# ON THE EFFECTIVE FREENESS OF THE DIRECT IMAGES OF PLURICANONICAL BUNDLES

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ABSTRACT. We give effective bounds on the generation of pushforwards of log-pluricanonical bundles for klt pairs twisted by ample line bundles. This gives a partial answer to a conjecture proposed by Popa and Schnell. We prove two types of statements: first, more in the spirit of the general conjecture, we show generic global generation with predicted bound when the dimension of the variety if less than or equal to 4 and more generally, with a quadratic Angehrn-Siu type bound. Secondly, assuming that the relative canonical bundle is relatively semi-ample, we make a very precise statement. In particular, when the morphism is smooth, it solves the conjecture with the same bounds, for certain pluricanonical bundles.

**1. Introduction.** The main purpose of this paper is to give a partial answer to a version of the Fujita-type conjecture proposed by Popa and Schnell [PS14, Conjecture 1.3], on the global generation of pushforwards of pluricanonical bundles twisted by ample line bundles. All varieties considered below are over the field of complex numbers.

*Notation* 1. We fix
$$N = \begin{cases} n & \text{; when } n \leqslant 4 \\ \binom{n+1}{2} & \text{; otherwise} \end{cases}$$
in what follows. Our results also work if $N$ was taken to be the effective bounds arising from the works of Helmke [Hel97, Hel99].

**Conjecture 1.1** (Popa-Schnell)**.** *Let $f : X \to Y$ be a morphism of smooth projective varieties, with $\dim Y = n$, and let $L$ be an ample line bundle on $Y$. Then, for every $k \geqslant 1$, the sheaf*
$$f_* \omega_X^{\otimes k} \otimes L^{\otimes l}$$
*is globally generated for $l \geqslant k(n+1)$.*

In [PS14], Popa and Schnell proved the conjecture in the case when $L$ is an ample and globally generated line bundle, and observed that it holds in general when $\dim Y = 1$. With the additional assumption that $L$ is globally generated, they could use Kollár and Ambro-Fujino type vanishing along with Castelnuovo-Mumford regularity to conclude global generation. We remove the global generation assumption on $L$, by making a generation statement at general points with quadratic bounds.

**Theorem A.** *Let $f \colon X \to Y$ be a surjective morphism of projective varieties, with $Y$ smooth and $\dim Y = n$. Let $L$ be an ample line bundle on $Y$. Consider a klt pair $(X, \Delta)$ with $\Delta$ a $\mathbb{Q}$-divisor, such that for some integer $k \geqslant 1$, $k(K_X + \Delta)$ is linearly equivalent to a Cartier divisor $P$. Then the sheaf*
$$f_* \mathcal{O}_X(P) \otimes L^{\otimes l}$$
*is generated by global sections at a general point $y \in Y$ for all $l \geqslant k(N+1)$ with $N$ as in Notation 1.*

As a particular case of Theorem A, we have the following corollary, which is a generic version of Conjecture 1.1 with Angehrn-Siu type bound.

**Corollary B.** *Let $f : X \to Y$ be a surjective morphism of smooth projective varieties, with $\dim Y = n$. Let $L$ be an ample line bundle on $Y$. Then for all $k \geqslant 1$, the sheaf*

$$f_* \omega_X^{\otimes k} \otimes L^{\otimes l}$$

*is generated by global sections at a general point $y \in Y$ for all $l$ as in Theorem A.*

According to [PS14, §4], this could be interpreted as an effective version of Viehweg's weak-positivity for $f_* \omega_{X/Y}^{\otimes k}$ [Vie83] (also see [Kol86, Theorem 3.5(i)]).

One can in fact describe the locus on which global generation holds, but not in a very explicit fashion. This suffices however in order to deduce the next Theorem, where assuming semiampleness of the canonical bundle along the smooth fibres, we prove that the global generation holds at the smooth (regular) values of $f$ in $Y$. The relative semiampleness hypothesis was removed by Deng [Den17], later was improved by Iwai [Iwa17] when $\dim Y \geqslant 5$ (see 1.1 below).

**Theorem C.** *Let $f: X \to Y$ be a surjective morphism of smooth projective varieties, with $\dim Y = n$. Suppose $f$ is smooth outside of a closed subvariety $B \subset Y$. Assume that for some $k \geqslant 1$, $\omega_X^{\otimes k}$ is relatively free over $Y \setminus B$, and let $L$ be an ample line bundle on $Y$. Then the sheaf*

$$f_* \omega_X^{\otimes k} \otimes L^{\otimes l}$$

*is generated by global sections at $y$, for all $y \notin B$ for all $l \geqslant k(N+1)$.*

*Remark* 1.2. Note, for instance, that this applies when $f : X \to Y$ is a projective surjective morphism with generalised Calabi-Yau fibres (i.e. $\omega_F = \mathcal{O}_F$ for any smooth fibre $F$ of $f$), or with fibres having nef and big canonical bundle (i.e. they are minimal varieties of general type). Indeed, in the second case there is an integer $s \gg 0$ such that $f^* f_* \omega_X^{\otimes s} \to \omega_X^{\otimes s}$ is surjective (see for instance [Fuj09, Theorem 1.3]).

In particular, if $f$ is smooth, i.e. $B = \emptyset$, Theorem C solves Conjecture 1.1 for the pluri-canonical bundles that are relatively globally generated, in $\dim X \leqslant 4$ and more generally with Angehrn-Siu type bound.

1.1. *A Discussion on Recent Results.* Since the first draft of this manuscript, several papers have significantly improved the results in this paper in dimension bigger than 4. In a joint work with Murayama [DM19], using the *weak positivity* of $f_* \mathcal{O}_Y(k(K_{X/Y} + \Delta))$, the author proved effective global generation at general points with a bound of $l \geqslant k(n+1) + n^2 - n$ for *log-canonical pairs*. In the same paper and also in a work of Iwai [Iwa17], slightly better quadratic bound was shown for klt $\mathbb{Q}$-pairs, improving the results of the current paper in high dimensions. In the situation of Theorem C, Iwai showed this generation at regular values without any assumptions on relative freeness of $\omega_X^{\otimes k}$, improving a similar statement by Deng [Den17]. The algebraic methods in this paper rely on Kawamata's arguments in [Kaw02], which in turn uses the arguments stemming from the work of Bombieri [Bom73], Kawamata [Kaw84] and Shokurov [Sho85], involving the problem of finding suitable singular divisors passing through the point at which one aims to show global generation. This enables us to obtain the bounds similar to the known cases of the Fujita conjecture. On the other hand, because of the cyclic cover techniques we use here, we require that the relative base loci of the pluricanonical bundles behave "nicely" along the fibres. The analytic methods get around this by directly lifting sections of pluricanonical bundles from the fibres of the map.

**1.2. *An Effective Vanishing Theorem*.** The proof of Theorem C leads to an effective vanishing theorem (see Theorem 4.1), in case of smooth morphisms, for the pushforwards of pluricanonical bundles that are relatively free. This is in the flavour of [PS14, Theorem 1.7], but with the global generation assumption on $L$ removed. This vanishing theorem has been improved in [DM19] for $n > 4$.

**1.3. *Proof Strategy*.** The proof of Theorem A is, in part, inspired by arguments in [PS14, Theorem 1.4]. However, since we do not assume that $L$ is globally generated, we need to follow a different path, avoiding Castelnuovo-Mumford regularity. To do this, we need to argue locally around each point and to appeal to the following *local version* of Kawamata's effective freeness result (see [Kaw02, Theorem 1.7]), another main source of inspiration for this paper.

**Proposition 1.3.** *Let $f \colon X \to Y$ be a surjective morphism of smooth projective varieties, with $\dim Y = n$, such that $f$ is smooth outside of a closed subvariety $B$ in $Y$. Let $\Delta$ be a $\mathbb{Q}$-divisor on $Y$ with simple normal crossing support and coefficients in $(0,1)$ and let $H$ be a semiample $\mathbb{Q}$-divisor on $Y$ such that there is a Cartier divisor $P$ satisfying*

$$P - (K_X + \Delta) \sim_{\mathbb{Q}} f^*H.$$

*Fixing a point $y \in Y \setminus B$, assume moreover that each strata of $(X, \mathrm{Supp}(\Delta))$ intersects the fibre above $y$ transversely or not at all. Furthermore, let $A$ be a nef and big line bundle on $Y$ satisfying $A^n > N^n$ and $A^d \cdot V > N^d$ for any irreducible closed subvariety $V \subset Y$ of dimension $d$ that contains $y$ and for $N$ as in Notation 1. Then*

$$f_* \mathcal{O}_X(P) \otimes A$$

*is generated by global sections at $y$.*

*Remark* 1.4.
  (1) When $\Delta = 0$, $H = \mathcal{O}_Y$ and $B$ is a simple normal crossing divisor, the result is known for all $y \in Y$. This is Kawamata's freeness result (see Theorem 2.1 below). Kawamata's proof relies on the existence of an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} \lambda A$ for some $0 < \lambda < 1$, such that the pair $(Y, D)$ has an isolated log canonical singularity at a given point $y \in Y$. Existence of such divisors is known, when $A$ satisfies the intersection properties as in the hypothesis of Proposition 1.3 (see [AS95], [Kol97, Theorem 5.8]). Slightly better bounds are known due to Helmke ([Hel97], [Hel99]). Our methods also work with $N$ replaced by Helmke's bounds.
  (2) The proof proceeds by reducing to the case $\Delta = 0$ and then to the situation in Kawamata's result i.e. when $B$ has simple normal crossing support. We perform the first reduction using an inductive procedure of removing the coefficients of the components of $\Delta$ via Kawamata coverings [Laz04, Theorem 4.1.12]. For details see §2.

**2. Generalisation of Kawamata's Freeness Result.** A key input in the proof of Proposition 1.3 is Kawamata's theorem. We state it here:

**Theorem 2.1** ([Kaw02, Theorem 1.7]). *Let $f: X \to Y$ be a surjective morphism of smooth projective varieties, with $\dim Y = n$, such that $f$ is smooth outside of a simple normal crossing divisor $\Sigma \subset Y$. Furthermore, let $A$ be a nef and big line bundle on $Y$ and fixing a point $y \in Y$ assume that, $A^n > N^n$ and $A^d \cdot V > N^d$ for any irreducible closed subvariety $V \subset Y$ of dimension $d$ that contains $y$ and for $N$ as in Notation 1. Then*

$$f_* \omega_X \otimes A$$

*is generated by global sections at $y$.*

We are now ready to prove a generic version of the above allowing a simple normal crossing klt pair.

*Proof of Proposition 1.3.* Since $H$ is semiample, so is $f^*H$ and therefore by Bertini's theorem (see Remark III.10.9.2 [Har77] and [Jou83]), we can pick a fractional $\mathbb{Q}$ divisor $D \sim_{\mathbb{Q}} f^*H$ with smooth support such that $\Delta + D$ still has simple normal crossings support, $\mathrm{Supp}(D)$ is not contained in the support of the $\Delta$ and intersects the fibre over $y$ transversely or not at all and $\Delta + D$ has coefficient in $(0,1)$. We rename $\Delta + D$ by $\Delta$.

We now proceed inductively by removing the components of $\Delta$.

**Step 1. Kawamata Covering of $\Delta$.** If $\Delta = 0$ we move to Step 2.

Otherwise let $\Delta = \frac{l}{k} D_1 + D_2$ with $l, k \in \mathbb{Z}_{>0}$, $l < k$ and $D_1$ smooth irreducible. We choose a Bloch-Gieseker cover $p: Z \to X$ along $D_1$, so that $p^*D_1 \sim kM$ for some Cartier divisor (possibly non-effective) $M$ on $Z$ and so that the components of $p^*\Delta$ and the fibre $(f \circ p)^{-1}(y)$ are smooth and intersect each other transversely or not at all [Laz04, Lemma 4.1.11]. Moreover since $p$ is flat and $f$ is smooth over a neighbourhood around $y$, we can conclude that there is a open neighbourhood $U$ around $y$ such that $f \circ p$ is still smooth over $U$ [Har77, Ex. III.10.2].

Set $g = f \circ p$ and denote by $B \subset Y$, the branch locus of $g$. Further note that $y \notin B$.

Now, $\omega_X$ is a direct summand of $p_* \omega_Z$ via the trace map. Therefore

$$f_* \mathcal{O}_X(P) \otimes A$$

is a direct summand of

$$g_* \mathcal{O}_Z(K_Z + lM + p^*D_2) \otimes A.$$

Hence it is enough to show that the latter is generated by global sections at $y$.

To do this we take the $k^{\text{th}}$ cyclic cover $q : X_1 \to Z$ of $p^*D_1$. The smoothness of the components of $\mathrm{Supp}(p^*\Delta)$ and of $g^{-1}(y)$, and the intersection properties carry over to $Y_1$, i.e. $(g \circ q)^{-1}(y)$ and $q^*p^*D_i$ are smooth and intersect each other transversely or not at all [Laz04, Remark 4.1.8]. Furthermore, $g \circ q$ is still smooth over $y$, and hence over an open subset $U$ around $y$. In other words $y$ is not in the branch locus (denoted $B$ again) of $g \circ q$. For the ease of notation set $f_1 := g \circ q$. Note that, (see for instance, [EV92, §1])

$$q_* \omega_{X_1} \simeq \bigoplus_{i=0}^{k-1} \omega_Z(p^*D_1 - iM) \simeq \bigoplus_{i=0}^{k-1} \omega_Z((k-i)M).$$

The last isomorphism is due to the fact that $p^*D_1 \sim kM$. Further, since $k > l$, the direct sum on the right hand side contains the term $\omega_Z(lM)$ when $i = k - l$.

Therefore it is enough to show that,

$$f_{1*} \mathcal{O}_{X_1}(K_{Y_1} + q^*p^*D_2) \otimes A$$

is generated by global sections at $y$.

Proceeding inductively this way, it is enough to show that

$$f_{s*}\omega_{X_s} \otimes A$$

is globally generated at $y$, where $f_s : X_s \to X$ is the composition of Kawamata covers along the components of $\Delta$ (here $s$ is the number of components of $\Delta$). We rename $f_s$ by $f$ and $X_s$ by $X$. We again call the non-smooth locus of $f_s$ by $B$ and note that $y \notin B$.

**Step 2. Base Case of the Induction**. Take a birational modification $Y'$ of $Y$ such that $\mu^{-1}(B)_{\mathrm{red}} =: \Sigma$ in $Y'$, as in the diagram below, is a simple normal crossing divisor and $Y' \setminus \Sigma \simeq Y \setminus \mathrm{Supp}(B)$. In particular, $\mu$ is an isomorphism around $y$. Let $X' \to X$ be a resolution of the largest irreducible component of the fibre product $Y' \times_Y X$. The situation is described in the following commutative diagram and a pictorial illustration.
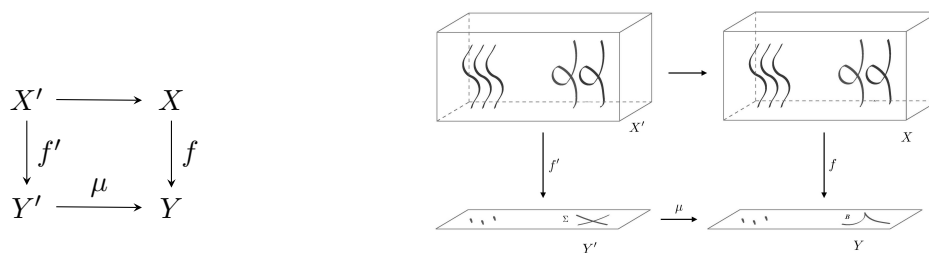


Note that $f'$ satisfies the hypothesis of Kawamata's theorem (Theorem 2.1) around $\mu^{-1}(y)$. Indeed, since $\mu$ is an isomorphism over a neighbourhood $U$ around $y$, $\mu^*A$ satisfies the intersection properties, as in the hypothesis, at the point $\mu^{-1}(y)$. Moreover $f'$ is smooth outside of the simple normal crossing divisor $\Sigma$. Therefore by Theorem 2.1, we obtain that $f'_*\omega_{X'} \otimes \mu^*A$ is generated by global sections at $\mu^{-1}(y)$. Additionally we have,

$$\mu_*(f'_*\omega_{X'} \otimes \mu^*A) \simeq f_*\omega_X \otimes A.$$

Therefore the sheaf $f_*\omega_X \otimes A$ is generated by global sections at $y$.

$\square$

*Remark* 2.2 (Local version of Kawamata's theorem). When $\Delta = 0$ and $H = \mathcal{O}_Y$, by Sźabo's Lemma (see e.g. [KK13, Theorem 10.45(1)]), we can choose $\mu$ in Step 2 of the above proof to be an isomorphism outside the simple normal crossing locus of $B$ to obtain a local version of Kawamata's theorem. Said differently, the proof shows that for any morphism $f : X \to Y$ between smooth projective varieties, if $y \in Y$ has a Zariski neighbourhood $U$ such that the morphism $f : f^{-1}(U) \to U$ is smooth outside a simple normal crossing divisor then $f_*\omega_X \otimes L^{\otimes l}$ is globally generated at $y$, for all $l \geqslant N + 1$ with $N$ as in Notation 1.

**3. Proof of the Main Theorems.** For its simplistic nature, we first include the proof of Theorem C. Inspired by [PS14], the strategy is to reduce generation problem for pluricanonical bundles to that of canonical bundles on pairs. We show that such a pair can be carefully chosen, so it satisfies the properties in the hypothesis of Proposition 1.3.

*Proof of Theorem C.* Let $\mathscr{I} \subseteq \mathcal{O}_X$ be the the relative base ideal of $\omega_Y^{\otimes k}$, i.e. there is a surjection $f^*f_*\omega_X^{\otimes k} \twoheadrightarrow \mathscr{I} \otimes \omega_X^{\otimes k}$ via the adjunction morphism. We first take a log resolution $\mu : \widetilde{X} \to X$ of $\mathscr{I}$, so that $\mu$ is an isomorphism outside the cosupport of $\mathscr{I}$ and the image of the adjunction morphism is $\omega_{\widetilde{X}}^{\otimes k}(-E)$, for an effective divisor $E$ with simple normal crossing support. Renaming $\widetilde{X}$ by $X$, we have the following surjection:

$$f^*f_*\omega_X^{\otimes k} \twoheadrightarrow \omega_X^{\otimes k}(-E)$$

By the relative freeness of $\omega_X^{\otimes k}$ over $Y \setminus B$, $f(E) \subset B$.

Fix a point $y \in Y \setminus B$. We pick a positive integer $m$ which is smallest with the property that the sheaf $f_* \omega_X^{\otimes k} \otimes L^{\otimes m}$ is generated by global sections on $U$. Then $f^* f_* \left( \omega_X^{\otimes k} \otimes f^* L^{\otimes m} \right)$ is also generated by global sections on $f^{-1}(U)$. Therefore by the surjectivity of the adjunction morphism, we have
$$\omega_X^{\otimes k}(-E) \otimes f^* L^{\otimes m}$$
is globally generated on $f^{-1}(U)$. As a consequence, we can pick a divisor $D \in \left| \omega_X^{\otimes k}(-E) \otimes f^* L^{\otimes m} \right|$ such that $D$ is smooth outside of $f^{-1}(B)$ and intersects the fibre $f^{-1}(y)$ transversely.

After replacing $X$ with a birational modification that is an isomorphism outside of $f^{-1}(B)$, we may assume that $D = D' + F$, where $D'$ is smooth, intersects the fibre $f^{-1}(y)$ transversely and does not share any component with $E$. Furthermore, $F$ is supported on $f^{-1}(B)$ and $D' + F + E$ has simple normal crossing support.

We write
$$kK_X + mf^* L \sim D' + F + E.$$
Multiplying both sides by $\frac{k-1}{k}$ and adding $K_X + lf^* L$ for any integer $l$, we can rewrite
$$kK_X + lf^* L \sim_{\mathbb{Q}} K_X + \frac{k-1}{k}(D' + F + E) + \left( l - \frac{k-1}{k}m \right) f^* L.$$

Now consider the effective divisor $E' := \left\lfloor \frac{k-1}{k}(E + F) \right\rfloor$ and denote
$$\Delta := \frac{k-1}{k}(D' + F + E) - E'.$$
We can rewrite $\mathbb{Q}$-linear equivalence as
$$kK_X - E' + lf^* L \sim_{\mathbb{Q}} K_X + \Delta + \left( l - \frac{k-1}{k}m \right) f^* L.$$
It is now enough to show that
$$f_* \mathcal{O}_X \left( K_X + \Delta + \left( l - \frac{k-1}{k}m \right) f^* L \right)$$
is generated by global sections at $y$ for all $l > \frac{k-1}{k}m + N$. Indeed, this would imply that the left hand side of the equation also satisfies similar global generation bounds, i.e. $f_* \mathcal{O}_X \left( kK_X - E' + lf^* L \right)$ is globally generated at $y$ for all $l > \frac{k-1}{k}m + N$. But note that $E'$ is supported on $f^{-1}(B)$ and $y \notin B$. Therefore the stalks
$$f_* \mathcal{O}_X \left( kK_X - E' + lf^* L \right)_y \simeq f_* \mathcal{O}_X \left( kK_X + lf^* L \right)_y$$
are isomorphic. Moreover the global sections of the former sheaf embeds into the global sections of the latter sheaf. Said differently, this would imply that
$$f_* \mathcal{O}_X \left( kK_X + lf^* L \right)$$
is globally generated on $U$ for all $l > \dfrac{k-1}{k}m + N$. From our choice of $m$, we must have that $m \leqslant \frac{k-1}{k}m + N + 1$. This is the same as $m \leqslant k(N+1)$. As a consequence,
$$f_* \omega_X^{\otimes k} \otimes L^{\otimes l}$$
is generated by global sections on $Y \setminus B$ for all $l \geqslant (k-1)(N+1) + N + 1 = k(N+1)$.

It now remains to show that
$$f_* \mathcal{O}_X \left( K_X + \Delta + \left( l - \frac{k-1}{k}m \right) f^* L \right)$$

is generated by global sections at $y$ when $l - \frac{k-1}{k}m > N$. This follows from Proposition 1.3 taking

$$H := \Big( \Big\lceil \frac{k-1}{k}m \Big\rceil - \frac{k-1}{k}m \Big)L$$

and

$$A := \Big( l - \Big\lceil \frac{k-1}{k}m \Big\rceil \Big)L.$$

Indeed, $L$ is ample, $H$ is semiample and $A$ satisfies the Angehrn-Siu type intersection properties by the choice of $l$ above. Moreover $\Delta$ has simple normal crossing support with coefficients in $(0,1)$ and its components intersect the fibre $f^{-1}(y)$ transversely or not at all. $\qquad\square$

The proof of Theorem A is fairly similar, except due to the lack of nice behaviour of $\omega_X^{\otimes k}$ over $Y \setminus B$, one needs to carefully choose the locus $U$ of global generation.

*Proof of Theorem A.* Following the proof of [PS14, Theorem 1.7], we first take a log resolution $\mu : \widetilde{X} \to X$ of the base ideal of the adjunction morphism $f^*f_*\mathcal{O}_X(P) \xrightarrow{\pi} \mathcal{O}_X(P)$ and the pair $(X, \Delta)$. Write:

$$K_{\widetilde{X}} - \mu^*(K_X + \Delta) = Q - N$$

where $Q$ and $N$ are effective $\mathbb{Q}$-divisors with simple normal crossing support, with no common components, moreover $N$ has coefficients strictly smaller than 1, and $Q$ is supported on the exceptional locus. Define:

$$\widetilde{P} := \mu^*P + k\lceil Q \rceil$$

and

$$\widetilde{\Delta} := N + \lceil Q \rceil - Q.$$

Then by definition, $\widetilde{P} \sim_{\mathbb{Q}} k(K_{\widetilde{X}} + \widetilde{\Delta})$. Moreover, since $Q$ is exceptional, we have the isomorphism $\mu_*\mathcal{O}_{\widetilde{X}}(\widetilde{P}) \simeq \mu_*\mathcal{O}_X(P)$. We rename $\widetilde{X}$ by $X$, $\widetilde{P}$ by $P$ and $\widetilde{\Delta}$ by $\Delta$, so that the image of the adjunction morphism $\pi$ is given by $\mathcal{O}_X(P - E)$, for an effective divisor $E$ and so that $X$ is smooth and the divisor $\Delta + E$ has simple normal crossing support.

Next, the strategy is to find a suitable open set $U \subseteq Y$ on which we prove global generation. For this purpose, write $\Delta = \sum_i a_i \Delta_i$, where $\Delta_i$'s are the irreducible components of $\Delta$. Let $E_j$'s denote the irreducible components of $E$. Set,

$$\mathbf{c} := k \times \text{l.c.m. of the denominators of } a_i.$$

Similar to the construction in the proof of Proposition 1.3, we inductively take $\mathbf{c}^{th}$ Kawamata covers of $\Delta_i$'s and $E_j$'s and denote the composition of these covers by $p : X' \to X$. We choose these covers so that $p^*\Delta_i = \mathbf{c}\Delta_i'$ and $p^*E_j = \mathbf{c}E_j'$ for irreducible divisors $\Delta_i'$ and $E_j'$. We further ensure that $p^*(\Delta + E)$ has simple normal crossing support.

Denote by $B$, the non-smooth locus of $f \circ p$. Assign $U := Y \setminus B$, consider the following cartesian diagram:

$$
\begin{array}{ccc}
f^{-1}(U) =: V & \xhookrightarrow{\;i_V\;} & X \\
\downarrow{\scriptstyle f_V} & & \downarrow{\scriptstyle f} \\
U & \xhookrightarrow{\;\;i\;\;} & Y
\end{array}
$$

and denote $C := f^{-1}(B)$.

Fix $y \in U$. Now, pick a positive integer $m$ which is the smallest with the property that the sheaf $f_*\mathcal{O}_X(P) \otimes L^{\otimes m}$ is generated by global sections at each point on $U$. Therefore by adjunction, $\mathcal{O}_X(P - E) \otimes f^*L^{\otimes m}$ is globally generated on $V$ and hence so is $p^*\big(\mathcal{O}_X(P - E) \otimes$

$f^*L^{\otimes m}$) on $X' \backslash p^{-1}(C)$. By Bertini's theorem, we can pick $D \in \left| \mathcal{O}_X(P-E) \otimes f^*L^{\otimes m} \right|$ so that $D$ is smooth outside of $C$ and such that $p^*D$ is also smooth outside $p^{-1}(C)$. We further ensure that the divisor $p^*D$ intersects the smooth fibre $f^{-1}(y)$ transversely. To simplify notations, we denote $p^{-1}(C)$ by $C$ again.

We can write:
$$kP + mf^*L \sim D + E$$

By a similar arithmetic as in the proof of Theorem C we obtain,
$$k\big(K_X + \Delta\big) \sim_{\mathbb{Q}} K_X + \Delta + \frac{k-1}{k}(D+E) - \frac{k-1}{k}mf^*L,$$
and hence for any integer $l$,
$$k\big(K_X + \Delta\big) + lf^*L \sim_{\mathbb{Q}} K_X + \Delta + \frac{k-1}{k}(D+E) + \Big(l - \frac{k-1}{k}m\Big)f^*L.$$

Now, since $E$ is the relative base locus of the adjunction morphism of $\mathcal{O}_X(P)$, for every effective Cartier divisor $E'$ such that $E - E'$ is effective we have
$$f_*\mathcal{O}_X(P - E') \simeq f_*\mathcal{O}_X(P).$$
We would like to such integral divisors, $E'$ so that
$$\Delta + \frac{k-1}{k}E - E'$$
has coefficients strictly smaller than 1. We do so as follows. Write
$$E = \sum_i s_i \Delta_i + \widetilde{E}$$
and
$$\Delta = \sum_i a_i \Delta_i$$
where $\widetilde{E}$ and $\Delta$ do not have any common component. Note that, by hypothesis, $0 < a_i < 1$ and $s_i \in \mathbb{Z}_{\geqslant 0}$. We want to pick non-negative integers $b_i$, such that
$$0 \leqslant a_i + \frac{k-1}{k}s_i - b_i < 1$$
and
$$b_i \leqslant s_i.$$
Denote by
$$\gamma_i := a_i + \frac{k-1}{k}s_i$$
and note that $\gamma_i < 1 + s_i$. We pick $b_i$ as follows. For some integer $j$ with $0 \leqslant j \leqslant s_i$, we can write $s_i - j + 1 > \gamma_i \geqslant s_i - j$. Then we pick
$$b_i = s_i - j.$$
Now let
$$E' := \sum_i b_i \Delta_i + \left\lfloor \frac{k-1}{k}\widetilde{E} \right\rfloor.$$
Then assign
$$\widetilde{\Delta} := \Delta + \frac{k-1}{k}E - E' = \sum_i \alpha_i \widetilde{\Delta}_i$$

and note that $\widetilde{\Delta}$ is a divisor with simple normal crossing support with coefficients $0 < \alpha_i < 1$. Then we rewrite the above $\mathbb{Q}$-linear equivalence of divisors as:

$$P - E' + lf^*L \sim_{\mathbb{Q}} K_X + \widetilde{\Delta} + \frac{k-1}{k}D + \left(l - \frac{k-1}{k}m\right)f^*L.$$

It is now enough to show that the pushforward of the right hand side of the above $\mathbb{Q}$-linear equivalence is globally generated at $y$, for all $l > \dfrac{k-1}{k}m + N$. Indeed, in that case the left hand side would satisfy similar global generation bounds and by the discussion above

$$f_*\mathcal{O}_X(P - E') \otimes L^{\otimes l} \simeq f_*\mathcal{O}_X(P) \otimes L^{\otimes l}.$$

Said differently, this implies that

$$f_*\mathcal{O}_X(P) \otimes L^{\otimes l}$$

is globally generated on $U$ for all $l > \dfrac{k-1}{k}m + N$. From our choice of $m$, we must have that $m \leqslant \frac{k-1}{k}m + N + 1$. This is the same as $m \leqslant k(N+1)$. Therefore,

$$f_*\mathcal{O}_X(P) \otimes L^{\otimes l}$$

is generated by global sections on $U$ for all $l \geqslant (k-1)(N+1) + N + 1 = k(N+1)$.

It now remains to show that

$$f_*\mathcal{O}_X\left(K_X + \widetilde{\Delta} + \frac{k-1}{k}D + \left(l - \frac{k-1}{k}m\right)f^*L\right)$$

is globally generated at $y$. To do so, we resort to Proposition 1.3. However the divisor $\widetilde{\Delta} + \frac{k-1}{k}D$ may not satisfy the hypothesis of Proposition 1.3, as $D$ may be quite singular along $C$. Therefore we cannot apply Proposition 1.3 directly. Since we are only interested in generic global generation though, we can get around these problems. The rest of the proof is devoted to this.

By definition, $\mathbf{c}\alpha_i$ is an integer and by construction, $p$ is a composition of $\mathbf{c}^{th}$ Kawamata coverings of the components $\widetilde{\Delta}_i$'s of $\widetilde{\Delta}$. Following an inductive argument similar to the one in the proof of Proposition 1.3, we see that

$$f_*\mathcal{O}_X\left(K_X + \widetilde{\Delta} + \frac{k-1}{k}D + \left(l - \frac{k-1}{k}m\right)f^*L\right)$$

is a direct summand of

$$(f \circ p)_*\mathcal{O}_{X'}\left(K_{Y'} + \frac{k-1}{k}D' + (f \circ p)^*\left(l - \frac{k-1}{k}m\right)L\right)$$

where $D' = p^*D$. Therefore it is enough to show that the latter is globally generated at $y$.

We are now almost in the situation of Proposition 1.3, except $D'$ may still be singular along $p^{-1}(C)$. We get around this using similar strategy as was used in the proof Theorem C. Let $\mu : X'' \to X'$ be a log resolution of $D'$ such that $\mu$ is an isomorphism outside of $p^{-1}(C)$. Then write

$$\mu^*D' = \widetilde{D} + F$$

where $\widetilde{D}$ is smooth, intersects the fibre over $y$ transversely and $F$ is supported on $\mu^{-1}(p^{-1}(C))$, denoted by $C$ again. We replace, $X''$ by $X'$, rename the divisor $\mu^*D'$ by $D'$. Therefore, we can assume that $D'$ has simple normal crossing support.

To deal with the fact that $F$ may not be klt, consider the effective Cartier divisor $F' = \left\lfloor \dfrac{k-1}{k}F \right\rfloor$. Since, $\mathrm{Supp}(F')$ is contained in the $C$ and $y \notin B$, the stalks

$$(f \circ p)_* \mathcal{O}_{X'}\Big(K_{X'} + \frac{k-1}{k}D' + \Big(l - \frac{k-1}{k}m\Big)(f \circ p)^*L\Big)_y \simeq$$
$$(f \circ p)_* \mathcal{O}_{X'}\Big(K_{X'} + \frac{k-1}{k}D' - F' + \Big(l - \frac{k-1}{k}m\Big)(f \circ p)^*L\Big)_y$$

are isomorphic. Moreover the global sections of the latter sheaf embed into the global sections of the former sheaf.

Letting
$$\widetilde{\Delta} := \frac{k-1}{k}D' - F',$$
it is now enough to show that,
$$(f \circ p)_* \mathcal{O}_{Y'}\Big(K_{Y'} + \widetilde{\Delta} + \Big(l - \frac{k-1}{k}m\Big)(f \circ p)^*L\Big)$$

is globally generated at $y$ for $l > \dfrac{k-1}{k}m + N$. The $\mathbb{Q}$-divisor $\widetilde{\Delta}$ satisfies the hypothesis in Proposition 1.3 and the required global generation follows from Proposition 1.3 taking

$$H := \Big(\Big\lceil \frac{k-1}{k}m \Big\rceil - \frac{k-1}{k}m\Big)L$$

and

$$A := \Big(l - \Big\lceil \frac{k-1}{k}m \Big\rceil\Big)L.$$

$\square$

*Remark* 3.1. Note that if $\Delta$ itself has simple normal crossing support and the relative base locus $E$ of $\mathcal{O}_X(k(K_X + \Delta))$ is a divisor so that $\Delta + E$ also has simple normal crossing support, then by construction, the loci of generation $U$ in the statement contains the largest open set in $Y$, over which $f$ restricted to each strata of $(X, \Delta + E)$ is smooth.

**4. An Effective Vanishing Theorem.** We deduce a pluricanonical version of Kollár's vanishing theorem for smooth morphisms satisfying certain properties. The proof essentially follows directly from the $\mathbb{Q}$-linear equivalences involved in the proof of Theorem C.

**Theorem 4.1** (Effective Vanishing Theorem). *Let $f : X \to Y$ be a smooth surjective morphism of smooth projective varieties, with $\dim Y = n$. Assume in addition that $\omega_X^{\otimes k}$ is relatively free for some $k \geqslant 1$, and let $L$ be an ample line bundle on $Y$. Then,*
$$H^i\Big(Y, f_*\omega_X^{\otimes k} \otimes L^{\otimes l}\Big) = 0$$
*for all $i > 0$ and $l \geqslant k(N+1) - N$ with $N$ as in Notation 1.*

*Proof.* Since $f$ is smooth, by Theorem C, we know that the sheaf $f_*\omega_X^{\otimes k} \otimes L^{\otimes l}$ is globally generated for all $l \geqslant k(N+1)$. Therefore by the surjectivity of the adjunction morphism $\omega_X^{\otimes k} \otimes f^*L^{\otimes k(N+1)}$ is globally generated as well. As a consequence, we can pick a smooth divisor $D \in \big|\omega_X^{\otimes k} \otimes f^*L^{\otimes k(N+1)}\big|$ such that $D$ intersects the fibre $f^{-1}(y)$ transversely .

Write:
$$kK_X + k(N+1)f^*L \sim D.$$

Multiplying by $\frac{k-1}{k}$ and adding $K_X + lf^*L$ for some integer $l$, we obtain as before

$$kK_X + lf^*L \sim_{\mathbb{Q}} K_X + \frac{k-1}{k}D + \big(l - (k-1)(N+1)\big)f^*L,$$

for any integer $l$. By applying Kollár's vanishing theorem [Kol95, Corollary 10.15] on the right hand side, we get

$$H^i\Big(Y, f_*\mathcal{O}_X\big(K_X + \frac{k-1}{k}D + \big(l - (k-1)(N+1)\big)f^*L\big)\Big) = 0$$

for all $i > 0$ and $l > (k-1)(N+1)$. Therefore, the left hand side satisfies similar vanishing properties

$$H^i\Big(Y, f_*\omega_X^{\otimes k} \otimes L^{\otimes l}\Big) = 0$$

for all $i > 0$ and $l \geqslant k(N+1) - N$ $\qquad\square$

*Remark* 4.2. The above bound is replaced in [DM19, Theorem 5.3] by $k(n+1) - n$ for all $n$. This is an improvement for $n > 4$.

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