

VANISHING FOR HODGE IDEALS ON TORIC VARIETIES

YAJNASENI DUTTA

ABSTRACT. In this article we construct a Koszul-type resolution of the p^{th} exterior power of the sheaf of holomorphic differential forms on smooth toric varieties and use this to prove a Nadel-type vanishing theorem for Hodge ideals associated to effective \mathbb{Q} -divisors on smooth projective toric varieties. This extends earlier results of Mustaa and Popa.

1. INTRODUCTION

In this paper we prove a Nadel-type vanishing statement for Hodge ideals on smooth projective toric varieties. This extends similar vanishing results in [MP16] and [MP18].

In a series of papers, Mustaa and Popa thoroughly studied Hodge ideals with the goal of understanding singularities and Hodge theoretic properties of hypersurfaces in smooth varieties. These ideals are indexed by non-negative integers, with the 0^{th} being a multiplier ideal (see Remark 1.4 below). In particular, a hypersurface D in a smooth variety X has log canonical singularities if and only if $I_0(D) \simeq \mathcal{O}_X$. For the sake of applications it is fundamental for such gadgets to satisfy vanishing theorems, just like multiplier ideals.

To formulate our statement, let X be a smooth projective toric variety of dimension n , with torus invariant divisors D_i , $i = 1, \dots, d$. Let D be an effective \mathbb{Q} -divisor on X such that there is an integer ℓ and a section s of some line bundle $M^{\otimes \ell}$ satisfying $D = \frac{1}{\ell}H$ where $H = (s = 0)$. Then we have the following

Definition 1.1 (Property $P_k(D)$). With the above assumption on D , we say that a line bundle L on a smooth toric variety X , satisfies property $P_k(D)$ if the divisors $L + D_{\text{red}} - D$ and $L + D_{\tau_1} + \dots + D_{\tau_p} + D_{\text{red}} - D$ for all $1 \leq p \leq k$ and for all $\tau_i \in \{1, \dots, d\}$ are ample. Moreover, when D and D_i 's are ample, we also allow $L \sim_{\mathbb{Q}} D - D_{\text{red}}$.

Remark/Notation 1.2. When D is reduced, $D = D_{\text{red}}$ and therefore Property $P_k(D)$ does not depend on D . In this case, we will use the notation P_k to emphasise the lack of dependence on D .

Theorem A. *With the notation as above, for a fixed integer k and a line bundle L on X satisfying property $P_k(D)$ we have*

$$H^i\left(X, \omega_X((k+1)D_{\text{red}}) \otimes L \otimes I_k(D)\right) = 0 \text{ for all } i > 0.$$

In [MP16, MP18] Mustaa and Popa showed such vanishing statements in general, for smooth projective varieties, as well as for projective spaces and for Abelian varieties with much weaker hypotheses (see [MP16, Theorem 25.3] and [MP18, Variant 12.5, Remark 12.4]). However their general statement required additional hypotheses. For instance, on smooth projective toric varieties, they require D to be *reduced $(k-1)$ -log canonical* (i.e. $I_p(D) \simeq \mathcal{O}_X(D_{\text{red}} - \lceil D \rceil)$ for all $p \leq k-1$) and the ample line bundle L to satisfy certain positivity properties with respect to D (see [MP16, Corollary 25.1]). Here we only need to assume certain positivity properties, depending essentially only on the toric data, on the line bundle L . Our statement recovers the statement for projective spaces as we will see in Example 1.

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Remark 1.3. The assumption that $\ell D \sim M^{\otimes \ell}$ is not necessary to define the Hodge ideal, but it is to ensure the gluing of the local \mathcal{D} -modules that are used to define the Hodge ideals for \mathbb{Q} -divisors (see Remark 2.4 for more details). Existence of such global objects is crucial for the purpose of the proof we present. Note that such assumptions can always be realised on a finite flat modification of X and D .

Remark 1.4. We know from [MP16, Proposition 10.1] (or in the \mathbb{Q} -divisor setting from [MP18, Proposition 9.1]) that $I_0(D) \simeq \mathcal{J}((1 - \epsilon)D)$, the multiplier ideal sheaf of $(1 - \epsilon)D$ for $0 < \epsilon \ll 1$. Note that when $k = 0$, Theorem A is the same as Nadel vanishing for this multiplier ideal sheaf.

On a smooth toric variety X of dimension n , we have the short exact sequence (see [CLS11, Theorem 8.1.6])

$$0 \rightarrow \Omega_X^1 \rightarrow \bigoplus_{i=1}^d \mathcal{O}_X(-D_i) \rightarrow \mathcal{O}_X^{\oplus d-n} \rightarrow 0 \quad (1)$$

where the D_i 's are the torus invariant divisors of X . Using this Euler type short exact sequence, we construct a resolution of the k^{th} exterior power of the sheaf of holomorphic differential forms Ω_X^k on X (see Lemma 2.2). Then an inductive argument, similar to that in [MP16, Theorem 25.3] for projective spaces, yields the above vanishing statement. We first look at some examples:

Examples.

- (1) *Projective spaces:* When $X = \mathbb{P}^n$ and $\deg D = d$, our statement recovers the statement in [MP16, Theorem 15.3](Or, [MP18, Variant 12.5]), namely

$$H^i(X, \mathcal{O}_X(\ell) \otimes \mathcal{O}_X(kD_{\text{red}}) \otimes I_k(D)) = 0 \text{ for all } i > 0$$

and $\ell \geq d - n - 1$. Indeed, torus invariant divisors on \mathbb{P}^n satisfy $\mathcal{O}_{\mathbb{P}^n}(D_i) \simeq \mathcal{O}_{\mathbb{P}^n}(1)$ and therefore are ample. In the above statement taking $\mathcal{O}_X(\ell) \simeq \omega_X \otimes L$, we see that when $\ell \geq d - n - 1$, $L - D_{\text{red}}$ satisfies property $P_k(D)$.

- (2) *Products of projective spaces:* When $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, and D is an effective \mathbb{Q} -divisor in the class (c, d) , we have

$$H^i(X, \mathcal{O}_X(a, b) \otimes \mathcal{O}_X(kD_{\text{red}}) \otimes I_k(D)) = 0 \text{ for all } i > 0$$

whenever $a \geq c - n_1$ and $b \geq d - n_2$. Indeed, the torus invariant divisors are all of type $(1, 0)$ or $(0, 1)$ and therefore nef. Then a line bundle $L \simeq \mathcal{O}_X(a', b')$ $L - D_{\text{red}}$ satisfies property $P_k(D)$ if $a' \geq c + 1$ and $b' \geq d + 1$. It is worth mentioning that when D is reduced, we get

$$H^i(X, \mathcal{O}_X((k+1)c - n_1, (k+1)d - n_2) \otimes I_k(D)) = 0 \text{ for all } i > 0.$$

- (3) *Hirzebruch surfaces:* When $X = F_r = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(r))$, the Hirzebruch surface with $r \geq 0$, we have $\text{Pic } X = \mathbb{Z}F \oplus \mathbb{Z}E$, where $E \sim E' + rF$, E' is the class of the section of self-intersection $-r$ and F is the class of a fibre. It is well-known that two of the four torus invariant divisors on X are linearly equivalent to F and the remaining two are equivalent to E and E' respectively. Note that all torus invariant divisors but E' are nef. Therefore, to satisfy property P_k , a line bundle $L \simeq \mathcal{O}_X(aF + bE)$ must satisfy that $a \geq kr$ and $b \geq 1$. Indeed, big and nef (resp. ample) bundles on X are given by $a, b \geq 0$ (resp. > 0) (see [BC13, Theorem 2.4(3)] or [CLS11, Example 6.1.16]) and property P_k above mandates $L + kE'$ to be ample. Then for a reduced curve D in the class (c, d) we have,

$$H^i(X, \mathcal{O}_X((k+1)(c+r) - 2, (k+1)d - 1)) = 0 \text{ for all } i > 0.$$

Applications. As an application we address the classical problem of finding the number of conditions imposed on certain spaces of hypersurfaces by the isolated singular points on a given singular hypersurface.

Corollary B. *Let D be a reduced effective divisor on a smooth projective toric variety X of dimension n . Let $S_m(D)$ denote the set of all isolated singular points in D with multiplicity at least m . Then $S_m(D)$ imposes independent conditions on the hypersurfaces in*

$$H^0\left(X, \mathcal{O}_X\left(\left(\left\lfloor \frac{n}{m} \right\rfloor + 1\right)D - \sum_i D_i\right) \otimes L\right)$$

for any line bundle L satisfying property $P_{\lfloor \frac{n}{m} \rfloor}$.

In particular, when $X = \mathbb{P}^n$ we recover [MP16, Corollary H]. We discuss some examples in §4.

We can extend this analysis to the study of $(j-1)$ -jets along $S_m(D)$, just as was done in [MP16, Corollary 27.3]. Recall that a line bundle M is said to separate $(j-1)$ -jets along a set of points $S \subset X$ if the map

$$H^0(X, M) \twoheadrightarrow \bigoplus_{p \in S} \mathcal{O}_X / \mathfrak{m}_p^j$$

is surjective. In particular, S imposes independent conditions on $H^0(X, M)$ is equivalent to the statement that M separates 0-jets along S . For $m \geq 3$, define

$$k_{m,j} = \begin{cases} \left\lfloor \frac{j+n-m}{m} \right\rfloor & \text{if } j \leq m-1 \\ \left\lfloor \frac{j+n-m}{m-2} \right\rfloor & \text{if } j \geq m \end{cases} \quad (2)$$

Then we obtain the following:

Corollary C. *With the assumptions and notation in Corollary B, for $m \geq 3$ the space of hypersurfaces*

$$H^0\left(\mathcal{O}_X\left((k_{m,j} + 1)D - \sum_i D_i\right) \otimes L\right)$$

separate $(j-1)$ -jets along $S_m(D)$ for any line bundle L satisfying the property $P_{k_{m,j}}$.

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2. PRELIMINARIES

2.1. Eagon-Northcott complexes. We first discuss the general theory behind the construction of a Koszul-type complex that will be used in the proof of Theorem A. The main reference for this section is [Laz04, Appendix B]. The following lemma is surely well-known to experts, nonetheless we include the proof for completeness.

Consider a short exact sequence of vector bundles on a variety X :

$$0 \rightarrow \Omega \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0 \quad (3)$$

such that $\text{rk } \mathcal{E} = e$, $\text{rk } \mathcal{F} = f$ with $e > f$. Hence, $\text{rk } \Omega = e - f$. We denote $\bigwedge^p \Omega =: \Omega^p$. We then have the following resolution of Ω^p in terms of \mathcal{E} and \mathcal{F} (see also [loc. cit. Appendix B.2]):

Lemma 2.1. *The following complex is exact and gives a resolution for Ω^p :*

$$0 \rightarrow \bigwedge^e \mathcal{E} \otimes S^{e-p-f} \mathcal{F}^\vee \otimes \left(\bigwedge^f \mathcal{F}\right)^\vee \rightarrow \cdots \rightarrow \bigwedge^{p+f+j} \mathcal{E} \otimes S^j \mathcal{F}^\vee \otimes \left(\bigwedge^f \mathcal{F}\right)^\vee \rightarrow \cdots \rightarrow \bigwedge^{p+f} \mathcal{E} \otimes \left(\bigwedge^f \mathcal{F}\right)^\vee$$

where $S^j \mathcal{F}^\vee$ denotes the j^{th} symmetric power of \mathcal{F}^\vee .

Proof. Since the morphism $\mathcal{E} \rightarrow \mathcal{F}$ is a map of vector bundles, we obtain by [Laz04, Theorem B.2.2.], that the p^{th} Eagon-Northcott complex (EN_p)

$$\begin{aligned} 0 \rightarrow \bigwedge^e \mathcal{E} \otimes S^{e-p-f} \mathcal{F}^\vee \otimes \left(\bigwedge^f \mathcal{F} \right)^\vee \rightarrow \cdots \rightarrow \bigwedge^{p+f+j} \mathcal{E} \otimes S^j \mathcal{F}^\vee \otimes \left(\bigwedge^f \mathcal{F} \right)^\vee \rightarrow \cdots \rightarrow \bigwedge^{p+f} \mathcal{E} \otimes \left(\bigwedge^f \mathcal{F} \right)^\vee \rightarrow \\ \rightarrow \bigwedge^p \mathcal{E} \xrightarrow{\phi} \bigwedge^{p-1} \mathcal{E} \otimes \mathcal{F} \rightarrow \cdots \rightarrow E \otimes S^{p-1} \mathcal{F} \rightarrow S^p \mathcal{F} \rightarrow 0 \end{aligned}$$

is exact.

To determine the kernel of $\phi: \bigwedge^p \mathcal{E} \rightarrow \bigwedge^{p-1} \mathcal{E} \otimes \mathcal{F}$ above, we need to analyse the construction of (EN_p) . Consider the projective bundle $\mathbb{P}(\mathcal{F}) := \underline{\text{Proj}}(\bigoplus_i S^i \mathcal{F})$ with the map $\pi: \mathbb{P}(\mathcal{F}) \rightarrow X$ and the map of vector bundles

$$\pi^* \mathcal{E}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{F})}.$$

Denote by $K := \text{Ker}(\pi^* \mathcal{E}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{F})})$ the kernel vector bundle. Then

$$\text{Ker}(\phi) = \text{Ker} \left(\pi_* \bigwedge^p \pi^* \mathcal{E} \xrightarrow{\phi} \pi_* \bigwedge^{p-1} \pi^* \mathcal{E}(1) \right) \simeq \pi_* \left(\bigwedge^p K \right)(p).$$

But the last term is isomorphic to Ω^p . Indeed, snake lemma applied to the following exact grid:

$$\begin{array}{ccccccc} & & & & 0 & & 0 \\ & & & & \downarrow & & \downarrow \\ & & & & \pi^* \Omega(-1) & & K \\ & & & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi^* \mathcal{E}(-1) & \xlongequal{\quad} & \pi^* \mathcal{E}(-1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega_{\mathbb{P}(\mathcal{F})/X}^1 & \longrightarrow & \pi^* \mathcal{F}(-1) & \longrightarrow & \mathcal{O}_{\mathbb{P}(\mathcal{F})} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ & & \Omega_{\mathbb{P}(\mathcal{F})/X}^1 & & 0 & & 0 \end{array}$$

gives us the short exact sequence

$$0 \rightarrow \pi^* \Omega \rightarrow K(1) \rightarrow \Omega_{\mathbb{P}(\mathcal{F})/X}(1) \rightarrow 0.$$

(In the above diagram, the horizontal short exact sequence is the relative Euler sequence and the middle vertical sequence is the pullback of (3) twisted by $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-1)$, under a smooth morphism.)

Now for a fixed p we have a filtration

$$\left(\bigwedge^p K \right)(p) = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_{p+1} = 0$$

such that

$$F_k / F_{k+1} = \pi^* \Omega^k \otimes \Omega_{\mathbb{P}(\mathcal{F})/X}^{p-k}(p-k).$$

We assert that

$$\pi_* \Omega_{\mathbb{P}(\mathcal{F})/X}^{p-k}(p-k) = 0 \text{ for all } p-k \geq 0.$$

Granted this, $\pi_* F_{k+1} \simeq \pi_* F_k$ for all $k \leq p$. In particular, $\Omega^p \simeq \pi_* F_p \simeq \pi_* F_0 \simeq \pi_* \left(\bigwedge^p K \right)(p)$.

The assertion follows from the Koszul resolution of $\pi_*\Omega_{\mathbb{P}(\mathcal{F})/X}^k(k)$, namely:

$$\bigwedge^f \pi^* \mathcal{F}(-k+f) \rightarrow \cdots \rightarrow \bigwedge^{k+1} \pi^* \mathcal{F}(-1) \rightarrow \Omega_{\mathbb{P}(\mathcal{F})/X}^k(k) \rightarrow 0$$

and the equality $R^i \pi_* \bigwedge^{k+j} \pi^* \mathcal{F}(-j) = 0$ for all i and for all $1 \leq j \leq f-1$ (see [Har77, Exer. III.8.4.(c)]). \square

We apply the above lemma to smooth toric varieties to obtain a resolution of Ω_X^p .

Lemma 2.2. *Let X be a smooth toric variety with torus invariant divisors D_i , $i = 1, \dots, d$ then, Ω_X^p admits the following Koszul-type resolution:*

$$0 \rightarrow \bigoplus \omega_X \rightarrow \cdots \rightarrow \bigoplus_{\sigma \in S_{n-p-j}} \bigoplus \omega_X(D_{\sigma_1} + \cdots + D_{\sigma_{n-p-j}}) \rightarrow \cdots \rightarrow \bigoplus_{\sigma \in S_{n-p}} \bigoplus \omega_X(D_{\sigma_1} + \cdots + D_{\sigma_{n-p}}) \rightarrow \Omega_X^p \rightarrow 0.$$

Here S_j is the set of all ordered sequences of length j in $\{1, \dots, d\}$.

Proof. Lemma 2.1 applied to the short exact sequence (1) gives the following resolution:

$$0 \rightarrow \bigoplus \bigwedge^d \left(\bigoplus_{i=1}^d \mathcal{O}_X(-D_i) \right) \rightarrow \cdots \rightarrow \bigoplus \bigwedge^{d-n+p+j} \left(\bigoplus_{i=1}^d \mathcal{O}_X(-D_i) \right) \rightarrow \cdots \rightarrow \bigoplus \bigwedge^{d-n+p} \left(\bigoplus_{i=1}^d \mathcal{O}_X(-D_i) \right) \rightarrow \Omega_X^p \rightarrow 0.$$

Since $\omega_X \simeq \mathcal{O}_X(-\sum_{i=1}^d D_i)$ [CLS11, Theorem 8.2.3.], we can rewrite each term in the above long exact sequence as the corresponding one from the long exact sequence in the statement. \square

2.2. Preliminaries on Hodge ideals. Let D be a reduced effective divisor on a smooth variety X . Hodge ideals associated to D arise as a measure of the deficit between the Hodge filtration and the pole-order filtration on the Hodge module $\mathcal{O}_X(*D) := \bigcup_{k \geq 0} \mathcal{O}_X(kD)$. If D is smooth, one has $F_k \mathcal{O}_X(*D) = \mathcal{O}_X((k+1)D)$, in other words $I_k(D) = \mathcal{O}_X$.

Similarly, when D is an effective \mathbb{Q} -divisor and H is an integral divisor such that $D = (1-\beta)H$ with $\beta < 1$, one can define Hodge ideals associated to D . However in this case one needs to consider the rank 1 free $\mathcal{O}_X(*D_{\text{red}})$ -module generated by h^β , where h is a local equation of H . It turns out that these local \mathcal{D} -modules appear as direct summand of certain Hodge modules, namely the pushforward of the sheaf of meromorphic functions with poles along the preimage of D_{red} , under some local cyclic cover along h (see [MP18, Lemma 2.6]). Therefore, one can make sense of a filtration on $\mathcal{O}_X(*D_{\text{red}})h^\beta$ induced from the Hodge filtration of the ambient Hodge module. Moreover, this filtration coincides with the Hodge filtration on $\mathcal{O}_X(*D)$ whenever D is reduced. We refer the reader to [MP18] for details.

Definition 2.3. Let D , H and h be as before. Then for an integer $k \geq 0$, the k^{th} -Hodge ideal associated to D is defined by the following equation:

$$F_k \mathcal{O}_X(*D_{\text{red}})h^\beta = I_k(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(kD_{\text{red}} + H)h^\beta.$$

The filtration F coincides with the Hodge filtration when D is reduced.

Remark 2.4. The local \mathcal{D} -modules $\mathcal{O}_X(*D_{\text{red}})h^\beta$ do not, in general, glue to a global object. However by [MP18, Remark 2.14], $I_k(D)$ is independent of the choice of h and β and hence is defined globally. The additional assumptions on D , in the statement of Theorem A, ensure a global object. In other words, when we have $\beta = \frac{\ell-1}{\ell}$ and $\mathcal{O}_X(H) \simeq M^{\otimes \ell}$ for some integer ℓ and some line bundle M on X , the local \mathcal{D} -modules $\mathcal{O}_X(*D)h^\beta$ glue. Following [MP18, §5], we denote this global \mathcal{D} -module by \mathcal{M}_1 . As an \mathcal{O}_X -module, we have

$$\mathcal{M}_1 \simeq M \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D_{\text{red}}).$$

For the proof of Theorem A that we present, it is crucial that we work with a globally defined \mathcal{D} -module.

The induced Hodge filtration on \mathcal{M}_1 is given by (see, for instance, the proof of [MP18, Theorem 12.1])

$$F_k \mathcal{M}_1 \simeq M(-D_{\text{red}}) \otimes_{\mathcal{O}_X} \mathcal{O}_X((k+1)D_{\text{red}}) \otimes I_k(D)$$

to which one can associate its filtered deRham complex $\text{DR}(\mathcal{M}_1)$.

Definition 2.5 (The de-Rham complex). For any non-negative integer k , the k^{th} filtered piece of the de-Rham complex of \mathcal{M}_1 is defined by

$$F_k \text{DR}(\mathcal{M}_1) = [F_k \mathcal{M}_1 \rightarrow \Omega_X^1 \otimes F_{k+1} \mathcal{M}_1 \rightarrow \cdots \rightarrow \Omega_X^n \otimes F_{k+n} \mathcal{M}_1]$$

and the associated graded complex is given by

$$gr_k^F \text{DR}(\mathcal{M}_1) = [gr_k^F \mathcal{M}_1 \rightarrow \Omega_X^1 \otimes gr_{k+1}^F \mathcal{M}_1 \rightarrow \cdots \rightarrow \Omega_X^n \otimes gr_{k+n}^F \mathcal{M}_1]$$

as complexes in degrees $-n, \dots, 0$.

The higher hypercohomologies of the associated graded complex of Hodge modules satisfy a vanishing statement due to Saito (see [Sai90, §2.g], also see [Sch14], [Pop16]). As a result, one also obtains vanishing statements for the associated graded complex of the filtered direct summand \mathcal{M}_1 . We therefore have:

Lemma 2.6 (Saito vanishing). *Let A be an ample line bundle on X . Then,*

$$\mathbb{H}^i(X, gr_k^F \text{DR}(\mathcal{M}_1) \otimes A) = 0 \text{ for all } i > 0, k \geq 0$$

The following vanishing is a consequence of Artin vanishing (see [MP16, Proposition 22.1] for instance):

Lemma 2.7. *Let X be a smooth projective variety and D an effective divisor on X such that $X \setminus D_{\text{red}}$ is affine (when D is ample for instance), then*

$$\mathbb{H}^i(X, gr_k^F \text{DR}(\mathcal{M}_1)) = 0 \text{ for all } i > 0, k \geq 0.$$

3. PROOF OF THE VANISHING THEOREM

We will prove the following:

Variante 3.1 (Variant of Theorem A). *With the hypotheses of Theorem A, we have for all integers $k \geq 0$ the following equivalent statements hold.*

- (1) $H^i(X, \omega_X((k+1)D_{\text{red}}) \otimes L \otimes I_k(D)) = 0$ for all $i > 0$.
- (2) $H^i(X, \omega_X \otimes gr_k^F \mathcal{M}_1 \otimes A) = 0$ for all $i > 0$ where $A \otimes M(-D_{\text{red}}) = L$.

The proof uses the vanishing Lemmas 2.6 and 2.7, followed by an inductive argument involving a spectral sequence. The base case of the induction is Nadel vanishing for $I_0(D)$, which by Remark 1.4, is the multiplier ideal $\mathcal{J}((1-\epsilon)D)$. The argument is, in spirit, very similar to the proof for projective spaces as in [MP16, Theorem 25.3].

Proof. The filtration on \mathcal{M}_1 is given by $F_k \mathcal{M}_1 \simeq \mathcal{O}_X((k+1)D_{\text{red}}) \otimes M(-D_{\text{red}}) \otimes I_k(D)$ and therefore we have a short exact sequence:

$$0 \rightarrow \omega_X(kD_{\text{red}}) \otimes L \otimes I_{k-1}(D) \rightarrow \omega_X((k+1)D_{\text{red}}) \otimes L \otimes I_k(D) \rightarrow \omega_X \otimes gr_k^F \mathcal{M}_1 \otimes A \rightarrow 0$$

where $A := L(D_{\text{red}} - D)$. Since by hypothesis A is ample and $D \sim_{\mathbb{Q}} M$, Nadel vanishing theorem gives,

$$H^i(X, \omega_X(D_{\text{red}}) \otimes L \otimes I_0(D)) = 0 \text{ for all } i > 0.$$

Therefore, by induction the above statements are equivalent.

We aim to show the second statement. We have,

$$gr_k^F \text{DR}(\mathcal{M}_1)[-n] = [gr_k^F \mathcal{M}_1 \rightarrow \Omega_X^1 \otimes gr_{k-n+1}^F \mathcal{M}_1 \rightarrow \cdots \rightarrow \Omega_X^{n-1} \otimes gr_{-1+k}^F \mathcal{M}_1 \rightarrow \omega_X \otimes gr_k^F \mathcal{M}_1] \quad (4)$$

in degrees 0 to n .

Consider the spectral sequence associated to the complex $C^\bullet := gr_k^F \text{DR}(\mathcal{M}_1)[-n] \otimes A$:

$$E_1^{p,q} := H^q(X, C^p) \implies \mathbb{H}^{p+q}(X, C^\bullet).$$

Now by Saito vanishing (Lemma 2.6), we have $\mathbb{H}^{p+q}(X, C^\bullet) = 0$ when $p+q \geq n+1$. Also note that, because of the length of C^\bullet , the spectral sequence degenerates at level $n+1$, therefore, $E_{n+1}^{p,q} \simeq E_\infty^{p,q}$. The latter, $E_\infty^{p,q}$, a quotient of $\mathbb{H}^{p+q}(X, C^\bullet)$, also vanishes for $p+q \geq n+1$. As the vanishing statement that we aim for, is $E_1^{n,i} = 0$ for all $i > 0$. It is thus enough to show that $E_1^{n,i} \simeq \cdots \simeq E_{n+1}^{n,i}$. In other words, it is enough to show that, $E_r^{n-r, i+r-1} = 0$ for all r with $1 \leq r \leq n$. Indeed, for each $r \geq 1$, the differential going out of $E_r^{n-r, i}$ maps to $E_r^{n-r, i-r+1}$, which is 0 because of the length of C^\bullet . In fact, we will show that,

$$E_1^{n-r, i+r-1} = H^{i+r-1}(X, \Omega_X^{n-r} \otimes A \otimes gr_{k-r}^F \mathcal{M}_1) = 0.$$

By Lemma 2.2, Ω_X^{n-r} has the resolution:

$$\begin{aligned} 0 \rightarrow \bigoplus \omega_X \rightarrow \cdots \rightarrow \bigoplus_{\sigma \in S_{r-j}} \bigoplus \omega_X \otimes \mathcal{O}_X(D_{\sigma_1} + \cdots + D_{\sigma_{r-j}}) \\ \rightarrow \cdots \rightarrow \bigoplus_{\sigma \in S_r} \bigoplus \omega_X \otimes \mathcal{O}_X(D_{\sigma_1} + \cdots + D_{\sigma_r}) \rightarrow \Omega_X^{n-r} \rightarrow 0 \end{aligned}$$

where S_j is the set of all ordered sequences of length j in $\{1, \dots, d\}$. Tensoring this long exact sequence with $gr_{k-r}^F \mathcal{M}_1 \otimes A$, we see that it is enough to show,

$$H^{i+r-1}(X, \omega_X \otimes A(D_{\sigma_1} + \cdots + D_{\sigma_j}) \otimes gr_{k-r}^F \mathcal{M}_1) = 0$$

for all $j \leq r$. But this follows by induction, since $L' := L(D_{\sigma_1} + \cdots + D_{\sigma_j})$ with $j \leq r$ satisfies property $P_{k-r}(D)$ by our hypotheses.

Further if D and the D_i 's are ample, we can take $A \simeq \mathcal{O}_X$. In this case, to begin the induction, we resort to the Artin-type vanishing in Lemma 2.7 and obtain:

$$\mathbb{H}^{p+q}(X, C^\bullet) = 0 \text{ for all } p+q \geq n+1.$$

The latter steps of the induction use Saito vanishing (Lemma 2.6) with $A' := \mathcal{O}_X(D_{\sigma_1} + \cdots + D_{\sigma_j})$, which are automatically ample due to the ampleness assumption on the D_i 's. The remaining argument follows the first part of the proof verbatim. \square

4. APPLICATIONS

In this section the divisor D is reduced and effective.

We now apply Theorem A to deduce Corollaries B and C. In [MP16, Corollary 27.2] similar results for projective spaces were deduced. The key point is to understand how deep the Hodge ideals sit inside the maximal ideal of an isolated singular point. This has already been studied in [Theorem E, Corollary 19.4 *loc. cit.*]. Without proof, we collect their results in the following:

Lemma 4.1. *Let D be a reduced hypersurface on a smooth variety X of dimension n . Let x be an isolated singular point of D with $\text{mult}_x(D) = m$. Let k be an integer such that $\frac{n}{k+1} < m < \frac{n}{k}$, then*

$$I_k(D) \subseteq \mathfrak{m}_x^{(k+1)m-n}.$$

Further if $m \geq \frac{n}{k}$, then

$$I_k(D) \subseteq \mathfrak{m}_x^\ell \quad \text{where } \ell = \max \{m-1, (k+1)(m-2) - n - 2\}$$

Proof of Corollary B. Recall that $S_m(D)$ is the set of all isolated singular points on D with multiplicity at least m . From Lemma 4.1 we know that for $k = \lfloor \frac{n}{m} \rfloor$ and for all $p \in S_m(D)$, $I_k(D)_p \subseteq \mathfrak{m}_p$. Therefore, for a line bundle L satisfying P_k , we have:

$$H^1\left(X, \mathcal{O}_X\left(\left(\left\lfloor \frac{n}{m} \right\rfloor + 1\right)D - \sum_i D_i\right) \otimes L \otimes I_k(D)\right) = 0.$$

This yields the following surjection

$$H^0\left(X, \mathcal{O}_X\left(\left(\left\lfloor \frac{n}{m} \right\rfloor + 1\right)D - \sum_i D_i\right) \otimes L\right) \twoheadrightarrow \bigoplus_{p \in S_m} \mathcal{O}_X/\mathfrak{m}_p$$

thereby proving the statement. \square

Note that the choice of k is optimal for this proof. For example, by [MP16, Theorem D], when p is an ordinary isolated singular point of D and $k \leq \lfloor \frac{n}{m} \rfloor - 1$, locally around p we have

$$I_k(D)_p \simeq \mathcal{O}_{X,p}.$$

The proof of separation of $(j-1)$ -jets follows similarly. For the sake of completeness, we include it here.

Proof of Corollary C. First note that $I_{k_{m,j}}(D) \subseteq \mathfrak{m}_p^j$ for all $p \in S_m(D)$. Indeed, if $m' > m$, then $k_{m,j} = k_{m',j'}$ for $j' \geq j$ and therefore, if $\text{mult}_{p'} D = m' > m$ for some point $p' \in S_m(D)$, then $I_{k_{m,j}}(D) \subseteq \mathfrak{m}_{p'}^{j'} \subseteq \mathfrak{m}_{p'}^j$. Now for a line bundle L satisfying $P_{k_{m,j}}$, we have the vanishing in Theorem A for $I_{k_{m,j}}(D)$ and therefore we obtain a surjection:

$$H^0\left(X, \mathcal{O}_X\left((k_{m,j} + 1)D - \sum_i D_i\right) \otimes L\right) \twoheadrightarrow \bigoplus_{p \in S_m} \mathcal{O}_X/\mathfrak{m}_p^j.$$

Hence, the Corollary. \square

Examples.

- (1) Let $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$ and denote $n := n_1 + n_2$. Let D be a reduced effective divisor of type (c, d) . Since the toric divisors on X are nef, any ample line bundle, and in particular $L = \mathcal{O}_X(1, 1)$, satisfies property P_k for all k . Therefore, $S_m(D)$ imposes independent conditions on the space of hypersurfaces

$$H^0\left(X, \left(\left(\left\lfloor \frac{n}{m} \right\rfloor + 1\right)c - n_1, \left(\left\lfloor \frac{n}{m} \right\rfloor + 1\right)d - n_2\right)\right).$$

In particular, isolated singular points on a surface in the class (c, d) on $X = \mathbb{P}^2 \times \mathbb{P}^1$ impose conditions on $|\mathcal{O}_X(2c - 2, 2d - 1)|$. This can be compared with the Severi-type bound as in [PW06, Main Theorem] for \mathbb{P}^3 , namely isolated singular points on a surface of degree d in \mathbb{P}^3 impose independent conditions on hypersurfaces of degree at least $2d - 5$.

- (2) Let $X = F_r$, the Hirzebruch surface with $r \geq 0$, as in Example 3 from the introduction. Note that the line bundle $L = \mathcal{O}_X((rF + E))$ satisfies P_1 . Moreover, we have $K_X \sim (r - 2)F - 2E$. Then for any reduced effective singular divisor D in the class (c, d) , $S_2(D)$ imposes independent conditions on the space of hypersurfaces

$$H^0(X, \mathcal{O}_X((2(c + r) - 2)F + (2d - 1)E)).$$

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DEPARTMENT OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208-2730, USA

E-mail address: ydutta@math.northwestern.edu

URL: <http://www.math.northwestern.edu/~ydutta/>