

R version 3.6.0 (2019-04-26) -- "Planting of a Tree"
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Platform: x86_64-apple-darwin15.6.0 (64-bit)

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Natural language support but running in an English locale

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[R.app GUI 1.70 (7657) x86_64-apple-darwin15.6.0]

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```
>
> ##### NIST #####
>
> table11 <- matrix(c(6378, 3282, 3189, 43897356),
+   2, 2, byrow = TRUE,
+   dimnames = list(Alice = c("d", "n"), Bob = c("d", "n")))
> table12 <- matrix(c(6794, 2821, 23243, 43276943),
+   2, 2, byrow = TRUE,
+   dimnames = list(Alice = c("d", "n"), Bob = c("d", "n")))
> table21 <- matrix(c(6486, 21334, 2843, 43338281),
+   2, 2, byrow = TRUE,
+   dimnames = list(Alice = c("d", "n"), Bob = c("d", "n")))
> table22 <- matrix(c(106, 27539, 30040, 42502788),
+   2, 2, byrow = TRUE,
+   dimnames = list(Alice = c("d", "n"), Bob = c("d", "n")))
>
> table11
  Bob
Alice d      n
d 6378 3282
n 3189 43897356
> table12
  Bob
Alice d      n
d 6794 2821
n 23243 43276943
> table21
  Bob
Alice d      n
d 6486 21334
n 2843 43338281
> table22
  Bob
Alice d      n
d 106 27539
n 30040 42502788
>
> tables <- cbind(as.vector(t(table11)), as.vector(t(table12)), as.vector(t(table21)), as.vector(t(table22)))
> dimnames(tables) = list(outcomes = c("dd", "dn", "nd", "nn"),
+   settings = c(11, 12, 21, 22))
> tables
  settings
outcomes 11      12      21      22
dd      6378     6794     6486     106
dn      3282     2821     21334    27539
nd      3189     23243    2843     30040
nn     43897356 43276943 43338281 42502788
>
> Ns <- apply(tables, 2, sum)
> Ns
 11      12      21      22
43910205 43309801 43368944 42560473
>
> rawProbsMat <- tables / outer(rep(1,4), Ns)
> rawProbsMat
  settings
outcomes 11      12      21      22
dd 1.452510e-04 1.568698e-04 1.495540e-04 2.490574e-06
dn 7.474345e-05 6.513537e-05 4.919188e-04 6.470558e-04
nd 7.262549e-05 5.366684e-04 6.555382e-05 7.058192e-04
nn 9.997074e-01 9.992413e-01 9.992930e-01 9.986446e-01
>
> VecNames <- as.vector(t(outer(colnames(rawProbsMat), rownames(rawProbsMat), paste, sep = "")))
```

```

> VecNames
[1] "11dd" "11dn" "11nd" "11nn" "12dd" "12dn" "12nd" "12nn" "21dd" "21dn" "21nd" "21nn" "22dd" "22dn" "22nd" "22nn"
>
> rawProbsVec <- as.vector(rawProbsMat)
> names(rawProbsVec) <- VecNames
>
> VecNames
[1] "11dd" "11dn" "11nd" "11nn" "12dd" "12dn" "12nd" "12nn" "21dd" "21dn" "21nd" "21nn" "22dd" "22dn" "22nd" "22nn"
> rawProbsVec
      11dd      11dn      11nd      11nn      12dd      12dn      12nd      12nn      21dd
1.452510e-04 7.474345e-05 7.262549e-05 9.997074e-01 1.568698e-04 6.513537e-05 5.366684e-04 9.992413e-01 1.495540e-04
      21dn      21nd      21nn      22dd      22dn      22nd      22nn
4.919188e-04 6.555382e-05 9.992930e-01 2.490574e-06 6.470558e-04 7.058192e-04 9.986446e-01
>
> Aplus <- c(1, 1, 0, 0)
> Aminus <- - Aplus
> Bplus <- c(1, 0, 1, 0)
> Bminus <- - Bplus
> zero <- c(0, 0, 0, 0)
> NSa1 <- c(Aplus, Aminus, zero, zero)
> NSa2 <- c(zero, zero, Aplus, Aminus)
> NSb1 <- c(Bplus, zero, Bminus, zero)
> NSb2 <- c(zero, Bplus, zero, Bminus)
> NS <- cbind(NSa1 = NSa1, NSa2 = NSa2, NSb1 = NSb1, NSb2 = NSb2)
> rownames(NS) <- VecNames
> NS
      NSa1 NSa2 NSb1 NSb2
11dd    1    0    1    0
11dn    1    0    0    0
11nd    0    0    1    0
11nn    0    0    0    0
12dd   -1    0    0    1
12dn   -1    0    0    0
12nd    0    0    0    1
12nn    0    0    0    0
21dd    0    1   -1    0
21dn    0    1    0    0
21nd    0    0   -1    0
21nn    0    0    0    0
22dd    0   -1    0   -1
22dn    0   -1    0    0
22nd    0    0    0   -1
22nn    0    0    0    0
>
> # "The number of valid trials is N = 173 149 423"
> sum(NS)
[1] 173149423
>
> cov11 <- diag(rawProbsMat[ , "11"]) - outer(rawProbsMat[ , "11"], rawProbsMat[ , "11"])
> cov12 <- diag(rawProbsMat[ , "12"]) - outer(rawProbsMat[ , "12"], rawProbsMat[ , "12"])
> cov21 <- diag(rawProbsMat[ , "21"]) - outer(rawProbsMat[ , "21"], rawProbsMat[ , "21"])
> cov22 <- diag(rawProbsMat[ , "22"]) - outer(rawProbsMat[ , "22"], rawProbsMat[ , "22"])
>
> Cov <- matrix(0, 16, 16)
> rownames(Cov) <- VecNames
> colnames(Cov) <- VecNames
> Cov[1:4, 1:4] <- cov11/NS["11"]
> Cov[5:8, 5:8] <- cov12/NS["12"]
> Cov[9:12, 9:12] <- cov21/NS["21"]
> Cov[13:16, 13:16] <- cov22/NS["22"]
>
> InvCovNN <- solve(covNN)
>
> J <- c(c(1, 0, 0, 0), c(0, -1, 0, 0), c(0, 0, -1, 0), c(-1, 0, 0, 0))
> names(J) <- VecNames
> sum(J * rawProbsVec)
[1] 1.207121e-05
>
> varJ <- t(J) %%% Cov %%% J
> sum(J * rawProbsVec) / sqrt(varJ)
      [,1]
[1,] 4.778576
> pnorm(sum(J * rawProbsVec) / sqrt(varJ), lower.tail = FALSE, log = TRUE)/log(10)
      [,1]
[1,] -6.054184
>
> covNN <- t(NS) %%% Cov %%% NS
> covJN <- t(J) %%% Cov %%% NS
> covNJ <- t(covJN)
> varJ - covJN %%% InvCovNN %%% covNJ
      [,1]
[1,] 2.786584e-12
> varJ
      [,1]
[1,] 6.381229e-12
> sqrt(varJ / (varJ - covJN %%% InvCovNN %%% covNJ))
      [,1]

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[1,] 1.513269
> covJN %%% InvCovNN
      NSa1      NSa2      NSb1      NSb2
[1,] 0.3660023 0.04544781 0.3795165 0.04234141
>
> Jopt <- J - covJN %%% InvCovNN %%% t(NS)
> Jopt
      11dd      11dn      11nd 11nn      12dd      12dn      12nd 12nn      21dd      21dn      21nd 21nn
[1,] 0.2544812 -0.3660023 -0.3795165 0 0.3236609 -0.6339977 -0.04234141 0 0.3340686 -0.04544781 -0.6204835 0
      22dd      22dn      22nd 22nn
[1,] -0.9122108 0.04544781 0.04234141 0
>
> sum(J * rawProbsVec)
[1] 1.207121e-05
>
> sum(Jopt * rawProbsVec)
[1] 1.274879e-05
>
> varJ / (varJ - covJN %%% InvCovNN %%% covNJ)
      [,1]
[1,] 2.289982
> varJopt <- varJ - covJN %%% InvCovNN %%% covNJ
> (varJ - covJN %%% InvCovNN %%% covNJ) / varJ
      [,1]
[1,] 0.4366846
> sqrt((varJ - covJN %%% InvCovNN %%% covNJ) / varJ)
      [,1]
[1,] 0.6608212
>
> pnorm(sum(J * rawProbsVec) / sqrt(varJ), lower.tail = FALSE, log = TRUE)/log(10)
      [,1]
[1,] -6.054184
> pnorm(sum(Jopt * rawProbsVec) / sqrt(varJopt), lower.tail = FALSE, log = TRUE)/log(10)
      [,1]
[1,] -13.9546
>
> S <- c(c(1, -1, -1, 1), c(1, -1, -1, 1), c(1, -1, -1, 1), -c(1, -1, -1, 1))
> names(S) <- VecNames
> sum(S * rawProbsVec)
[1] 2.000092
>
> varS <- t(S) %%% Cov %%% S
> covNN <- t(NS) %%% Cov %%% NS
> covSN <- t(S) %%% Cov %%% NS
> covNS <- t(covSN)
>
> varCHSH <- varS
>
> varCHSHopt <- varS - covSN %%% InvCovNN %%% covNS
> varS
      [,1]
[1,] 2.47335e-10
> sqrt(varCHSH / varCHSHopt)
      [,1]
[1,] 2.355303
> covSN %%% solve(covNN)
      NSa1      NSa2      NSb1      NSb2
[1,] -0.5359906 -1.818209 -0.4819342 -1.830634
> Sopt <- S - covSN %%% InvCovNN %%% t(NS)
> Sopt
      11dd      11dn      11nd 11nn      12dd      12dn      12nd 12nn      21dd      21dn      21nd 21nn      22dd
[1,] 2.017925 -0.4640094 -0.5180658 1 2.294644 -1.535991 0.8306344 1 2.336275 0.8182087 -1.481934 1 -4.648843
      22dn      22nd 22nn
[1,] -0.8182087 -0.8306344 -1
>
> CHSH <- sum(S * rawProbsVec)
> CHSH
[1] 2.000092
>
> CHSHopt <- sum(Sopt * rawProbsVec)
> CHSHopt
[1] 2.000051
>
> pnorm((CHSH - 2) / sqrt(varCHSH), lower.tail = FALSE, log = TRUE)/log(10)
      [,1]
[1,] -8.685507
> pnorm((CHSHopt - 2) / sqrt(varCHSHopt), lower.tail = FALSE, log = TRUE)/log(10)
      [,1]
[1,] -13.9546
>
>

```