THE HARDER-NARASIMHAN FILTRATION OF A MULTI-VALUED VECTOR SPACE

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ABSTRACT. An s-valued vector space over a field K is a tuple $\overline{V} = (V, w_1, \dots, w_s)$, consisting of a finite dimensional K-vector space V and valuations $w_1, \dots, w_s$ on V. Such spaces (in an other but equivalent formulation) were introduced by Faltings and Wüstholz [6] in their then new proof of W.M. Schmidt's Subspace Theorem from Diophantine approximation. An important ingredient of their proof was that the tensor product of two semistable s-valued vector spaces is again semistable, and they proved this using an analogous existing result for vector bundles of Narasimhan and Seshadri [10]. Later, various other proofs of this fact were given, all of them highly non-elementary. The most down-to-earth proof was given by Faltings himself, in [5], where he used modules over the formal power series ring $K[[t]]$.

In the present paper, we have worked out Faltings' arguments from this last paper in detail, and translated them into elementary linear algebra. We proved various generalizations of the semistability result of Faltings and Wüstholz. We recall the definition of weighted Harder-Narasimhan filtration and corresponding Harder-Narasimhan valuation of an s-valued vector space from [6], and show among other things that taking the Harder-Narasimhan valuation commutes with taking exterior powers, symmetric powers, base extensions and tensor products. This contains as a special case the semistability result of Faltings and Wüstholz mentioned above, and moreover that exterior powers, symmetric powers and base extensions of semistable s-valued vector spaces are semistable. Further, we give a procedure to compute the Harder-Narasimhan valuation of an s-valued vector space. Our results are valid over fields K of any characteristic.

1. INTRODUCTION AND RESULTS

Let K be a field (of any characteristic) and V a K-vector space. A *valuation* on V is a function $w : V \to \mathbb{R} \cup \{\infty\}$ such that

$$(1.1) \qquad \begin{cases} w(x) = \infty \Longleftrightarrow x = 0, \quad w(\lambda x) = w(x) \text{ for } x \in V,\ \lambda \in K^*, \\ \qquad w(x+y) \geqslant \min(w(x), w(y)) \text{ for } x, y \in V. \end{cases}$$

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Vector spaces with valuations were studied among others by Fuchs [7]. An *s-valued K-vector space* is a tuple $\overline{V} = (V, w_1, \ldots, w_s)$, where V is a K-vector space and $w_1, \ldots, w_s$ are valuations on V. A *multi-valued K-vector space* is an s-valued K-vector space for any $s \geqslant 1$.

We assume throughout this paper that V is finite-dimensional and non-zero. Let w be a valuation on V. If $\alpha_1 > \alpha_2 > \cdots$ is any sequence of values assumed by w on $V \setminus \{0\}$, then the sets $F_i := \{x \in V : w(x) \geqslant \alpha_i\}$ $(i = 1, 2, \ldots)$ form a strictly increasing sequence of linear subspaces of V, which necessarily has to be finite. Hence w assumes only finitely many values on $V \setminus \{0\}$, say $\alpha_1 > \cdots > \alpha_r$. We call the corresponding sequence of subspaces

$$(0) = F_0 \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_r = V, \quad \text{where } F_i = \{x \in V : w(x) \geqslant \alpha_i\} \tag{1.2}$$

the (unweighted) *filtration* of w, and the tuple

$$\big((0) = F_0 \subsetneqq F_1 \subsetneqq \cdots \subsetneqq F_r = V,\ \alpha_1 > \cdots > \alpha_r\big) \tag{1.3}$$

the *weighted filtration* of w. Conversely, the weighted filtration (1.3) uniquely determines w, as $w(x) = \alpha_i$ for $x \in F_i \setminus F_{i-1}$. In contrast to the literature, we work with multi-valued vector spaces instead of multi-filtered vector spaces (vector spaces endowed with a finite number of weighted filtrations) since in our set-up, valuations are more convenient. But it should be kept in mind that both notions are equivalent.

In the 1970s, W.M. Schmidt [12] (see also [13]) and later in a more general form Schlickewei [11] proved a central theorem in Diophantine approximation, the *Subspace Theorem.* Roughly speaking, this asserts that the solutions of a particular system of Diophantine inequalities with unknowns from $\mathbb{P}^n(K)$ with K an algebraic number field, lie in finitely many proper linear subspaces of $\mathbb{P}^n(K)$.

In their landmark paper [6], Faltings and Wüstholz gave an entirely new proof of this Subspace Theorem, which depends heavily on multi-valued vector spaces (in fact, Faltings and Wüstholz used multi-filtered vector spaces). They observed that for finite dimensional s-valued vector spaces there are a semistability theory, and thus a Harder-Narasimhan filtration. They attached a multi-valued vector space to a system of Diophantine inequalities as considered in the Subspace Theorem, and pointed out, that the Harder-Narasimhan filtration of this space plays an important role in a more refined analysis of this system of inequalities. See Ballaÿ [1, Chap. 3] for another approach to their proof.

There is a natural notion of tensor product of s-valued vector spaces. One of the key tools in the proof of Faltings and Wüstholz in [6] is their Theorem 4.1, asserting that if $\overline{V}$, $\overline{W}$ are semistable s-valued vector spaces over a field K of characteristic 0, then their tensor product $\overline{V} \otimes_K \overline{W}$ is also semistable. Faltings and Wüstholz proved this using semistability theory for vector bundles over algebraic curves, developed by Narasimhan and Seshadri [10]. Shortly afterwards, Totaro [15] gave another proof, also valid only for fields of characteristic 0, in which he linked the semistability of an s-valued vector space $\overline{V} = (V, w_1, \ldots, w_s)$ to the existence of a suitable metric on V. Then Faltings [5] gave a new proof, valid for fields K of any characteristic, based on an argument using modules over the power series ring $K[[t]]$, inspired by work of Lafaille [9]. Finally, Fujimori [8] gave a proof, based on Schmidt's Subspace Theorem, in which he more or less converted the arguments of Faltings and Wüstholz (this was allowed since Schmidt and Schlickewei had already given a proof of the Subspace Theorem independent of multi-valued vector spaces). In Fujimori's proof one has to assume that K is an algebraic number field.

It is rather unsatisfactory that the semistability result of Faltings and Wüstholz, which is in essence just linear algebra, could so far be proved only using techniques going far beyond linear algebra. In fact, Faltings' $K[[t]]$-modules argument from [5] can be translated into terms of elementary linear algebra, but it uses a limit argument for sequences of valuations. What remains open is to give a fully combinatorial proof, avoiding this limit argument.

In the present paper, we have worked out in detail Faltings' $K[[t]]$-modules argument, translated into elementary linear algebra. This approach is valid for fields K of any characteristic. Our main result is a central theorem, which relates the Harder-Narasimham valuation of a given multi-valued K-vector space to a particular binary operator on the collection of valuations on the ambient vector space. From this central theorem we deduce the semistability result of Faltings and Wüstholz for tensor products. More generally, for not necessarily semistable s-valued vector spaces we show that the Harder-Narasimhan valuation commutes with tensor products in the sense that the Harder-Narasimhan valuation of the tensor product of two s-valued vector spaces is the (to be defined) tensor product of their Harder-Narasimhan valuations. Likewise we show that the Harder-Narasimhan valuation commutes with exterior powers, symmetric powers and base extensions. Using our result on exterior powers we describe a(n unfortunately very inefficient) method to compute the Harder-Narasimhan valuation of a given multi-valued vector space.

We first recall the necessary definitions, and then state our theorems.

1.1. **Definitions.** Throughout this paper, K is any field. We say that an s-valued K-vector space $\overline{V} = (V, w_1, \ldots, w_s)$ is non-zero if V is non-zero and define the dimension of $\overline{V}$ to be that of V. By $\otimes$ we always denote the tensor product with respect to K. A *morphism* from an s-valued K-vector space $\overline{V} = (V, w_1, \ldots, w_s)$ to another s-valued K-vector space $\overline{V'} = (V', w'_1, \ldots, w'_s)$ is a K-linear map $\varphi : V \to V'$ such that for $i = 1, \ldots, s$ we have $w'_i \circ \varphi \geqslant w_i$, i.e., $w'_i(\varphi(x)) \geqslant w_i(x)$ for $x \in V$. Note that φ is an isomorphism precisely if φ is bijective and $w'_i \circ \varphi = w_i$ for $i = 1, \ldots, s$. Clearly, the composition of two morphisms of s-valued K-vector spaces is another such morphism.

In what follows, V is a non-zero, finite-dimensional K-vector space. For a valuation w on V with weighted filtration (1.3) we define

$$w(V) := \sum_{i=1}^{r} \alpha_i (\dim F_i - \dim F_{i-1}).$$

Then the *slope* of an s-valued vector space $\overline{V} = (V, w_1, \ldots, w_s)$ is defined by

$$\mu(\overline{V}) := \frac{1}{\dim V} \sum_{i=1}^{s} w_i(V).$$

Let U be a linear subspace of V. Denote by x^U the image of x under the canonical map $V \to V/U$. A valuation w on V induces valuations $w|_U$ on U, which is the restriction of w to U, and w^U on V/U, given by

$$w^U(y) := \max\{w(x) : x \in V,\, x^U = y\}. \tag{1.4}$$

Now for a given s-valued vector space $\overline{V} = (V, w_1, \ldots, w_s)$ we define the s-valued subspace $\overline{U} := (U, w_1|_U, \ldots, w_s|_U)$ and s-valued quotient $\overline{V}/\overline{U} := (V/U, w_1^U, \ldots, w_s^U)$.

A (by default finite dimensional) s-valued vector space $\overline{V}$ is called *semistable* if $\mu(\overline{U}) \leqslant \mu(\overline{V})$ for every non-zero linear subspace U of V. A not necessarily semistable s-valued vector space $\overline{V}$ has a *maximal destabilizing subspace* V_1, which is such that $\mu(\overline{U}) \leqslant \mu(\overline{V_1})$ for every non-zero linear subspace U of V and such that all subspaces U with $\mu(\overline{U}) = \mu(\overline{V_1})$ are contained in V_1. This leads to the *weighted Harder-Narasimhan filtration*

$$\big((0) = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V,\ \mu_1 > \cdots > \mu_r \big),$$

where V_i is such that V_i/V_{i-1} is the maximal destabilizing subspace of $\overline{V}/\overline{V_{i-1}}$ and $\mu_i := \mu(\overline{V_i}/\overline{V_{i-1}})$ for $i = 1, \ldots, r$. The corresponding *Harder-Narasimham valuation*

$w_{\overline{V}}^{HN}$ of $\overline{V}$ is given by $w_{\overline{V}}^{HN}(x) := \mu_i$ for $x \in V_i \setminus V_{i-1}$, $i = 1, \dots, r$. For further details we refer to Section 2.

Given a non-zero, finite-dimensional K-vector space V, we denote by $\mathcal{W}(V)$ the collection of valuations on V. For $w \in \mathcal{W}(V)$, define

$$\min w := \min\{w(x) : x \in V\}, \quad \max w := \max\{w(x) : x \in V \setminus \{0\}\}.$$

Let $\mathcal{W}$ be a subcollection of $\mathcal{W}(V)$ that is bounded from below, i.e., there is $C \in \mathbb{R}$ with $\min w \geqslant C$ for $w \in \mathcal{W}$. Then the infimum of $\mathcal{W}$, given by

$$(\inf \mathcal{W})(x) := \inf\{w(x) : w \in \mathcal{W}\} \quad \text{for } x \in V,$$

defines a valuation on V.

Let $V_1, \dots, V_k, W$ be finite-dimensional K-vector spaces and $\rho : V_1 \times \cdots \times V_k \to W$ a multi-linear map such that $\rho(V_1 \times \cdots \times V_k)$ generates W. Further, let w_i be a valuation on V_i, for $i = 1, \dots, k$. We define a valuation $\rho(w_1, \dots, w_k)$ on W by

$$\rho(w_1, \dots, w_k) \;:=\; \inf\Big\{w \in \mathcal{W}(W) : w(\rho(x_1, \dots, x_k)) \geqslant \sum_{j=1}^{k} w_j(x_j) \text{ for all } x_1 \in V_1, \dots, x_k \in V_k\Big\}. \tag{1.5}$$

This is a well-defined valuation on W. For let $\mathcal{W}$ denote the collection of valuations on the right-hand side. First, $\mathcal{W}$ is non-empty, for instance it contains the valuation that is equal to $\sum_{j=1}^{k} \max w_j$ on $W \setminus \{0\}$. Second, $\mathcal{W}$ is bounded from below. For W consists of sums of elements $\rho(x_1, \dots, x_k)$ with $x_j \in V_j$ for $j = 1, \dots, k$ and so by (1.1), any $w \in \mathcal{W}$ assumes its minimum at such an element. Now clearly, $\min w \geqslant \sum_{j=1}^{k} \min w_j$ for $w \in \mathcal{W}$.

Notice that with $\rho : V \to V/U : x \mapsto x^U$, the definitions (1.5) and (1.4) coincide.

By specializing (1.5) to $W = V_1 \otimes \cdots \otimes V_k$ (tensor product), $\rho : (x_1, \dots, x_k) \mapsto x_1 \otimes \cdots \otimes x_k$ we get a valuation $\rho(w_1, \dots, w_k) =: w_1 \otimes \cdots \otimes w_k$ on $V_1 \otimes \cdots \otimes V_k$. Taking $V_i = V, w_i = w$ for $i = 1, \dots, k$ where $1 \leqslant k \leqslant n$, $W = \wedge^k V$ (k-th exterior power), $\rho : (x_1, \dots, x_k) \mapsto x_1 \wedge \cdots \wedge x_k$ we get a valuation $\wedge^k w$ on $\wedge^k V$. Lastly, taking $V_i = V$, $w_i = w$ for $i = 1, \dots, k$ where $k \geqslant 1$, $W = \mathrm{S}^k V$ (k-th symmetric power), $\rho : (x_1, \dots, x_k) \mapsto x_1 \cdots x_k$, we get a valuation $\mathrm{S}^k w$ on $\mathrm{S}^k V$.

Let again V be a finite-dimensional K-vector space and w a valuation on V. For any extension field L of K we define a valuation $w \otimes L$ on the base extension $V \otimes L$ by

$$w \otimes L := \inf\{w' \in \mathcal{W}(V \otimes L) : w'(x \otimes \xi) \geqslant w(x) \text{ for all } x \in V, \xi \in L\}. \tag{1.6}$$

Lastly, given two finite-dimensional K-vector spaces V, V' and valuations w on V and w' on V' we define a valuation on the (external) direct sum $V\oplus V'$ by

$$(w\oplus w')(x,y) := \min(w(x), w'(y)) \text{ for } (x,y)\in V\oplus V'.$$

Now the k-th exterior power, k-th symmetric power and base extension of a finite-dimensional s-valued vector space $\overline{V} = (V, w_1, \dots, w_s)$ are given by

$$\wedge^k\overline{V} := (\wedge^k V, \wedge^k w_1, \dots, \wedge^k w_s), \quad \mathrm{S}^k\,\overline{V} := (\mathrm{S}^k\,V, \mathrm{S}^k\,w_1, \dots, \mathrm{S}^k\,w_s),$$
$$\overline{V}\otimes L := (V\otimes L, w_1\otimes L, \dots, w_s\otimes L),$$

while the direct sum and tensor product of two s-valued vector spaces $\overline{V} = (V, w_1, \dots, w_s)$ and $\overline{V'} = (V', w_1', \dots, w_s')$ are given by

$$\overline{V}\oplus\overline{V'} := (V\oplus V', w_1\oplus w_1', \dots, w_s\oplus w_s'), \quad \overline{V}\otimes\overline{V'} := (V\otimes V', w_1\otimes w_1', \dots, w_s\otimes w_s').$$

1.2. **Results.** We start with formulating our central result, which we proved by following the ideas of Faltings from [5]. From this central theorem we will deduce our other results.

Let V be a non-zero, finite-dimensional K-vector space. On its collection of valuations $\mathcal{W}(V)$ we define a binary operator $*$ as follows:

$$w_1 * w_2 := \inf\{w\in\mathcal{W}(V) :\ w \geqslant w_1 + w_2\} \text{ for } w_1, w_2\in\mathcal{W}(V); \tag{1.7}$$

this defines a valuation on V since $\min w \geqslant \min w_1 + \min w_2$ for every valuation w in the collection on the right-hand side. The $*$-operator is clearly commutative, but in case that $\dim V \geqslant 2$ it is non-associative.

We define a metric on $\mathcal{W}(V)$ by

$$|w_1 - w_2| := \max_{x\in V\setminus\{0\}} |w_1(x) - w_2(x)| \text{ for } w_1, w_2\in\mathcal{W}(V).$$

Our central theorem reads as follows.

Theorem 1.1. *Let $\overline{V} = (V, w_1, \dots, w_s)$ be a non-zero, finite-dimensional s-valued vector space, where $s \geqslant 2$. Define the sequence of valuations $(v_m)_{m=0}^{\infty}$ on V recursively by $v_0(x) := 0$ for $x\in V\setminus\{0\}$ and*

$$v_m := (\cdots((v_{m-1} * w_1) * w_2)\cdots) * w_s \quad \text{for } m = 1, 2, \dots.$$

Then there is $C > 0$ such that

$$|v_m - m w_{\overline{V}}^{HN}| \leqslant C \quad \text{for all } m \geqslant 0.$$

It is not too hard to give a direct proof of the following result, but we have chosen to deduce it from Theorem 1.1. In fact, in Section 6 we will deduce a more general result, i.e., Corollary 6.2.

Corollary 1.2. *Let $\overline{V} = (V, w_1, w_2)$ be a non-zero, finite-dimensional two-valued vector space. Then $w_{\overline{V}}^{HN} = w_1 * w_2$.*

Unfortunately, our method of proof of Theorem 1.1 is ineffective, in the sense that it gives only the existence of a constant C, but not a method to compute it.

We considered some toy examples, i.e., two-dimensional three-valued vector spaces $\overline{V} = (V, w_1, w_2, w_3)$, and discovered that in each of them the sequence $(v_m - mw_{\overline{V}}^{HN})_{m=0}^{\infty}$ is ultimately periodic. Further, it turned out that by varying w_1, w_2, w_3, the pre-period can be made arbitrarily long, whereas the length of the period remains bounded. Inspired by this, we would like to pose the following problem:

Problem 1.3. *Let $\overline{V} = (V, w_1, \dots, w_s)$ be an n-dimensional s-valued vector space. Is it true that the sequence $(v_m - mw_{\overline{V}}^{HN})_{m=0}^{\infty}$ is ultimately periodic, with an upper bound for the period depending only on n and s?*

The results stated below will be deduced by combining Theorem 1.1 with properties of the $*$-operator. For some of these results there are more direct proofs.

Our first consequence asserts that the Harder-Narasimhan valuation commutes with exterior powers, symmetric powers, base extensions, direct sums, and tensor products.

Theorem 1.4. **(i)** *Let $\overline{V}$ be a non-zero, finite-dimensional s-valued K-vector space. Then the following identities of valuations hold:*

$$w_{\wedge^k \overline{V}}^{HN} = \wedge^k w_{\overline{V}}^{HN} \text{ on } \wedge^k V \text{ for every } k \in \{1, \dots, \dim V\}; \tag{1.8}$$

$$w_{\mathrm{S}^k \overline{V}}^{HN} = \mathrm{S}^k\, w_{\overline{V}}^{HN} \text{ on } \mathrm{S}^k\, V \text{ for every positive integer } k; \tag{1.9}$$

$$w_{\overline{V} \otimes L}^{HN} = w_{\overline{V}}^{HN} \otimes L \text{ on } V \otimes L \text{ for every extension field } L \text{ of } K. \tag{1.10}$$

(ii) *Let $\overline{V}$, $\overline{V'}$ be two non-zero, finite-dimensional s-valued K-vector spaces. Then the following identities of valuations hold:*

$$w_{\overline{V} \oplus \overline{V'}}^{HN} = w_{\overline{V}}^{HN} \oplus w_{\overline{V'}}^{HN} \text{ on } V \oplus V'; \tag{1.11}$$

$$w_{\overline{V} \otimes \overline{V'}}^{HN} = w_{\overline{V}}^{HN} \otimes w_{\overline{V'}}^{HN} \text{ on } V \otimes V'. \tag{1.12}$$

A valuation on a K-vector space V is called constant if it is constant on $V \setminus \{0\}$. It is trivial that an s-valued K-vector space $\overline{V}$ is semistable if and only if its

Harder-Narasimhan valuation is constant. Further, from (1.5), (1.6) it is clear that exterior powers, symmetric powers, base extensions and tensor products of constant valuations are again constant. This leads at once to the following:

Corollary 1.5. **(i)** *Let* $\overline{V}$ *be a non-zero, finite-dimensional, semistable* s*-valued* K*-vector space. Then* $\wedge^k\overline{V}$ *(for every* $k\in\{1,\dots,\dim V\}$*),* $\mathrm{S}^k\,\overline{V}$ *(for every positive integer* k*) and* $\overline{V}\otimes L$ *(for every extension field* L *of* K*), are all semistable.*
(ii) *Let* $\overline{V}$*,* $\overline{V'}$ *be two non-zero, finite-dimensional semistable* s*-valued* K*-vector spaces. Then* $\overline{V}\otimes\overline{V'}$ *is semistable.*

Identity (1.8) was proved implicitly in a paper with Ferretti [4], where it is an important ingredient. The proof given there, in the spirit of Fujimori's, uses a special case of the Subspace Theorem, and thus it works only if the ground field K is an algebraic number field. The arguments in the present paper do not go beyond linear algebra, and work for any field K of any characteristic.

The next result that we derive from Theorem 1.1 shows that the Harder-Narasimhan valuation is compatible with morphisms of s-valued vector spaces. For a direct proof (in the general framework of Harder-Narasimhan categories), we refer to Chen [2, Thm. 5.7].

Theorem 1.6. *Let* $\overline{V}$ *and* $\overline{V'}$ *be two non-zero finite-dimensional* s*-valued* K*-vector spaces and* φ *a morphism from* $\overline{V}$ *to* $\overline{V'}$*. Then* $w^{HN}_{\overline{V'}}\circ\varphi\geqslant w^{HN}_{\overline{V}}$*.*

Given a collection $\mathcal{A}$ of linear subspaces of V, the $(+,\cap)$*-algebra generated by* $\mathcal{A}$ is the smallest collection $\mathcal{U}$ of linear subspaces of V such that

(i) $\mathcal{A}\subseteq\mathcal{U}$ and $\{0\},V\in\mathcal{U}$;
(ii) for all $U_1,U_2\in\mathcal{U}$ we have $U_1+U_2\in\mathcal{U}$, $U_1\cap U_2\in\mathcal{U}$.

Theorem 1.7. *Let* $\overline{V}=(V,w_1,\dots,w_s)$ *be a finite-dimensional* s*-valued* K*-vector space. Then the subspaces in the Harder-Narasimhan filtration of* $\overline{V}$ *belong to the* $(+,\cap)$*-algebra generated by the subspaces occurring in the unweighted filtrations of* $w_1,\dots,w_s$*.*

Let $\mathcal{A}$ consist of the spaces in the unweighted filtrations of $w_1,\dots,w_s$. We can decompose the $(+,\cap)$-algebra generated by $\mathcal{A}$ as $\cup_{m\geqslant 0}\mathcal{A}_m$, where the collections $\mathcal{A}_m$ $(m=0,1,,\dots,)$ are defined inductively by

$$\mathcal{A}_0:=\mathcal{A}\cup\{\emptyset,V\},\quad \mathcal{A}_{m+1}:=\{U_1+U_2,\,U_1\cap U_2\,:\,U_1,U_2\in\cup_{l\leqslant m}\mathcal{A}_l\}\ \text{for } m\geqslant 0.$$

We define the *depth* of $U \in \cup_{m \geqslant 0} \mathcal{A}_m$ to be the smallest m such that $U \in \mathcal{A}_m$. We are interested in effective upper bounds for the depths of the spaces of the Harder-Narasimhan filtration of $\overline{V}$, but unfortunately, our arguments do not provide these. The importance of such effective depth bounds would be that they allow us to compute the Harder-Narasimhan valuation of $\overline{V}$. Ideas from Vojta [16] suggest the following

Problem 1.8. *Given an n-dimensional s-valued K-vector space $\overline{V}$, can the depths of the spaces in the Harder-Narasimhan filtration of $\overline{V}$ be bounded above in terms of n and s only?*

In Section 7 we describe an algorithm to compute the Harder-Narasimhan valuation of an s-weighted vector space $\overline{V}$, based on other principles than effective depth bounds. The idea is that the first space V_1 in the Harder-Narasimhan filtration of $\overline{V}$ can be easily computed if $\dim V_1 = 1$. Then one can make a reduction to this special case by applying our result Theorem 1.4, (1.8) on exterior powers. In case the underlying field K is an algebraic number field, we give (Theorem 7.4) explicit upper bounds for the heights of the subspaces occurring in the Harder-Narasimhan filtration of $\overline{V}$.

The remainder of our paper is organized as follows. In Section 2 we have collected some basic facts. In Section 3 we deduce some properties of the $*$-operator introduced above. In Section 4 we prove some convergence results for sequences of valuations. In Section 5 we prove Theorem 1.1 and in Section 6 we deduce Theorems 1.5–1.7 and Corollary 1.2. In Section 7 we describe our method to compute the Harder-Narasimhan valuation of an s-valued vector space, and give upper bounds for the heights of the spaces in the Harder-Narasimhan filtration in case K is a number field.

2. Basic facts

For convenience of the reader we have recalled the proofs of some well-known facts about multi-valued vector spaces. Throughout this paper, K is any field. For a subset $\mathcal{A}$ of a K-vector space V we denote by $\operatorname{span} \mathcal{A}$ the K-linear subspace of V generated by $\mathcal{A}$. The collection of valuations on V is denoted by $\mathcal{W}(V)$. A valuation w on V is said to be constant if there is $\mu \in \mathbb{R}$ such that $w(x) = \mu$ for $x \in V \setminus \{0\}$. In this situation we will be sloppy and write $w = \mu$. More generally, given reals λ, μ on V with $\lambda \geqslant 0$, we define the valuation $\lambda w + \mu$ on V by $(\lambda w + \mu)(x) := \lambda w(x) + \mu$ for $x \in V \setminus \{0\}$.

2.1. **Weights, subspaces, quotients, degrees, slopes.** In what follows, V is a non-zero K-vector space of finite dimension.

Let w be a valuation on V. Let $\alpha_1 > \cdots > \alpha_r$ be the distinct values assumed by w on $V \setminus \{0\}$. Then the (unweighted) *filtration* of w is the strictly increasing sequence of linear subspaces of V,

$$(0) = F_0 \underset{\neq}{\subseteq} F_1 \underset{\neq}{\subseteq} \cdots \underset{\neq}{\subseteq} F_r = V \quad \text{where } F_i := \{x \in V : w(x) \geqslant \alpha_i\}, \tag{1.2}$$

and the *weighted filtration* of w is

$$\Big((0) = F_0 \underset{\neq}{\subseteq} F_1 \underset{\neq}{\subseteq} \cdots \underset{\neq}{\subseteq} F_r = V;\ \alpha_1 > \cdots > \alpha_r\Big). \tag{1.3}$$

This weighted filtration uniquely determines w. With the help of (1.3) we define the weight of V,

$$\begin{aligned} w(V) \ &:= \ \sum_{i=1}^{r} \alpha_i(\dim F_i - \dim F_{i-1}) \\ &= \ \sum_{i=1}^{r-1} (\alpha_i - \alpha_{i+1}) \dim F_i + \alpha_r \dim V. \end{aligned}$$

For $\beta \in \mathbb{R} \cup \{\infty\}$, not necessarily in the value set of w, define the linear subspace of V,

$$F_\beta^{(w)} := \{x \in V : w(x) \geqslant \beta\}.$$

Let $\beta_0 := \infty$ and let $\{\beta_1 > \cdots > \beta_t\}$ be any finite set of reals containing the values assumed by w on $V \setminus \{0\}$. Then

$$w(x) = \beta_i \ \text{ for } x \in F_{\beta_i}^{(w)} \setminus F_{\beta_{i-1}}^{(w)},\ i = 1, \ldots, t, \tag{2.1}$$

$$w(V) = \sum_{i=1}^{t-1} (\beta_i - \beta_{i+1}) \dim F_{\beta_i}^{(w)} + \beta_t \dim V. \tag{2.2}$$

We define $w(V) := 0$ if $V = (0)$; then in this case (2.1) and (2.2) are trivially true.

The following lemma will be useful. We assume henceforth that V is non-zero. Given two valuations w_1, w_2 on V we write $w_1 \leqslant w_2$ or $w_2 \geqslant w_1$ if $w_1(x) \leqslant w_2(x)$ for all $x \in V$, and $w_1 < w_2$ or $w_2 > w_1$ if $w_2 \geqslant w_1$ and $w_1 \neq w_2$.

Lemma 2.1. *Let w_1, w_2 be valuations on V with $w_1 < w_2$. Then $w_1(V) < w_2(V)$.*

Proof. Let $\{\beta_1 > \cdots > \beta_t\}$ be the union of the sets of values assumed by w_1 and w_2, respctively, on $V \setminus \{0\}$. By $w_1 < w_2$ and (2.1) we have $F_{\beta_i}^{(w_1)} \subseteq F_{\beta_i}^{(w_2)}$ for $i = 1, \ldots, t$, with strict inclusion for at least one i. Now (2.2) applied with w_1 and w_2 gives $w_1(V) < w_2(V)$. $\square$

Let U be a non-zero linear subspace of V. For $x \in V$, denote by x^U the image of x under the canonical map $V \to V/U$. The restriction $w|_U$ of w to U defines a valuation on U, while w^U, given by

$$w^U(y) := \max\{w(x) : x \in V,\, x^U = y\} \text{ for } y \in V/U$$

defines a valuation on V/U. Noting that $\{x \in U : w(x) \geqslant \alpha_i\} = U \cap F_i$ $(i = 1, \dots, r)$, while

$$\begin{aligned} \{y \in V/U : w^U(y) \geqslant \alpha_i\} &= \{x^U : x \in V,\, \exists z \in U \text{ with } w(x+z) \geqslant \alpha_i\} \\ &= (F_i + U)/U \quad (i = 1, \dots, r), \end{aligned}$$

it follows at once from (2.2) that

$$w|_U(U) = \sum_{i=1}^{r-1} (\alpha_i - \alpha_{i+1}) \dim(U \cap F_i) + \alpha_r \dim U, \tag{2.3}$$

$$w^U(V/U) = \sum_{i=1}^{r-1} (\alpha_i - \alpha_{i+1}) \dim((U + F_i)/U) + \alpha_r \dim(V/U), \tag{2.4}$$

and thus,

$$w^U(V/U) = w(V) - w|_U(U). \tag{2.5}$$

For convenience, for a linear subspace U of V we write $w(U)$ instead of $w|_U(U)$. Then for any two linear subspaces U_1, U_2 of V we have

$$w(U_1 + U_2) + w(U_1 \cap U_2) \geqslant w(U_1) + w(U_2). \tag{2.6}$$

This follows easily from (2.3) and from

$$\begin{aligned} \dim((U_1 + U_2) \cap F) &\geqslant \dim((U_1 \cap F) + (U_2 \cap F)) \\ &= \dim(U_1 \cap F) + \dim(U_2 \cap F) - \dim(U_1 \cap U_2 \cap F) \end{aligned}$$

for any linear subspace F of V, with equality if $F = V$.

Let $\overline{V} = (V, w_1, \dots, w_s)$ be a non-zero, finite-dimensional s-valued K-vector space. The *degree* and *slope* of $\overline{V}$ are given by respectively

$$d(\overline{V}) := \sum_{i=1}^{s} w_i(V), \qquad \mu(\overline{V}) := \frac{d(\overline{V})}{\dim V}.$$

Let U be a linear subspace of V. Then the corresponding s-valued subspace $\overline{U}$ of $\overline{V}$ and s-valued quotient $\overline{V}/\overline{U}$ are given by

$$\overline{U} := (U, w_1|_U, \ldots, w_s|_U), \quad \overline{V}/\overline{U} := (V/U, w_1^U, \ldots, w_s^U).$$

From (2.5) we infer at once that

$$d(\overline{V}/\overline{U}) = d(\overline{V}) - d(\overline{U}). \tag{2.7}$$

Further, by (2.6) we have for any two linear subspaces U_1, U_2 of V,

$$d(\overline{U_1 + U_2}) + d(\overline{U_1 \cap U_2}) \geqslant d(\overline{U}_1) + d(\overline{U}_2). \tag{2.8}$$

2.2. **Semistability, Harder-Narasimhan valuation.** Let $\overline{V} = (V, w_1, \ldots, w_s)$ be a non-zero s-valued K-vector space of dimension n. We say that $\overline{V}$ is *semistable* if $\mu(\overline{U}) \leqslant \mu(\overline{V})$ for every non-zero linear subspace U of V. In this case, the Harder-Narasimhan valuation of $\overline{V}$ is given by $w_{\overline{V}}^{HN}(x) := \mu(\overline{V})$ for $x \in V \setminus \{0\}$.

Henceforth, we do not require that $\overline{V}$ is semistable and construct the Harder-Narasimhan valuation in this general case. The basic tool is the following lemma.

Lemma 2.2. *There is a unique, non-zero linear subspace $D = D(\overline{V})$ of V (called the **maximal destabilizing subspace of** $\overline{V}$) such that for every non-zero linear subspace U of V we have*

$$\mu(\overline{U}) \leqslant \mu(\overline{D}) \quad \text{if } U \subseteq D, \quad \mu(\overline{U}) < \mu(\overline{D}) \quad \text{if } U \not\subseteq D.$$

Proof. It is clear that a subspace D as in the lemma is unique. We prove the existence of such a subspace. Let μ be the maximum of the quantities $\mu(\overline{U})$, taken over all non-zero linear subspaces U of V. Inequality (2.8) implies that if U_1, U_2 are any two non-zero linear subspaces of V with $\mu(\overline{U}_1) = \mu(\overline{U}_2) = \mu$, then

$$\begin{aligned} d(\overline{U_1 + U_2}) &\geqslant d(\overline{U}_1) + d(\overline{U}_2) - d(\overline{U_1 \cap U_2}) \\ &\geqslant \mu\Big(\dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)\Big) = \mu \dim(U_1 + U_2), \end{aligned}$$

hence $\mu(\overline{U_1 + U_2}) = \mu$. Now let D be the sum of all non-zero linear subspaces U of V with $\mu(U) = \mu$. Then clearly, $\mu(\overline{D}) = \mu \geqslant \mu(\overline{U})$ for every non-zero linear subspace U of V, with strict inequality if $U \not\subseteq D$. □

We note that by (2.7) we have for $D = D(\overline{V})$,

$$\mu(\overline{U}/\overline{D}) = \frac{d(\overline{U}) - d(\overline{D})}{\dim U - \dim D} = \frac{\mu(\overline{U}) \dim U - \mu(\overline{D}) \dim D}{\dim U - \dim D} < \mu(\overline{D}) \tag{2.9}$$

$$\text{for every linear subspace } U \text{ of } V \text{ with } U \underset{\neq}{\supset} D.$$

Now let $\overline{V}$ be an s-valued vector space, which is not necessarily semistable. We construct a filtration

$$(0) = V_0 \underset{\neq}{\subset} V_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} V_r = V \tag{2.10}$$

by taking

$$V_1 = D(\overline{V}), \quad V_2/V_1 = D(\overline{V}/\overline{V_1}), \quad V_3/V_2 = D(\overline{V}/\overline{V_2}), \dots .$$

It is clear that

$$\overline{V_1},\ \overline{V_2}/\overline{V_1},\ \dots,\ \overline{V}/\overline{V_{r-1}} \text{ are semistable}$$

and moreover, by (2.9), using $\overline{V_3}/\overline{V_2} \cong (\overline{V_3}/\overline{V_1})/(\overline{V_2}/\overline{V_1})$ etc.,

$$\mu(\overline{V_1}) > \mu(\overline{V_2}/\overline{V_1}) > \cdots > \mu(\overline{V}/\overline{V_{r-1}}).$$

We call (2.10) the *Harder-Narasimhan filtration* of $\overline{V}$ and

$$\Big((0) = V_0 \underset{\neq}{\subset} V_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} V_r = V,\ \mu(\overline{V_1}) > \mu(\overline{V_2}/\overline{V_1}) > \cdots > \mu(\overline{V}/\overline{V_{r-1}})\Big)$$

the *weighted Harder-Narasimhan filtration* of $\overline{V}$. The associated *Harder-Narasimhan valuation* on V is then defined by

$$w^{HN}_{\overline{V}}(x) := \mu(\overline{V_i}/\overline{V_{i-1}}) \text{ for } x \in V_i \setminus V_{i-1},\ i = 1, \dots, r.$$

Remark 2.3. Let $\overline{V} = (V, w_1, \dots, w_s)$ be an s-valued vector space. The following facts can be easily verified:

(i) If $s = 1$, $\overline{V} = (V, w_1)$, then $w^{HN}_{\overline{V}} = w_1$.

(ii) $w^{HN}_{\overline{V}/\overline{V_1}} = (w^{HN}_{\overline{V}})^{V_1}$.

(iii) For the weight of V with respect to the Harder-Narasimhan valuation of $\overline{V}$ we have $w^{HN}_{\overline{V}}(V) = d(\overline{V})$.

(iv) Let $\lambda \in \mathbb{R}_{\geqslant 0}$, $\mu_1, \dots, \mu_s \in \mathbb{R}$ and $\overline{V}' = (V, \lambda w_1 + \mu_1, \dots, \lambda w_s + \mu_s)$. Then $w^{HN}_{\overline{V}'} = \lambda w^{HN}_{\overline{V}} + \mu_1 + \cdots + \mu_s$.

2.3. **Adapted bases.** Let V be a non-zero, finite-dimensional vector space over a field K, let w be a valuation on V, and let $(0) = F_0 \underset{\neq}{\subset} F_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} F_r = V$ be its filtration (see (1.2)). Further, let $\{f_j : j \in I\}$ with I a finite index set be a basis of V.

We say that $\{f_j : j \in I\}$ is a basis of V *adapted to* w if it contains precisely $\dim F_i$ vectors from F_i, for $i = 1, \dots, r$.

Let β_j $(j \in I)$ be reals. We say that the valuation w *is given by* (f_j, β_j) $(j \in I)$ if whenever we express a non-zero $x \in V$ as $\sum_{j \in I} \xi_j f_j$ with $\xi_j \in K$, we have

$$w(x) = \min\{\beta_j : j \in I, \xi_j \neq 0\}.$$

Lemma 2.4. *Let V be a non-zero, finite-dimensional vector space over a field K, w a valuation on V, and $\{f_j : j \in I\}$ with I a finite index set a basis of V.*
(i) $\sum_{j \in I} w(f_j) \leqslant w(V)$*;*
(ii) $\{f_j : j \in I\}$ *is adapted to* $w \Leftrightarrow \sum_{j \in I} w(f_j) = w(V)$*;*
(iii) $\{f_j : j \in I\}$ *is adapted to* $w \Leftrightarrow w$ *is given by* $(f_j, w(f_j))$ $(j \in I)$*, i.e., if* $x = \sum_{j \in I} \xi_j f_j$ *with* $\xi_j \in K$*, not all* 0*, then* $w(x) = \min\{w(f_j) : \xi_j \neq 0\}$.

Proof. We assume without loss of generality that the given basis of V is $\{f_1, \dots, f_n\}$ and that $w(f_1) \geqslant \cdots \geqslant w(f_n)$. Let (1.3) be the weighted filtration of w. For $i = 0, \dots, r$, let $e_i := \#\Big(\{f_1, \dots, f_n\} \cap F_i\Big)$. Then $e_i \leqslant d_i := \dim F_i$ for $i = 0, \dots, r$, and $e_0 = d_0 = 0$, $e_r = d_r = n$. Further, $w(f_j) = \alpha_i$ for $e_{i-1} < j \leqslant e_i$, $i = 1, \dots, r$.

(i), (ii) We have $\sum_{j=1}^{n} w(f_j) = \sum_{i=1}^{r} (e_i - e_{i-1})\alpha_i = \sum_{i=1}^{r-1} e_i(\alpha_i - \alpha_{i+1}) + n\alpha_r$. This is $\leqslant w(V)$, and equal to $w(V)$ precisely if $e_i = d_i$ for $i = 1, \dots, r-1$, i.e., if $\{f_j : j \in I\}$ is adapted to w.

(iii) w is given by $(f_j, w(f_j))$ $(j \in I) \Leftrightarrow F_i \subseteq \mathrm{span}\{f_j : j \leqslant e_i\}$ for $i = 1, \dots, r \Leftrightarrow e_i = d_i$ for $i = 1, \dots, r$. □

Lemma 2.5. *Let U be a proper, non-zero linear subspace of V. Further, let $\{f_1, \dots, f_n\}$ be a basis of V such that $\{f_1, \dots, f_m\}$ is a basis of U. Then the following two assertions are equivalent:*
(i) $\{f_1, \dots, f_n\}$ *is adapted to* w*;*
(ii) $\{f_1, \dots, f_m\}$ *is adapted to* $w|_U$*,* $\{f_{m+1}^U, \dots, f_n^U\}$ *is a basis of V/U adapted to* w^U*, and* $w^U(f_i^U) = w(f_i)$ *for* $i = m+1, \dots, n$.

Proof. By (2.5), Lemma 2.4 (i) and the definition of w^U we have

$$w(V) = w|_U(U) + w^U(V/U) \geqslant \sum_{i=1}^{m} w(f_i) + \sum_{i=m+1}^{n} w^U(f_i^U) \geqslant \sum_{i=1}^{n} w(f_i).$$

So $w(V) = \sum_{i=1}^n w(f_i)$ if and only if $w|_U(U) = \sum_{i=1}^m w(f_i)$, $w^U(V/U) = \sum_{i=m+1}^n w(f_i^U)$ and $w^U(f_i^U) = w(f_i)$ for $i = m+1, \ldots, n$. Apply Lemma 2.4 (ii). □

To our knowledge, the following important observation occurred for the first time in a paper by Corvaja and Zannier [3, Lemma 3.2], but it was known before. [1]

Lemma 2.6. *Let V be a non-zero n-dimensional K-vector space and w_1, w_2 two valuations on V. Then V has a basis $\{f_1, \ldots, f_n\}$ adapted to both w_1, w_2.*

Proof. We proceed by induction on n. For $n = 1$ the assertion is obviously true. Let $n \geqslant 2$ and suppose Lemma 2.6 is true for all vector spaces of dimension $< n$. If both valuations w_1, w_2 are constant (i.e., on $V \setminus \{0\}$), then any basis will do. Suppose w_1 is not constant. Then w_1 has a filtration $(0) \subsetneqq \cdots \subsetneqq F_{r-1} \subsetneqq F_r = V$, where F_{r-1} is non-zero and strictly smaller than V. By the induction hypothesis, $U := F_{r-1}$ has a basis $\{f_1, \ldots, f_m\}$ adapted to both $w_1|_U$ and $w_2|_U$. Choose $f_{m+1}, \ldots, f_n \in V \setminus U$ such that $\{f_{m+1}^U, \ldots, f_n^U\}$ is a basis of V/U adapted to w_2^U and $w_2^U(f_i^U) = w_2(f_i)$ for $i = m+1, \ldots, n$. Then by Lemma 2.5, the set $\{f_1, \ldots, f_n\}$ is a basis of V adapted to w_2. But by its very construction this basis contains precisely $\dim F_i$ vectors from F_i for $i = 1, \ldots, r-1$, and also $\dim F_r = n$ vectors from $F_r = V$. Hence this basis is adapted to w_1 as well. □

Remark 2.7. This can not be extended to more than two valuations.

2.4. **Exterior powers, symmetric powers, base extensions, direct sums and tensor products.** For integers k, n with $1 \leqslant k \leqslant n$ we denote by $\mathcal{I}_{n,k}$ the collection of integer tuples $(i_1, \ldots, i_k)$ with $1 \leqslant i_1 < \cdots < i_k \leqslant n$. For positive integers k, n, we denote by $\mathcal{J}_{n,k}$ the collection of integer tuples $(i_1, \ldots, i_k)$ with $1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n$. As before, K is any field.

Lemma 2.8. (i) *Let V be a K-vector space of dimension $n > 0$, w a valuation on V, and $\{f_1, \ldots, f_n\}$ a basis of V adapted to w. Then the valuations $\wedge^k w$ on $\wedge^k V$ ($1 \leqslant k \leqslant n$), $\mathrm{S}^k w$ on $\mathrm{S}^k V$ ($k \geqslant 1$) and $w \otimes L$ on the L-vector space $V \otimes L$ (where L*

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[1] It was mentioned to me several years earlier by Roberto Ferretti (personal communication).

is any extension field of K) are given by respectively

$$\big(f_{i_1}\wedge\cdots\wedge f_{i_k},\, w(f_{i_1})+\cdots+w(f_{i_k})\big)\ \big((i_1,\dots,i_k)\in\mathcal{I}_{n,k}\big);$$
$$\big(f_{i_1}\cdots f_{i_k},\, w(f_{i_1})+\cdots+w(f_{i_k})\big)\ \big((i_1,\dots,i_k)\in\mathcal{J}_{n,k}\big);$$
$$\big(f_i\otimes 1,\, w(f_i)\big)\ (i=1,\dots,n).$$

(ii) *Let V, V' be K-vector spaces of dimensions n, m, respectively, w a valuation on V, w' a valuation on V', $\{f_1,\dots,f_n\}$ a basis of V adapted to w, and $\{g_1,\dots,g_m\}$ a basis of V' adapted to w'. Then the valuations $w\oplus w'$ on $V\oplus V'$ and $w\otimes w'$ on $V\otimes V'$ are given by respectively*

$$\big(\,(f_i,0),\, w(f_i)\,\big)\ (i=1,\dots,n)\ \text{ and }\big(\,(0,g_j),\, w'(g_j)\,\big)\ (j=1,\dots,m);$$
$$\big(f_i\otimes g_j,\, w(f_i)+w'(g_j)\big)\ \big(i=1,\dots,n,\ j=1,\dots,m\big).$$

Proof. We prove only the result for the tensor product; the proofs of the other results in the lemma are entirely similar. Denote by u the valuation on $V\otimes V'$ given by $\big(f_i\otimes g_j,\, w(f_i)+w'(g_j)\big)\ \big(i=1,\dots,n,\ j=1,\dots,m\big)$. Let $x=\sum_{i\in I}\xi_i f_i$, $y=\sum_{j\in J}\eta_j g_j$ be non-zero elements of V,V', where $\xi_i,\eta_j\in K^*$ for all $i\in I, j\in J$. Then

$$\begin{aligned} u(x\otimes y) &= \min_{(i,j)\in I\times J}(w(f_i)+w'(g_j)) \\ &\geqslant \big(\min_{i\in I} w(f_i)\big)+\big(\min_{j\in J} w'(g_j)\big)=w(x)+w'(y). \end{aligned}$$

In view of definition (1.5), applied to the tensor product, this means that $u\geqslant w\otimes w'$. To prove the reverse inequality, let $z\in V\otimes V'$ and write $z=\sum_{(i,j)\in H}\xi_{i,j}f_i\otimes g_j$, where H is a subset of $\{1,\dots,n\}\times\{1,\dots,m\}$ and $\xi_{i,j}\in K^*$ for $(i,j)\in H$. Then by definition (1.5),

$$(w\otimes w'))(z)\geqslant \min_{(i,j)\in H}(w\otimes w')(f_i\otimes g_j)\geqslant \min_{(i,j)\in H}(w(f_i)+w'(g_j))=u(z).$$

Hence $w\otimes w'\geqslant u$. □

Let $\overline{V}=(V,w_1,\dots,w_s)$ be an s-valued K-vector space of dimension $n>0$. Recall that the k-th exterior power ($1\leqslant k\leqslant n$), k-th symmetric power ($k\geqslant 1$) and tensor product with an extension field L of K of $\overline{V}$ are given by

$$\wedge^k\overline{V}:=(\wedge^k V,\wedge^k w_1,\dots,\wedge^k w_s),\quad \mathrm{S}^k\,\overline{V}:=(\mathrm{S}^k\,V,\mathrm{S}^k\,w_1,\dots,\mathrm{S}^k\,w_s),$$
$$\overline{V}\otimes L:=(V\otimes L,w_1\otimes L,\dots,w_s\otimes L).$$

Further, the direct sum, respectively tensor product over K, of two s-valued K-vector spaces $\overline{V} = (V, w_1, \dots, w_s)$, $\overline{V'} = (V', w'_1, \dots, w'_s)$ are given by

$$\overline{V} \oplus \overline{V'} := (V \oplus V', w_1 \oplus w'_1, \dots, w_s \oplus w'_s), \quad \overline{V} \otimes \overline{V'} = (V \otimes V', w_1 \otimes w'_1, \dots, w_s \otimes w'_s).$$

Using Lemmas 2.8 and 2.4, the following is not hard to show.

Corollary 2.9. *Let $\overline{V}$ be an s-valued K-vector space of dimension $n > 0$. Then*

(i) $d(\wedge^k \overline{V}) = \binom{n-1}{k-1} d(\overline{V})$, $\mu(\wedge^k \overline{V}) = k\mu(\overline{V})$ *for* $k \in \{1, \dots, n\}$;

(ii) $d(\mathrm{S}^k\, \overline{V}) = \frac{k}{n}\binom{n+k-1}{k-1} d(\overline{V})$, $\mu(\mathrm{S}^k\, \overline{V}) = k\mu(\overline{V})$ *for* $k \geqslant 1$;

(iii) $d(\overline{V} \otimes L) = d(\overline{V})$, $\mu(\overline{V} \otimes L) = \mu(\overline{V})$ *for any extension field L of K.*

Further, for any two finite dimensional s-valued K-vector spaces $\overline{V}$, $\overline{V'}$ we have

(iv) $d(\overline{V} \oplus \overline{V'}) = d(\overline{V}) + d(\overline{V'})$;

(v) $d(\overline{V} \otimes \overline{V'}) = \dim V' \cdot d(\overline{V}) + \dim V \cdot d(\overline{V'})$, $\mu(\overline{V} \otimes \overline{V'}) = \mu(\overline{V}) + \mu(\overline{V'})$.

3. THE $*$-OPERATOR

Let K be a field and V a K-vector space of finite dimension $n > 0$.

Recall that the $*$-operator on the collection $\mathcal{W}(V)$ of valuations of V is defined by

$$w_1 * w_2 := \inf\{w \in \mathcal{W}(V) : w \geqslant w_1 + w_2\} \text{ for } w_1, w_2 \in \mathcal{W}(V). \tag{1.7}$$

This binary operator is commutative, but if $\dim V \geqslant 2$ it is non-associative. To illustrate this, take two linearly independent vectors $f_1, f_2 \in V$, put $f_3 := f_1 + f_2$, define $U_i := \mathrm{span}\{f_i\}$ for $i = 1, 2, 3$, $U := \mathrm{span}\{f_1, f_2\}$, and for $i = 1, 2, 3$ define a valuation w_i on V by

$$w_i(x) = 1 \text{ for } x \in U_i \backslash \{0\},\ w_i(x) = 0 \text{ for } x \in V \setminus U_i.$$

It can be shown that $(w_1 * w_2) * w_3 \neq w_1 * (w_2 * w_3)$ by comparing their weighted filtrations: the weighted filtrations of $(w_1 * w_2) * w_3$, $w_1 * (w_2 * w_3)$ are

$$\Big((0) \underset{\neq}{\subset} U_3 \underset{\neq}{\subset} U \underset{\neq}{\subset} V,\ 2 > 1 > 0\Big), \quad \Big((0) \underset{\neq}{\subset} U_1 \underset{\neq}{\subset} U \underset{\neq}{\subset} V,\ 2 > 1 > 0\Big) \text{ if } \dim V \geqslant 3,$$
$$\Big((0) \underset{\neq}{\subset} U_3 \underset{\neq}{\subset} V,\ 2 > 1\Big), \quad \Big((0) \underset{\neq}{\subset} U_1 \underset{\neq}{\subset} V,\ 2 > 1\Big) \text{ if } \dim V = 2.$$

Below, we deduce some properties of the $*$-operator. Recall Lemma 2.6.

Lemma 3.1. *Let V be a K-vector space of dimension $n > 0$ and w_1, w_2 valuations on V. Further, let $\{f_1, \ldots, f_n\}$ be a basis of V adapted to w_1 and w_2. Then $w_1 * w_2$ is given by $(f_i, w_1(f_i) + w_2(f_i))$ $(i = 1, \ldots, n)$.*

Proof. We write $x \in V \setminus \{0\}$ as $\sum_{i \in I_x} \xi_i f_i$, where $I_x \subseteq \{1, \ldots, n\}$ and $\xi_i \in K^*$ for $i \in I_x$. Let u be the valuation given by $(f_i, w_1(f_i) + w_2(f_i))$ $(i = 1, \ldots, n)$. Then for $x \in V \setminus \{0\}$ we have

$$u(x) = \min_{i \in I_x}(w_1(f_i) + w_2(f_i)) \geqslant \min_{i \in I_x} w_1(f_i) + \min_{i \in I_x} w_2(f_i) = w_1(x) + w_2(x),$$

hence $u \geqslant w_1 * w_2$. Conversely, we have for $x \in V \setminus \{0\}$,

$$(w_1 * w_2)(x) \geqslant \min_{i \in I_x}(w_1 * w_2)(f_i) \geqslant \min_{i \in I_x}(w_1(f_i) + w_2(f_i)) = u(x),$$

hence $w_1 * w_2 \geqslant u$. □

Lemma 3.2. **(i)** *Let V be a K-vector space of dimension $n > 0$ and w_1, w_2 valuations on V. Then the following identities of valuations hold:*

$$\begin{aligned} \wedge^k(w_1 * w_2) &= (\wedge^k w_1) * (\wedge^k w_2) \quad \text{on } \wedge^k V \text{ for each } k \in \{1, \ldots, n\}, \\ \mathrm{S}^k(w_1 * w_2) &= (\mathrm{S}^k w_1) * (\mathrm{S}^k w_2) \quad \text{on } \mathrm{S}^k V \text{ for every positive integer } k, \\ (w_1 * w_2) \otimes L &= (w_1 \otimes L) * (w_2 \otimes L) \quad \text{on } V \otimes L \text{ for every extension field } L \text{ of } K. \end{aligned}$$

(ii) *Let w_1, w_2 be valuations on V and w_1', w_2' valuations on another non-zero finite-dimensional K-vector space V'. Then the following identities of valuations hold:*

$$\begin{aligned} (w_1 * w_2) \oplus (w_1' * w_2') &= (w_1 \oplus w_1') * (w_2 \oplus w_2') \quad \text{on } V \oplus V', \\ (w_1 * w_2) \otimes (w_1' * w_2') &= (w_1 \otimes w_1') * (w_2 \otimes w_2') \quad \text{on } V \otimes V'. \end{aligned}$$

Proof. Straightforward using Lemmas 2.8 and 3.1. We prove only the identity for the tensor product; the other identities are obtained in the same manner. Define the valuations $u_1 := (w_1 * w_2) \otimes (w_1' * w_2')$, $u_2 := (w_1 \otimes w_1') * (w_2 \otimes w_2')$. Let $\{f_1, \ldots, f_n\}$ be a basis of V adapted to both w_1 and w_2, hence also to $w_1 * w_2$, and let $\{g_1, \ldots, g_m\}$ be a basis of V' adapted to w_1' and w_2', hence to $w_1' * w_2'$. Write $I := \{1, \ldots, n\}$, $J := \{1, \ldots, m\}$. Then $\{f_i \otimes g_j : (i, j) \in I \times J\}$ is a basis of $V \otimes V'$, adapted to both u_1 and u_2, and moreover, $u_1(f_i \otimes g_j) = u_2(f_i \otimes g_j) = w_1(f_i) + w_2(f_i) + w_1'(g_j) + w_2'(g_j)$, for $i \in I$, $j \in J$. Together with Lemma 2.4 (iii) this implies $u_1 = u_2$. □

Lemma 3.3. *Let V be a non-zero finite-dimensional K-vector space.*

(i) *Let w_1, w_2 be valuations on V and λ, μ_1, μ_2 reals with $\lambda > 0$. Then $(\lambda w_1 + \mu_1) * (\lambda w_2 + \mu_2) = \lambda(w_1 * w_2) + \mu_1 + \mu_2$.*

(ii) *Let w_1, w_2 be valuations on V. Then $(w_1 * w_2)(V) = w_1(V) + w_2(V)$.*
(iii) *Let $\varphi : V \to V'$ be a linear map from V to another finite-dimensional K-vector space V', and let w_1, w_2 be valuations on V and w'_1, w'_2 valuations on V' such that $w'_i \circ \varphi \geqslant w_i$ for $i = 1, 2$. Then $(w'_1 * w'_2) \circ \varphi \geqslant w_1 * w_2$ for $i = 1, 2$.*
(iv) *Let w_1, w_2, w'_1, w'_2 be valuations on V with $w'_1 \geqslant w_1$, $w'_2 \geqslant w_2$. Then $w'_1 * w'_2 \geqslant w_1 * w_2$.*
(v) *Let w_1, w_2, w'_1, w'_2 be valuations on V. Then $|w_1 * w_2 - w'_1 * w'_2| \leqslant |w_1 - w'_1| + |w_2 - w'_2|$.*

Proof. (i) Trivial from definition.

(ii) Choose a basis $\{f_1, \ldots, f_n\}$ of V adapted to both w_1, w_2, take the sum over $i = 1, \ldots, n$ of $(w_1 * w_2)(f_i) = w_1(f_i) + w_2(f_i)$ and apply Lemma 2.4 (ii).

(iii) Applying definition (1.7) we get $(w'_1 * w'_2) \circ \varphi \geqslant w'_1 \circ \varphi + w'_2 \circ \varphi \geqslant w_1 + w_2$ and subsequently (iii).

(iv) Apply (iii) with $V' = V$ and φ the identity.

(v) Put $c_i := |w_i - w'_i|$ for $i = 1, 2$. Then $w_i \leqslant w'_i + c_i$ for $i = 1, 2$, hence by (iv),(i),

$$w_1 * w_2 \leqslant (w'_1 + c_1) * (w'_2 + c_2) = w'_1 * w'_2 + c_1 + c_2.$$

Likewise, $w'_1 * w'_2 \leqslant w_1 * w_2 + c_1 + c_2$. Hence $|w_1 * w_2 - w'_1 * w'_2| \leqslant c_1 + c_2$. □

Lemma 3.4. *Let V be a non-zero, finite-dimensional K-vector space and w_1, w_2 valuations on V. Then the subspaces in the filtration of $w_1 * w_2$ lie in the $(+, \cap)$-algebra generated by the subspaces in the filtrations of w_1 and w_2.*

Proof. Let $\alpha_1 > \cdots > \alpha_r$ be the values assumed by w_1, let $\beta_1 > \cdots > \beta_s$ be those assumed by w_2, let $\gamma_1 > \cdots > \gamma_t$ be those assumed by $w_1 * w_2$, and let $F_i := \{x \in V : w_1(x) \geqslant \alpha_i\}$, $G_j := \{x \in V : w_2(x) \geqslant \beta_j\}$ and $H_k := \{x \in V : (w_1 * w_2)(x) \geqslant \gamma_k\}$ be the corresponding subspaces in the filtrations of w_1, w_2 and $w_1 * w_2$, respectively. Take a basis $\{f_1, \ldots, f_n\}$ of V adapted to w_1 and w_2. Then by Lemma 3.1,

$$\begin{aligned} H_k &= \operatorname{span}\{f_l : w_1(f_l) + w_2(f_l) \geqslant \gamma_k\} \\ &= \sum_{\alpha_i + \beta_j \geqslant \gamma_k} \Big(\operatorname{span}\{f_l : w_1(f_l) \geqslant \alpha_i\} \cap \operatorname{span}\{f_l : w_2(f_l) \geqslant \beta_j\} \Big) \\ &= \sum_{\alpha_i + \beta_j \geqslant \gamma_k} \Big(F_i \cap G_j \Big), \end{aligned}$$

which clearly implies Lemma 3.4. □

Let again V be a non-zero finite-dimensional K-vector space and w a valuation on V. For a proper linear subspace U of V we define

$$\delta(U,w) := \max\big(0,\ \min\{w(x)-w(y) :\ x\in U,\ y\in V\setminus U\}\big). \tag{3.1}$$

One easily shows that if w has weighted filtration given by (1.3), then

$$\delta(U,w) = \begin{cases} \infty & \text{if } U=(0), \\ \alpha_i-\alpha_{i+1}>0 & \text{if } U=F_i \text{ for some } i\in\{1,\ldots,r-1\}, \\ 0 & \text{if } U\neq (0),F_1,\ldots,F_{r-1}. \end{cases} \tag{3.2}$$

Thus, if $U\subsetneq V$ then $\delta(U,w)>0$ if $U=(0)$ or U is in the filtration of w, and $\delta(U,w)=0$ otherwise. A consequence of this is, that if U is in the filtration of w and $x\in V\setminus U$, then $w^U(x^U)=w(x)$.

The next lemma gives a sufficient condition under which the $*$-operator commutes with taking restrictions or quotients. For a valuation w on a non-zero K-vector space V, we define $|w| := \max\{|w(x)| :\ x\in V\setminus\{0\}\}$.

Lemma 3.5. *Let V be a non-zero, finite-dimensional K-vector space and w_1,w_2 valuations on V and let U be a proper, non-zero linear subspace of V such that $\delta(U,w_1)>2|w_2|$. Then*

(i) $\delta(U,w_1*w_2)\geqslant\delta(U,w_1)-2|w_2|>0$,
(ii) $(w_1*w_2)|_U=(w_1|_U)*(w_2|_U)$, $(w_1*w_2)^U=w_1^U*w_2^U$,
(iii) $(w_1*w_2)(x)=(w_1^U*w_2^U)(x^U)$ *for* $x\in V\setminus U$.

Here in (ii), (iii), the $*$-operators on the left-hand sides are those on $\mathcal{W}(V)$, while the $*$-operators on the right-hand sides are those on $\mathcal{W}(U)$, $\mathcal{W}(V/U)$, respectively.

Proof. (i) By Lemma 3.3 (v) we have $|(w_1*w_2)(x)-w_1(x)|\leqslant|w_2|$ for $x\in V\setminus\{0\}$. So $(w_1*w_2)(x)-(w_1*w_2)(y)\geqslant w_1(x)-w_1(y)-2|w_2|$ for $x\in U$, $y\in V\setminus U$. Apply (3.1).

(ii) Choose a basis $\{f_1,\ldots,f_n\}$ of V adapted to both w_1, w_2. Our assumption implies that U is in the filtration of w_1. So $\{f_1,\ldots,f_n\}$ contains a basis of U, $\{f_1,\ldots,f_m\}$, say. By Lemmas 2.5 and 3.1, $\{f_1,\ldots,f_m\}$ is adapted to $w_1|_U$, $w_2|_U$ and $(w_1*w_2)|_U$. Now both $(w_1*w_2)|_U$ and $(w_1|_U)*(w_2|_U)$ are given by $\big(f_i,\ w_1(f_i)+w_2(f_i)\big)$ $(i=1,\ldots,m)$, hence are equal. The second assertion of (ii) can be proved in the same manner.

(iii) By (i), U is in the filtration of w_1*w_2. Hence for $x\in V\setminus U$ we have $(w_1*w_2)(x)=(w_1*w_2)^U(x^U)=(w_1^U*w_2^U)(x^U)$. □

4. Sequences of valuations

We will prove some convergence results for sequences of valuations. As before, K is any field and V a non-zero, finite-dimensional K-vector space.

We need an auxiliary result from linear algebra. Denote by $\mathcal{S}$ the collection of subsets of V of the shape

$$(V_1 \setminus W_1) \cup (V_2 \setminus W_2) \cup \cdots \cup (V_r \setminus W_r) \tag{4.1}$$

where $r \geqslant 1$ and $V_1, W_1, \dots, V_r, W_r$ are linear subspaces of V such that

$$(0) \subseteq W_1 \subseteq V_1 \subseteq W_2 \subseteq V_2 \subseteq \cdots \subseteq W_r \subseteq V_r \subseteq V.$$

Lemma 4.1. *Any non-decreasing sequence $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ of sets from $\mathcal{S}$ is eventually constant.*

Proof. Notice that in (4.1) we can delete $V_i \setminus W_i$ if $V_i = W_i$, while if $V_i = W_{i+1}$ we can shorten (4.1) using $(V_i \setminus W_i) \cup (V_{i+1} \setminus W_{i+1}) = V_{i+1} \setminus W_i$. By repeatedly applying this, we see that any non-empty element $\mathcal{F}$ of $\mathcal{S}$ can be expressed in the form (4.1) where $r \geqslant 1$ and $V_1, W_1, \dots, V_r, W_r$ are linear subspaces of V such that

$$(0) \subseteq W_1 \underset{\neq}{\subseteq} V_1 \underset{\neq}{\subseteq} W_2 \underset{\neq}{\subseteq} V_2 \underset{\neq}{\subseteq} \cdots \underset{\neq}{\subseteq} W_r \underset{\neq}{\subseteq} V_r \subseteq V.$$

Then $\operatorname{span} \mathcal{F} = V_r$. Further,

$$V_r \setminus \mathcal{F} = (W_r \setminus V_{r-1}) \cup \cdots \cup (W_2 \setminus V_1) \cup W_1,$$

hence $\operatorname{span}(V_r \setminus \mathcal{F}) = W_r \underset{\neq}{\subseteq} \operatorname{span} \mathcal{F}$.

We now prove Lemma 4.1, where we proceed by induction on the dimension of V. If $\dim V = 1$ our lemma is clear. Suppose $\dim V > 1$. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be a non-decreasing sequence from $\mathcal{S}$. Then $\operatorname{span} \mathcal{F}_1 \subseteq \operatorname{span} \mathcal{F}_2 \subseteq \cdots$. Hence there is i_0 such that for $i \geqslant i_0$, $\operatorname{span} \mathcal{F}_i = V_0$ is independent of i. Further, $\operatorname{span}(V_0 \setminus \mathcal{F}_{i_0}) \supseteq \operatorname{span}(V_0 \setminus \mathcal{F}_{i_0+1}) \supseteq \cdots$. Hence there is $i_1 \geqslant i_0$ such that for $i \geqslant i_1$, $\operatorname{span}(V_0 \setminus \mathcal{F}_i) = W_0$ is independent of i, while moreover, $W_0 \underset{\neq}{\subseteq} V_0$. This means that for $i \geqslant i_1$ we have $\mathcal{F}_i = (V_0 \setminus W_0) \cup \mathcal{G}_i$, where $\mathcal{G}_i$ belongs to $\mathcal{S}$ and is contained in W_0. Now apply the induction hypothesis to the sequence $\{\mathcal{G}_i\}$. □

Lemma 4.2. *Let $(w_{1,m})_{m=0}^{\infty}$, $(w_{2,m})_{m=0}^{\infty}$ be two sequences of valuations on V such that*

$$w_{1,m} - w_{2,m} \geqslant w_{1,m+1} - w_{2,m+1} \text{ pointwise on } V \setminus \{0\} \text{ for } m \geqslant 0,$$

$$\text{for every } x \in V \setminus \{0\} \text{ there is } m \geqslant 0 \text{ such that } w_{1,m}(x) \leqslant w_{2,m}(x).$$

Then there is m_0 *such that* $w_{1,m} \leqslant w_{2,m}$ *for* $m \geqslant m_0$.

Proof. For $m = 0, 1, \ldots$, define

$$\mathcal{F}_m := \{x \in V \setminus \{0\} : w_{1,m}(x) \leqslant w_{2,m}(x)\}.$$

We first show that $\mathcal{F}_m$ belongs to $\mathcal{S}$ for $m = 1, 2, \ldots$. Fix m and let the weighted filtration of $w_{1,m}$ be given by (1.3). Thus, $w_{1,m}$ assumes the values $\alpha_1 > \cdots > \alpha_r$ on $V \setminus \{0\}$ and $w_{1,m}(x) = \alpha_i$ if and only if $x \in F_i \setminus F_{i-1}$. Define the subspaces $G_i := \{x \in V : w_{2,m}(x) \geqslant \alpha_i\}$ for $i = 1, \ldots, r$. Then

$$\mathcal{F}_m = \bigcup_{i=1}^{r} \Big((F_i \setminus F_{i-1}) \cap G_i \Big) = \bigcup_{i=1}^{r} \Big((F_i \cap G_i) \setminus (F_{i-1} \cap G_i) \Big)$$

which is indeed a set in $\mathcal{S}$ since $(0) = F_0 \underset{\neq}{\subseteq} F_1 \underset{\neq}{\subseteq} \cdots \underset{\neq}{\subseteq} F_r$ and $G_1 \subseteq \cdots \subseteq G_r$.

Our assumptions on the sequences $(w_{1,m})_{m=0}^{\infty}$, $(w_{2,m})_{m=0}^{\infty}$ imply that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ and $\bigcup_{m=0}^{\infty} \mathcal{F}_m = V \setminus \{0\}$. By the previous lemma there is m_0 such that $\mathcal{F}_m = \mathcal{F}_{m_0}$ for all $m \geqslant m_0$. Hence $\mathcal{F}_m = V \setminus \{0\}$ for $m \geqslant m_0$, which means precisely that $w_{1,m} \leqslant w_{2,m}$ for $m \geqslant m_0$. □

Given a sequence of valuations $(w_m)_{m=0}^{\infty}$ and a valuation w on V, we write $w_m \downarrow w$ if $w_0 \geqslant w_1 \geqslant \cdots$ and $\lim_{m\to\infty} w_m(x) = w(x)$ for $x \in V \setminus \{0\}$.

Lemma 4.3. *Let* $(w_{1,m})_{m=0}^{\infty}$, $(w_{2,m})_{m=0}^{\infty}$ *be sequences of valuations, and* w_1, w_2 *valuations on* V *such that* $w_{i,m} \downarrow w_i$ *for* $i = 1, 2$. *Then* $w_{1,m} * w_{2,m} \downarrow w_1 * w_2$.

Proof. By Lemma 3.3 (iv) the sequence $(w_{1,m} * w_{2,m})_{m=0}^{\infty}$ is non-increasing. We prove that the limit is $w_1 * w_2$. Let $\varepsilon > 0$. For every $x \in V \setminus \{0\}$ and $i = 1, 2$ there is m_i such that $w_{i,m_i}(x) \leqslant w_i(x) + \varepsilon$. So by Lemma 4.2, there is m_0 such that $w_{i,m} \leqslant w_i + \varepsilon$ for $i = 1, 2$, $m \geqslant m_0$. Now parts (iv),(i) of Lemma 3.3 yield

$$w_1 * w_2 \leqslant w_{1,m} * w_{2,m} \leqslant (w_1 + \varepsilon) * (w_2 + \varepsilon) = w_1 * w_2 + 2\varepsilon \quad \text{for } m \geqslant m_0.$$

This proves Lemma 4.3. □

Lemma 4.4. *Let* $(w_m)_{m=0}^{\infty}$ *be a sequence of valuations on* V *with* $w_0 \geqslant w_1 \geqslant \cdots$. *Then*

$$U := \{x \in V : \lim_{m\to\infty} w_m(x) > -\infty\}$$

is a linear subspace of V, *and if* $U \neq (0)$, $U \neq V$, *then* $\lim_{m\to\infty} \delta(U, w_m) = \infty$.

Proof. It is obvious that U is a linear subspace of V. Suppose $U \neq (0)$, $U \neq V$. We have to prove that for every $A > 0$ there is m_0 such that $\delta(U, w_m) > A$ for all $m \geqslant m_0$, or equivalently, $w_m(x) - w_m(y) > A$ for all $x \in U$, $y \in V \setminus U$, $m \geqslant m_0$.

We obtain a valuation u on U by setting $u(x) := \lim_{m\to\infty} w_m(x)$ for all non-zero $x \in U$. Hence $w_m(x) \geqslant C$ for $x \in U$, $m \geqslant 1$, where C is the minimum of u. Let $A > 0$ and define valuations w'_m $(m = 0, 1, \ldots)$ on V by

$$w'_m(x) := w_m(x) \text{ for } x \in U, \quad w'_m(x) := C - A \text{ for } x \in V \setminus U.$$

Clearly $w_m - w'_m$ $(m = 0, 1, \ldots)$ is non-increasing, and for every $x \in V \setminus \{0\}$ there is an integer m such that $w_m(x) \leqslant w'_m(x)$. So by Lemma 4.2 there is m_0 such that $w_m \leqslant w'_m$ for $m \geqslant m_0$, implying $w_m(y) \leqslant C - A$ for $y \in V \setminus U$, $m \geqslant m_0$. This implies $w_m(x) - w_m(y) > C - (C - A) = A$ for $x \in U$, $y \in V \setminus U$, $m \geqslant m_0$. □

5. Proof of Theorem 1.1

We first prove two lemmas and an important proposition. For a non-zero s-valued K-vector space $\overline{V} = (V, w_1, \ldots, w_s)$ we define an operator $[\cdot; \overline{V}]$ on the collection $\mathcal{W}(V)$ of valuations on V by

$$[w; \overline{V}] := (\cdots((w * w_1) * w_2) \cdots) * w_s \quad \text{for } w \in \mathcal{W}(V).$$

Notice that for a linear subspace U of V with $(0) \subsetneq U \subsetneq V$ this gives

$$\begin{aligned} [w'; \overline{U}] &:= (\cdots((w' * (w_1|_U)) * (w_2|_U)) * \cdots) * (w_s|_U) \quad \text{for } w' \in \mathcal{W}(U), \\ [w''; \overline{V}/\overline{U}] &:= (\cdots((w'' * w_1^U) * w_2^U) * \cdots) * w_s^U \quad \text{for } w'' \in \mathcal{W}(V/U). \end{aligned}$$

In the two lemmas below we have collected some properties of these operators. Henceforth, we fix an s-valued K-vector space $\overline{V} = (V, w_1, \ldots, w_s)$ of finite dimension $n > 0$.

Lemma 5.1. (i) *Let* $w \in \mathcal{W}(V)$. *Then* $[w; \overline{V}](V) = w(V) + d(\overline{V})$.
(ii) *Let* $\overline{V}' := (V, w_1 + \mu_1, \ldots, w_s + \mu_s)$ *for some* $\mu_1, \ldots, \mu_s \in \mathbb{R}$. *Then* $[w; \overline{V}'] = [w; \overline{V}] + (\mu_1 + \cdots + \mu_s)$ *for* $w \in \mathcal{W}(V)$.
(iii) *Let* $u_1, u_2 \in \mathcal{W}(V)$ *with* $u_1 \geqslant u_2$. *Then* $[u_1; \overline{V}] \geqslant [u_2; \overline{V}]$.
(iv) *Let* $u_1, u_2 \in \mathcal{W}(V)$. *Then* $|\,[u_1; \overline{V}] - [u_2; \overline{V}]\,| \leqslant |u_1 - u_2|$.
(v) *Let* $(u_m)_{m=0}^{\infty}$ *be a sequence of valuations on* V *and* u *another valuation on* V *such that* $u_m \downarrow u$. *Then* $[u_m; \overline{V}] \downarrow [u; \overline{V}]$.

Proof. Part (i) follows from $\sum_{i=1}^{s} w_i(V) = d(\overline{V})$ and a repeated application of Lemma 3.3 (ii), part (ii) follows by repeatedly applying Lemma 3.3 (i), and parts (iii), (iv), (v) by repeatedly applying Lemma 3.3 (iv), (v) and Lemma 4.3. □

Lemma 5.2. *Let w be a valuation on V and let U be a linear subspace of V such that $(0) \subsetneq U \subsetneq V$ and $\delta(U, w) > 2(|w_1| + \cdots + |w_s|)$.*
(i) $[w;\overline{V}]|_U = [w|_U;\overline{U}]$;
(ii) $[w;\overline{V}]^U = [w^U;\overline{V}/\overline{U}]$;
(iii) $w^U(x^U) = w(x)$, $[w^U;\overline{V}/\overline{U}](x^U) = [w;\overline{V}](x)$ *for* $x \in V \setminus U$.

Proof. Repeated application of Lemma 3.5. □

The hard core of the proof of Theorem 1.1 (and thus of this paper) is the following proposition. Both this proposition and its proof are translations into the terminology of our paper of ideas of Faltings [5].

Proposition 5.3. *Assume that $\overline{V}$ is semistable. Then there exists a valuation u on V such that $[u;\overline{V}] = u + \mu(\overline{V})$.*

Proof. We start with a reduction. Notice that the s-valued vector space $\overline{V}' := (V, w_1 - \mu(\overline{V})/s, \ldots, w_s - \mu(\overline{V})/s)$ is semistable, has $\mu(\overline{V}') = 0$ and satisfies $[u;\overline{V}'] = [u;\overline{V}] - \mu(\overline{V})$ for $u \in \mathcal{W}(V)$ by Lemma 5.1 (ii). Once we have shown that there is $u \in \mathcal{W}(V)$ with $[u;\overline{V}'] = u$, it follows that $[u,\overline{V}] = u + \mu(\overline{V})$. So no generality is lost if we assume

$$\mu(\overline{V}) = 0 \tag{5.1}$$

and show that there is a valuation u on V with $[u;\overline{V}] = u$.

So assume (5.1). Pick any valuation u_0 on V and define valuations $u_1, u_2, , \ldots,$ recursively by

$$u_{m+1} := \min(u_m, [u_m;\overline{V}]) \text{ for } m \geqslant 0,$$

where $\min(w, w')$ denotes the pointwise minimum of two valuations w, w'; this is clearly a valuation on V.

We note that since $u_0 \geqslant u_1 \geqslant \cdots$, the limit $\lim_{m\to\infty} u_m(x)$ exists for every $x \in V \setminus \{0\}$ but it may be $-\infty$. Define

$$U := \{x \in V : \lim_{m\to\infty} u_m(x) > -\infty\}.$$

Then U is a linear subspace of V. We distinguish three cases.

Case I. $U = V$.
Then $u(x) := \lim_{m\to\infty} u_m(x)$ is finite for every $x \in V \setminus \{0\}$. Clearly, u defines a valuation on V, and $u_m \downarrow u$. By Lemma 5.1 (v), we have $[u_m; \overline{V}] \downarrow [u; \overline{V}]$. By letting $m \to \infty$ in $[u_m; \overline{V}] \geqslant u_{m+1}$ we obtain $[u; \overline{V}] \geqslant u$. On the other hand, by Lemma 5.1 (i) and (5.1) we have $[u; \overline{V}](V) = u(V)$. Now Lemma 2.1 implies that $[u; \overline{V}] = u$.

Case II. $U = (0)$.
We show that this is impossible. We first observe that for all $m \geqslant 0$,

$$[u_m; \overline{V}] - u_{m+1} \geqslant [u_{m+1}; \overline{V}] - u_{m+2} \text{ pointwise on } V \setminus \{0\}. \tag{5.2}$$

Indeed, substituting $u_{m+2} = \min(u_{m+1}, [u_{m+1}; \overline{V}])$, we see that (5.2) is equivalent to

$$[u_m; \overline{V}] - u_{m+1} \geqslant \max(0, [u_{m+1}; \overline{V}] - u_{m+1}) \text{ pointwise on } V \setminus \{0\}$$

and this is satisfied since $[u_m; \overline{V}] \geqslant u_{m+1}$ and since $[u_{m+1}; \overline{V}] \leqslant [u_m; \overline{V}]$ by Lemma 5.1 (iii).

Assume that $U = (0)$. Then for every $x \in V \setminus \{0\}$ there is $m \geqslant 0$ such that $u_{m+1}(x) < u_m(x)$; hence $u_{m+1}(x) = [u_m; \overline{V}](x)$ for this m. Together with (5.2) and Lemma 4.2 this implies that there is m_0 such that $[u_m; \overline{V}] \leqslant u_{m+1}$ for $m \geqslant m_0$, so certainly $[u_m; \overline{V}] \leqslant u_m$ for $m \geqslant m_0$. On the other hand, by Lemma 5.1 (i) and (5.1) we have $[u_m; \overline{V}](V) = u_m(V)$, so by Lemma 2.1 we have $[u_m; \overline{V}] = u_m$ for $m \geqslant m_0$. But then, $u_m = u_{m_0}$ for $m \geqslant m_0$, implying $U = V$, contradicting our assumption. So case II cannot occur.

Case III. $(0) \subsetneq U \subsetneq V$.
We will derive a contradiction by reducing this to Case II. We first observe that by Lemma 4.4, there is m_0 such that $\delta(U, u_m) > 2\sum_{i=1}^{s} |w_i|$ for every $m \geqslant m_0$.

We first deal with $\overline{U} = (U, w_1|_U, \ldots, w_s|_U)$. Define a valuation on U by $u' := \lim_{m\to\infty} u_m|_U$. Then by Lemmas 5.1 (v) and 5.2 (i) we have

$$[u'; \overline{U}] = \lim_{m\to\infty} [u_m|_U; \overline{U}] = \lim_{m\to\infty} [u_m; \overline{V}]|_U,$$

and letting $m \to \infty$ in the inequality $u_{m+1}|_U \leqslant [u_m; \overline{V}]|_U$ yields $u' \leqslant [u'; \overline{U}]$. So $u'(U) \leqslant [u'; \overline{U}](U)$ by Lemma 2.1. On the other hand, by Lemma 5.1 (i), applied with $\overline{U}$ instead of $\overline{V}$ we have

$$[u; \overline{U}](U) = u'(U) + d(\overline{U}),$$

hence $\mu(\overline{U}) = \frac{d(\overline{U})}{\dim U} \geqslant 0$. But then $\mu(\overline{U}) = 0$ by (5.1) and the semistability of $\overline{V}$.

We now proceed with $\overline{V}/\overline{U} = (V/U, w_1^U, \ldots, w_s^U)$. An easy computation shows that $\mu(\overline{V}/\overline{U}) = 0$ and $\overline{V}/\overline{U}$ is semistable. By Lemma 5.2 (iii) we have $u_{m+1}^U = \min(u_m^U, [u_m^U; \overline{V}/\overline{U}])$ for $m \geqslant m_0$ and $\lim_{m\to\infty} u_m^U(x^U) = -\infty$ for $x^U \in V/U$, $x^U \neq 0$. Hence we are in the same situation as in case II, but with $\overline{V}/\overline{U}$ instead of $\overline{V}$. This leads again to a contradiction. So also case III cannot occur. This completes the proof of Proposition 5.3. □

Proof of Theorem 1.1. We assume that $w_{\overline{V}}^{HN}$ has weighted filtration

$$\Big((0) = V_0 \underset{\neq}{\subset} V_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} V_r = V; \mu_1 > \cdots > \mu_r \Big). \tag{5.3}$$

We prove our theorem by induction on r. First let $r = 1$. Then $\overline{V}$ is semistable and $\mu(\overline{V}) = \mu_1$. Let u be the valuation from Proposition 5.3. By applying $[\cdot; \overline{V}]$ m times to u, using Lemma 5.1 (ii), we obtain $u + m\mu_1$ and subsequently, by applying Lemma 5.1 (iv) m times, $|v_m - (u + m\mu_1)| \leqslant |0 - u| = |u|$. Hence

$$|v_m - mw_{\overline{V}}^{HN}| = |v_m - m\mu_1| \leqslant 2|u| \text{ for } m > 0.$$

This settles the case $r = 1$.

Next, let $r \geqslant 2$. We define sequences of valuations $(v'_m)_{m=0}^{\infty}$ on V_1 and $(v''_m)_{m=0}^{\infty}$ on V/V_1 such that $v'_0 = 0$, $v''_0 = 0$, $v'_m = [v'_{m-1}; \overline{V_1}]$, $v''_m = [v''_{m-1}; \overline{V/V_1}]$ for $m = 1, 2, \ldots$. By what we just showed there is a constant C' such that

$$|v'_m - m\mu_1| \leqslant C' \text{ for } m \geqslant 0. \tag{5.4}$$

Further, by the induction hypothesis, there is $C'' > 0$ such that

$$|v''_m - mw_{\overline{V/V_1}}^{HN}| \leqslant C'' \text{ for } m \geqslant 0. \tag{5.5}$$

These inequalities imply that for $m \geqslant 0$,

$$\begin{aligned} v'_m(x) &\geqslant m\mu_1 - C' \text{ for } x \in V_1 \setminus \{0\}, \\ v''_m(y) &\leqslant m\mu_2 + C'' \text{ for } y \in (V/V_1) \setminus \{0\}, \end{aligned}$$

where in the last inequality we have used that $w_{\overline{V/V_1}}^{HN} = (w_{\overline{V}}^{HN})^{V_1} \leqslant \mu_2$. Since $\mu_1 > \mu_2$ there is m_0 such that

$$\begin{aligned} v'_m(x) - v''_m(y) > 2(|w_1| + \cdots + |w_s|) \\ \text{for } x \in V_1,\ y \in (V/V_1) \setminus \{0\},\ m \geqslant m_0. \end{aligned} \tag{5.6}$$

We now define functions u_m $(m \geqslant m_0)$ on V by

$$u_m(x) := \begin{cases} v'_m(x) & \text{for } x \in V_1, \\ v''_m(x^{V_1}) & \text{for } x \in V \setminus V_1. \end{cases}$$

By (5.6) these functions define valuations on V with

$$\begin{cases} u_m|_{V_1} = v'_m, \quad u_m^{V_1} = v''_m, \\ \delta(V_1, u_m) > 2(|w_1| + \cdots + |w_s|) \quad \text{for } m \geqslant m_0. \end{cases} \tag{5.7}$$

Inequalities (5.4), (5.5) together with $(w_{\overline{V}}^{HN})|_{V_1} = \mu_1$, $(w_{\overline{V}}^{HN})^{V_1} = w_{\overline{V/V_1}}^{HN}$ imply

$$|u_m - m w_{\overline{V}}^{HN}| \leqslant \max(C', C'') \quad \text{for } m \geqslant m_0.$$

Thanks to (5.7) we can apply Lemma 5.2 and deduce $u_{m+1} = [u_m; \overline{V}]$ for $m \geqslant m_0$. Together with Lemma 5.1 (iv) this yields

$$|v_m - u_m| \leqslant |v_{m_0} - u_{m_0}| \quad \text{for } m \geqslant m_0.$$

This leads finally to

$$|v_m - m w_{\overline{V}}^{HN}| \leqslant |v_{m_0} - u_{m_0}| + \max(C', C'') \quad \text{for } m \geqslant m_0,$$

which clearly implies Theorem 1.1. □

6. Proofs of Corollary 1.2 and Theorems 1.4, 1.6, 1.7

Let K be a field and V a finite-dimensional, non-zero K-vector space. Given a valuation w on V and a sequence $(w_m)_{m=0}^{\infty}$ of valuations on V, we write $w_m \to w$ uniformly on V if $|w_m - w| \to 0$ as $m \to \infty$.

We start with an immediate consequence of Theorem 1.1.

Corollary 6.1. *Let $\overline{V}$ and $(v_m)_{m=0}^{\infty}$ be as in Theorem 1.1. Then $\frac{1}{m} v_m \to w_{\overline{V}}^{HN}$ uniformly on V.*

Proof. Divide the inequality in Theorem 1.1 by m and let $m \to \infty$. □

We deduce the following result, which, in view of Lemma 3.1, contains Corollary 1.2 as a special case.

Corollary 6.2. *Let $\overline{V} = (V, w_1, \ldots, w_s)$ be an s-valued K-vector space of dimension $n > 0$. Assume that V has a basis $\{f_1, \ldots, f_n\}$ adapted to $w_1, \ldots, w_s$. Then*

$$w_{\overline{V}}^{HN} = (\cdots(w_1 * w_2)\cdots) * w_s.$$

In particular, if $s = 2$ *then* $w^{HN}_{\overline{V}} = w_1 * w_2$.

Proof. By repeatedly applying Lemma 3.1 one deduces that for all $m \geqslant 1$, v_m is given by $(f_i,\, m\sum_{j=1}^{s} w_j(f_i))$ $(i = 1, \ldots, n)$; hence $v_m = mv_1$. Apply Corollary 6.1. □

Our last auxiliary result is the following simple lemma.

Lemma 6.3. (i) *Let* V *be a* K*-vector space of dimension* $n > 0$ *and let* w *be a valuation and* $(w_m)_{m=0}^{\infty}$ *a sequence of valuations on* V *such that* $w_m \to w$ *uniformly on* V. *Then*

$$\begin{aligned}
&\wedge^k w_m \to \wedge^k w \text{ uniformly on } \wedge^k V \text{ for each } k \in \{1, \ldots, n\};\\
&\mathrm{S}^k\, w_m \to \mathrm{S}^k\, w \text{ uniformly on } \mathrm{S}^k\, V \text{ for every positive integer } k;\\
&w_m \otimes L \to w \otimes L \text{ uniformly on } V \otimes L \text{ for every extension field } L \text{ of } K.
\end{aligned}$$

(ii) *Let* V, V' *be two non-zero, finite dimensional* K*-vector spaces. Let* w, $(w_m)_{m=0}^{\infty}$ *be a valuation and sequence of weights on* V *such that* $w_m \to w$ *uniformly on* V, *and* w', $(w'_m)_{m=0}^{\infty}$ *a weight and sequence of valuations on* V' *such that* $w'_m \to w'$ *uniformly on* V'. *Then*

$$\begin{aligned}
&w_m \oplus w'_m \to w \oplus w' \text{ uniformly on } V \oplus V';\\
&w_m \otimes w'_m \to w \otimes w' \text{ uniformly on } V \otimes V'.
\end{aligned}$$

Proof. We prove only the statement concerning the tensor product, the proofs of the other assertions being similar. For $m \geqslant 0$, let $c_m := |w_m - w|$, $c'_m := |w'_m - w'|$. By (1.5) we have $w_m \otimes w'_m \geqslant w \otimes w' - (c_m + c'_m)$, and likewise, $w \otimes w' \geqslant w_m \otimes w'_m - (c_m + c'_m)$; hence $|w_m \otimes w'_m - w \otimes w'| \leqslant c_m + c'_m \to 0$ as $m \to \infty$. □

Proof of Theorem 1.4. We just have to combine Corollary 6.1 with Lemmas 3.2 and 6.3. We only detail the proof of (1.12).

Let K be a field and $\overline{V} = (V, w_1, \ldots, w_s)$, $\overline{V'} = (V', w'_1, \ldots, w'_s)$ two non-zero, finite dimensional s-valued K-vector spaces. Let v_m be the valuations from Theorem 1.1. In a similar manner we define valuations v'_m on V' (with w'_i replacing w_i for all i) and u_m on $V \otimes V'$ (with $w_i \otimes w'_i$ replacing w_i for all i). Then $u_m = v_m \otimes v'_m$ for $m = 1, 2, \ldots$ by a repeated application of Lemma 3.2. From Corollary 6.1 one infers $\frac{1}{m}u_m \to w^{HN}_{\overline{V \otimes V'}}$, while on the other hand by Corollary 6.1 and Lemma 6.3, $\frac{1}{m}u_m = (\frac{1}{m}v_m) \otimes (\frac{1}{m}v'_m) \to w^{HN}_{\overline{V}} \otimes w^{HN}_{\overline{V'}}$. This proves (1.12). The assertions (1.8)–(1.11) can be proved in precisely the same manner. □

Proof of Theorem 1.6. Let $\overline{V} = (V, w_1, \ldots, w_s)$ and $\overline{V}' = (V', w'_1, \ldots, w'_s)$ be two finite dimensional s-valued K-vector spaces and φ a morphism from $\overline{V}$ to $\overline{V}'$; this means that $w'_i \circ \varphi \geqslant w_i$ for $i = 1, \ldots, s$. Let v_m $(m = 0, 1, 2, \ldots)$ be the valuations on V from Theorem 1.1, and let the valuations v'_m on V' be defined in the same way, replacing w_i by w'_i for $i = 1, \ldots, s$. By repeatedly applying Lemma 3.3 (iii), it follows that $v'_m \circ \varphi \geqslant v_m$ for all m, and then Theorem 1.6 follows by dividing by m and applying Corollary 6.1. □

Proof of Theorem 1.7. Let again $\overline{V} = (V, w_1, \ldots, w_s)$ be an n-dimensional, s-valued K-vector space. Denote by $\mathcal{U}$ the $(+, \cap)$-algebra generated by the subspaces in the filtrations of $w_1, \ldots, w_s$. From Lemma 3.4 it follows that if w', w'' are any two valuations on V whose filtrations consist of subspaces from $\mathcal{U}$, then also the subspaces in the filtration of $w' * w''$ belong to $\mathcal{U}$. This implies that for $m = 1, 2, \ldots$, the subspaces in the filtrations of the valuations v_m from Theorem 1.1 belong to $\mathcal{U}$.

Let the weighted Harder-Narasimhan filtration of $\overline{V}$ be given by (5.3) and let C be the constant from Theorem 1.1. We may assume that $r \geqslant 2$. Put $\varepsilon := \min_{1 \leqslant i \leqslant r-1}(\mu_i - \mu_{i+1})$ and let m be an integer with $m > 3C/\varepsilon$. Then for $i = 1, \ldots, r-1$, $x \in V_i$, $y \in V \setminus V_i$ we have

$$v_m(x) - v_m(y) \geqslant m w^{HN}_{\overline{V}}(x) - m w^{HN}_{\overline{V}}(y) - 2C \geqslant m(\mu_i - \mu_{i+1}) - 2C > C,$$

that is, $\delta(V_i, v_m) > 0$. We conclude that $V_1, \ldots, V_{r-1}$ are in the filtration of v_m, hence belong to $\mathcal{U}$. □

7. Effective computation of the Harder-Narasimhan valuation

Let K be a given field and V a finite-dimensional K-vector space. We show that if the s-valued K-vector space $\overline{V} = (V, w_1, \ldots, w_s)$ is explicity given in some sense then its Harder-Narasimhan valuation can be computed in principle. We do not claim practical efficiency.

Here, the input and output of a computation are finite tuples from $K \amalg \mathbb{R}$, and a computation is built up from finitely many applications of an arithmetic operation on K or $\mathbb{R}$ $(+, -, \times, /)$ and finitely many if-then-else commands, where the condition to be checked is either whether a given K-valued expression is 0, or whether a given $\mathbb{R}$-valued expression is $\geqslant 0$. We say that a particular object is effectively computable from a given input if it is representable by a finite tuple from $K \amalg \mathbb{R}$ that can be computed from the input by means of a computation as above.

We fix a basis $B = \{e_1, \dots, e_n\}$ of V and perform computations in V by representing an element of V by means of its coordinates with respect to B. Linear subspaces of V are described by means of a basis, of which each element is given by its coordinates with respect to B. By standard procedures from linear algebra one can compute the intersection and sum of two given linear subspaces of V.

Let $w_1, \dots, w_s$ be valuations on V, and $\overline{V} = (V, w_1, \dots, w_s)$ the corresponding s-valued K-vector space. We assume that w_i $(i = 1, \dots, s)$ is given by $(f_{i,j}, \alpha_{i,j})$ $(j = 1, \dots, n)$, i.e., $B_i := \{f_{i,1}, \dots, f_{i,n}\}$ is a basis of V and if $x = \sum_{j=1}^{n} \xi_j f_{i,j}$ with $\xi_j \in K$, not all 0, then $w_i(x) = \min\{\alpha_{i,j} : \xi_{i,j} \neq 0\}$. From these defining data, one can compute their respective weighted filtrations,

$$\Big((0) = F_0^{(w_i)} \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} F_{r_i}^{(w_i)} = V; \alpha_{i,1} > \cdots > \alpha_{i,r_i}\Big) \quad (i = 1, \dots, s). \tag{7.1}$$

Our algorithm is based on two lemmas, which we state and prove below. Given a filtration $(0) \underset{\neq}{\subset} F_1 \underset{\neq}{\subset} \cdots \underset{\neq}{\subset} F_r$ of linear subspaces of a given vector space, we call F_i the i-th space of this filtration.

Lemma 7.1. *Let V_1 be the first space in the Harder-Narasimhan filtration of $\overline{V}$. Suppose that* $\dim V_1 = 1$. *Then there are indices $j_i \in \{1, \dots, r_i\}$ for $i = 1, \dots, s$ such that*

$$V_1 = \bigcap_{i=1}^{s} F_{j_i}^{(w_i)}. \tag{7.2}$$

Proof. Let $V_1 = \mathrm{span}\{x\}$. For $i = 1, \dots, s$, let j_i be the smallest index j from $\{1, \dots, r_i\}$ such that $x \in F_j^{(w_i)}$. Thus, $V_1 \subseteq \bigcap_{i=1}^{s} F_{j_i}^{(w_i)}$. Conversely, let $y \in \bigcap_{i=1}^{s} F_{j_i}^{(w_i)}$ with $y \neq 0$. Then

$$\mu(\overline{\mathrm{span}\{y\}}) = \sum_{i=1}^{s} w_i(y) \geqslant \sum_{i=1}^{s} \alpha_{i,j_i} = \sum_{i=1}^{s} w_i(x) = \mu(\overline{V_1}).$$

Hence $\mu(\overline{\mathrm{span}\{y\}}) = \mu(\overline{V_1})$, and so $\mathrm{span}\{y\} \subseteq V_1$. Identity (7.2) follows. □

We make a reduction to the case $\dim V_1 = 1$ using exterior powers. We need the following lemma.

Lemma 7.2. *Suppose that the i-th space V_i of the Harder-Narasimhan filtration of $\overline{V}$ has dimension k. Then the one-dimensional space $\wedge^k V_i$ is the first space in the Harder-Narasimhan filtration of $\wedge^k \overline{V}$.*

Proof. Let $\{f_1, \dots, f_n\}$ be a basis of V adapted to $w_{\overline{V}}^{HN}$, ordered such that $w_{\overline{V}}^{HN}(f_1) \geqslant \cdots \geqslant w_{\overline{V}}^{HN}(f_n)$. This means that in the sequence $f_1, \dots, f_n$, the first vectors form a basis of V_1, the next vectors augment this to a basis of V_2, etc. Hence $\{f_1, \dots, f_k\}$ is a basis of V_i and $w_{\overline{V}}^{HN}(f_k) > w_{\overline{V}}^{HN}(f_{k+1})$. Now by (1.8) $w_{\wedge^k \overline{V}}^{HN} = \wedge^k w_{\overline{V}}^{HN}$, and so by Lemma 2.8, $\{f_{i_1} \wedge \cdots \wedge f_{i_k} : (i_1, \dots, i_k) \in \mathcal{I}_{n,k}\}$ is a basis of $\wedge^k V$ adapted to $w_{\wedge^k \overline{V}}^{HN}$. The first space in the Harder-Narasimhan filtration of $\wedge^k \overline{V}$ has a basis consisting of those vectors $f_{i_1} \wedge \cdots \wedge f_{i_k}$ with maximal $w_{\wedge^k \overline{V}}^{HN}$-value. Now clearly,

$$w_{\wedge^k \overline{V}}^{HN}(f_1 \wedge \cdots \wedge f_k) = \sum_{i=1}^{k} w_{\overline{V}}^{HN}(f_i) > w_{\wedge^k \overline{V}}^{HN}(f_{i_1} \wedge \cdots \wedge f_{i_k})$$

for any $(i_1, \dots, i_k) \in \mathcal{I}_{n,k}$ different from $(1, \dots, k)$. Hence $\wedge^k V_i = \operatorname{span}\{f_1 \wedge \cdots \wedge f_k\}$ is the first space in the Harder-Narasimhan filtration of $\wedge^k \overline{V}$. ☐

Before describing our algorithm to compute the Harder-Narasimhan valuation we prove another lemma. Given a basis $B_0 = \{f_1, \dots, f_n\}$ of V, let $\wedge^k B_0$ be the basis of $\wedge^k V$ consisting of the elements $f_{i_1} \wedge \cdots \wedge f_{i_k}$ $((i_1, \dots, i_k) \in \mathcal{I}_{n,k})$. We will express elements of $\wedge^k V$ by means of their coordinates with respect to $\wedge^k B$, where $B = \{e_1, \dots, e_n\}$ is the given basis of V.

Lemma 7.3. *let $1 \leqslant k \leqslant n$. Then for any given non-zero $x \in \wedge^k V$ it can be checked whether there is a k-dimensional linear subspace U of V with $\wedge^k U = \operatorname{span}\{x\}$, and if so, compute a basis of U.*

Proof. We have to check whether there are linearly independent $x_1, \dots, x_k \in V$ such that x is a scalar multiple of $x_1 \wedge \cdots \wedge x_k$ and if so, compute such $x_1, \dots, x_k$. This can be done as follows. We may assume that the basis $\{x_1, \dots, x_k\}$ to be found is special, that is, if $[x_1, \dots, x_k]$ is the $n \times k$-matrix whose j-th column consists of the coordinates of x_j with respect to B, then one of the $k \times k$-submatrices of $[x_1, \dots, x_k]$ is the unit matrix. This being the case, one of the coordinates of $x_1 \wedge \cdots \wedge x_k$ with respect to $\wedge^k B$ is equal to ± 1 and moreover, the coordinates of $x_1, \dots, x_k$ all occur, except maybe for the sign, among the coordinates of $x_1 \wedge \cdots \wedge x_k$. So the coordinates of $x_1, \dots, x_k$ can be easily determined from $x_1 \wedge \cdots \wedge x_k$. Now what we have to do is computing all scalar multiples of x with one of the coordinates with respect to $\wedge^k B$ equal to ± 1, and check whether one of these multiples equals $x_1 \wedge \cdots \wedge x_k$ for some special set $\{x_1, \dots, x_k\}$. ☐

Description of the algorithm.

The input of our algorithm is an n-dimensional s-valued K-vector space $\overline{V} =$

$(V, w_1, \dots, w_s)$, given explicitly by means of a basis $B_i := \{f_{i,1}, \dots, f_{i,n}\}$ adapted to w_i, and the quantities $w_i(f_{i,1}), \dots, w_i(f_{i,n})$, for $i = 1, \dots, s$. The output will be the weighted Harder-Narasimhan filtration of $\overline{V}$.

We first construct a finite collection $\mathcal{S}$ of subspaces of V, guaranteed to contain the spaces of the Harder-Narasimhan filtration of $\overline{V}$.

The construction is as follows. For $i = 1, \dots, s$, let G_i run through all the subspaces of V spanned by a subset of B_i and consider all intersections $G_1 \cap \cdots \cap G_s$. Let $\mathcal{S}_1$ be the collection of those intersections that have dimension 1. Clearly, the spaces in $\mathcal{S}_1$ can be computed. Next, for $k = 2, \dots, n$, $i = 1, \dots, s$, let $G_{i,k}$ run through the subspaces of $\wedge^k V$ spanned by a subset of $\wedge^k B_i$, and consider all intersections $G_{1,k} \cap \cdots \cap G_{s,k}$. Among these intersections, select those that are of dimension 1 and are of the shape $\wedge^k U$ for some k-dimensional linear subspace U of V. Then let $\mathcal{S}_k$ consist of those spaces U thus obtained. By Lemma 7.3, the spaces in $\mathcal{S}_k$ can be computed. Lastly, let $\mathcal{S} = \cup_{k=1}^{n} \mathcal{S}_k$.

If the first space V_1 in the Harder-Narasimhan filtration of $\overline{V}$ has dimension 1, then by Lemma 7.1 it belongs to $\mathcal{S}_1$. If the i-th space V_i of the Harder-Narasimhan filtration of $\overline{V}$ has dimension k, then by Lemmas 2.8 (i), 7.2 and 7.1 it belongs to $\mathcal{S}_k$. Hence $\mathcal{S}$ contains the spaces of the Harder-Narasimhan filtration of $\overline{V}$.

We now compute the spaces in the Harder-Narasimhan filtration of $\overline{V}$. We compute the slope $\mu(\overline{U})$ of each of the spaces U in $\mathcal{S}$. From the spaces in $\mathcal{S}$ one selects those with maximal slope, and among these the one of largest dimension. This is the first space V_1 in the Harder-Narasimhan filtration of $\overline{V}$ (recall that V_1 contains all spaces having maximal slope; hence it is the single one of largest dimension among all spaces of maximal slope). Next, we obtain the second space V_2 by considering all spaces $U \underset{\neq}{\supset} V_1$ from $\mathcal{V}$ for which $\mu(\overline{U}/\overline{V}_1)$ is maximal and taking from these the space of largest dimension, etc. This will eventually give us the complete Harder-Narasimhan filtration of $\overline{V}$ and together with the already computed slopes $\mu(\overline{V_i}/\overline{V_{i-1}})$, the weighted Harder-Narasimhan filtration and thus, the Harder-Narasimhan valuation. □

Henceforth, we assume that K is an algebraic number field. We give an explicit upper bound for the heights of the subspaces in the Harder-Narasimhan filtration of a given s-valued K-vector space.

Denote by M_K the set of places (equivalence classes of absolute values) of K. For $v \in M_K$, we choose the absolute value $|\cdot|_v$ representing v such that its restriction to $\mathbb{Q}$ is either the ordinary absolute value given by $|x|_\infty = \max(x, -x)$, or the p-adic absolute value $|\cdot|_p$ with $|p|_p = p^{-1}$. The place v is called infinite ($v \mid \infty$) if $|\cdot|_v$ extends the ordinary absolute value, and finite ($v \nmid \infty$) otherwise. The absolute values $|\cdot|_v$ satisfy the Product formula $\prod_{v\in M_K} |x|_v^{d_v} = 1$ for $x \in K^*$, where d_v is the local degree of v, i.e. $d_v := [K_v : \mathbb{Q}_p]$, where $p \in \{\infty\} \cup \{\text{primes}\}$ is such that $|\cdot|_v$ extends $|\cdot|_p$ and K_v, $\mathbb{Q}_p$ denote the respective completions.

Let V be an n-dimensional K-vector space with basis $B = \{e_1, \dots, e_n\}$. We define norms $|x|_{B,v}$ ($v \in M_K$) and a height $H_B(x)$ for $x \in V$ by expressing x as $\sum_{i=1}^n \xi_i e_i$ with $\xi_1, \dots, \xi_n \in K$ and putting

$$\begin{aligned} |x|_{B,v} &:= \Big(\sum_{i=1}^n |\xi_i|_v^2\Big)^{1/2} \quad \text{if } v|\infty; \\ |x|_{B,v} &:= \max\big(|\xi_1|_v, \dots, |\xi_n|_v\big) \quad \text{if } v \nmid \infty \end{aligned}$$

and

$$H_B(x) := \prod_{v\in M_K} |x|_{B,v}^{d_v/d},$$

where $d := [K : \mathbb{Q}]$. By the Product formula, $H_B(\alpha x) = H_B(x)$ for $x \in V$, $\alpha \in K^*$.

Let $k \in \{1, \dots, n\}$. From the basis $B = \{e_1, \dots, e_n\}$ of V chosen above, we construct a basis $\wedge^k B := \{e_{i_1} \wedge \cdots \wedge e_{i_k} : (i_1, \dots, i_k) \in \mathcal{I}_{n,k}\}$ of $\wedge^k V$.

We define the height $H_B(U)$ of a linear subspace U of V by putting $H_B(U) := 1$ if $U = (0)$ or $U = V$, and

$$H_B(U) := H_{\wedge^k B}(x_1 \wedge \cdots \wedge x_k)$$

otherwise, where $k = \dim U$ and $\{x_1, \dots, x_k\}$ is any basis of U. This does not depend on the choice of the basis, since the vector $x_1 \wedge \cdots \wedge x_k$ is determined uniquely by U up to a scalar factor.

Theorem 7.4. *Let K be an algebraic number field, V an n-dimensional K-vector space, and $\overline{V} = (V, w_1, \dots, w_s)$ an s-valued vector space. Choose a basis $B = \{e_1, \dots, e_n\}$ of V and for each $i = 1, \dots, s$, choose a basis $\{f_{i,1}, \dots, f_{i,n}\}$ of V adapted to w_i. Put*

$$H := \max\{H_B(f_{i,j}) : 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant n\}.$$

Then for the spaces $V_1, \dots, V_r$ in the Harder-Narasimhan filtration of $\overline{V}$, we have

$$H_B(V_i) \leqslant H^{4^n} \text{ for } i = 1, \dots, r.$$

Proof. We start with some inequalities for heights of subspaces of V. Let $x_1,\dots,x_k$ be elements of V. By Hadamard's inequality for the infinite places v and the ultrametric inequality for the finite places v, we have for any $x_1,\dots,x_k\in V$,

$$|x_1\wedge\cdots\wedge x_k|_{\wedge^k B,v}\leqslant |x_1|_{B,v}\cdots|x_k|_{B,v} \text{ for } v\in M_K,$$

and so

$$H_{\wedge^k B}(x_1\wedge\cdots\wedge x_k)\leqslant H_B(x_1)\cdots H_B(x_k). \tag{7.3}$$

In particular, if U is a linear subspace of V with basis $\{x_1,\dots,x_k\}$,

$$H_B(U)\leqslant H_B(x_1)\cdots H_B(x_k). \tag{7.4}$$

More generally, by a result of Struppeck and Vaaler [14], we have for any two linear subspaces U_1,U_2 of V,

$$H_B(U_1\cap U_2)\leqslant H_B(U_1\cap U_2)H_B(U_1+U_2)\leqslant H_B(U_1)H_B(U_2). \tag{7.5}$$

Write as before V_i for the i-th space in the Harder-Narasimhan filtration of $\overline{V}$. First assume that $\dim V_1=1$. The space V_1 is the intersection of at most $n-1$ spaces from those in (7.2), and all of them have dimension at most $n-1$. These spaces are all generated by vectors from the bases $\{f_{i,1},\dots,f_{i,n}\}$ chosen above, and so by (7.4) have height with respect to B at most H^{n-1}. A repeated application of (7.5) then gives

$$H_B(V_1)\leqslant H^{(n-1)^2}. \tag{7.6}$$

We now deal with the general case. Let $i\in\{1,\dots,r\}$ and suppose that V_i has dimension k. By taking the exterior products of all k-element subsets of our chosen basis $\{f_{j,1},\dots,f_{j,n}\}$ adapted to w_j we obtain a basis adapted to $\wedge^k w_j$, for $j=1,\dots,s$. By (7.3), the vectors from this basis have height with respect to $\wedge^k B$ at most H^k. Clearly, $\wedge^k V_i$ has dimension 1, and by Lemma 7.2, it is the first space in the Harder-Narasimhan filtration of $\wedge^k\overline{V}$. Now applying (7.6) with $\wedge^k\overline{V}$, $\binom{n}{k}$, H^k instead of $\overline{V}$, n, H, we obtain

$$H_B(V_i)=H_{\wedge^k B}(\wedge^k V_i)\leqslant H^{k(\binom{n}{k}-1)^2}\leqslant H^{4^n}.$$

Here we have used $\sqrt{k}\binom{n}{k}\leqslant 2^n$ for $k=1,\dots,n$, which is an easy consequence of Stirling's formula. $\square$

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