AN EXPLICIT VERSION OF FALTINGS' PRODUCT THEOREM AND AN IMPROVEMENT OF ROTH'S LEMMA

Jan-Hendrik Evertse

University of Leiden, Department of Mathematics and Computer Science
P.O. Box 9512, 2300 RA Leiden, The Netherlands, email evertse@wi.leidenuniv.nl

To Professor Wolfgang Schmidt on his 60th birthday

Introduction.

In his paper 'Diophantine approximation on abelian varieties' [1], Faltings proved among others the following conjecture of Weil and Lang: if A is an abelian variety over a number field k and X a subvariety of A not containing a translate of a positive dimensional abelian subvariety of A, then X contains only finitely many k-rational points. One of Faltings' basic tools was a new non-vanishing result of his, also proved in [1], the so-called (arithmetic version of the) *Product theorem.* It has turned out that this Product theorem has a much wider range of applicability in Diophantine approximation. For instance, recently Faltings and Wüstholz gave an entirely new proof [2] of Schmidt's Subspace theorem [15] based on the Product theorem.

Faltings' Product theorem is not only very powerful for deriving new qualitative finiteness results in Diophantine approximation but, in an explicit form, it can be used also to derive significant improvements of existing *quantitative* results. In the present paper, we work out an explicit version of the arithmetic version of the Product theorem; except for making explicit some of Faltings' arguments from [1] this did not involve anything new. By using the same techniques we improve Roth's lemma from [12]. Roth's lemma was used by Roth in his theorem on the approximation of algebraic numbers by rationals [12] and later by Schmidt in his proof of the Subspace theorem [15].

In two subsequent papers we shall apply our improvement of Roth's lemma to derive significant improvements on existing explicit upper bounds for the number of subspaces in the Subspace theorem, due to Schmidt [16] and Schlickewei [14] and for the number of solutions of norm form equations [17] and S-unit equations [13].

At the conference on Diophantine problems in Boulder in honour of W.M. Schmidt (26 June - 1 July, 1994), Wüstholz announced that his student R. Ferretti had independently obtained results similar to our Theorems 1 and 2. These results have been published in [3]. Part of the arguments used in the proof of Theorem 1 had already been worked out by van der Put [11] in his lecture at the conference

'Diophantine approximation and Abelian varieties, Soesterberg, The Netherlands, 12-15 April 1992.

As the Product theorem appears to have applications outside arithmetic algebraic geometry, we have tried to make this paper accessible to non-geometers with a modest knowledge of algebraic geometry.

§1. Statement of the results.

Let $\mathbf{n} = (n_1, ..., n_m)$ be a tuple of positive integers. For $h = 1, ..., m$, denote by $\mathbf{X}_h$ the block of $n_h + 1$ variables $X_{h0}, ..., X_{h,n_h}$. For a ring R, denote by $R[\mathbf{X}]$ or $R[\mathbf{X}_1, ..., \mathbf{X}_m]$ the polynomial ring in the $(n_1+1)+...+(n_m+1)$ variables X_{hj} $(h = 1, ..., m,\ j = 0, ..., n_h)$. For a tuple of non-negative integers $\mathbf{d} = (d_1, ..., d_m)$, denote by $\Gamma_R^{\mathbf{n}}(\mathbf{d})$ the R-module of polynomials in $R[\mathbf{X}]$ which are homogeneous of degree d_1 in the block $\mathbf{X}_1$, ..., homogeneous of degree d_m in $\mathbf{X}_m$ i.e. the R-module generated by the monomials

$$\mathbf{X}^{\mathbf{i}} := \prod_{h=1}^{m} \prod_{j=0}^{n_h} X_{hj}^{i_{hj}} \quad \text{with} \quad \sum_{j=0}^{n_h} i_{hj} = d_h \quad \text{for} \quad h = 1, ..., m.$$

Let $\Gamma_R^{\mathbf{n}} := \cup_{\mathbf{d} \in (\mathbb{Z}_{\geq 0})^m} \Gamma_R^{\mathbf{n}}(\mathbf{d})$ be the set of polynomials which are homogeneous in each block $\mathbf{X}_h$ for $h = 1, ..., m$. An $\mathbf{n}$-*ideal* of $R[\mathbf{X}]$ is an ideal generated by polynomials from $\Gamma_R^{\mathbf{n}}$. An essential $\mathbf{n}$-prime ideal of $R[\mathbf{X}]$ is an $\mathbf{n}$-ideal which is a prime ideal and which does not contain any of the ideals $(X_{h0}, ..., X_{h,n_h})$ $(h = 1, ..., m)$.

Let k be an algebraically closed field and denote by $\mathbb{P}^n(k)$ the n-dimensional projective space over k. Every point $P \in \mathbb{P}^n(k)$ can be represented by an up to a scalar multiple unique non-zero vector $\mathbf{x} = (x_0, ..., x_n) \in k^{n+1}$ of homogeneous coordinates. Let again $\mathbf{n} = (n_1, ..., n_m)$ be a tuple of positive integers. Define the multi-projective space $\mathbb{P}^{\mathbf{n}}(k)$ as the cartesian product

$$\mathbb{P}^{\mathbf{n}}(k) := \mathbb{P}^{n_1}(k) \times ... \times \mathbb{P}^{n_m}(k).$$

In what follows, $\mathbb{P}^n(k)$ with a non-bold face superscript denotes the n-dimensional (single-) projective space, and $\mathbb{P}^{\mathbf{n}}(k)$ with a bold-face superscript a multi-projective space. For $f \in \Gamma_k^{\mathbf{n}}$ and for $P = (P_1, ..., P_m) \in \mathbb{P}^{\mathbf{n}}(k)$ with $P_h \in \mathbb{P}^{n_h}(k)$ for $h = 1, ..., m$ we say that $f(P) = 0$ (or $\neq 0$) if $f(\mathbf{x}_1, ..., \mathbf{x}_m) = 0$ (or $\neq 0$) for any vectors of homogeneous coordinates $\mathbf{x}_1, ..., \mathbf{x}_m$, representing $P_1, ..., P_m$ respectively. This is well-defined. A (Zariski-) closed subset of $\mathbb{P}^{\mathbf{n}}(k)$ is a set

$$\{P \in \mathbb{P}^{\mathbf{n}}(k) : \quad f_1(P) = 0, ..., f_r(P) = 0\}$$

(abbreviated $\{f_1 = 0, ..., f_r = 0\}$), where $f_1, ..., f_r \in \Gamma_k^{\mathbf{n}} \backslash \{0\}$. A closed subset X of $\mathbb{P}^{\mathbf{n}}(k)$ is called *reducible* if it is the union of two closed subsets A, B of $\mathbb{P}^{\mathbf{n}}(k)$ with $A \subsetneq X, B \subsetneq X$, and irreducible otherwise. Every closed subset X of $\mathbb{P}^{\mathbf{n}}(k)$ can be expressed uniquely as

$$X = Z_1 \cup ... \cup Z_r,$$

where $Z_1, ..., Z_r$ are irreducible closed subsets of $\mathbb{P}^{\mathbf{n}}(k)$ such that $Z_i \not\subseteq Z_j$ for $i, j \in \{1, ..., r\}, i \neq j$ (cf. [18], p. 23). $Z_1, ..., Z_r$ are called the *irreducible components* of X. We agree here that we shall use the term 'subvariety' exclusively for a projective subvariety, i.e. a closed irreducible subset.

There is a one-to-one correspondence between subvarieties of $\mathbb{P}^{\mathbf{n}}(k)$ and essential $\mathbf{n}$-prime ideals I of $k[\mathbf{X}]$:

$$I \leftrightarrow V(I) = \{P \in \mathbb{P}^{\mathbf{n}}(k) : \ f(P) = 0 \quad \text{for all} \quad f \in I\}.$$

We say that the subvariety V of $\mathbb{P}^{\mathbf{n}}(k)$ is defined over a subfield k_1 of k if its corresponding prime ideal can be generated by polynomials with coefficients from k_1. An important class of subvarieties of $\mathbb{P}^{\mathbf{n}}(k)$ we will encounter are the *product varieties*

$$Z_1 \times ... \times Z_m = \{(P_1, ..., P_m) : P_h \in Z_h \quad \text{for} \quad h = 1, ..., m\}$$

where Z_h is a subvariety of $\mathbb{P}^{n_h}(k)$ for $h = 1, ..., m$. It is a theorem, cf. [18], pp. 61/62, that the cartesian product of subvarieties of $\mathbb{P}^{n_1}(k), ..., \mathbb{P}^{n_m}(k)$, respectively, is a subvariety of $\mathbb{P}^{\mathbf{n}}(k)$.

Let $F \in \Gamma_k^{\mathbf{n}}$. For a tuple of non-negative integers $\mathbf{i} = (i_{hj} : h = 1, ..., m, j = 0, ..., n_h)$ define the partial derivative of F:

$$F_{\mathbf{i}} := \left(\prod_{h=0}^{m} \prod_{j=0}^{n_h} \frac{\partial^{i_{hj}}}{\partial X_{hj}^{i_{hj}}} \right) F.$$

Let $\mathbf{d} = (d_1, ..., d_m)$ be a tuple of positive integers. For a tuple $\mathbf{i}$ as above, put

$$(\mathbf{i}/\mathbf{d}) := \sum_{h=1}^{m} \frac{1}{d_h} (i_{h0} + ... + i_{h,n_h}).$$

The *index* of F with respect to $P \in \mathbb{P}^{\mathbf{n}}(k)$ and $\mathbf{d}$, notation $i_{\mathbf{d}}(F, P)$, is the largest number σ such that

$$F_{\mathbf{i}}(P) = 0 \quad \text{for all} \quad \mathbf{i} \quad \text{with} \quad (\mathbf{i}/\mathbf{d}) \leq \sigma.$$

The index of F at P is some kind of weighted multiplicity of F at P. The index is independent of the choice of homogeneous coordinates on $\mathbb{P}^{n_h}$ for $h = 1, ..., m$.

Namely, if for $h = 1, ..., m$, $Y_{h0}, ..., Y_{h,n_h}$ are linearly independent linear forms in $\mathbf{X}_h$, then the differential operators $\partial/\partial Y_{hj}$ are linear combinations of the $\partial/\partial X_{hj}$ and vice versa, hence the index does not change when in its definition the operators $\partial/\partial X_{hj}$ $(j = 0, ..., n_h)$ are replaced by $\partial/\partial Y_{hj}$ $(j = 0, ..., n_h)$ for $h = 1, ..., m$.

For $\sigma \geq 0$, define the closed subset of $\mathbb{P}^{\mathbf{n}}(k)$,

$$\begin{aligned} Z_\sigma = Z_\sigma(F, \mathbf{d}) &:= \{P \in \mathbb{P}^{\mathbf{n}}(k) : i_{\mathbf{d}}(F, P) \geq \sigma\} \\ &= \{P \in \mathbb{P}^{\mathbf{n}}(k)) : F_{\mathbf{i}}(P) = 0 \quad \text{for all} \quad \mathbf{i} \quad \text{with} \quad (\mathbf{i}/\mathbf{d}) \leq \sigma\}. \end{aligned}$$

Z_σ need not be irreducible. The Product theorem of Faltings [1], Thm. 3.1 states that if Z is an irreducible component of Z_σ and also of $Z_{\sigma+\epsilon}$ for some $\epsilon > 0$, and if the quotients $d_1/d_2, ..., d_{m-1}/d_m$ are sufficiently large in terms of ϵ and m, then Z is a product variety. Below we have stated this result in an explicit form. The *degree* $\deg Z$ of a subvariety Z of $\mathbb{P}^n$ is the number of points in the intersection of Z with a generic linear projective subspace L of $\mathbb{P}^n$ such that $\dim Z + \dim L = n$. The *codimension* of Z is $n - \dim Z$.

Theorem 1. *Let k be an algebraically closed field of characteristic* 0*. Further, let m be an integer ≥ 2, $\mathbf{n} = (n_1, ..., n_m)$, $\mathbf{d} = (d_1, ..., d_m)$ tuples of positive integers and σ, ϵ reals such that $\sigma \geq 0, 0 < \epsilon \leq 1$ and*

$$(1.1) \qquad \frac{d_h}{d_{h+1}} \geq \left(\frac{mM}{\epsilon}\right)^M \quad \textit{for} \quad h = 1, ..., m-1$$

where

$$M := n_1 + ... + n_m.$$

Finally, let $F \in \Gamma_k^{\mathbf{n}}(\mathbf{d}) \backslash \{0\}$, and let Z be an irreducible component of both $Z_\sigma(F, \mathbf{d})$ and $Z_{\sigma+\epsilon}(F, \mathbf{d})$.
Then Z is a product variety

$$(1.2) \qquad Z = Z_1 \times ... \times Z_m,$$

where Z_h is a subvariety of $\mathbb{P}^{n_h}(k)$ for $h = 1, ..., m$. Further, if F has its coefficients in a subfield k_0 of k, then $Z_1, ..., Z_m$ are defined over an extension k_1 of k_0 with

$$(1.3) \qquad [k_1 : k_0] \deg Z_1 ... \deg Z_m \leq \left(\frac{ms}{\epsilon}\right)^s,$$

where $s = \sum_{i=1}^m \mathrm{codim} Z_i$.

The idea behind the proof of Theorem 1 is roughly as follows. Any irreducible component Z of both Z_σ and $Z_{\sigma+\epsilon}$ must have in some sense large multiplicity (analogously, if for a polynomial f in one variable all derivatives of f up to some order vanish at P then P has large multiplicity). On the other hand, using intersection theory one shows that the multiplicity of Z_σ can be that large only if this component is a product variety.

Now let $k = \overline{\mathbb{Q}}$ be the field of algebraic numbers. We need estimates for the heights of $Z_1, ..., Z_m$ in terms of the height of F. First we define the height of $\mathbf{x} = (x_0, ..., x_n) \in \overline{\mathbb{Q}}^{n+1} \backslash \{\mathbf{0}\}$. Take any number field K containing $x_0, ..., x_n$. Denote by O_K the ring of integers of K and let $\sigma_1, ..., \sigma_f, f = [K : \mathbb{Q}]$ be the embeddings of K into $\mathbb{C}$. Choose $\alpha \in O_K \backslash \{0\}$ such that $\alpha x_0, ..., \alpha x_n \in O_K$, let $\mathfrak{a} = \alpha x_0 O_K + ... + \alpha x_n O_K$ be the ideal generated by $\alpha x_0, ..., \alpha x_n$, and $N\mathfrak{a} = \#(O_K/\mathfrak{a})$ the norm of $\mathfrak{a}$. Then the height of $\mathbf{x}$ is defined by

$$
(1.4) \qquad H(\mathbf{x}) := \left\{ \frac{1}{N\mathfrak{a}} \prod_{j=1}^{f} \left(\sum_{i=0}^{n} |\sigma_j(\alpha x_i)|^2 \right)^{1/2} \right\}^{1/f}.
$$

It is easy to show that this does not depend on the choices of α and K. The height of a non-zero polynomial $F \in \overline{\mathbb{Q}}[X_1, ..., X_n]$ is defined by $H(F) = H(\mathbf{x})$ where $\mathbf{x}$ is the vector of non-zero coefficients of F.
It is obvious that $H(\lambda \mathbf{x}) = H(\mathbf{x})$ for every $\lambda \in \overline{\mathbb{Q}}^*$. Hence we can define a height on $\mathbb{P}^n(\overline{\mathbb{Q}})$ by $H(P) = H(\mathbf{x})$ where $\mathbf{x} \in \overline{\mathbb{Q}}^{n+1} \backslash \{\mathbf{0}\}$ is any vector representing P. By using the arithmetic intersection theory of Gillet and Soulé [5] for schemes over Spec $\mathbb{Z}$, Faltings defined a height $h(Z)$ for subvarieties Z of $\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})$, cf, [1], pp. 552/553 and [7] for more details. This height is always ≥ 0. Further, for points $P \in \mathbb{P}^n(\overline{\mathbb{Q}})$ we have

$$
(1.5) \qquad h(P) = \log H(P).
$$

Philippon [10] and Soulé [19] gave an explicit expression for the Faltings height of Z in terms of the Chow form of Z. This is the up to a constant unique polynomial F_Z in the $r+1$ blocks of $n+1$ variables $\zeta_0 = (\zeta_{0d}, ..., \zeta_{0n}), ..., \zeta_r = (\zeta_{r0}, ..., \zeta_{rn})$, where $r = \dim Z$ such that F_Z has degree deg Z in each block ζ_i $(i = 0, ..., r)$ and such that $F_Z(\zeta_0, ..., \zeta_r) = 0$ if and only if Z and the $r+1$ linear hyperplanes $(\zeta_i, \mathbf{X}) = 0$ $(i = 0, ..., r)$ have a point in common (cf. [18] pp. 65-66). From the investigations of Philippon and Soulé it follows that

$$
(1.6) \qquad |h(Z) - \log H(F_Z)| \leq c(n) \mathrm{deg} Z,
$$

where $c(n)$ is effectively computable in terms of n.
Below we give an explicit version of [1], Theorem 3.3.

Theorem 2. *Let* $m, \mathbf{n}, \mathbf{d}, \sigma, \epsilon, F, Z, Z_1, ..., Z_m, k_0, k_1, s = \sum_{h=1}^{m} \mathrm{codim}\ Z_h$ *be as in Theorem 1, except that* $k = \overline{\mathbb{Q}}$. *Then*

$$
\begin{aligned}
& [k_1 : k_0] \, \mathrm{deg} Z_1 ... \deg Z_m \left(\sum_{h=1}^{m} \frac{1}{\deg Z_h} \cdot d_h h(Z_h) \right) \\
(1.7) \qquad & \leq 2(s/\epsilon)^s m^M \cdot M^2 (d_1 + ... + d_m + \log H(F)).
\end{aligned}
$$

As mentioned in the Introduction results similar to Theorems 1 (cf. [3]) and 2 were independently obtained by Ferretti.

The following corollary of Theorems 1 and 2 is useful.

Corollary. *Let m be an integer $\geq 2, \mathbf{n} = (n_1, ..., n_m), \mathbf{d} = (d_1, ..., d_m)$ tuples of positive integers and ϵ a real such that $0 \leq \epsilon \leq M+1$ and*

$$\frac{d_h}{d_{h+1}} \geq \left(\frac{mM(M+1)}{\epsilon}\right)^M \quad \textit{for} \quad h = 1, ..., m-1,$$

where again $M := n_1 + ... + n_m$. Further, let $F \in \Gamma^{\mathbf{n}}_{\overline{\mathbb{Q}}}(\mathbf{d})\backslash\{0\}$. Then each irreducible component of Z_ϵ is contained in a product variety

$$Z_1 \times ... \times Z_m \subsetneqq \mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})$$

where for $h = 1, ..., m$, Z_h is a subvariety of $\mathbb{P}^{n_h}(\overline{\mathbb{Q}})$. Further, if F has its coefficients in an algebraic number field k_0, then $Z_1, ..., Z_m$ are defined over an extension k_1 of k_0 with

$$(1.9) \qquad [k_1 : k_0] \deg Z_1 ... \deg Z_m \leq \left(\frac{m(M+1)s}{\epsilon}\right)^s,$$

$$\textit{where} \quad s = \sum_{h=1}^{m} \text{codim} \; Z_h,$$

$$(1.10) \qquad [k_1 : k_0] \deg Z_1 ... \deg Z_m \left(\sum_{h=1}^{m} \frac{1}{\deg Z_h} \cdot d_h h(Z_h)\right)$$

$$\leq 2\left(\frac{(M+1)s}{\epsilon}\right)^s m^M M^2 (d_1 + ... + d_m + \log H(F)).$$

Proof. Put $\epsilon' := \epsilon/(M+1)$. Consider the sequence of closed subsets of $\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})$:

$$\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}}) = Z_0 \supseteq Z_{\epsilon'} \supseteq Z_{2\epsilon'} \supseteq ... \supseteq Z_{(M+1)\epsilon'} = Z_\epsilon.$$

For $i = 0, ..., M+1$, choose an irreducible component W_i of $Z_{i\epsilon'}$ such that

$$\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}}) = W_0 \supseteq W_1 \supseteq ... \supseteq W_{(M+1)} = Z.$$

By [18], p. 54, $\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})$ has dimension $n_1 + ... + n_m = M$ and if V_1, V_2 are two subvarieties of $\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})$ with $V_1 \subsetneqq V_2$ then $\dim V_1 < \dim V_2$. It follows that there is an $i \in \{0, ..., M\}$ with $W_i = W_{i+1}$. Clearly, $W := W_i = W_{i+1} \subsetneqq \mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})$ as it is contained in $\{F = 0\}$. Further, W is an irreducible component of both $Z_{i\epsilon'}$ and $Z_{i\epsilon' + \epsilon'}$. By (1.8), the conditions of Theorems 1 and 2 are satisfied with $i\epsilon', \epsilon'$

replacing σ, ϵ. Hence $W = Z_1 \times ... \times Z_m$, where Z_h is a subvariety of $\mathbb{P}^{n_h}(\overline{\mathbb{Q}})$, for $h = 1, ..., m$. Inequalities (1.9), (1.10) follow by replacing ϵ by $\epsilon/(M+1)$ in (1.3), (1.7), respectively. $\square$

Using the techniques of the proofs of Theorems 1 and 2 one can prove the following sharpening of a non-vanishing result of Roth from 1955 [12], now known as Roth's lemma. Roth used this in his proof of his famous theorem, also in [12], that for every algebraic number α and every $\kappa > 2$ there are only finitely many rationals x/y with $x, y \in \mathbb{Z}, y > 0$ and $|\alpha - x/y| < y^{-\kappa}$. In fact, from the Corollary with $n_1 = ... = n_m = 1$ one can derive Theorem 3 below with instead of (1.11) the more restrictive condition $d_h/d_{h+1} \geq (2m^3/\epsilon)^m$ for $h = 1, ..., m-1$.

Theorem 3. *(Roth's lemma). Let m be an integer ≥ 2, let $\mathbf{d} = (d_1, ..., d_m)$ be a tuple of positive integers, let $F \in \overline{\mathbb{Q}}[X_{10}, X_{11}; ...; X_{m0}, X_{m1}]$ be a non-zero polynomial which is homogeneous of degree d_h in the pair of variables (X_{h0}, X_{h1}) for $h = 1, ...m$ and let ϵ be a real with $0 < \epsilon \leq m+1$ such that*

$$\frac{d_h}{d_{h+1}} \geq 2m^3/\epsilon \quad \textit{for} \quad h = 1, ..., m-1. \tag{1.11}$$

Further, let $P = (P_1, ..., P_m)$ where $P_1, ..., P_m$ are points in $\mathbb{P}^1(\overline{\mathbb{Q}})$ with

$$H(P_h)^{d_h} > \left\{ e^{d_1 + ... + d_m} H(F) \right\}^{(3m^3/\epsilon)^m} \quad \textit{for } h = 1, ..., m, \tag{1.12}$$

where $e = 2.7182....$ Then

$$i_{\mathbf{d}}(F, P) < \epsilon \ .$$

The original lemma proved by Roth in 1955 [12] differs from Theorem 3 in that instead of (1.11) it has the more restrictive condition

$$d_h/d_{h+1} \geq (10^m/\epsilon)^{2^m} \quad \text{for} \quad h = 1, ..., m-1. \tag{1.13}$$

Roth's lemma with (1.13) was also used by Schmidt in his proof of the Subspace theorem and by Schmidt and Schlickewei in their proofs of quantitative versions of the Subspace Theorem. In our improvements of the results of Schmidt and Schlickewei mentioned in the introduction, it was crucial that (1.13) could be replaced by (1.11).

Remark. (inspired by a suggestion of the referee). We have formulated the Product theorem and its consequences for multi-homogeneous polynomials. There are affine analogues for polynomials which are not multi-homogeneous. For instance, for $h = 1, ..., m$, let $\mathbf{Y}_h = (Y_{h1}, ..., Y_{h,n_h})$ be a block of affine variables, and let

$f \in k[\mathbf{Y}_1, ..., \mathbf{Y}_m]$ be a polynomial whose total degree in the block $\mathbf{Y}_h$ is at most d_h, for $h = 1, ..., m$. Denote by $\mathbf{i}, \mathbf{k}$ tuples $(i_{hj} : h = 1, ..., m, \ j = 0, ..., n_h)$, $(k_{hj} : h = 1, ..., m, \ j = 1, ..., n_h)$, respectively. Define the index of f at a point p as the largest number σ such that $f_{\mathbf{k}}(p) = 0$ for all tuples $\mathbf{k}$ with $\sum_{h=1}^m d_h^{-1}(k_{h1} + ... + k_{h,n_h}) \leq \sigma$, where $f_{\mathbf{k}} = (\prod_{h=1}^m \prod_{j=1}^{n_h} \partial^{k_{hj}} / \partial Y_{hj}^{k_{hj}}) f$. For $h = 1, ..., m$, define a block of variables $\mathbf{X}_h = (X_{h0}, ..., X_{hm})$ such that $Y_{hj} = X_{hj}/X_{h0}$ for $j = 1, ..., n_h$. Let $F = \prod_{h=1}^m X_{h0}^{d_h} f$ be the multi-homogeneous polynomial in $\mathbf{X}_1, ..., \mathbf{X}_m$ corresponding to f. One obtains an analogue of Theorem 1 for f (the same statement with everywhere "affine varieties" replacing "projective varieties") simply by applying Theorem 1 to F. We have to check that the index of f at $p = (p_{11}, ..., p_{1,n_1}; ...; p_{m1}, ..., p_{m,n_m})$, defined using the variables Y_{hj}, is equal to the index of F at $P = (1, p_{11}, ..., p_{1,n_1}; ...; 1, ..., p_{m,n_m})$ defined using the variables X_{hj}. This follows by observing first that $f_{\mathbf{k}} = H^{-1} F_{\mathbf{i}}$, where H is a product of powers of X_{h0} $(h = 1, ..., m)$ and $\mathbf{i}$ is the same tuple as $\mathbf{k}$ augmented with $i_{h0} := 0$ for $h = 1, ..., m$, and second, in view of Euler's identity $\partial H / \partial X_{h0} = X_{h0}^{-1}(e_h H - \sum_{j=1}^{n_h} X_{hj} \partial H / \partial X_{hj})$ for polynomials H homogeneous of degree e_h in $\mathbf{X}_h$, that for each tuple $\mathbf{i}$, $F_{\mathbf{i}}$ is a linear combination of $f_{\mathbf{k}}$ over tuples $\mathbf{k}$ with $k_{hj} \leq i_{hj}$ for $h = 1, ..., m, \ j = 1, ..., n_h$, the coefficients being rational functions whose denominators are products of powers of X_{h0} $(h = 1, ..., m)$.

§2. Intersection theory.

Most of the results from intersection theory we need can be found in [4], Chaps, 1,2 and in [9]. As in §1, k denotes an algebraically closed field and $\mathbf{n} = (n_1, ..., n_m)$ a tuple of positive integers. The block $\mathbf{X}_h$ of $n_h + 1$ variables, the ring $k[\mathbf{X}] = k[\mathbf{X}_1, ..., \mathbf{X}_m]$ and the sets $\Gamma_k^{\mathbf{n}}(\mathbf{d})$ will have the meaning of §1. We write $\mathbb{P}^{\mathbf{n}}, \Gamma^{\mathbf{n}}$, $\Gamma^{\mathbf{n}}(\mathbf{d})$ for $\mathbb{P}^{\mathbf{n}}(k), \Gamma_k^{\mathbf{n}}, \Gamma_k^{n}(\mathbf{d})$.
For every subvariety Z of $\mathbb{P}^{\mathbf{n}}$ there is a unique essential $\mathbf{n}$-prime ideal I of $k[\mathbf{X}]$ such that $Z = V(I) = \{P \in \mathbb{P}^{\mathbf{n}} : f(P) = 0 \text{ for every } f \in I\}$. The *local ring* of Z is defined by

$$(2.1) \qquad \mathcal{O}_Z := \left\{ \frac{f}{g} : \exists \mathbf{d} \in (\mathbb{Z}_{\geq 0})^m \ \text{ with } f, g \in \Gamma^{\mathbf{n}}(\mathbf{d}), \ g \notin I \right\}.$$

For any $\mathbf{n}$-ideal J of $k[\mathbf{X}]$ we put $J\mathcal{O}_Z := \{f/g : \exists \mathbf{d} \in (\mathbb{Z}_{\geq 0})^m \text{ with } f, g \in \Gamma^{\mathbf{n}}(\mathbf{d}), f \in J, g \notin I\}$. Then $M_Z := I\mathcal{O}_Z$ is the maximal ideal of $\mathcal{O}_Z$. The residue field $k(Z) := \mathcal{O}_Z / M_Z$ is called the *function field* of Z. The dimension of Z is $\dim Z := \text{trdeg}_k \, k(Z)$. In particular, $\dim \mathbb{P}^{\mathbf{n}} = M := n_1 + ... + n_m$. The *codimension* of Z is $\text{codim} Z := M - \dim Z$; if W is a subvariety of Z then the codimension of W in Z is codim $(W, Z) = \dim Z - \dim W$.

A *cycle* in $\mathbb{P}^{\mathbf{n}}$ is a finite formal linear combination with integer coefficients of subvarieties V of $\mathbb{P}^{\mathbf{n}}, Z = \sum n_V V$, say. The *components* of Z are the subvarieties

V for which $n_V \neq 0$, and n_V is called the multiplicity of V in Z. Z is called *effective* if all $n_V \geq 0$. Denote by $\mathcal{Z}_k = \mathcal{Z}_k(\mathbb{P}^{\mathbf{n}})$ the abelian group of cycles in $\mathbb{P}^{\mathbf{n}}$ all whose components have dimension k and put $\mathcal{Z}_k := (0)$ for $k < 0$. We denote by Z cycles as well as varieties.
For a ring A and an A-module M, we define the *length* $l_A(M)$ to be the integer l for which there exists a sequence of A-modules

$$M = M_0 \supsetneqq M_1 \supsetneqq \ldots \supsetneqq M_l = (0)$$

such that $M_{i-1}/M_i \cong A/p_i$ for $i = 1, \ldots, l$ where p_i is a maximal ideal of A (cf. [4], p. 406); $l_A(M)$ is independent of the choice of $M_0, \ldots, M_l$. Now let $Z = V(I)$ be a subvariety of $\mathbb{P}^{\mathbf{n}}$ and $f \in \Gamma^{\mathbf{n}} \backslash \{0\}$ such that f does not vanish identically on Z, i.e. $f \notin I$. We define the divisor of f restricted to Z by attaching certain multiplicities to the irreducible components of $Z \cap \{f = 0\}$. These irreducible components are all of codimension 1 in V (cf. [21], p. 196). For each subvariety W of Z with $\operatorname{codim}(W, Z) = 1$, the number

$$\operatorname{ord}_W(f|Z) := l_{\mathcal{O}_W}(\mathcal{O}_W/(I + (f))\mathcal{O}_W) \tag{2.2}$$

is a finite, non-negative integer and $\operatorname{ord}_W(f|Z) > 0$ if and only if $I + (f)$ is contained in the prime ideal of W, i.e. if W is an irreducible component of $Z \cap \{f = 0\}$. Now define

$$\operatorname{div}(f|Z) = \sum_W \operatorname{ord}_W(f|Z) \cdot W, \tag{2.3}$$

where the sum is taken over all subvarieties W of codimension 1 in Z. By [3], App. A3, $\operatorname{ord}_W(fg|Z) = \operatorname{ord}_W(f|Z) + \operatorname{ord}_W(g|Z)$ and hence $\operatorname{div}(fg|(Z) = \operatorname{div}(f|Z) + \operatorname{div}(g|Z)$ whenever f, g do not identically vanish on Z. By abuse of terminology, we say that f does not identically vanish on a cycle $Z = \sum n_V V$ if for each component V of Z, f does not identically vanish on V. In that case we define $\operatorname{div}(f|Z) = \sum n_V \operatorname{div}(f|V)$. Note that $\operatorname{div}(f|Z)$ is effective if Z is effective. We write $\operatorname{div}(f)$ if $Z = \mathbb{P}^{\mathbf{n}}$.

Two cycles $Z_1, Z_2 \in \mathcal{Z}_t(\mathbb{P}^{\mathbf{n}})$ are called *rationally equivalent* if $Z_1 - Z_2$ is a linear combination of cycles $\operatorname{div}(f|V) - \operatorname{div}(g|V)$, where V is a $(t+1)$-dimensional subvariety of $\mathbb{P}^{\mathbf{n}}$ and $f, g \in \Gamma^{\mathbf{n}}(\mathbf{d})$ for some $\mathbf{d} \in (\mathbb{Z}_{\geq 0})^n$. Addition of cycles induces addition of rational equivalence classes. Note that all divisors $\operatorname{div}(f)$ with $f \in \Gamma^{\mathbf{n}}(\mathbf{d})(\mathbf{d} \in (\mathbb{Z}_{\geq 0})^m)$ are rationally equivalent; denote by $\mathcal{O}(\mathbf{d})$ the rational equivalence class of $\operatorname{div}(f), f \in \Gamma^{\mathbf{n}}(\mathbf{d})$. Clearly, $\mathcal{O}(\mathbf{d}_1) + \mathcal{O}(\mathbf{d}_2) = \mathcal{O}(\mathbf{d}_1 + \mathbf{d}_2)$. We define $\mathcal{O}(\mathbf{d})$ for $\mathbf{d} \in \mathbb{Z}^m$ by additivity. Put $Pic(\mathbf{n}) = \{\mathcal{O}(\mathbf{d}) : \mathbf{d} \in \mathbb{Z}^m\}, Pic^+(\mathbf{n}) = \{\mathcal{O}(\mathbf{d}) : \mathbf{d} \in (\mathbb{Z}_{\geq 0})^m\}$. If $\mathcal{M} = \mathcal{O}(\mathbf{d}) \in Pic^+(\mathbf{n})$, then write $\Gamma(\mathcal{M})$ or $\Gamma_k(\mathcal{M})$ for $\Gamma_k^{\mathbf{n}}(\mathbf{d})$.
For a zero-dimensional cycle $Z = \sum_P n_P P$ we define its degree:

$$\deg Z := \sum_P n_P.$$

Then we have:

Lemma 1. *For $t = 0, ..., M$ there is a unique function (intersection number) from $\mathcal{Z}_t(\mathbb{P}^{\mathbf{n}}) \times Pic(\mathbf{n})^t$ to $\mathbb{Z} : (Z, \mathcal{M}_1, ..., \mathcal{M}_t) \mapsto (Z \cdot \mathcal{M}_1 ... \mathcal{M}_t)$ with the following properties:*
(i). $(Z \cdot \mathcal{M}_1 ... \mathcal{M}_t)$ is additive in $Z, \mathcal{M}_1, ..., \mathcal{M}_t$ and invariant under permutations of $\mathcal{M}_1, ..., \mathcal{M}_t$;
(ii). $(Z \cdot \mathcal{M}_1 ... \mathcal{M}_t) = 0$ if Z is rationally equivalent to 0;
(iii). if $Z \in \mathcal{Z}_0(\mathbb{P}^{\mathbf{n}})$ then $(Z) = \deg Z$;
(iv). if $\mathcal{M}_1 \in Pic^+(\mathbf{n})$ then there is an $f \in \Gamma(\mathcal{M}_1)$ not identically vanishing on Z and for every such f we have $(Z \cdot \mathcal{M}_1, ..., \mathcal{M}_t) = (\mathrm{div}(f|Z) \cdot \mathcal{M}_2 ... \mathcal{M}_t)$.

Proof. This comprises some of the results from [4], Chaps. 1, 2. Rationally equivalent cycles in $\mathcal{Z}_0$ have the same degree and if $Z, Z' \in \mathcal{Z}_t$ are rationally equivalent and $f, f' \in \Gamma(\mathcal{M}_1)$, then $\mathrm{div}(f|Z), \mathrm{div}(f|Z')$ are rationally equivalent. Hence the intersection number can be defined inductively by (iii), (iv). □

We write $(\mathcal{M}_1 ... \mathcal{M}_m)$ for $(\mathbb{P}^{\mathbf{n}} \cdot \mathcal{M}_1 ... \mathcal{M}_m)$. If among $\mathcal{M}_1, ..., \mathcal{M}_t$, $\mathcal{N}_i$ appears e_i times for $i = 1, ..., s$, where $e_1 + ... + e_s = t$ then we write $(Z \cdot \mathcal{N}_1^{e_1} ... \mathcal{N}_s^{e_s})$ for $Z \cdot \mathcal{M}_1 ... \mathcal{M}_t)$. The *degree* of $Z \in \mathcal{Z}_t$ is defined by deg $Z := (Z \cdot \mathcal{O}(1)^t)$.

Remarks (i). By induction on the dimension it follows easily that if $Z \in \mathcal{Z}_t$ is effective and $\mathcal{M}_1, ..., \mathcal{M}_t \in Pic^+(\mathbf{n})$ then $(Z \cdot \mathcal{M}_1 ... \mathcal{M}_t) \geq 0$. Moreover, if Z is a subvariety of $\mathbb{P}^{\mathbf{n}}$ and $f_1 \in \Gamma(\mathcal{M}_1), ..., f_t \in \Gamma(\mathcal{M}_t)$ are 'generic', then $(Z \cdot \mathcal{M}_1, ..., \mathcal{M}_t)$ is precisely the cardinality of the set of points $V \cap \{f_1 = ... = f_t = 0\}$.
(ii). Let k_0 be a perfect subfield of k, i.e. every finite extension of k_0 is separable. A k_0-subvariety of $\mathbb{P}^{\mathbf{n}}$ is a set $\{P \in \mathbb{P}^{\mathbf{n}} : f(P) = 0$ for every $f \in I\}$ where I is an essential $\mathbf{n}$-prime ideal of $k_0[\mathbf{X}]$. Every such k_0-subvariety Z is a union of equal dimensional subvarieties of $\mathbb{P}^{\mathbf{n}}, Z = Z_1 \cup ... \cup Z_q$, and we put dim $Z :=$ dim Z_1; now if $\dim Z = k$ and $\mathcal{M}_1, ..., \mathcal{M}_t \in Pic^+(\mathbf{n})$ then we define

$$(Z \cdot \mathcal{M}_1 ... \mathcal{M}_t) := \sum_{i=1}^{q} (Z_i \cdot \mathcal{M}_1 ... \mathcal{M}_t). \tag{2.4}$$

This is extended by linearity to k_0-cycles, i.e. finite formal sums of k_0-subvarieties.

We need some further properties of the intersection number. Let $\mathbf{e}_1 = (1, 0, ..., 0)$, $\mathbf{e}_2 = (0, 1, ..., 0), ..., \mathbf{e}_m = (0, ..., 0, 1)$ and put $\mathcal{L}_h = \mathcal{O}(\mathbf{e}_h)$ for $h = 1, ..., m$. Further, fix $\mathbf{d} = (d_1, ..., d_m) \in (\mathbb{Z}_{>0})^m$ and put $\mathcal{L} := \mathcal{O}(\mathbf{d}) = d_1\mathcal{L}_1 + ... + d_m\mathcal{L}_m$. If $Z_h = \sum_{V_h} n_{V_h} V_h (h = 1, ..., m)$ is a cycle in $\mathbb{P}^{n_h}$ then of course we define

$$Z_1 \times ... \times Z_m = \sum n_{V_1} n_{V_2} ... n_{V_m} V_1 \times ... \times V_m.$$

Further, we denote by π_h the projection to the h-th factor $\mathbb{P}^{\mathbf{n}} \to \mathbb{P}^{n_h}$ and by π_h^* the inclusion ("pull back") $k[\mathbf{X}_h] \hookrightarrow k[\mathbf{X}_1, ..., \mathbf{X}_m] = k[\mathbf{X}]$.

Lemma 2. *Let $Z_h \in \mathcal{Z}_{\delta_h}(\mathbb{P}^{n_h})$ $(h = 1, ..., m)$ and $Z = Z_1 \times ... \times Z_m$. Put $\delta = \delta_1 + ... + \delta_m$.*
(i). Suppose that $f \in \Gamma^{n_1}$ does not vanish identically on Z_1. Then $\pi_1^* f$ does not vanish identically on Z and

$$\operatorname{div}(\pi_1^* f | Z) = \operatorname{div}(f | Z_1) \times Z_2 \times ... \times Z_m \ . \tag{2.5}$$

(ii). Let $e_1, ..., e_m$ be non-negative integers with $e_1 + ... + e_m = \delta$. Then

$$\begin{aligned}(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) &= \deg Z_1 ... \deg Z_m \quad \text{if} \quad (e_1, ..., e_m) = (\delta_1, ..., \delta_m) \\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

(iii). $(Z \cdot \mathcal{L}^\delta) = (\delta!/\delta_1!...\delta_m!) d_1^{\delta_1} ... d_m^{\delta_m} \deg Z_1 ... \deg Z_m$. In particular $(\mathcal{L}^M) = C := (M!/n_1!...n_m!) d_1^{n_1} ... d_m^{n_m}$.

Proof (i). [4], p. 35, ex. 2.3.1. This is analogous to the set-theoretic statement that if $Z_1, ..., Z_m$ are varieties then $Z \cap \{\pi_1^* f = 0\} = (Z_1 \cap \{f = 0\}) \times Z_2 \times ... \times Z_m$.

(ii). This follows easily from (i) by induction on δ. Another way is as follows. For $h = 1, ..., m$ assume that Z_h is a subvariety of $\mathbb{P}^{n_h}$, take generic linear forms $f_{hj} \in k[\mathbf{X}_h]$ for $j = 1, ..., e_h$ and put $W_h = Z_h \cap \{f_{h1} = 0, ..., f_{h,e_h} = 0\}$. Then by remark (i) above $(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m})$ is the cardinality of the set $W = W_1 \times ... \times W_m$. This cardinality is zero if $(e_1, ... e_m) \neq (\delta_1, ..., \delta_m)$ since then one of the sets W_h is empty; while otherwise this cardinality is $\prod_{h=1}^m \# W_h = \prod_{h=1}^m \deg Z_h$.

(iii). By additivity we have

$$\begin{aligned}(Z \cdot \mathcal{L}^\delta) &= (Z \cdot (d_1 \mathcal{L}_1 + ... + d_m \mathcal{L}_m)^\delta) \\ &= \sum_{e_1 + ... + e_m = d} \frac{\delta!}{e_1! ... e_m!} d_1^{e_1} ... d_m^{e_m} (Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \\ &= \frac{\delta!}{\delta_1! ... \delta_m!} d_1^{\delta_1} ... d_m^{\delta_m} \deg Z_1 ... \deg Z_m\end{aligned}$$

□

Lemma 3. *Suppose that $m \geq 2$. Let Z be a δ-dimensional subvariety of $\mathbb{P}^{\mathbf{n}}$ that can not be expressed as a product $Z = Z_1 \times ... \times Z_m$ with $Z_h \subseteq \mathbb{P}^{n_h}$ for $h = 1, ..., m$. Then there are at least two tuples of non-negative integers $(e_1, ..., e_m)$ with $e_1 + ... + e_m = \delta$ and $(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) > 0$.*

Proof. cf. [11], p. 79. The idea is as follows. By [18], p. 45, Thm 2, if X is a closed subset of $\mathbb{P}^{\mathbf{n}}$ and $f : X \to \mathbb{P}^{\mathbf{n}}$ a morphism, then $f(X)$ is closed, and f maps subvarieties of X to subvarieties of $f(X)$. We apply this with the projections $\pi_h : \mathbb{P}^{\mathbf{n}} \to \mathbb{P}^{n_h}$. Put $Z_h := \pi_h(Z), \delta_h := \dim Z_h$ for $h = 1, ..., m$. Since Z is not a product, Z is a proper subvariety of $Z_1 \times ... \times Z_m$ and therefore, $\delta = \dim Z < \dim Z_1 \times ... \times Z_m = \delta_1 + ... + \delta_m$. We prove by induction on m the following assertion: for each $h \in \{1, ..., m\}$ there is a tuple $(e_1, ..., e_m)$ as in the statement of Lemma 3 with $e_h = \delta_h$. This implies Lemma 3 since $\delta_1 + ... + \delta_m > \delta$. This assertion is obviously true if $m = 1$. Suppose that the assertion holds for $m = r - 1$ where $r > 1$. We prove the assertion for $m = r, h = 1$ which clearly suffices. In the induction step we proceed by induction on δ_1. If $\delta_1 = 0$ then $Z = Q \times W$ where $Q \in \mathbb{P}^{n_1}$ and W is a subvariety of $\mathbb{P}^{n_2} \times ... \times \mathbb{P}^{n_m}$ and the assertion follows by applying the induction hypothesis to W. If $\delta_1 > 0$ then choose a linear form $f \in k[\mathbf{X}_1]$ that does not identically vanish on Z_1. Then $g := \pi_1^* f$ does not identically vanish on Z. Clearly, π_1 maps the irreducible components of $Z \cap \{g = 0\}$ to those of $Z_1 \cap \{f = 0\}$ and the latter have dimension $\delta_1 - 1$. By applying the second induction hypothesis to the irreducible components of $Z \cap \{g = 0\}$, we infer that there are non-negative integers $e_1, ..., e_m$ with $e_1 + ... + e_m = \delta$ and $e_1 = \delta_1$ such that $(\mathrm{div}(g|Z) \cdot \mathcal{L}_1^{e_1 - 1} ... \mathcal{L}_m^{e_m}) > 0$. Hence $(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) > 0$. This proves the assertion. □

Lemma 4. *Let A be a set of polynomials from $\Gamma^{\mathbf{n}}(\mathbf{d}) \backslash \{0\}$ and I the ideal generated by A. Let $Z_1, ..., Z_t$ be irreducible components of codimension t of $X := \{P \in \mathbb{P}^{\mathbf{n}} : f(P) = 0$ for $f \in A\}$. Then for all tuples of non-negative integers $(e_1, ..., e_m)$ with $e_1 + ... + e_m = M - t$ one has*

$$\sum_{i=1}^{r} m_{Z_i} (Z_i \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \leq (\mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m} \cdot \mathcal{L}^t),$$

where $m_{Z_i} := l_{\mathcal{O}_{Z_i}}(\mathcal{O}_{Z_i} / I \mathcal{O}_{Z_i})$ for $i = 1, ..., r$.

Proof. This is essentially Prop. 2.3 of [1] and Lemma 6.4, p. 76 of [9]. We give some details of the proof to which we have to refer later. For a subvariety Z of $\mathbb{P}^{\mathbf{n}}$ and $f \in \Gamma^{\mathbf{n}} \backslash \{0\}$ not vanishing identically on Z, define the truncated divisor

$$\mathrm{div}^X(f|Z) := \sum_{W \not\subset X} \mathrm{ord}_W(f|Z) W,$$

where the sum is taken over all irreducible components of $\mathrm{div}(f|Z)$ which are not an irreducible component of X. This is extended by linearity to cycles. Put $Z_0 := \mathbb{P}^{\mathbf{n}}$ and choose inductively $f_1, ..., f_t \in I$ and define cycles $C_1, ..., C_t$ as follows:

(2.6) for $j = 1, ..., t$, f_j does not vanish identically on C_{j-1},
each Z_i $(i = 1, ..., r)$ is a subvariety of one of the irreducible components of $\mathrm{div}(f_j | C_{j-1})$, and
$C_j := \mathrm{div}^X(f_j | C_{j-1})$;

in the next lemma we explicitly construct such f_j. Clearly, the irreducible components of C_j have codimension j. Therefore $Z_1, ..., Z_r$ are irreducible components of C_t. We need some more advanced results from intersection theory to estimate the multiplicity m_{Z_i,C_t} of Z_i in C_t from below. By [4], Ex. 7.1.10, p. 123, m_{Z_i,C_t} is equal to $l_{\mathcal{O}_{Z_i}}(\mathcal{O}_{Z_i}/I'\mathcal{O}_{Z_i})$, where $I' = (f_1, ..., f_t)$. (Note that $\mathbb{P}^{\mathbf{n}}$ is smooth whence that all local rings $\mathcal{O}_{Z_i}$ are Cohen-McAulay rings). Further, since $I' \subseteq I$ we have $l_{\mathcal{O}_{Z_i}}(\mathcal{O}_{Z_i}/I'\mathcal{O}_{Z_i}) \geq l_{\mathcal{O}_{Z_i}}(\mathcal{O}_{Z_i}/I\mathcal{O}_{Z_i}) = m_{Z_i}$. Hence $m_{Z_i,C_t} \geq m_{Z_i}$. It follows that

$$C_t = \sum_{i=1}^{r} m_{Z_i} Z_i + \quad \text{(effective cycle)}. \tag{2.7}$$

Further, by (2.6) we have

$$\operatorname{div}(f_j | C_{j-1}) = C_j + \quad \text{(effective cycle)} \quad \text{for} \quad j = 1, ..., t. \tag{2.8}$$

Now by (2.7), (2.8) and $f_j \in \Gamma(\mathcal{L})$ we have

$$\sum_{i=1}^{r} m_{Z_i}(Z_i \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \leq (C_t \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e^m})$$
$$\leq (C_{t-1} \cdot \mathcal{L} \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \leq (C_{t-2} \cdot \mathcal{L}^2 \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \leq ... \leq (\mathcal{L}^t \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}). \qquad \square$$

Lemma 5. *It is possible to choose $f_1, ..., f_t$ as in (2.6) such that*

$$f_i = \sum_{j=1}^{C} n_{ij} g_{ij} \quad \text{for } i = 1, ..., t, \tag{2.9}$$

where $C = (M!/n_1!...n_m!) d_1^{n_1} ... d_m^{n_m}$ and $g_{ij} \in A, n_{ij} \in \mathbb{Z}, |n_{ij}| \leq C$ for $i = 1, ..., t, j = 1, ..., C$.

Proof. From (2.6) and Lemma 1 it follows that $(C_i \cdot \mathcal{L}^{M-i}) \leq (C_{i-1} \cdot \mathcal{L}^{M-i+1})$ for $i = 1, ..., t$, whence

$$(C_i \cdot \mathcal{L}^{M-i}) \leq (\mathcal{L}^M) = C \quad \text{for} \quad i = 1, ..., t.$$

Letting $C_i = \sum_{k=1}^{u} a_k V_k$, where the V_k are the components of C_i and $a_k > 0$, we see that $u \leq \sum_{k=1}^{u} a_k (V_k \cdot \mathcal{L}^{M-i}) = (C_i \cdot \mathcal{L}^{M-i}) \leq C$. Hence each C_i has at most u irreducible components. Suppose we have already chosen $f_1, ..., f_s$ $(0 \leq s \leq t-1)$ such that (2.6) and (2.9) are satisfied for $i = 1, ..., s$. Let $V_1, ..., V_{u'}$ be the components of C_s which are not an irreducible component of X. Then for $j = 1, ..., u'$, there is a $g_j \in A$ which does not vanish identically on V_j. We construct $h_1, ..., h_{u'}$ such that for $j = 1, ..., u'$, h_j is not identically zero on $V_1, ..., V_j$ inductively as follows: Take $h_1 = g_1$. Suppose that h_j has been constructed. There are $\mathbf{x}_1 \in V_1, ..., \mathbf{x}_j \in V_j$ such that $h_j(\mathbf{x}_i) \neq 0$ for $i = 1, ..., j$; further, there is $\mathbf{x}_{j+1} \in V_{j+1}$ with $g_{j+1}(\mathbf{x}_{j+1}) \neq 0$. Now there is an $a \in \{0, ..., u'\}$ with $(h_j + a g_{j+1})(\mathbf{x}_i) \neq 0$ for $i = 1, ..., j+1$; take $h_{j+1} := h_j + a g_{j+1}$. Obviously, $f_{s+1} := h_{u'}$ does not identically vanish on C_s and $f_1, ..., f_{s+1}$ satisfy (2.6), (2.9). By repeating this process we arrive at $f_1, ..., f_t$ satisfying (2.6), (2.9). $\square$

§3. The Faltings height.

From [1] we have collected some properties of the Faltings height of varieties over $\overline{\mathbb{Q}}$. We use the following notation. The extension of a ring homomorphism $\psi : R_1 \to R_2$ to $R_1[X_1, ..., X_t] \to R_2[X_1, ..., X_t]$, defined by applying ψ to the coefficients of $f \in R_1[X_1, ..., X_t]$ is denoted also by ψ. The ring of integers of a number field K is denoted by O_K. For a non-zero prime ideal $\wp$ of O_K, let

$$\mathbb{F}_\wp := O_K/\wp, \quad N\wp := \#\mathbb{F}_\wp.$$

The multi-projective space $\mathbb{P}^{\mathbf{n}}(\mathbb{C})$, as well as its algebraic subvarieties, can be given the structure of a complex analytic variety. This implies that we can integrate differential forms over these varieties (cf. [6], Chap. 0). To every divisor class $\mathcal{M} = \mathcal{O}(\mathbf{d}) \in Pic(\mathbf{n})$ we associate a $(1,1)$-differential form $c_1(\mathcal{M})$, its so-called *Chern form*:

if $m = 1, \mathcal{M} = \mathcal{O}(1), \mathbf{n} = (n)$, then $c_1(\mathcal{M}) := \omega_n = \frac{\sqrt{1}}{2\pi}\partial\overline{\partial}\log(|Z_0|^2 + ... + |Z_n|^2)$ is the $(1,1)$- form associated to the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$, where $Z_0, ..., Z_n$ are the homogeneous coordinates on $\mathbb{P}^n$, cf. [6], p. 30 for an explicit formula;
if $m \geq 1, \mathbf{d} = (d_1, ..., d_m)$, then $c_1(\mathcal{M}) = d_1\pi_1^*\omega_{n_1} + ... + d_m\pi_m^*\omega_{n_m}$, where π_h is the projection $\mathbb{P}^{\mathbf{n}} \to \mathbb{P}^{n_h}$ and $\pi_h^*\omega_{n_h}$ is the pull back of ω_{n_h} from $\mathbb{P}^{n_h}$ to $\mathbb{P}^{\mathbf{n}}$ (i.e. $\pi_h^*\omega_{n_h}$ is defined by precisely the same formula as ω_{n_h} in terms of the homogeneous coordinates of $\mathbb{P}^{n_h}$ but it is considered as a differential form on $\mathbb{P}^{\mathbf{n}}$).

(t,t)-forms can be integrated over t-dimensional subvarieties of $\mathbb{P}^{\mathbf{n}}(\mathbb{C})$. For a cycle $Z = \sum n_V V \in \mathcal{Z}_t(\mathbb{P}^{\mathbf{n}}(\mathbb{C}))$ and a (t,t)-form ρ on $\mathbb{P}^{\mathbf{n}}(\mathbb{C})$, we set $\int_Z \rho := \sum n_V \int_V \rho$. By *Wirtinger's theorem* (cf. [6], p. 171 or [7], Prop. 3.6), we have for $Z \in \mathcal{Z}_t(\mathbb{P}^{\mathbf{n}}(\mathbb{C}))$ and $\mathcal{M}_1, ..., \mathcal{M}_t \in Pic(\mathbf{n})$,

$$\int_Z c_1(\mathcal{M}_1) \wedge ... \wedge c_1(\mathcal{M}_t) = (Z.\mathcal{M}_1...\mathcal{M}_t). \tag{3.1}$$

The form ω_n is *positive* on $\mathbb{P}^n$ (cf. [6], p. 31). This implies that if $Z \in \mathcal{Z}_t(\mathbb{P}^{\mathbf{n}}(\mathbb{C}))$ is effective, if $\mathcal{M}_1, ..., \mathcal{M}_t \in Pic^+(\mathbf{n})$, and if f is a real function which is non-negative everywhere on the components of Z, then

$$\int_Z f \cdot c_1(\mathcal{M}_1) \wedge ... \wedge c_1(\mathcal{M}_t) \geq 0. \tag{3.2}$$

If $Z = \sum n_P P$ is a zero-dimensional cycle in $\mathbb{P}^{\mathbf{n}}(\mathbb{C})$ and f is a function on $\mathbb{P}^{\mathbf{n}}(\mathbb{C})$ then we write $\int_Z f$ for $\sum n_P f(P)$. For $f \in \Gamma_{\mathbb{C}}^{\mathbf{n}}(\mathbf{d})$ we define a function $||f||$ on $\mathbb{P}^{\mathbf{n}}(\mathbb{C})$ as follows: let $\mathbf{Z}_h = (Z_{h0}, ..., Z_{h,n_h})$ be the complex homogeneous coordinates in $\mathbb{P}^{n_h}(\mathbb{C})$, $\mathbf{Z} = (\mathbf{Z}_1, ..., \mathbf{Z}_m), ||\mathbf{Z}_h|| = (\sum_{j=0}^{n_h} |Z_{hj}|^2)^{1/2}$; then put

$$||f||(\mathbf{Z}) = \frac{|f(\mathbf{Z})|}{||\mathbf{Z}_1||^{d_1}...||\mathbf{Z}_m||^{d_m}},$$

where $(d_1, ..., d_m) = \mathbf{d}$.

Let Z be a subvariety of $\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})$, defined over an algebraic number field K. Suppose that $f \in \Gamma^{\mathbf{n}}$ has its coefficients in O_K and does not vanish identically on Z. Let $\mathcal{M}_1, ..., \mathcal{M}_t \in Pic(\mathbf{n})$. Define for each embedding $\sigma : K \hookrightarrow \mathbb{C}$:

$$\begin{aligned} \kappa_\sigma &= \kappa_\sigma(Z, f, \mathcal{M}_1, ..., \mathcal{M}_t) \\ &= -\frac{1}{[K:\mathbb{Q}]} \int_{Z \times_\sigma \mathbb{C}} \log ||\sigma(f)|| c_1(\mathcal{M}_1) \wedge ... \wedge c_1(\mathcal{M}_t), \end{aligned} \tag{3.3}$$

where $Z \times_\sigma \mathbb{C} = \{P \in \mathbb{P}^{\mathbf{n}}(\mathbb{C}) : \sigma(g)(P) = 0$ for every $g \in K[\mathbf{X}]$ vanishing identically on $Z\}$.

Now let $\wp$ be a non-zero prime ideal of O_K. Let $I = \{f \in O_K[\mathbf{X}] : f(P) = 0$ for all $P \in Z\}$. This is an essential prime ideal of $O_K[\mathbf{X}]$ with $I \cap O_K = 0$. Let $J_{1,\wp}, ..., J_{g,\wp}$ be the minimal $\mathbf{n}$-prime ideals of $O_K[\mathbf{X}]$ containing $I + \wp O_K[\mathbf{X}]$. Then

$$W_{i\wp} = \left\{ \overline{P} \in \mathbb{F}^{\mathbf{n}}(\mathbb{F}_\wp) : \overline{g}(\overline{P}) = 0 \ \text{ for } \ \overline{g} \in J_{i\wp}/\wp O_K[\mathbf{X}] \right\}$$

is an $\mathbb{F}_\wp$-subvariety of $\mathbb{P}^n(\mathbb{F}_\wp)$ for $i = 1, ..., g$; $W_{1\wp}, ..., W_{g\wp}$ may be considered as the irreducible components of the reduction of Z mod $\wp$. Define the local ring

$$\mathcal{O}_{W_{i\wp}} = \left\{ \frac{h}{g} : h, g \in \Gamma^{\mathbf{n}}_{O_K}(\mathbf{d}) \text{ for some } \mathbf{d} \in (\mathbb{Z}_{\geq 0})^m, \quad g \notin J_{i\wp} \right\},$$

put $\mathrm{ord}_{W_{i\wp}}(f|Z) := l_{\mathcal{O}_{W_{i\wp}}}(\mathcal{O}_{W_{i\wp}}/(I + (f))\mathcal{O}_{W_{i\wp}})$ (which is finite since $I + (f)$ is a primary ideal for the maximal ideal of $\mathcal{O}_{W_{i\wp}}$), and define the $\wp$-divisor of f restricted to Z,

$$\mathrm{div}_\wp(f|Z) = \sum_{i=1}^{g} \mathrm{ord}_{W_{i\wp}}(f|Z) W_{i\wp}.$$

For all but finitely many $\wp$ we have $\mathrm{div}_\wp(f|Z) = 0$. Now put

$$\kappa_\wp = \kappa_\wp(Z, f, \mathcal{M}_1, ..., \mathcal{M}_t) = \frac{\log N\wp}{[K:\mathbb{Q}]} (\mathrm{div}_\wp(f|Z) \cdot \mathcal{M}_1 ... \mathcal{M}_t), \tag{3.4}$$

where the latter intersection number is for $\mathbb{F}_\wp$- cycles. Finally put

$$\kappa_K(Z, f, \mathcal{M}_1, ..., \mathcal{M}_t) := \sum_\sigma \kappa_\sigma + \sum_\wp \kappa_\wp$$

where the sums are over all embeddings $\sigma : K \hookrightarrow \mathbb{C}$ and all nonzero prime ideals $\wp$ of O_K. By linearity we define $\kappa_\sigma, \kappa_\wp, \kappa_K$ for cycles in $\mathcal{Z}_t$ with components defined over K.

The next result is due to Faltings [1]; implicitly, it implies that κ_K is independent of K.

Lemma 6. *There are unique functions* $h : \mathcal{Z}_t(\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})) \times Pic(\mathbf{n})^{t+1} \to \mathbb{R}$ *for* $t = 0, ..., M$ *(heights) with the following properties:*
(i). $h(Z, \mathcal{M}_0, ..., \mathcal{M}_t)$ *is additive in* $Z, \mathcal{M}_0, ..., \mathcal{M}_t$ *and invariant under permutations of* $\mathcal{M}_0, ..., \mathcal{M}_t$*;*
(ii). for $Z \in \mathcal{Z}_t(\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})), \mathcal{M}_0 \in Pic^+(\mathbf{n}), \mathcal{M}_1, ..., \mathcal{M}_t \in Pic(\mathbf{n})$*, choose any number field* K *over which all components of* Z *are defined, and choose any* $f \in \Gamma(\mathcal{M}_0)$ *with coefficients in* O_K *such that* f *does not identically vanish on* Z*. Then*

$$h(Z, \mathcal{M}_0, ..., \mathcal{M}_t) = h(\operatorname{div}(f|Z), \mathcal{M}_1, ..., \mathcal{M}_t) + \kappa_K(Z, f, \mathcal{M}_1, ..., \mathcal{M}_t). \tag{3.5}$$

Remark. (3.5) holds true also for $t = 0$, by agreeing that then $\operatorname{div}(f|Z) = 0$.

Proof. Put $R := O_K$, $\mathbb{P}^{\mathbf{n}}_R = \mathbb{P}^{n_1}_R \times_{\mathrm{Spec}R} ... \times_{\mathrm{Spec}R} \mathbb{P}^{n_m}_R$. A subvariety of $\mathbb{P}^{\mathbf{n}}_R$ is by definition an integral closed subscheme of $\mathbb{P}^{\mathbf{n}}_R$ and a cycle in $\mathbb{P}^{\mathbf{n}}_R$ a finite formal linear combination with integer coefficients of subvarieties of $\mathbb{P}^{\mathbf{n}}_R$. In [1], Faltings defined a logarithmic height for cycles in $\mathbb{P}^{\mathbf{n}}_R$ by means of the arithmetic intersection theory on $\mathbb{P}^{\mathbf{n}}_R$ developed by Gillet and Soulé [5], and he gave a sketchy proof of the analogue of our Lemma 6 for cycles in $\mathbb{P}^{\mathbf{n}}_R$. A more detailed proof of this analogue was given by Gubler [7], Props. 4.3, 5.3.

It is straightforward to translate Gubler's results into Lemma 6 by going through the definition of a scheme. Similar to [8], Ex. 3.12 on p. 92, 5.16 on pp. 119-120 and Ex. 5.10 on p. 125, there is a one-to-one correspondence $I \leftrightarrow V(I)$ between essential $\mathbf{n}$-prime ideals of $R[\mathbf{X}]$ and subvarieties of $\mathbb{P}^{\mathbf{n}}_R$, such that $V(I)$ is a subvariety of $V(J) \Leftrightarrow I \supseteq J$. Further, for subvarieties $V(I)$ of $\mathbb{P}^{\mathbf{n}}_R$ we have that either $I \cap R = (0)$ in which case $V(I)$ is flat (over Spec R) (cf. [8], p. 257, Prop. 9.7) or $I \cap R$ is a non-zero prime ideal $\wp$ of R, in which case $V(I)$ maps to $\wp$ (under $V(I) \to$ Spec R).

Now let Z be a subvariety of $\mathbb{P}^{\mathbf{n}}$ defined over K, and let $I = \{f \in R[\mathbf{X}] : f(P) = 0$ for $P \in Z\}$. Then $\tilde{Z} := V(I)$ is a flat subvariety of $\mathbb{P}^{\mathbf{n}}_R$. Now the height $h(Z, .)$ defined in Lemma 6 is equal to the height $h(\tilde{Z}, .)$ defined by Gubler (and $1/[K : \mathbb{Q}]$ times the height defined by Faltings). Faltings and Gubler, Prop. 4.3 have a similar recurrence relation as (3.5) for the height of flat subvarieties $\tilde{Z}$ of $\mathbb{P}^{\mathbf{n}}_R$, with instead of κ_K only the sum of infinite components κ_σ. The divisor $\operatorname{div}(f|Z)$ might have also non-flat components and the terms $\kappa_\wp$ in (3.5) are precisely the contributions of the heights of these non-flat components. By Prop. 5.3 of Gubler, the Faltings height for subvarieties of $\mathbb{P}^{\mathbf{n}}_R$ is invariant under base extensions from R to the ring of integers of any finite extension of K. This implies that in Lemma 6, the height does not depend on the choice of the field K. □

If $Z \in \mathcal{Z}_t(\mathbb{P}^{\mathbf{n}})$ and among $\mathcal{M}_0, ..., \mathcal{M}_t$, $\mathcal{N}_i$ appears e_i times for $i = 1, ..., s$ where $e_1 + ... + e_s = t + 1$, then we write $h(Z, \mathcal{N}_1^{e_1} ... \mathcal{N}_s^{e_s})$ for $h(Z, \mathcal{M}_0, ..., \mathcal{M}_t)$. Further, for $Z \in \mathcal{Z}_t(\mathbb{P}^n)$ we put $h(Z) := h(Z, \mathcal{O}(1)^{t+1})$. We write again $\mathbb{P}^{\mathbf{n}}$ for $\mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}})$.

Lemma 7. *(i). For $P \in \mathbb{P}^n$ we have $h(P) = \log H(P)$.*
(ii). $h(\mathbb{P}^n) = \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{j} 1/l$.
(iii). If $Z \in \mathcal{Z}_t(\mathbb{P}^{\mathbf{n}})$ is effective and $\mathcal{M}_0, ..., \mathcal{M}_t \in Pic^+(\mathbf{n})$, then $h(Z, \mathcal{M}_0, ..., \mathcal{M}_t) \geq 0$.
(iv). Let $Z, \mathcal{M}_0, ..., \mathcal{M}_t$ be as in (iii) and $f \in \Gamma(\mathcal{M}_0)$ such that f does not identically vanish on Z. Then

$$h(\mathrm{div}(f|Z), \mathcal{M}_1, ..., \mathcal{M}_t) \leq h(Z, \mathcal{M}_0, ..., \mathcal{M}_t) + \log H(f).(Z.\mathcal{M}_1 ... \mathcal{M}_t).$$

Proof. (i). In a sufficiently large number field K we can choose the coordinates $\mathbf{x} = (x_0, ..., x_n)$ of P such that $x_0, ..., x_n \in O_K$ and the ideal generated by these coordinates is (1). Then there are $\alpha_0, ..., \alpha_n \in O_K$ with $\alpha_0 x_0 + ... + \alpha_n x_n = 1$. Take $f(\mathbf{X}) = \alpha_0 X_0 + ... + \alpha_n X_n$. Then f does not vanish at $P, \mathrm{div}(f|\mathcal{M}) = 0, \kappa_\wp(P, f) = 0$ for each prime ideal $\wp \neq (0)$ of O_K and

$$\kappa_\sigma(P, f) = -\frac{1}{[K:\mathbb{Q}]} \log \frac{|\sigma(f)(\mathbf{x}))|}{(\sum_{i=0}^{n} |\sigma(x_i)|^2)^{1/2}} = \log\left\{ \left(\sum_{i=0}^{n} |\sigma(x_i)|^2 \right)^{1/2[K:\mathbb{Q}]} \right\}.$$

Hence

$$h(P) = h(\mathrm{div}(f|P)) = \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \kappa_\sigma(P, f) = \log H(P).$$

(ii). cf. [7], Prop. 4.4. This can be proved by induction on n. Take $f = X_0$. Then $h(\mathbb{P}^n) = h(\mathrm{div}(f|\mathbb{P}^n)) + \kappa = h(V) + \kappa$, where $V = \{X_0 = 0\}$ and

$$\kappa = -\int_{\mathbb{P}^n(\mathbb{C})} \log \frac{|z_0|}{(|z_0|^2 + ... + |z_n|^2)^{1/2}} \cdot \omega_n.$$

By the induction hypothesis, $h(V) = h(\mathbb{P}^{n-1}) = \frac{1}{2} \sum_{j=1}^{n-1} \sum_{l=1}^{j} 1/l$ and, by a straightforward but elaborate integration, $\kappa = \frac{1}{2} \sum_{l=1}^{n} 1/l$.

(iv). We assume that Z is a subvariety of $\mathbb{P}^{\mathbf{n}}$ which is no restriction. Choose a number field K such that Z and the components of $\mathrm{div}(f|Z)$ are defined over K and the coefficients of f belong to K. By enlarging K if need be, we may assume that the ideal $\mathfrak{a}$ generated by the coefficients of f is principal, $\mathfrak{a} = (\lambda)$, say. Since $\mathrm{div}(f|Z)$ and $H(f)$ do not change when f is replaced by $\lambda^{-1} f$, we may assume that $\mathfrak{a} = (1)$ and shall do so in the sequel. Suppose $\mathcal{M}_0 = \mathcal{O}(\mathbf{d})$, with $\mathbf{d} = (d_1, ..., d_m) \in (\mathbb{Z}_{\geq 0})^m$. Let $\mathcal{J}$ be the set of tuples of non-negative integers $\mathbf{i} = (i_{hj} : h = 1, ..., m, j = 0, ..., n_h)$ with $\sum_{j=0}^{n_h} i_{hj} = d_h$ for $h = 1, ..., m$. Then

$$f = \sum_{\mathbf{i} \in \mathcal{J}} a(\mathbf{i}) \prod_{h=1}^{m} \prod_{j=0}^{n_h} X_{hj}^{i_{hj}} \quad \text{with } a(\mathbf{i}) \in K.$$

Let σ be an embedding : $K \hookrightarrow \mathbb{C}$ and $A_\sigma := \left(\sum_{\mathbf{i}\in\mathcal{J}} |\sigma(a(\mathbf{i}))|^2\right)^{1/2}$. By Schwarz' inequality we have for $\mathbf{z} = (\mathbf{z}_1, ..., \mathbf{z}_m)$ with $\mathbf{z}_h = (z_{h0}, ..., z_{h,n_h}) \in \mathbb{C}^{n_h+1}$ for $h = 1, ..., m$ that

$$|\sigma(f)(\mathbf{z})| = |\sum_{i\in\mathcal{J}} \sigma(a(\mathbf{i})) \prod_{h=1}^{m} \prod_{j=0}^{n_m} z_{h_j}^{i_{h_j}}| \leq A_\sigma ||\mathbf{z}_1||^{d_1} ... ||\mathbf{z}_m||^{d_m}.$$

Hence $||\sigma(f)|| \leq A_\sigma$. Together with (3.2), (3.1) this implies that

$$\kappa_\sigma \geq -\frac{1}{[K:\mathbb{Q}]} \log A_\sigma \int_Z c_1(\mathcal{M}_1) \wedge ... \wedge c_1(\mathcal{M}_t) \geq -\frac{1}{[K:\mathbb{Q}]} \log A_\sigma (Z \cdot \mathcal{M}_1 ... \mathcal{M}_t),$$

where $\kappa_\sigma = \kappa_\sigma(Z, f, \mathcal{M}_1, ..., \mathcal{M}_t)$. Further, for every non-zero prime ideal $\wp$ of O_K we have $\kappa_\wp = \kappa_\wp(Z, f, \mathcal{M}_1, ..., \mathcal{M}_t) \geq 0$. Since $\mathfrak{a} = (1)$ we have $H(f) = (\prod_\sigma A_\sigma)^{1/[K:\mathbb{Q}]}$. It follows that

$$\begin{aligned} h(\mathrm{div}(f|Z), \mathcal{M}_1, ..., \mathcal{M}_t) &= h(Z, \mathcal{M}_0, ..., \mathcal{M}_t) - \sum_\sigma \kappa_\sigma - \sum_\wp \kappa_\wp \\ &\leq h(Z, \mathcal{M}_0, ..., \mathcal{M}_t) + (\log H(f)).(Z \cdot \mathcal{M}_1 ... \mathcal{M}_t). \end{aligned}$$

(iii). Apply (iv) with f a monomial. Then $\log H(f) = 0$; hence $h(\mathrm{div}(f|Z), \mathcal{M}_1, ..., \mathcal{M}_t) \leq h(\mathcal{M}_0, ..., \mathcal{M}_t)$. Now (iii) follows easily by induction on t. □

Lemma 8. *(i). Let $Z = Z_1 \times ... \times Z_m$ where $Z_h \in \mathcal{Z}_{\delta_h}(\mathbb{P}^{n_h})$ for $h = 1, ..., m$ and put $\delta = \delta_1 + ... + \delta_m$. Further, let $e_1, ..., e_m$ be non-negative integers with $e_1 + ... + e_m = \delta + 1$. Then*

$$h(Z, \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) = h(Z_h). \prod_{j\neq h} \deg Z_j$$

if for some $h \in \{1, ..., m\}$ we have $(e_1, ..., e_m) = (\delta_1, ..., \delta_{h-1}, \delta_h + 1, ..., \delta_m)$ and

$$h(Z, \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) = 0 \quad \textit{otherwise.}$$

(ii). $h(Z, \mathcal{L}^{\delta+1}) = d_1^{\delta_1} ... d_m^{\delta_m} \sum_{h=1}^m \frac{(\delta+1)!}{\delta_1! ... (\delta_h+1)! ... \delta_m!} \left(d_h h(Z_h) \prod_{j\neq h} \deg Z_j\right)$.

Proof. This was stated without proof by Faltings [1]. We assume that $e_1 - \delta_1 \geq ... \geq e_m - \delta_m$ and that Z_h is a δ_h-dimensional subvariety of $\mathbb{P}^{n_h}$ for $h = 1, ..., m$ which are no restrictions. For convenience of notation, put $c = 1$ if $(e_1 - \delta_1, ..., e_m - \delta_m) = (1, 0, ..., 0)$ and $c = 0$ otherwise.

We proceed by induction on δ_1. Note that $e_1 - \delta_1 \geq 1$; hence $e_1 \geq 1$. Choose a number field K and a linear form $f \in O_K[\mathbf{X}_1]$ such that f does not vanish identically on Z_1, and such that $Z_1, ..., Z_m$ and the components of $\mathrm{div}(f|Z_1), \mathrm{div}(\pi_1^* f|Z)$ are defined over K. Consider the quantities

$$U := h(Z, \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) - ch(Z_1) \prod_{h=2}^{m} \deg Z_h,$$

$$u := h\Big(\mathrm{div}(\pi_1^* f|Z), \mathcal{L}_1^{e_1 - 1} \mathcal{L}_2^{e_2} ... \mathcal{L}_m^{e_m}\Big) - ch(\mathrm{div}(f|Z_1)) \prod_{h=2}^{m} \deg Z_h.$$

If $\delta_1 = 0$ then $\mathrm{div}(\pi_1^* f|Z) = 0, \mathrm{div}(f|Z_1) = 0$ hence $u = 0$; if $\delta_1 > 0$ then also $u = 0$ by the induction hypothesis. We have to show that $U = 0$.

By Lemma 6 (ii) we have

$$U = U - u = \sum_{\sigma} \lambda_\sigma + \sum_{\wp} \lambda_\wp, \tag{3.7}$$

where

$$\lambda_v = \kappa_v(Z, \pi_1^* f, \mathcal{L}_1^{e_1 - 1} ... \mathcal{L}_m^{e_m}) - c. \deg Z_2 ... \deg Z_m \kappa_v(Z_1, f, \mathcal{O}(1)^{e_1 - 1})$$

for $v \in \{\sigma\} \cup \{\wp\}$, where σ stands for the embeddings of K into $\mathbb{C}$ and $\wp$ for the non-zero prime ideals of O_K. If $(e_1 - \delta_1, ..., e_m - \delta_m) = (1, 0, ..., 0)$, then by (3.1) we have for each embedding $\sigma : K \hookrightarrow \mathbb{C}$,

$$\begin{aligned}
&\kappa_\sigma\Big(Z, \pi_1^* f, \mathcal{L}_1^{e_1-1} ... \mathcal{L}_m^{e_m}\Big) \\
&\qquad = -\frac{1}{[K : \mathbb{Q}]} \int_{Z \times_\sigma \mathbb{C}} \log ||\sigma(\pi_1^* f)|| c_1(\mathcal{L}_1)^{e_1 - 1} \wedge ... \wedge c_1(\mathcal{L}_m)^{e_m} \\
&\qquad = -\frac{1}{[K : \mathbb{Q}]} \int_{Z_1 \times_\sigma \mathbb{C}} \log ||\sigma(f)|| \omega_{n_1}^{e_1 - 1} \cdot \prod_{h=2}^{m} \int_{Z_h \times_\sigma \mathbb{C}} \omega_{n_h}^{e_h} \\
&\qquad = \kappa_\sigma(Z_1, f_1, \mathcal{O}(1)^{e_1 - 1}) \deg Z_2 ... \deg Z_m.
\end{aligned}$$

If $(e_1 - \delta_1, ..., e_m - \delta_m) \neq (1, 0, ..., 0)$ then $\kappa_\sigma(Z, \pi_1^* f, \mathcal{L}_1^{e_1 - 1} ... \mathcal{L}_m^{e_m}) = 0$; namely in that case either $e_1 - 1 > \delta_1$ or $e_h > \delta_h$ for some $h \geq 2$ which implies that the restriction of the differential form $c_1(\mathcal{L}_1)^{e_1 - 1} \wedge ... \wedge c_1(\mathcal{L}_m)^{e_m}$ to Z_h has degree larger than $2 \dim Z_h$ which is the dimension of Z_h over $\mathbb{R}$. It follows that in both cases,

$$\lambda_\sigma = 0 \text{ for each embedding } \sigma : K \hookrightarrow \mathbb{C}. \tag{3.8}$$

Let p be any prime number and for each prime ideal $\wp$ of O_K dividing p, put $d_\wp := [\mathbb{F}_\wp : \mathbb{F}_p]$. Then

$$\sum_{\wp|p} \lambda_\wp = n_p(f). \log p,$$

where

$$n_p(f) = \frac{1}{[K:\mathbb{Q}]} \sum_{\wp|p} f_\wp \Big\{ \Big(\mathrm{div}_\wp(\pi_1^* f|Z) \cdot \mathcal{L}_1^{e_1-1} ... \mathcal{L}_m^{e_m} \Big) - c \deg Z_2 ... \deg Z_m (\mathrm{div}_\wp(f|Z_1).\mathcal{O}(1)^{e_1-1}) \Big\}.$$

By (3.7), (3.8) we have

$$U = \sum_p n_p(f) \log p; \tag{3.9}$$

hence the right-hand side of (3.9) is independent of the choice of f and K. But by the unique prime decomposition in $\mathbb{Z}$ the numbers $\log p$ (p prime) are $\mathbb{Q}$-linearly independent; therefore the rational numbers $n_p(f)$ are independent of the choice of f and K.

We show that for every prime number p we can choose f with $n_p(f) = 0$. Let $I = \{g \in O_K[\mathbf{X}] : g \text{ vanishes identically on } Z\}$ and $J_1, ..., J_g$ the minimal $\mathbf{n}$-prime ideals of $O_K[\mathbf{X}]$ containing at least one of the ideals $I + \wp O_K[\mathbf{X}]$ with $\wp|p$. Let $I' = \{g' \text{ vanishes identically on } Z_1\}$, and $J'_1, ..., J'_{g'}$ the minimal homogeneous prime ideals of $O_K[\mathbf{X}_1]$ containing at least one of the ideals $I' + \wp O_K[\mathbf{X}_1]$ with $\wp|p$. Choose a linear form $f \in O_K[\mathbf{X}_1]$ with $f \notin \pi_1^{*-1}(J_1) \cup ... \cup \pi_1^{*-1}(J_g) \cup J'_1 \cup ... \cup J'_{g'}$. Such an f exists since each of the ideals in the union is a homogeneous prime ideal not containing $(X_{10}, ..., X_{1,n_1})$. Thus, $\mathrm{div}_\wp(\pi_1^* f|Z) = 0, \mathrm{div}_\wp(f|Z_1) = 0$ for every $\wp|p$ which implies that $n_p(f) = 0$. Now (3.9) implies that $U = 0$. This completes the proof of (i).

(ii). By the additivity of the height and (i) we have

$$\begin{aligned} h(Z_1, \mathcal{L}^{\delta+1}) &= h(Z, (d_1\mathcal{L}_1 + ... + d_m\mathcal{L}_m)^{\delta+1}) \\ &= \sum_{e_1+...+e_m=\delta+1} \frac{(\delta+1)!}{e_1!...e_m!} d_1^{e_1} ... d_m^{e_m} h(Z, \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \\ &= \sum_{h=1}^{m} \frac{(\delta+1)!}{\delta_1!...(\delta_h+1)!...\delta_m!} \cdot d_1^{\delta_1} ... d_h^{\delta_h+1} ... d_m^{\delta_m} \cdot \big(h(Z_h) \prod_{j \neq h} \deg Z_j \big) \end{aligned}$$

which is (ii). □

Finally, we need an analogue of Lemma 4 for heights. For a polynomial $f(X_1, ..., X_r) = \sum_{\mathbf{i}} a(\mathbf{i}) X_1^{i_1} ... X_r^{i_r}$ with coefficients in a number field K and for each embedding $\sigma : K \hookrightarrow \mathbb{C}$, put

$$H_\sigma(f) = \left(\sum_{\mathbf{i}} |\sigma(a(\mathbf{i}))|^2 \right)^{1/2}.$$

Lemma 9. *Let $\mathbf{d} = (d_1, ..., d_m) \in (\mathbb{Z}_{\geq 0})^m$ and A a subset of $\Gamma^{\mathbf{n}}_{\overline{\mathbb{Q}}}(\mathbf{d})\backslash\{\mathbf{0}\}$ such that every polynomial $f \in A$ has algebraic integer coefficients in some number field K and such that $H_\sigma(f) \leq H_\sigma$ for each embedding $\sigma : K \hookrightarrow \mathbb{C}$. Put $H := (\prod_\sigma H_\sigma)^{1/[K:\mathbb{Q}]}$. Further, let $Z_1, ..., Z_r$ be irreducible components of $X := \{P \in \mathbb{P}^{\mathbf{n}}(\overline{\mathbb{Q}}) : f(P) = 0$ for $f \in A\}$ of codimension t. Then*

$$\sum_{i=1}^{r} m_{Z_i} h(Z_i, \mathcal{L}^{M-t+1}) \leq \frac{M!}{n_1!...n_m!} d_1^{n_1}...d_m^{n_m} \left\{ M^2(d_1 + ... + d_m) + t \log H \right\}.$$

Proof. Let $f_1, ..., f_t$ be polynomials satisfying (2.6) and Lemma 5, and $C_0, ..., C_t$ the cycles defined by (2.6) ; so $C_0 = \mathbb{P}^{\mathbf{n}}$. From the definition of the height of a polynomial and the fact that the quantities $H_\sigma(f)$ satisfy the triangle inequality it follows that

$$H(f_i) \leq C^2 H \quad \text{for} \quad i = 1, ..., t, \quad \text{where} \quad C = \frac{M!}{n_1!...n_m!} d_1^{n_1}...d_m^{n_m}. \tag{3.10}$$

By Lemma 7 (iv) we have

$$h(\mathbb{P}^n) = \frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{j} \frac{1}{l} \leq \frac{1}{2}(n + \log n!).$$

Together with Lemma 8 (ii) this implies

$$\begin{aligned} h(\mathbb{P}^{\mathbf{n}}, \mathcal{L}^{M+1}) &= d_1^{n_1}...d_m^{n_m} \sum_{h=1}^{m} \frac{(M+1)!}{n_1!...(n_h+1)!...n_m!} \cdot d_h h(\mathbb{P}^{n_h}) \\ &\leq \frac{1}{2} C \cdot (M+1) \sum_{h=1}^{m} \frac{n_h + \log n_h!}{n_h + 1} d_h \\ &\leq \frac{1}{4} C \cdot (M+1) \sum_{h=1}^{m} n_h d_h \leq \frac{1}{4} C M^2 (d_1 + ... + d_m) . \end{aligned} \tag{3.11}$$

By (2.8) we have $\mathrm{div}(f_j|C_{j-1}) = C_j +$ (effective cycle) for $j = 1, ..., t$. By Lemma 1 (iv) we have $(C_j \cdot \mathcal{L}^{M-j}) \leq (C_{j-1} \cdot \mathcal{L}^{M-j+1}) \leq ... \leq (C_0 \cdot \mathcal{L}^M) = C$ for $j = 0, ..., t$. Further, by Lemma 7 (iii), (iv) and (3.10),

$$\begin{aligned} h(C_j, \mathcal{L}^{M-j+1}) &\leq h(C_{j-1}, \mathcal{L}^{M-j+2}) + (\log C^2 H) \cdot (C_{j-1} \mathcal{L}^{M-j+1}) \\ &\leq h(C_{j-1}, \mathcal{L}^{M-j+2}) + C \log(C^2 H) \end{aligned} \tag{3.12}$$

for $j = 1, ..., t$. Now from $C_t = \sum_{i=1}^r m_{Z_i} Z_i +$ (effective cycle), (3.12), (3.11), $C \leq \sum_{j_1+...+j_m=M} (M!/j_1!...j_m!) d_1^{j_1}...d_m^{j_m} = (d_1 + ... + d_m)^M$, $\log(d_1 + ... + d_m) \leq (\log 3/3)(d_1 + ... + d_m)$ and $t \leq M$ it follows that

$$\sum_{i=1}^r m_{Z_i} h(Z_i, \mathcal{L}^{M-t+1}) \leq h(C_t, \mathcal{L}^{M-t+1}) \leq h(\mathbb{P}^{\mathbf{n}}, \mathcal{L}^{M+1}) + Ct \log(C^2 H)$$

$$\leq C \left\{ \frac{1}{4} M^2 (d_1 + ... + d_m) + 2t \log C + t \log H \right\}$$

$$\leq C \left\{ \frac{1}{4} M^2 (d_1 + ... + d_m) + 2tM \log(d_1 + ... + d_m) + t \log H \right\}$$

$$\leq C \left\{ (\frac{1}{4} + 2\frac{\log 3}{3}) M^2 (d_1 + ... + d_m) + t \log H \right\}$$

$$\leq C \left\{ M^2 (d_1 + ... + d_m) + t \log H \right\},$$

which is Lemma 9. □

§4. Proof of Theorems 1 and 2.

We use the notation of Theorem 1: k is an algebraically closed field of characteristic $0, m$ an integer $\geq 2, \mathbf{n} = (n_1, ..., n_m), \mathbf{d} = (d_1, ..., d_m)$ are tuples of positive integers and σ, ϵ reals with $\sigma \geq 0, 0 < \epsilon \leq 1$ and

$$\text{(1.1)} \qquad \frac{d_h}{d_{h+1}} \geq \left(\frac{mM}{\epsilon} \right)^M \quad \text{for } h = 1, ..., m-1,$$

where $M := n_1 + ... + n_m$. We write $\mathbb{P}^{\mathbf{n}}$ for $\mathbb{P}^{\mathbf{n}}(k)$. Further, F is a polynomial from $\Gamma_k^{\mathbf{n}}(\mathbf{d}) \backslash \{0\}$, and Z is an irreducible component of both $Z_\sigma(F, \mathbf{d})$ and $Z_{\sigma+\epsilon}(F, \mathbf{d})$. Let A be the set of polynomials

$$\text{(4.1)} \qquad \prod_{h=1}^m \prod_{j=0}^{n_h} X_{hj}^{c_{hj}} \cdot \left(\prod_{h=1}^m \prod_{j=0}^{n_h} \frac{1}{i_{hj}!} \frac{\partial^{i_{hj}}}{\partial X_{hj}^{i_{hj}}} F \right)$$

for all tuples of nonnegative integers $\mathbf{i} = (i_{hj} : h = 1, ..., m, j = 0, ..., n_h), \mathbf{c} = (c_{hj} : h = 1, ..., m, j = 0, ..., n_h)$ with

$$(\mathbf{i}/\mathbf{d}) \leq \sigma, \qquad \sum_{j=0}^{n_h} (c_{hj} + i_{hj}) = d_h \quad \text{for } h = 1, ..., m$$

and let I be the ideal in $k[\mathbf{X}]$ generated by A. Note that $A \subset \Gamma_k^{\mathbf{n}}(\mathbf{d})$, and that $X := Z_\sigma(F, \mathbf{d}) = \{P \in \mathbb{P}^{\mathbf{n}} : f(P) = 0 \text{ for } P \in A\}$.

Let k_0 be a subfield of k containing the coefficients of F, and k_1 the smallest extension of k_0 over which Z is defined. Thus, $Z = V(J)$ with $J = (f_1, ..., f_u)$, where $f_1, ..., f_u \in k_1[\mathbf{X}]$. Letting $\sigma_1 =$ identity, $\sigma_2, ..., \sigma_g$ $(g = [k_1 : k_0])$ be the injective k_0-homomorphisms from k_1 into k, put $J^{(j)} := (\sigma_j(f_1), ..., \sigma_j(f_u)), Z^{(j)} = V(J^{(j)})$ for $j = 1, ..., g$. Since $A \subset k_0[\mathbf{X}], Z^{(1)}, ..., Z^{(g)}$ are irreducible components of X. Each σ_j induces a isomorphism $\overline{\sigma}_j$ from $\mathcal{O}_Z$ to $\mathcal{O}_{Z^{(j)}}$ mapping the maximal ideal M_Z to $M_{Z^{(j)}}$. It follows that the fields $k(Z^{(j)})$ and $k(Z)$ are isomorphic, whence that $\dim Z^{(j)} = \dim Z$ for $j = 1, ..., g$. Further, since I is generated by polynomials from $k_0[\mathbf{X}], \overline{\sigma}_j$ induces an isomorphism from $\mathcal{O}_Z/I\mathcal{O}_Z$ to $\mathcal{O}_{Z^{(j)}}/I\mathcal{O}_{Z^{(j)}}$. Therefore,

$$m_{Z^{(j)}} = m_Z \tag{4.2}$$

(cf. Lemma 4). Let $s := \text{codim}Z$ and let $e_1, ..., e_m$ be non-negative integers with $e_1 + ... + e_m = M - s$. Let $\mathcal{L}_1, ..., \mathcal{L}_m$ have the same meaning as in §2, and put $\mathcal{L} := d_1\mathcal{L}_1 + ... + d_m\mathcal{L}_m$. By applying Lemma 1 (iv) with polynomials from $k_0[\mathbf{X}]$, we infer that

$$(Z^{(j)} \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) = (Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \quad \text{for} \quad j = 1, ..., g.$$

Together with (4.2) and Lemma 4, this implies that

$$[k_1 : k_0] m_Z (Z.\mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \leq (\mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}.\mathcal{L}^s). \tag{4.3}$$

We shall estimate m_Z from below, using differential operators similar to Wüstholz [20]. Here it will be crucial that Z is also an irreducible component of $Z_{\sigma+\epsilon}(F, \mathbf{d})$. If Z is not a product variety then by Lemma 3 there are at least two tuples $(e_1, ..., e_m)$ for which $(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) > 0$. Using (1.1) and the lower bound for m_Z, we show that for some tuple $(e_1, ..., e_m)$, the left-hand side of (4.3) is larger than the right-hand side, thus arriving at a contradiction.

Lemma 10. *For $i = 1, ..., m$, let $p_i : \mathbb{P}^{\mathbf{n}} \to \mathbb{P}^{n_i} \times ... \times \mathbb{P}^{n_m}$ be the projection onto the last $m - i + 1$ factors of $\mathbb{P}^{\mathbf{n}}$ and put $\delta_i := \dim p_i(Z) - \dim p_{i+1}(Z)$ for $i = 1, ..., m$, where $\dim p_{m+1}(Z) := 0$. Let $s := \text{codim}Z$. Then*

$$m_Z = l_{\mathcal{O}_Z}(\mathcal{O}_Z/I\mathcal{O}_Z) \geq (\epsilon/s)^s d_1^{n_1-\delta_1} ... d_m^{n_m-\delta_m}.$$

Proof. We follow the arguments of van der Put [11] and Wüstholz [20]. For convenience of the reader, we have worked out more details.

Choose $P \in Z$ such that Z is smooth in P (i.e. the tangent space of Z at P has dimension equal to that of Z) and for $i = 1, ..., m$, $p_i(Z)$ is smooth in $p_i(P)$ and the map p_i is smooth at P (i.e. the linear map of tangent spaces dp_i corresponding to p_i is surjective). Such a point P exists since by [8], Lemma 10.5, p. 271, the set of such points is a non-empty Zariski open subset of Z. After applying a linear transformation if need be, we may assume that $P = (P_1, ..., P_m)$ with $P_h = (1 : 0 : ... : 0) \in \mathbb{P}^{n_h}$ for $h = 1, ..., m$. Now define the affine variety

$$A := \mathbb{A}^M = \mathbb{A}^{n_1} \times ... \times \mathbb{A}^{n_m} = \mathbb{P}^{\mathbf{n}} \cap \left\{ X_{10} \neq 0, ..., X_{m0} \neq 0 \right\}.$$

On A we choose the affine coordinates $Y_{hj} = X_{hj}/X_{h0}$ ($h = 1, ..., m, j = 1, ..., n_h$). Let $k[\mathbf{Y}]$ be the polynomial ring in these coordinates. Put $F^{'}(\mathbf{Y}) :=$ $F(1, Y_{11}, ..., Y_{1,n}, ..., 1, Y_{m1}, ..., Y_{m,n_m})$ and let I'_μ be the ideal generated by the polynomials

$$\prod_{h=1}^{m} \prod_{j=1}^{n_m} \frac{\partial^{i_{hj}}}{\partial Y_{hj}^{i_{hj}}} F' \quad \text{with} \quad \sum_{h=1}^{m} \frac{1}{d_h} \left(\sum_{j=1}^{n_h} i_{hj} \right) \leq \mu;$$

by the Remark at the end of §1, this is the defining ideal of $Z_\mu(f, \mathbf{d}) \cap A$. Further, let $Z' := Z \cap A$ and $J' = \{f \in k[\mathbf{Y}] : f(P) = 0 \text{ for } P \in Z'\}$. Then J' is a minimal prime ideal containing I'_σ and also a minimal prime ideal containing $I'_{\sigma+\epsilon}$. The local ring of Z',

$$\hat{R} := \left\{ \frac{f}{g} : f, g \in k[\mathbf{Y}], g \notin J' \right\}$$

is isomorphic to $\mathcal{O}_Z$ and has maximal ideal $\hat{M} := J'\hat{R}$. Put $\hat{I} := I'\hat{R}$. Then $\hat{R}/\hat{I} \cong \mathcal{O}_Z/I\mathcal{O}_Z$. Therefore, $m_Z = l_{\hat{R}}(\hat{R}/\hat{I})$, so it suffices to show that

$$l_{\hat{R}}(\hat{R}/\hat{I}) \geq (\epsilon/s)^s d_1^{n_1 - d_1} ... d_m^{n_m - \delta_m}. \tag{4.4}$$

Since $\hat{M} = (f_1, ..., f_u)\hat{R}$, the tangent space of $Z^{'}$ at $\mathbf{0}$ is given by

$$T_{\mathbf{0}}(Z^{'}) = \Big\{ \mathbf{w} = (w_{hj} : h = 1, ..., m, j = 1, ..., n_h) \in k^M : \\ \sum_{h=1}^{m} \sum_{j=1}^{n_h} \frac{\partial f_l}{\partial Y_{hj}}(\mathbf{0}) w_{hj} = 0 \text{ for } l = 1, ..., u \Big\}.$$

The linear mapping dp_i induced by p_i from $T_{\mathbf{0}}$ to the tangent space $T_{p_i(\mathbf{0})}(p_i(Z^{'}))$ of $p_i(Z')$ at $p_i(\mathbf{0})$, can be given by $dp_i(\mathbf{w}) = (w_{hj} : h = i, ..., m, j = 1, ..., n_h)$. Our smoothness assumptions at the beginning of the proof imply that $\dim T_{\mathbf{0}}(Z') = \dim Z', \dim T_{p_i(\mathbf{0})}(p_i(Z^{'})) = \dim p_i(Z') = \delta_i + ... + \delta_m$, and that dp_i is surjective. Therefore,

$$\begin{cases} \dim \ker\, (dp_i) & = \dim T_{\mathbf{0}}(Z') - \dim T_{p_i(\mathbf{0})}(p_i(Z')) \\ & = \delta_1 + ... + \delta_{i-1} \quad \text{for } i = 2, ..., m, \\ \ker\, dp_1 & = (\mathbf{0}). \end{cases} \tag{4.5}$$

Note that

$$(4.6)\qquad \ker\,(dp_i) = \{\mathbf{w}\in k^M : \sum_{h=1}^{i-1}\sum_{j=1}^{n_h}\frac{\partial f_l}{\partial Y_{hj}}(\mathbf{0})w_{hj} = 0 \quad \text{for} \quad l = 1,...,u,$$
$$w_{hj} = 0 \quad \text{for} \quad h = i,...,m, j = 1,...,n_h\}.$$

By (4.5), (4.6) , the $(n_1 + ... + n_{i-1}) \times u$-matrix

$$A_i = \left(\frac{\partial f_l}{\partial Y_{hj}}\quad(\mathbf{0})\right)_{\substack{h=1,...,i-1,j=1,...,n_h\\ l=1,...,u}}$$

with the rows being indexed by (h, j) and the columns by l, has rank $(n_1 - \delta_1) + ... + (n_{i-1} - \delta_{i-1})$. Hence among the rows $(\partial f_l/\partial Y_{ij})(\mathbf{0})$ $(j = 1,...,n_i)$ of A_{i+1} there are precisely $n_i - \delta_i$ rows which are linearly independent of each other and also linearly independent of the rows of A_i; we assume w.l.o.g. that these rows are $(\partial f_l/\partial Y_{ij})(\mathbf{0})$ with $j = 1,...,n_i - \delta_i$ and $l = 1,...,u$. This gives altogether $(n_1 - \delta_1) + ... + (n_m - \delta_m) = s$ linearly independent rows $(\partial f_l/\partial Y_{hj})(\mathbf{0})$ $(h = 1,...,m, j = 1,...,n_h - \delta_h)$.

For convenience, write $Y_1,...,Y_s$ for the variables Y_{hj} $(h = 1,...,m,\ j = 1,...,n_h - \delta_h)$ and put $c_i = d_h$ whenever $Y_i = Y_{hj}$. Obviously (4.4) follows once we have shown that

$$(4.7)\qquad l_{\hat{R}}(\hat{R}/\hat{I}) \geq (\epsilon/s)^s c_1...c_s.$$

By what we have seen above, the matrix $((\partial f_l/\partial Y_j)(\mathbf{0}))_{j=1,...,s,l=1,...,u}$ has rank s. We assume w.l.o.g. that $\det((\partial f_l/\partial Y_j)(\mathbf{0}))_{i\leq j,l\leq s}$ is non-zero. Then

$$D(\mathbf{Y}) := \det\left(\frac{\partial f_l}{\partial Y_j}\right)_{i\leq j,l\leq s} \notin J^{'}.$$

Hence the elements of the inverse matrix $(g_{kl}) = (\partial f_l/\partial Y_j)^{-1}$ belong to $\hat{R}$. Define the rational functions

$$T_j := \sum_{l=1}^{s} g_{lj} f_l \quad (j = 1,...,s).$$

Further, define differential operators $\partial/\partial T_i$ by

$$\left(\frac{\partial}{\partial T_1},...,\frac{\partial}{\partial T_s}\right) = \left(\frac{\partial}{\partial Y_1},...,\frac{\partial}{\partial Y_s}\right)\left(\frac{\partial T_i}{\partial Y_j}\right)^{-1}_{i,j}.$$

$T_1,...,T_s$ belong to $\hat{M}$ since $g_{lj} \in \hat{R}$ and $f_l \in \hat{M}$. If $h \in \hat{R}$ then $\partial h/\partial Y_j \in \hat{R}$ for $j = 1,...,s$ namely if $h = f/g$ with $f, g \in k[\mathbf{Y}], g \notin J^{'}$, then $\partial h/\partial Y_j = g^{-2}\left\{g(\partial f/\partial Y_j) - f(\partial g/\partial Y_j)\right\} \in \hat{R}$. Further,

$$\frac{\partial T_i}{\partial Y_j} = \sum_{l=1}^{s} g_{li}\frac{\partial f_l}{\partial Y_j} + \sum_{l=1}^{s}\left(\frac{\partial g_{li}}{\partial Y_j}\right) f_l \equiv \delta_{ij} \bmod \hat{M}.$$

Hence $\det(\partial T_i/\partial Y_j)$ is a unit in $\hat{R}$ which implies that the elements of $(\partial T_i/\partial Y_j)^{-1}$ belong to $\hat{R}$. It follows that $\partial h/\partial T_j \in \hat{R}$ for $h \in \hat{R}$, $j = 1,...,s$. The operators $\partial/\partial T_i$ satisfy the usual rules for differentiaton, e.g., $\partial T_i/\partial T_j = \delta_{ij}$ and $\partial T_i^l/\partial T_i = lT_i^{l-1}$ for $l = 1, 2, ...$. We have the following crucial fact:

(4.8) $$\frac{\partial^{j_1+...+j_s} f}{\partial T_1^{j_1}...\partial T_s^{j_s}} \in \hat{M}$$

for every $f \in \hat{I}$ and all tuples of non-negative integers $(j_1, ..., j_s)$ with $j_1/c_1 + ... + j_s/c_s \leq \epsilon$.

Namely, let $f \in \hat{I}$ and $j_1/c_1 + ... + j_s/c_s \leq \epsilon$. f can be expressed as $g_1 f_1 + ... g_r f_r$ with $g_1, ..., g_r \in \hat{R}$, $f_1, ..., f_r \in I_\sigma$. Hence $\partial^{j_1+...+j_s} f/\partial T_1^{j_1}...\partial T_s^{j_s}$ can be expressed as $\sum h_{ik} D_i f_k$ with $h_{ik} \in \hat{R}$ and $D_i = \partial^{i_1+...+i_s}/\partial T_1^{i_1}...\partial T_s^{i_s}$ for certain $i_1 \leq j_1, ..., i_s \leq j_s$. Furthermore, $D_i f_k$ can be expressed as $\sum p_{lki} D_l^{'} f_k$ with $p_{lki} \in \hat{R}, D_l^{'} = \partial^{l_1+...+l_s}/\partial Y_1^{l_1}...\partial Y_s^{l_s}$ with $l_1 \leq i_1 \leq j_1, ..., l_s \leq i_s \leq j_s$. Since $I_{\sigma+\epsilon}^{'} \subseteq J^{'}$ we have $D_l^{'} f_k \in J^{'}$; this implies (4.8).

We are now ready to prove Lemma 10. Define an ordering on $(\mathbb{Z}_{\geq 0})^s$ by defining $\mathbf{i} < \mathbf{j}$ if the first non-zero coordinate of $\mathbf{j} - \mathbf{i}$ is > 0. For $\mathbf{i} = (i_1, ..., i_s)$, put $D^{\mathbf{i}} = \partial^{i_1+...+i_s}/\partial T_1^{i_1}...\partial T_s^{i_s}, \mathbf{T}^{\mathbf{i}} = T_1^{i_1}...T_s^{i_s}$. Let $\mathbf{i}_1, ..., \mathbf{i}_l$ be the tuples with $i_1/c_1 + ... + i_s/c_s \leq \epsilon$, ordered such that $\mathbf{i}_1 > \mathbf{i}_2 > ... > \mathbf{i}_l$. Define the ideals in $\hat{R}$:

$$J_0 = \hat{I} + (\mathbf{T}^{\mathbf{j}} : \text{ all } \mathbf{j} = (j_1, ..., j_s) \text{ with } j_1/c_1 + ... + j_s/c_s > \epsilon),$$
$$J_t = J_0 + (\mathbf{T}^{\mathbf{i}_1}, ..., \mathbf{T}^{\mathbf{i}_t}) \quad \text{for} \quad t = 1, ..., l.$$

We have

(4.9) $$J_0 \subsetneqq J_1 \subsetneqq ... \subsetneqq J_l.$$

Namely, suppose that for some t we have $J_{t+1} = J_t$. Then $\mathbf{T}^{\mathbf{i}_{t+1}} \in J_t$, i.e. $\mathbf{T}^{\mathbf{i}_{t+1}} = \sum_{\mathbf{i}} g_{\mathbf{i}} T^{\mathbf{i}} + f$, where the sum is taken over tuples $\mathbf{i} > \mathbf{i}_{t+1}$ and where $g_{\mathbf{i}} \in \hat{R}$ and $f \in \hat{I}$. Since $T_1, ..., T_s \in \hat{M}$ we have $D^{\mathbf{k}}\mathbf{T}^{\mathbf{i}} \in \hat{M}$ if $\mathbf{k} \neq \mathbf{i}$. Note that $D^{\mathbf{i}_{t+1}}\mathbf{T}^{\mathbf{i}_{t+1}}$ is a non-zero constant, whence does not belong to $\hat{M}$. On the other hand, $D^{\mathbf{i}_{t+1}}(\mathbf{g}_i\mathbf{T}^{\mathbf{i}})$ can be expressed as $\sum h_{\mathbf{i},\mathbf{k}} D^{\mathbf{k}}\mathbf{T}^{\mathbf{i}}$ with $h_{\mathbf{i},\mathbf{k}} \in \hat{R}$ and $\mathbf{k} \leq \mathbf{i}_{t+1} < \mathbf{i}$, hence $D^{\mathbf{i}_{t+1}}(g_{\mathbf{i}}\mathbf{T}^{\mathbf{i}}) \in \hat{M}$. Further, by (4.8), $D^{\mathbf{i}_{t+1}} f \in \hat{M}$. Thus we arrive at a contradiction and we must conclude that $J_{t+1} \supsetneqq J_t$. This proves (4.9). Consequently,

(4.10) $$\hat{R}/\hat{I} \supseteq \hat{R}/J_0 \supsetneqq \hat{R}/J_1 \supsetneqq ... \supsetneqq \hat{R}/J_l \supseteq (0).$$

Hence $l_{\hat{R}}(\hat{R}/\hat{I}) \geq l$. The tuples $(i_1, ..., i_s)$ with $0 \leq i_j \leq [\epsilon c_j/s]$ $(j = 1, ..., s)$ satisfy $i_1/c_1 + ... + i_s/c_s \leq \epsilon$. Hence

$$l \geq \prod_{j=1}^{s} \left(\left[\frac{\epsilon c_j}{s} \right] + 1 \right) \geq (\epsilon/s)^s c_1...c_s \ .$$

This proves (4.7) and hence Lemma 10. ☐

Proof of Theorem 1. Let $s = \text{codim}\, Z = M-(\delta_1+...+\delta_m)$, where $\delta_1, ..., \delta_m$ are the integers from Lemma 10. Let $(e_1, ..., e_m)$ be a tuple of non-negative integers with $e_1 + ... + e_m = M - s = \delta_1 + ... + \delta_m$ and $(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) > 0$. We have

$$\eta_i := (\delta_i + ... + \delta_m) - (e_i + ... + e_m) \geq 0 \quad \text{for } i = 2, ..., m.$$

Namely, take generic $f_{hj} \in \Gamma(\mathcal{L}_h)$ $(h = 1, ..., m,\ j = 1, ..., e_h)$ and put $W := Z \cap \{f_{hj} = 0 \text{ for } h = 1, ..., m,\ j = 1, ..., e_h\}$. Then W is not empty, hence $p_i(W)$ is not empty. Further, $p_i(W) \subseteq p_i(Z) \cap \{f_{hj} = 0 \text{ for } h = i, ..., m,\ j = 1, ..., e_h\}$. Hence $\dim p_i(Z) = \delta_i + ... + \delta_m \geq e_i + ... + e_m$.

From (4.3), Lemma 4 and Lemma 10 it follows that

$$
\begin{aligned}
\text{(4.11)}\quad [k_1 : k_0](Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) &\leq m_Z^{-1}(\mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m} . \mathcal{L}^s) \\
&= m_Z^{-1}(\mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m} \cdot (d_1\mathcal{L}_1 + ... + d_m\mathcal{L}_m)^s) \\
&= m_Z^{-1} \frac{s!}{(n_1 - e_1)! ... (n_m - e_m)!)} d_1^{n_1 - e_1} ... d_m^{n_m - e_m} \\
&\leq m_Z^{-1} m^s d_1^{n_1 - e_1} ... d_m^{n_m - e_m} \\
&\leq \left(\frac{ms}{\epsilon}\right)^s d_1^{\delta_1 - n_1} ... d_m^{\delta_m - n_m} . d_1^{n_1 - e_1} ... d_m^{n_m - e_m} \\
&= \left(\frac{ms}{\epsilon}\right)^s \cdot \left(\frac{d_2}{d_1}\right)^{\eta_2} ... \left(\frac{d_m}{d_{m-1}}\right)^{\eta_m}.
\end{aligned}
$$

Suppose that Z is not a product variety $Z_1 \times ... \times Z_m$ with Z_h a subvariety of $\mathbb{P}^{n_h}$ for $h = 1, ..., m$. Then by Lemma 3 there are at least two tuples $(e_1, ..., e_m)$ with $(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) > 0$ so there is such a tuple with $(e_1, ..., e_m) \neq (\delta_1, ..., \delta_m)$. But then, at least one of the numbers η_i is ≥ 1. Together with (4.11) and condition (1.1) on $d_1/d_2, ..., d_{m-1}/d_m$ this implies that

$$[k_1 : k_0](Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \leq \left(\frac{ms}{\epsilon}\right)^s \cdot \left(\frac{mM}{\epsilon}\right)^{-M(\eta_2 + ... + \eta_m)} < 1$$

which is impossible as $(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m})$ is a positive integer. It follows that Z is a product variety $Z_1 \times ... \times Z_m$ with Z_h a subvariety of $\mathbb{P}^{n_h}$ and that $e_h = \delta_h = \dim Z_h$ for $h = 1, ..., m$. Hence $\eta_1 = ... = \eta_m = 0$. By inserting this into (4.11) and using Lemma 2 (ii) we get

$$[k_1 : k_0] \deg Z_1 ... \deg Z_m = [k_1 : k_0](Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \leq \left(\frac{ms}{\epsilon}\right)^s.$$

This completes the proof of Theorem 1. ☐

Proof of Theorem 2. We use the notation introduced at the beginning of this section, except that $k=\overline{\mathbb{Q}}$. We assume that F has its coefficients in some number field K, and that the ideal generated by the coefficients of F is (1). Similarly as in the proof of Lemma 7 (iv) this is no restriction. The coefficients of each polynomial

$$F_{\mathbf{i}} := \prod_{h=1}^{m}\prod_{j=0}^{n_h} X_{hj}^{c_{hj}}\left(\prod_{h=1}^{m}\prod_{j=0}^{n_h}\frac{1}{i_{hj}!}\frac{\partial^{i_{hj}}}{\partial X_{hj}^{i_{hj}}}F\right)$$

are obtained by multiplying the coefficient of F attached to the monomial $\prod_{h=1}^{m}\prod_{j=0}^{n_h} X_{hj}^{l_{hj}}$ with $\prod_{h=1}^{m}\prod_{j=0}^{n_h}\binom{l_{hj}}{i_{hj}}$, which is an integer

$$\leq 2^{\sum_{h=1}^{m}\sum_{j=1}^{n_h} l_{hj}} \leq 2^{d_1+\dots+d_m}.$$

It follows that for each embedding $\sigma : K \hookrightarrow \mathbb{C}$,

$$H_\sigma(F_{\mathbf{i}}) \leq 2^{d_1+\dots+d_m} H_\sigma(F) =: H_\sigma. \tag{4.12}$$

Recall that the coefficients of F generate the ideal (1), so that

$$\left(\prod_{\sigma} H_\sigma\right)^{1/[K:\mathbb{Q}]} \leq 2^{d_1+\dots+d_m} H(F). \tag{4.13}$$

By applying Lemma 6 (iv) with f having coefficients in k_0 and using induction on the dimension, we see that

$$h(Z^{(i)},\mathcal{L}^{M-s+1}) = h(Z,\mathcal{L}^{M-s+1}) \quad \text{for} \quad i=1,\dots,[k_1:k_0].$$

Together with Lemma 9, (4.12), (4.13) this implies

$$\begin{aligned}
&(4.14)\ [k_1:k_0]m_Z h(Z,\mathcal{L}^{M-s+1})\\
&\qquad \leq \frac{M!}{n_1!\dots n_m!}d_1^{n_1}\dots d_m^{n_m}\cdot\Big\{(M^2+\log 2)(d_1+\dots+d_m)+s\log H(F)\Big\}\\
&\qquad \leq 2m^M M^2 d_1^{n_1}\dots d_m^{n_m}(d_1+\dots+d_m+\log H(F)).
\end{aligned}$$

We have shown that $Z = Z_1\times\dots\times Z_m$, where Z_h is a δ_h-dimensional subvariety of $\mathbb{P}^{n_h}$ for $h=1,\dots,m$. By Lemmas 10, 9 and 8 (ii) we have

$$\begin{aligned}
&[k_1:k_0]m_Z h(Z,\mathcal{L}^{M-s+1})\\
&\quad \geq [k_1:k_0](\epsilon/s)^s d_1^{n_1-\delta_1}\dots d_m^{n_m-\delta_n}\cdot d_1^{\delta_1}\dots d_m^{\delta_m}\deg Z_1\dots\deg Z_m\left(\sum_{h=1}^{m}\frac{d_h h(Z_h)}{\deg Z_h}\right)\\
&\quad = [k_1:k_0]\deg Z_1\dots\deg Z_m.(\epsilon/s)^s d_1^{n_1}\dots d_m^{n_m}\sum_{h=1}^{m}\frac{d_h h(Z_h)}{\deg Z_h}.
\end{aligned}$$

By comparing this with (4.14) we see that the term $d_1^{n_1}\dots d_m^{n_m}$ cancels and that

$$[k_1:k_0]\deg Z_1\dots\deg Z_m\left(\sum_{h=1}^{m}\frac{d_h h(Z_h)}{\deg Z_h}\right) \leq 2\left(\frac{s}{\epsilon}\right)^s m^M M^2(d_1+\dots+d_m+\log H(F)),$$

which is Theorem 2. □

§5. Proof of Theorem 3 (Roth's lemma).

Let m be an integer ≥ 2. Put $\mathbb{P} := \mathbb{P}^1(\overline{\mathbb{Q}}) \times ... \times \mathbb{P}^1(\overline{\mathbb{Q}})$ (m times) and denote by $\mathbb{P}^1_i$ the i-th factor of $\mathbb{P}$. Define the blocks of two variables $\mathbf{X}_h := (X_{h0}, X_{h1})$, for $\mathbf{d} = (d_1, ..., d_m) \in (\mathbb{Z}_{\geq 0})^m$ let $\Gamma(\mathbf{d})$ be the set of polynomials from $\overline{\mathbb{Q}}[\mathbf{X}_1, ..., \mathbf{X}_m]$ which are homogeneous of degree d_h in $\mathbf{X}_h$ for $h = 1, ..., m$ and let $\Gamma := \cup_{\mathbf{d}} \Gamma(\mathbf{d})$. As usual, we put $\mathcal{L}_h = \mathcal{O}(0, ..., 1, ..., 0)$ (1 on the h-th place).

Now let $0 < \epsilon \leq m+1$ and let $\mathbf{d} = (d_1, ..., d_m)$ be a tuple of positive integers satisfying (1.11). Further, let F be a non-zero polynomial from $\Gamma(\mathbf{d})$ and let $P = (P_1, ..., P_m)$ where $P_h \in \mathbb{P}^1$ for $h = 1, ..., m$. Assume that $i_{\mathbf{d}}(F, P) \geq \epsilon$. We shall show that for at least one h we have that P_h does not satisfy (1.12), i.e.

$$H(P_h)^{d_h} \leq \left(e^{d_1 + ... + d_m} H(F)\right)^{(3m^3/\epsilon)^m} . \tag{5.1}$$

This clearly implies Theorem 3.

Put $\epsilon' := \epsilon/(m+1)$. As in the proof of the Corollary, there is an $i \in \{0, ..., m\}$ such that $Z_{i\epsilon'}$ and $Z_{(i+1)\epsilon'}$ have a common irreducible component, Z, say, containing P. Put $s :=$ codim Z. As in Lemma 10, let p_i be the projection of $\mathbb{P}$ onto the product of its last $m - i + 1$ factors $\mathbb{P}^1_i \times ... \times \mathbb{P}^1_m$ and put $\delta_i := \dim p_i(Z) - \dim p_{i+1}(Z)$ for $i = 1, ..., m$, where $\dim p_{m+1}(Z) := 0$; note that $\delta_i \in \{0, 1\}$. Further, let π_h be the projection of $\mathbb{P}$ onto its h-th factor $\mathbb{P}^1_h$. Then either $\pi_h(Z) = \mathbb{P}^1_h$ or $\pi_h(Z)$ is a point in which case $\pi_h(Z) = P_h$. We shall show that for some h we have $\pi_h(Z) = P_h$ and that this P_h satisfies (5.1). To this end we need the following improvement of Lemma 3 for the case $\mathbf{n} = (1, ..., 1)$.

Lemma 11. *There are $e_1, ..., e_m \in \{0, 1\}$ with $e_1 + ... + e_m = \dim Z = m - s$, $(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) > 0$, $\eta_i := \sum_{j=i}^m (\delta_j - e_j) \geq 0$ for $i = 2, ..., m$ and*

$$\sum_{i=2}^m \eta_i \geq \left(\sum_{i=1}^m \dim \pi_i(Z)\right) - \dim Z .$$

Proof. For any subset $\mathbf{i} = \{i_1, ..., i_t\}$ of $\{1, ..., m\}$ denote by $\pi_{\mathbf{i}}$ the projection of $\mathbb{P}$ onto $\mathbb{P}^1_{i_1} \times ... \times \mathbb{P}^1_{i_t}$ and put $c_{\mathbf{i}} := \dim \pi_{\mathbf{i}}(Z)$.

We proceed by induction on m. For $m = 1$, Lemma 11 is trivial. Suppose that $m \geq 2$. For the moment, suppose also that $\pi_1(Z) = \mathbb{P}^1_1$. Let X be the set of points P in Z such that for some $\mathbf{i} \subseteq \{1, ..., m\}$ either $\pi_{\mathbf{i}}(Z)$ is not smooth at $\pi_{\mathbf{i}}(P)$ or the restriction $\pi_{\mathbf{i}}|Z$ of $\pi_{\mathbf{i}}$ to Z is not smooth at P. Then X is a proper, Zariski-closed subset of Z. For $Q = (p : q) \in \mathbb{P}^1_1$, let $f_Q = qX_{10} - pX_{11}$, $Z_Q = Z \cap \{f_Q = 0\}$. There are only finitely many $Q \in \mathbb{P}^1_1$ such that one of the irreducible components of

Z_Q is contained in X. Namely, X has only finitely many irreducible components and if some irreducible component Z' of Z_Q is contained in X, then Z' is an irreducible component of X since $\dim Z' = \dim Z - 1$. Now choose $Q \in \mathbb{P}^1_1$ such that no irreducible component of Z_Q is contained in X and let Z' be any irreducible component of Z_Q. We are going to apply the induction hypothesis to Z'.

We have to consider tangent spaces at an appropriate point. Choose $P \in Z'$ such that for each $\mathbf{i} \subseteq \{1, ..., m\}$, $\pi_{\mathbf{i}}(Z)$ is smooth at $\pi_{\mathbf{i}}(P)$, $\pi_{\mathbf{i}}|Z$ is smooth at P, $\pi_{\mathbf{i}}(Z')$ is smooth at $\pi_{\mathbf{i}}(P)$ and the restriction $\pi_{\mathbf{i}}|Z'$ is smooth at P. Such a P exists since Z' is not contained in X and the set of $P \in Z'$ such that for some $\mathbf{i} \subseteq \{1, ..., m\}$ either $\pi_{\mathbf{i}}(Z')$ is not smooth at $\pi_{\mathbf{i}}(P)$ or $\pi_{\mathbf{i}}|Z'$ is not smooth at P is a proper Zariski-closed subset of Z'.
We assume w.l.o.g. that $P = (1 : 0; ...; 1 : 0)$. Let $\mathbb{A} := \{X_{10} \neq 0, ..., X_{m0} \neq 0\}$ and define affine coordinates $Y_1 = X_{11}/X_{10}, ..., Y_m = X_{m1}/X_{m0}$. Thus, $Z' \cap \mathbb{A}$ is an irreducible component of $(Z \cap \mathbb{A}) \cap \{Y_1 = 0\}$. There are polynomials $f_1, ..., f_r \in \overline{\mathbb{Q}}[Y_1, ..., Y_m]$ such that $Z \cap \mathbb{A} = \{\mathbf{y} \in \mathbb{A} : f_1(\mathbf{y}) = ... = f_r(\mathbf{y}) = 0\}$. The tangent space of Z at P is given by

$$T := \{\mathbf{y} = (y_1, ..., y_m) \in \overline{\mathbb{Q}}^m : \sum_{j=1}^{m} (\partial f_i/\partial Y_j)(\mathbf{0}) y_j = 0 \text{ for } i = 1, ..., r\}.$$

Since $\pi_{\mathbf{i}}|Z$ is smooth at P, the linear map $d\pi_{\mathbf{i}}$ corresponding to $\pi_{\mathbf{i}}$, which is the projection $\mathbf{y} \mapsto (y_i : i \in \mathbf{i})$, maps T surjectively to the tangent space $T_{\mathbf{i}}$ of $\pi_{\mathbf{i}}(Z)$ at $\pi_{\mathbf{i}}(P)$. Since Z is smooth at P we have $\dim T = \dim Z$ and since $\pi_{\mathbf{i}}(Z)$ is smooth at $\pi_{\mathbf{i}}(P)$ we have $\dim T_{\mathbf{i}} = \dim \pi_{\mathbf{i}}(Z) = c_{\mathbf{i}}$.
Similarly, $d\pi_{\mathbf{i}}$ maps the tangent space T' of Z' at P surjectively to the tangent space $T'_{\mathbf{i}}$ of $\pi_{\mathbf{i}}(Z')$ at $\pi_{\mathbf{i}}(P)$ and $\dim T'_{\mathbf{i}} = \dim \pi_{\mathbf{i}}(Z')$. Since $Y_1 \equiv 0$ on $Z' \cap \mathbb{A}$ we have $y_1 \equiv 0$ on T'. Hence $T' \subseteq T \cap \{y_1 = 0\}$. Further, y_1 is not identically zero on T since $\dim f_{\{1\}}(T) = \dim Z_1 = 1$ and $\dim T' = \dim Z' = \dim Z - 1 = \dim T - 1$. Hence $T' = T \cap \{y_1 = 0\}$.
We consider $y_1, ..., y_m$ as linear functions on T. Thus, for $\mathbf{i} \subseteq \{1, ..., m\}$ we have $c_{\mathbf{i}} = \dim T_{\mathbf{i}} = \text{rank}\, \{y_i : i \in \mathbf{i}\}$.
We have the following crucial fact:

$$(5.2) \qquad \begin{aligned} \dim \pi_{\mathbf{i}}(Z') &= \dim \pi_{\mathbf{i}}(Z) \\ &\qquad \text{for each subset } \mathbf{i} \text{ of } \{1, ..., m\} \text{ with } c_{\{1\}\cup\mathbf{i}} > c_{\mathbf{i}}\ , \\ \dim \pi_{\mathbf{i}}(Z') &= \dim \pi_{\mathbf{i}}(Z) - 1 \\ &\qquad \text{for each subset } \mathbf{i} \text{ of } \{1, ..., m\} \text{ with } c_{\{1\}\cup\mathbf{i}} = c_{\mathbf{i}}\ . \end{aligned}$$

Namely, for $\mathbf{i} \subseteq \{1, ..., m\}$ let $V_{\mathbf{i}} = \ker\, d\pi_{\mathbf{i}} \cap T = T \cap \{y_i = 0 \text{ for } i \in \mathbf{i}\}$, $V'_{\mathbf{i}} = \ker\, d\pi_{\mathbf{i}} \cap T' = T' \cap \{y_i = 0 \text{ for } i \in \mathbf{i}\}$. Thus, $V'_{\mathbf{i}} = V_{\{1\}\cup\mathbf{i}}$. Further, put $e_{\mathbf{i}} = c_{\{1\}\cup\mathbf{i}} - c_{\mathbf{i}}$; then $e_{\mathbf{i}} \in \{0, 1\}$. Now for $\mathbf{i} \subseteq \{1, ..., m\}$ we have

$$\begin{aligned} \dim \pi_{\mathbf{i}}(Z') &= \dim T' - \dim V'_{\mathbf{i}} = \dim T - 1 - \dim V_{\{1\}\cup\mathbf{i}} \\ &= \dim \pi_{\{1\}\cup\mathbf{i}}(Z') - 1 = \dim \pi_{\mathbf{i}}(Z) + e_{\mathbf{i}} - 1 \end{aligned}$$

which is precisely (5.2).

We now complete the induction step. Put $\delta'_i := \dim p_i(Z') - \dim p_{i+1}(Z')$ for $i = 1, ..., m$, where $\dim p_{m+1}(Z') := 0$. Put also $c_i := \dim \pi_i(Z), c'_i := \dim \pi_i(Z')$. Recall that $Z' = Q \times W$, where $Q \in \mathbb{P}^1_1$ and W is a subvariety of $\mathbb{P}^1_2 \times ... \times \mathbb{P}^1_m$. By applying the induction hypothesis to W we infer that there are $e_2, ..., e_m \in \{0, 1\}$ such that

$$
\begin{aligned}
&e_2 + ... + e_m = \dim Z' = \dim Z - 1 \ , \\
&(Z' \cdot \mathcal{L}_2^{e_2} ... \mathcal{L}_m^{e_m}) > 0 \ , \\
&\eta'_i := \sum_{j=i}^{m} (\delta'_j - e_j) \geq 0 \ \text{ for } i = 3, ..., m \ , \\
&\eta'_3 + ... + \eta'_m \geq \sum_{j=2}^{m} c'_j - \dim Z' \ .
\end{aligned}
$$

Put $e_1 = 1$. Obviously, $(Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) > 0$. Let t be the largest index such that y_1 is linearly dependent on $\{y_i : i \geq t\}$. Then $\delta_t = \dim p_t(Z) - \dim p_{t+1}(Z) = \text{rank } \{y_i : i \geq t\} - \text{rank } \{y_i : i \geq t+1\} = 1$. Further, by (5.2) we have $\delta'_i = \delta_i, \eta'_i = \eta_i, c'_i = c_i$ for $i > t$, $\delta'_t = \delta_t - 1 = 0, \delta'_i = \delta_i$ for $2 \leq i < t$ and $\eta'_i = \eta_i - 1$ for $2 \leq i \leq t$, where $\eta'_2 := \sum_{j=2}^{m} (\delta'_j - e_j) = 0$. Further, $c'_i \geq c_i - 1$ for $i \leq t$ and $\dim Z' = \dim Z - 1$. It follows that

$$
\begin{aligned}
\sum_{j=2}^{m} \eta_j &= \sum_{j=3}^{m} \eta'_j \ + t - 1 \\
&\geq \left(\sum_{j=2}^{m} c'_j \right) + t - 1 - \dim Z' \geq \left(\sum_{j=1}^{m} c_j \right) - \dim Z \ .
\end{aligned}
$$

This completes the induction step for the case $\dim \pi_1(Z) > 0$. In the other case we have $Z = Q \times W$ where $Q \in \mathbb{P}^1_1$ and W is a subvariety of $\mathbb{P}^1_2 \times ... \times \mathbb{P}^1_m$ and then the induction step is completed by applying the induction hypothesis to W. This proves Lemma 11. □

Proof of Theorem 3. Suppose that the integers $d_1, ..., d_m$ satisfy (1.11), i.e. $d_h / d_{h+1} > 2m^3/\epsilon$ for $h = 1, ..., m-1$. Put $\mathcal{L} := d_1 \mathcal{L}_1 + ... + d_m \mathcal{L}_m$. Let $e_1, ..., e_m$ be the integers from Lemma 11.

Assume that $\pi_h(Z) = \mathbb{P}^1_h$ for $h = 1, ..., m$. Then by Lemma 11,

$$\eta_2 + ... + \eta_m \geq \ \text{codim } Z = s \ .$$

Together with (4.11) (cf. proof of Theorem 1 with $i\epsilon', \epsilon'$ replacing σ, ϵ, respectively) and $\eta_1 = 0$ this implies that

$$\begin{aligned}1 &\le (Z \cdot \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) \\ &\le \left(\frac{ms}{\epsilon'}\right)^s \left(\frac{d_2}{d_1}\right)^{\eta_2} ... \left(\frac{d_m}{d_{m-1}}\right)^{\eta_m} \\ &< \left(\frac{m(m+1)}{\epsilon}\right)^s \cdot \left(\frac{2m^3}{\epsilon}\right)^{-s} \le 1\end{aligned}$$

which is impossible. Therefore there is an $h \in \{1, ..., m\}$ with $\pi_h(Z) = P_h$.

We now show that P_h satisfies (5.1). By precisely the same argument as in the proof of Theorem 2 we have

$$\begin{aligned}(5.3)\quad & h(Z, \mathcal{L}^{m-s+1}) \\ &\le m_Z^{-1} \cdot m! d_1 ... d_m \big((m^2 + \log 2)(d_1 + ... + d_m) + s \log H(F)\big) \\ &\le \left(\frac{s}{\epsilon'}\right)^s 2m^2 m! \cdot d_1^{\delta_1} ... d_m^{\delta_m} (d_1 + ... + d_m + \log H(F)) \\ &\le \left(\frac{3m^3}{\epsilon}\right)^m d_1^{\delta_1} ... d_m^{\delta_m} (d_1 + ... + d_m + \log H(F)) \ .\end{aligned}$$

By Lemmas 6,7 we have

$$\begin{aligned}& h(Z, \mathcal{L}^{m-s+1}) \\ &\quad = \sum_{f_1 + ... + f_m = m-s+1} \frac{(m-s+1)!}{f_1! ... f_m!} d_1^{f_1} ... d_m^{f_m} h(Z, \mathcal{L}_1^{f_1} ... \mathcal{L}_m^{f_m}) \\ &\quad \ge d_1^{e_1} ... d_m^{e_m} \cdot d_h \cdot h(Z, \mathcal{L}_1^{e_1} ... \mathcal{L}_h^{e_h + 1} ... \mathcal{L}_m^{e_m}) \ .\end{aligned}$$

Together with (5.3) and $\eta_i \ge 0$ for $i = 2, ..., m$ this implies

$$\begin{aligned}(5.4)\quad & d_h \cdot h(Z, \mathcal{L}_1^{e_1} ... \mathcal{L}_h^{e_h+1} ... \mathcal{L}_m^{e_m}) \\ &\le \left(\frac{3m^3}{\epsilon}\right)^m d_1^{\delta_1 - e_1} ... d_m^{\delta_m - e_m} \big(d_1 + ... + d_m + \log H(F)\big) \\ &\le \left(\frac{3m^3}{\epsilon}\right)^m \left(\frac{d_2}{d_1}\right)^{\eta_2} ... \left(\frac{d_m}{d_{m-1}}\right)^{\eta_m} \big(d_1 + ... + d_m + \log H(F)\big) \\ &\le \left(\frac{3m^3}{\epsilon}\right)^m \big(d_1 + ... + d_m + \log H(F)\big)\end{aligned}$$

It is no restriction to assume that $P_h = (a : b)$ where a, b belong to some number field K and $(a, b) = (1)$. Then there are $\alpha, \beta \in O_K$ with $\alpha a + \beta b = 1$. Put $f = \alpha X_{h0} + \beta X_{h1}$. Then $\mathrm{div}(f|Z) = 0$ and $\kappa_\wp = \kappa_\wp(Z, f, \mathcal{L}_1^{e_1} ... \mathcal{L}_m^{e_m}) = 0$ for each non-zero prime ideal $\wp$ of O_K. Further, for each embedding $\sigma : K \hookrightarrow \mathbb{C}$ we have that $||\sigma(f)|| = (|\sigma(a)|^2 + |\sigma(b)|^2)^{-1/2}$ is constant on $Z \times_\sigma \mathbb{C}$. Hence, using (3.1),

$$\begin{aligned}
\kappa_\sigma &= \kappa_\sigma(Z, f, \mathcal{L}_1^{e_1} \dots \mathcal{L}_m^{e_m}) \\
&= -\frac{1}{[K:\mathbb{Q}]} \int_{Z \times_\sigma \mathbb{C}} \log ||\sigma(f)|| \cdot c_1(\mathcal{L}_1)^{e_1} \wedge \dots \wedge c_1(\mathcal{L}_m)^{e_m} \\
&= -\frac{1}{[K:\mathbb{Q}]} \log \left((|\sigma(a)|^2 + |\sigma(b)|^2)^{-1/2} \right) \int_{Z \times_\sigma \mathbb{C}} c_1(\mathcal{L}_1)^{e_1} \wedge \dots \wedge c_1(\mathcal{L}_m)^{e_m} \\
&= \log \left((|\sigma(a)|^2 + |\sigma(b)|^2)^{1/2[K:\mathbb{Q}]} \right) \cdot (Z \cdot \mathcal{L}_1^{e_1} \dots \mathcal{L}_m^{e_m}) \; .
\end{aligned}$$

By inserting this into (3.5) and using that $(a, b) = (1)$ we obtain

$$\begin{aligned}
& h(Z, \mathcal{L}_1^{e_1} \dots \mathcal{L}_h^{e_h+1} \dots \mathcal{L}_m^{e_m}) \\
&\qquad = \sum_\sigma \kappa_\sigma + \sum_\wp \kappa_\wp = \log H(P_h) \cdot (Z \cdot \mathcal{L}_1^{e_1} \dots \mathcal{L}_m^{e_m}) \\
&\qquad \geq \log H(P_h) \; .
\end{aligned}$$

Together with (5.4) this implies (5.1). This completes the proof of Theorem 3. $\square$

References.

[1] G. FALTINGS, *Diophantine approximation on abelian varieties*, Annals of Math. 133 (1991), 549-576.

[2] G. FALTINGS & G. WÜSTHOLZ, *Diophantine approximations on projective spaces*, Invent. Math. 116 (1994), 109-138.

[3] R. FERRETTI, *An effective version of Faltings' Product Theorem*, Forum matematicum, to appear.

[4] W. FULTON, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 2, Springer Verlag, Berlin etc., 1984.

[5] H. GILLET & C. SOULÉ, *Arithmetic intersection theory*, IHES, Publ. Math. 72 (1990), 93-174.

[6] P. GRIFFITHS & J. HARRIS, *Principles of algebraic geometry*, Wiley & Sons, New York, 1978.

[7] W. GUBLER, *Höhentheorie*, Math. Ann. 298 (1994), 427-456.

[8] R. HARTSHORNE, *Algebraic geometry*, Springer Verlag, Berlin etc., 1977.

[9] J. de JONG, *Ample line bundles and intersection theory*, in: Diophantine approximation and abelian varieties, Proc. conf. Soesterberg, Netherlands, 1992, B. Edixhoven, J.-H. Evertse, eds., Lecture Notes in Mathematics 1566, Springer Verlag, Berlin etc. 1993, pp. 69-76.

[10] P. PHILIPPON, *Sur des hauteurs alternatives, I*, Math. Ann. 289, (1991), 255-283.

[11] M. van der PUT, *The Product theorem*, in: Diophantine approximation and abelian varieties, B. Edixhoven, J.-H. Evertse, eds., Springer LNM 1566, 1993, pp. 77-82.

[12] K.F. ROTH, *Rational approximations to algebraic numbers*, Mathematika 2 (1995), 1-20, Corrigendum 168.

[13] H.P. SCHLICKEWEI, *An explicit upper bound for the number of solutions of the S-unit equation*, J. reine angew. Math. 406 (1990), 109-120.

[14] H.P. SCHLICKEWEI, *The quantitative Subspace Theorem for number fields*, Compos. Math. 82 (1992), 245-273.

[15] W.M. SCHMIDT, *Norm form equations*, Ann. Math. 96 (1972), 526-551.

[16] W.M. SCHMIDT, *The subspace theorem in diophantine approximations*, Compos. Math. 69 (1989), 121-173.

[17] W.M. SCHMIDT, *The number of solutions of norm form equations*, Trans. Am. Math. Soc. 317 (1990), 197-227.

[18] I.R. SHAFAREVICH, *Basic Algebraic Geometry*, Springer Verlag, Berlin etc., 1977.

[19] C. SOULÉ, *Geometrie d' Arakelov et theorie des nombres transcendants*, in Journées Arithmétiques de Luminy, 1989, G. Lachaud, ed. pp. 355-372, Astérisque 198-199-220, 1991, Soc. Math. France.

[20] G. WÜSTHOLZ, *Multiplicity estimates on group varieties*, Ann. Math. 129 (1989), 471-500.

[21] O. ZARISKI & P. SAMUEL, *Commutative Algebra, Vol. II*, Springer Verlag, Berlin etc., 1960.