)$ that is generated by the elements $v \otimes v - q(v)$ for all $v \in V$. The Clifford algebra $\mathrm{Cl}(V, q)$ is then defined by quotienting the tensor algebra $T(V)$ by the ideal I:

$$\mathrm{Cl}(V, q) := T(V)/I.$$

The product on $\mathrm{Cl}(V, q)$ is induced by the tensor algebra, and for all $x, y \in \mathrm{Cl}(V, q)$, we simply write the product as xy to avoid clutter.

Remark 5.6. Generally, it is not hard to find a generating set for the Clifford algebra by considering all possible combinations of the e_i. Furthermore, the Clifford algebra has dimension 2^m; see Theorem 1.8 of Chapter 5 in [Lam73].

Example 5.7. Suppose that (V, q) is a quadratic space over k, where $V = \langle v \rangle$ and $q(x) = ax^2$, for some $a \in k$. In this instance, one may identify $T(V)$ with the polynomial ring $k[x]$, and then I is simply the ideal $(x^2 - a)$. Hence, we find that $\mathrm{Cl}(V, q) \simeq k[x]/(x^2 - a)$. Now for general V. For those familiar with the exterior algebra $\wedge V$, it can be shown that $\mathrm{Cl}(V, 0)$ is isomorphic to $\wedge V$. Now suppose that $V = \langle e_1, e_2 \rangle$, and let $q(x, y) = ax^2 + by^2$, for some $a, b \in k^*$. Then one can show that $\mathrm{Cl}(V, q)$ is simply the quaternion algebra $(\frac{a,b}{k})$ generated by $1, e_1, e_2, e_1e_2$, as described in section 4.2.

We can grade the Clifford algebra by letting $\mathrm{Cl}(V, q)_0$ be the subalgebra generated by the even tensor products of the e_i and $\mathrm{Cl}(V, q)_1$ be the subspace generated by the odd tensor products of the e_i.

Remark 5.8. If V is m-dimensional and m is odd, then $\mathrm{Cl}(V, q)$ is not central and thus not contained in $\mathrm{Br}(k)$. This statement will be expanded on in Theorem 5.16. This is another reason why we introduce this theory for graded algebras, and leads us to the following definition.

Definition 5.9. For a graded algebra, we define the *graded center* as

$$\hat{Z}(A) := \{a \in \mathrm{h}(A) \mid \forall r \in \mathrm{h}(A) : ra = (-1)^{\partial r \partial a} ar\}.$$

Definition 5.10. A *graded ideal* I of a graded algebra A is an ideal such that it can be written as the direct sum

$$I = I \cap A_0 \oplus I \cap A_1$$

Additionally, I satisfies the property:

$$A_i \cdot I_j \subseteq I_{i+j} \quad \text{for all } i, j \in \mathbb{Z}_2.$$

Definition 5.11. Let A be a graded k-algebra.

1. A is *graded central* if the graded center of A is isomorphic to $\hat{Z}(A) \simeq k$.
2. A is called *graded simple* if it has no proper (two sided) graded ideals.
3. A is said to be a *central simple graded algebra* (*CSGA*), if A is graded central and graded simple.

Example 5.12. The Clifford algebra, and $\hat{\mathrm{End}}(V)$ as in Example 5.5 are CSGA.

5.2 Construction of the Graded Brauer Group

The tensor product of two Clifford algebras is not necessarily a Clifford algebra. Hence, we introduce the definition of the graded tensor product, which preserves the structure of Clifford algebras. This product will serve as a proper product for the graded Brauer group, similar to how the regular tensor product did for $\mathrm{Br}(k)$.

Definition 5.13. Let A and B be graded algebras. We define the *graded tensor product* of A and B to be the graded algebra $A \hat{\otimes}_k B$, where the i-th component is

$$\bigoplus_{j+l=i} A_j \otimes_k B_l, \quad \text{with subscript taken modulo 2.}$$

Furthermore, for all $a, a' \in \mathrm{h}(A)$ and $b, b' \in \mathrm{h}(B)$ the multiplication

$$(a \otimes b)(a' \otimes b') := (-1)^{\partial b \partial a'} aa' \otimes bb'$$

induces a multiplication structure on $A \hat{\otimes}_k B$.

Remark 5.14. The graded tensor product is associative.

Theorem 5.15. *Let A and B be two graded k-algebras. If both of them are graded central, then so is $A \hat{\otimes}_k B$. If A is a CSGA and B is graded simple, then $A \hat{\otimes}_k B$ is graded simple. In particular, if A and B are CSGAs, then $A \hat{\otimes}_k B$ is also a CSGA.*

Proof. The proof is similar to that of Theorem 4.12, and can be found in Theorem 2.3 of Chapter 4 in [Lam73]. □

The following statement is a structure theorem for Clifford algebras, covered in Theorem 2.4 and 2.5 of Chapter 5 in [Lam73]. The proof is omitted due to it's size.

Theorem 5.16. *Let V be an m-dimensional vector space, q a non-singular quadratic form on V, and let k^{*2} denote the set of elements in k^* that have a square root. If we let $\delta := (-1)^{\frac{m(m-1)}{2}} \det(q)$, where $\det(q)$ is the determinant of the matrix associated to q, then the following statements hold.*

1. *If m is odd then $Cl(V, q)_0$ is CSA over k and $Cl(V, q) \simeq Cl(V, q)_0 \hat{\otimes}_k k\langle\sqrt{\delta}\rangle$ [3]. Additionally, if $\delta \notin k^{*2}$, then $Cl(V, q)$ is a CSA over $k(\delta)$. But if $\delta \in k^{*2}$, then $Z(Cl(V, q)) \simeq k \times k$ and $Cl(V, q) \simeq Cl(V, q)_0 \times Cl(V, q)_0$.*
2. *If m is even, then $Cl(V, q)$ is CSA over k. Additionally, if $\delta \notin k^{*2}$, then $Cl(V, q)_0$ is CSA over $k(\delta)$. But if $\delta \in k^{*2}$, then $Z(Cl(V, q)_0) \simeq k \times k$ and if $Cl(V, q) \simeq M_t(D)$, for a central division k-algebra D. Then, t is a power of 2 and $Cl(V, q) \simeq \hat{M}_t(D)$ as graded algebras and $Cl(V, q)_0 \simeq M_{\frac{t}{2}}(D) \times M_{\frac{t}{2}}(D)$.*

Definition 5.17. Let A and B be CSGAs. Then A and B are *graded-Brauer equivalent*, denoted by $A \sim_{\mathrm{BW}} B$, if there exist finite dimensional graded vector spaces V and W, graded as in Example 5.5, such that we have graded k-algebra isomorphisms

$$A \hat{\otimes}_k \hat{\mathrm{End}}(V) \simeq B \hat{\otimes}_k \hat{\mathrm{End}}(W).$$

Remark 5.18. Similarly to $\sim_{Br}$, it is not hard to see that $\sim_{\mathrm{BW}}$ is also an equivalence relation. The equivalence class of a CSGA A, will be denoted by $[\![A]\!]$.

The product of two classes $[\![A_1]\!]$ and $[\![A_2]\!]$ is defined as $[\![A_1]\!] \cdot [\![A_1]\!] := [\![A_1 \hat{\otimes}_k A_2]\!]$. It is not hard to see that this product is independent of its representative. Thus, by Theorem 5.15 and Remark 5.14, this is a well-defined product. We now have a well-defined semigroup, that we will denote by $\mathrm{BW}(k)$. This semigroup $\mathrm{BW}(k)$ is a group by the following. The identity element is given by $[\![k]\!]$, by considering $k = k \oplus 0$ as a graded k-algebra. However, to establish the existence of inverses, we must define the opposite graded algebra.

[3] The definition of the CSGA $k\langle\sqrt{\delta}\rangle$ can be found at the end of Example 5.5.

Definition 5.19. Let A be an k-algebra. We define the *graded opposite algebra* of A to be $A^* := \{a^* : a \in A\}$, with grading $A_0^* := \{a^* : a \in A_0\}$ and $A_1^* := \{a^* : a \in A_1\}$. Where for each $a, b \in \mathrm{h}(A)$, the multiplication is induced by $a^* \dot{b}^* := (-1)^{\partial a \partial b}(ba)^*$.

Proposition 5.20. *Let A be an CSGA. Then, A^* is also a CSGA and we have $A \hat{\otimes} A^* \simeq \hat{End}(A)$ as graded algebras.*

Proof. See the proof of Proposition 4.1 in Chapter 4 of [Lam73]. □

Remark 5.21. Note that A^{op} is not the right candidate for the inverse, as the graded center part of Proposition 5.20 would have failed.

By Proposition 5.20, we conclude that $\mathrm{BW}(k)$ is indeed a group, as the inverse of an element is given by $[\![A]\!]^{-1} = [\![A^*]\!]$. Note that $\mathrm{BW}(k)$ is abelian. This group will be referred to as the graded Brauer group of k. We will consider a group that is tied closely to $\mathrm{Br}(k)$ and $\mathrm{BW}(k)$, called $Q(k)$, where we define $Q(k) := \mathbb{Z}_2 \times k^*/k^{*2}$ set theoretically. The product is defined by $(x, y)(x', y') := (x + x', (-1)^{xx'} yy')$. Then the identity element is $(0, 1)$, and the inverse is given by $(x, y)^{-1} = (x, (-1)^x y)$. This makes $Q(k)$ a well-defined abelian group. We will not go in to the details of where it comes from, due to the margin of this thesis. However, more can be found in Chapter 2 of [Lam73]. The following theorem allows us to consider the Brauer group as a subgroup of the graded Brauer group and shows that they are in fact different.

Theorem 5.22. *For any field k, there exist homomorphisms $i\colon Br(k) \to BW(k)$ and $j\colon BW(k) \to Q(k)$, such that we have an exact sequence:*

$$0 \to Br(k) \xrightarrow{i} BW(k) \xrightarrow{j} Q(k) \to 0.$$

Proof. The exact maps and proof are given in Theorem 4.4 of Chapter 4 in [Lam73]. □

Example 5.23. Theorem 5.22 can be used to find $\mathrm{BW}(k)$. Let k be an algebraically closed field, or a finite field of characteristic 2. Then $\mathrm{Br}(k)$ is trivial, and $k^*/k^{*2} \simeq 1$, therefore we have that $\mathrm{BW}(k) \simeq \mathbb{Z}_2$. If k is a finite field of characteristic not 2, then $\mathrm{BW}(k) \simeq Q(k)$. So then $\mathrm{BW}(k) = \langle (1, 1) \rangle \simeq \mathbb{Z}_4$ if -1 is not a square in k and $\mathrm{BW}(k) \simeq V_4$ if -1 is a square in k. Furthermore, if $k = \mathbb{R}$, then a computation shows that $\mathrm{Br}(k) = \langle [\![\mathbb{R}\langle\sqrt{1}\rangle]\!] \rangle \simeq \mathbb{Z}_8$. One way to compute this is shown in [WAL64].

5.3 Graded Brauer Equivalence Versus Morita Equivalence

One question that likely arises for the reader is whether we can establish a connection between the Morita equivalence and the graded Brauer equivalence, similarly to what we did for the Brauer equivalence in section 4.3. The first step is to ask ourselves what the Morita equivalence would even look like for graded rings. This has already been done before and can be found in [ART23] and in Chapter 2 of [Haz16]. We say that two rings are *graded Morita equivalent* if their categories of graded modules are equivalent. However, to my knowledge, there is no paper addressing any connection between the graded Morita equivalence and the graded Brauer equivalence.

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