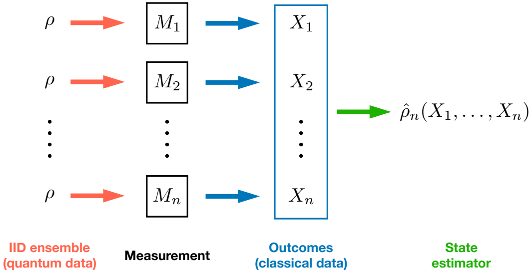
From quantum Fisher information to local asymptotic normality

Mădălin Guţă

School of Mathematical Sciences
University of Nottingham

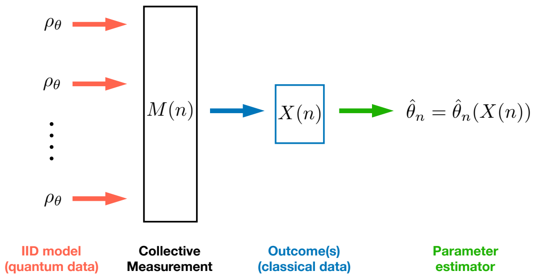
Quantum tomography



Partial answers to the key questions:

- measurement design: separate measurements
- estimation method: LS, PLS, ML, ...
- statistical model: completely unknown state or small rank state

General quantum parameter estimation setup with IID ensembles



- quantum IID model: n systems in state ρ_θ with unknown parameter $\theta \in \Theta$
- measurement: allow for general (collective) measurements
- estimation problem: find 'optimal procedures for achieving ultimate precision'
  - minimise estimation risk: $R(\hat{\theta}_n|\theta) = \mathbb{E}(d(\hat{\theta}_n, \theta))$
  - define suitable confidence regions (error bars)

Outline

- Quantum Fisher information and quantum Cramér-Rao bound
- Local Asymptotic Normality for quantum IID ensembles
- Local Asymptotic Normality for quantum Markov processes

Quantum Cramér-Rao bound

Theorem [Helstrom, Holevo, Belavkin, Braunstein&Caves]

Let $\mathcal{Q} = \{\rho^\theta : \theta \in \mathbb{R}^k\}$ be a 'smooth' quantum model.

For any unbiased measurement M with outcome $\hat{\theta} \in \mathbb{R}^k$ (i.e. $\mathbb{E}\hat{\theta} = \theta$)

$$\mathrm{Var}(\hat{\theta}) \geq F(\theta)^{-1} \quad \Longrightarrow \quad \mathbb{E}\|\hat{\theta} - \theta\|^2 \geq \mathrm{Tr} F(\theta)^{-1}$$

- $F(\theta)$ is the Quantum Fisher information matrix $F(\theta)_{i,j} := \mathrm{Tr}(\rho_\theta \mathcal{L}_{\theta,i} \circ \mathcal{L}_{\theta,j})$
- Symmetric logarithmic derivatives $\mathcal{L}_{\theta,j}$: selfadjoint solutions of $\frac{\partial \rho_\theta}{\partial \theta_j} = \rho_\theta \circ \mathcal{L}_{\theta,j}$

Quantum Cramér-Rao bound

Theorem [Helstrom, Holevo, Belavkin, Braunstein&Caves]

Let $\mathcal{Q} = \{\rho^\theta : \theta \in \mathbb{R}^k\}$ be a 'smooth' quantum model.

For any unbiased measurement M with outcome $\hat\theta \in \mathbb{R}^k$ (i.e. $\mathbb{E}\hat\theta = \theta$)

$$\mathrm{Var}(\hat\theta) \geq F(\theta)^{-1} \quad \Longrightarrow \quad \mathbb{E}\|\hat\theta - \theta\|^2 \geq \mathrm{Tr}F(\theta)^{-1}$$

- $F(\theta)$ is the Quantum Fisher information matrix $F(\theta)_{i,j} := \mathrm{Tr}(\rho_\theta \mathcal{L}_{\theta,i} \circ \mathcal{L}_{\theta,j})$
- Symmetric logarithmic derivatives $\mathcal{L}_{\theta,j}$: selfadjoint solutions of $\frac{\partial \rho_\theta}{\partial \theta_j} = \rho_\theta \circ \mathcal{L}_{\theta,j}$

- Quantum Fisher information as quadratic approximation for the Bures distance

$$d_b^2(\rho_\theta, \rho_{\theta+\delta\theta}) = \frac{1}{4}\delta\theta^T F(\theta)\delta\theta, \qquad d_b^2(\rho,\sigma) = 2[1 - \mathrm{Tr}(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})]$$

- one parameter pure state rotation model: $|\psi_\theta\rangle := e^{-i\theta G}|\psi\rangle, \qquad \langle\psi|G|\psi\rangle = 0$

$$F(\theta) \quad = \quad 4\left\|\frac{d\psi_\theta}{d\theta}\right\|^2 = 4\mathrm{Var}_\psi(G) = 4\left\langle \psi \,\middle|\, G^2 \,\middle|\, \psi \right\rangle$$

(non-) Achievability of the QCR bound

- $\theta \in \mathbb{R}$: bound achieved (locally) at θ_0 by measuring $\mathbf{X} = \theta_0 \mathbf{1} + \frac{\mathcal{L}_{\theta_0}}{F(\theta_0)}$

  - $$\mathbb{E}_\theta \mathbf{X} = \theta_0 + \frac{\mathrm{Tr}(\rho_\theta \mathcal{L}_{\theta_0})}{F(\theta_0)} = \theta_0 + \frac{\mathrm{Tr}(\rho_{\theta_0} \mathcal{L}_{\theta_0})}{F(\theta_0)} + \Delta\theta \frac{\mathrm{Tr}(\rho'_{\theta_0} \mathcal{L}_{\theta_0})}{F(\theta_0)} + O(\Delta\theta^2)$$
    $$= \theta_0 + \Delta\theta + O(\Delta\theta^2) = \theta + O(\Delta\theta^2)$$

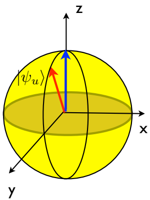
  - $$\mathrm{Var}_{\theta_0}(\mathbf{X}) = \mathbb{E}_{\theta_0}\left[(X - \mathbb{E}_{\theta_0} X)^2\right] = \frac{\mathrm{Tr}(\rho_{\theta_0} \mathcal{L}_{\theta_0}^2)}{F^2(\theta_0)} = \frac{1}{F_{\theta_0}}$$

- For n samples: measure separately (and adaptively) and average $\mathbf{X}(n) = \frac{1}{n} \sum_i \mathbf{X}^{(i)}$

- Standard MSE scaling: $\mathbb{E}\left[(\hat{\theta}_n - \theta)^2\right] \approx \frac{1}{nF(\theta)}$

- multidimensional θ: achievability of QFI is problematic if $[\mathcal{L}_{\theta,i}, \mathcal{L}_{\theta,j}] \neq 0$

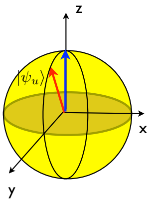
Example: estimating the direction of the spin vector



- One-dim. model: (small) rotation of $|\uparrow\rangle$

$$|\psi_u\rangle := \exp(iu\sigma_x)|\uparrow\rangle = \cos(u)|\uparrow\rangle + \sin(u)|\downarrow\rangle$$

Example: estimating the direction of the spin vector



- One-dim. model: (small) rotation of $|\uparrow\rangle$

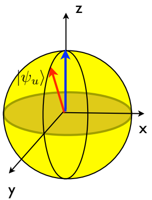
$$|\psi_u\rangle := \exp(iu\sigma_x)|\uparrow\rangle = \cos(u)|\uparrow\rangle + \sin(u)|\downarrow\rangle$$

- Quantum Fisher information $F = 4\langle\uparrow|\sigma_x^2|\uparrow\rangle = 4$

- SLD $\mathcal{L} = 2\sigma_y$ is the 'most informative' spin observable

$$\mathbb{E}\left(\frac{\mathcal{L}}{F}\right) = \frac{2\sin(2u)}{4} \approx u, \qquad \mathrm{Var}(\hat{u}) = \mathrm{Var}\left(\frac{\mathcal{L}}{F}\right) = \frac{1}{4} = \frac{1}{F}$$

Example: estimating the direction of the spin vector



- One-dim. model: (small) rotation of $|\uparrow\rangle$

$$|\psi_u\rangle := \exp(iu\sigma_x)|\uparrow\rangle = \cos(u)|\uparrow\rangle + \sin(u)|\downarrow\rangle$$

- Quantum Fisher information $F = 4\langle\uparrow|\sigma_x^2|\uparrow\rangle = 4$
- SLD $\mathcal{L} = 2\sigma_y$ is the 'most informative' spin observable

$$\mathbb{E}\left(\frac{\mathcal{L}}{F}\right) = \frac{2\sin(2u)}{4} \approx u, \qquad \mathrm{Var}(\hat{u}) = \mathrm{Var}\left(\frac{\mathcal{L}}{F}\right) = \frac{1}{4} = \frac{1}{F}$$

- Two parameter model $|\psi_{u_x,u_y}\rangle = \exp(i(u_y\sigma_x - u_x\sigma_y))|\uparrow\rangle$

- Since $[\sigma_x, \sigma_y] \neq 0$, optimal measurements for u_x and u_y are incompatible

Example: quantum Gaussian shift

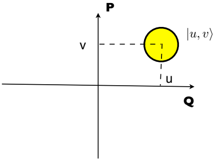
- Continuous variables system: canonical observables Q, P on $L^2(\mathbb{R})$

$$QP - PQ = i\mathbf{1} \qquad \text{(Heisenberg's commutation relations)}$$

- Vacuum (Gaussian) state $|\mathbf{0}\rangle \in L^2(\mathbb{R})$ with characteristic function

$$\phi(u,v) := \langle \mathbf{0} \,|\, \exp(-ivQ - iuP) \,|\, \mathbf{0}\rangle = \exp(-(u^2+v^2)/4)$$

- Coherent states $\quad |u,v\rangle := \exp(-ivQ - iuP) \,|\, \mathbf{0}\rangle$



- QFI $\quad F = 4\begin{pmatrix} \mathrm{Var}(P) & 0 \\ 0 & \mathrm{Var}(Q) \end{pmatrix} = 2 \cdot \mathbf{1}$

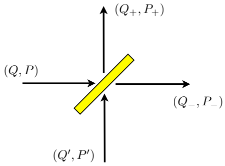
- Optimal measurements
  - one-parameter: $\hat{u} \sim N(u, 1/2)$ by measuring $Q \quad \Rightarrow \quad \mathbb{E}[|\hat{u} - u|^2] = \frac{1}{2}$
  - QCR bound not achievable: since Q, P are incompatible, (u,v) cannot be estimated optimally simultaneously. What is the optimal measurement?

Optimal measurement for Gaussian shift

- Idea: 'make' Q and P commute by 'adding quantum noise'

- Beamsplitter: combine (Q, P) with independent system (Q', P')

$$\begin{aligned} Q_{\pm} &:= Q \pm Q' \\ P_{\pm} &:= P \pm P' \end{aligned}$$



- Noisy coordinates commute:

$\Rightarrow [Q_+, P_-] = [Q + Q', P - P'] = 0$

- Heterodyne measurement (Q_+, P_-) gives estimator $(\hat{u}, \hat{v}) \sim N((u, v), \frac{\mathbf{1}}{2} + V')$

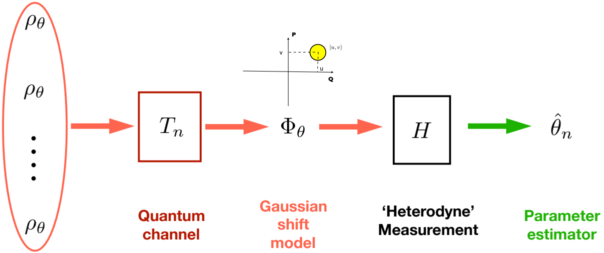
- MSE minimised when (Q', P') is in the 'minimum uncertainty' state $|\mathbf{0}\rangle$ with $V' = \frac{\mathbf{1}}{2}$

$$\mathbb{E}[|u - \hat{u}|^2 + |v - \hat{v}|^2] = 2$$

Outline

- Quantum Fisher information and quantum Cramér-Rao bound
- Local Asymptotic Normality for quantum IID ensembles
- Local Asymptotic Normality for quantum Markov processes

Optimal estimation using local asymptotic normality[1 2 3 4]



L. Le Cam

- LAN: sequence of IID models converges to a Gaussian shift model for $\theta = \theta_0 + u/\sqrt{n}$
- Operational formulation: there exist quantum channels T_n and S_n (dep. on θ_0) such that

$$\lim_{n\to\infty} \sup_{\|u\|\leq n^{\epsilon}} \left\| T_n\left(\rho^{\otimes n}_{\theta_0+u/\sqrt{n}}\right) - \Phi(u, V_0)\right\|_1 = 0$$

$$\lim_{n\to\infty} \sup_{\|u\|\leq n^{\epsilon}} \left\| \rho^{\otimes n}_{\theta_0+u/\sqrt{n}} - S_n(\Phi(u, V_0))\right\|_1 = 0$$

- LAN is used to derive minimax rates and optimal measurements

[1] J. Kahn, M.G., *Commun. Math. Phys.* (2009), M.G., B. Janssens and J.Kahn, *Commun. Math. Phys.* (2008)

[2] R.D. Gill, M.G., *I.M.S. Collections* (2012)

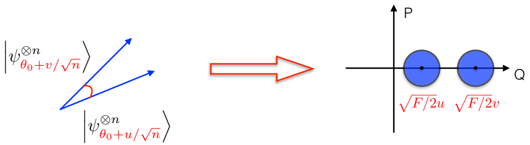
[3] C. Butucea, M.G. and M. Nussbaum *Ann. Statist.* (2018)

[4] M.G., J. Kiukas, *J. Math. Phys.* (2017), M.G., J. Kiukas, *Commun. Math. Phys.* (2015), C. Catana, L. Bouten, M.G. *J.*

Convergence to Gaussian model for i.i.d. ensembles of pure states

- Quantum data: ensemble of n identically prepared systems

$$|\psi_\theta\rangle^{\otimes n} := \left(e^{i\theta G}|\psi\rangle\right)^{\otimes n}, \qquad \langle\psi|G|\psi\rangle = 0$$



- Local asymptotic normality (Gaussian approximation):

Write $\theta = \theta_0 + u/\sqrt{n}$ for θ an "uncertainty neighbourhood" of size $n^{-1/2}$ around θ_0

The overlaps of such joint states converge to those of a Gaussian shift model with QFI = F

$$\left\langle \psi^{\otimes n}_{\theta_0+u/\sqrt{n}} \middle| \psi^{\otimes n}_{\theta_0+v/\sqrt{n}} \right\rangle = \underbrace{\left\langle \psi | e^{i(u-v)G/\sqrt{n}} \middle| \psi \right\rangle^n}_{(1-\langle\psi|G^2|\psi\rangle/2n+\dots)^n} \longrightarrow e^{(u-v)^2F/8} = \left\langle \sqrt{F/2}\,u \middle| \sqrt{F/2}\,v \right\rangle$$

Gaussian approximation for pure states

- n identically prepared spins

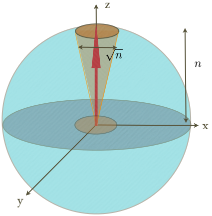
$$\left|\psi_{\frac{u_x}{\sqrt{n}},\frac{u_y}{\sqrt{n}}}\right\rangle := \exp\left(i\frac{u_y\sigma_x - u_x\sigma_y}{\sqrt{n}}\right)|\uparrow\rangle$$

- Collective observables $L_{x,y,z} := \sum_{i=1}^{n} \sigma^{(i)}_{x,y,z}$

- Quantum Central Limit Theorem

$$u_x, u_y = 0 \Longrightarrow \begin{cases} \frac{L_x}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1) \\ \frac{L_y}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1) \end{cases}$$

$$\left[\frac{L_x}{\sqrt{n}}, \frac{L_y}{\sqrt{n}}\right] = \frac{2i}{n} L_z \xrightarrow{l.l.n.} 2i\mathbf{1}$$



Gaussian approximation for pure states

- n identically prepared spins

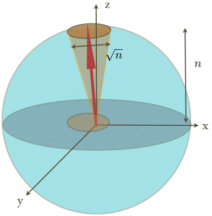
$$\left|\psi_{\frac{u_x}{\sqrt{n}},\frac{u_y}{\sqrt{n}}}\right\rangle := \exp\left(i\frac{u_y\sigma_x - u_x\sigma_y}{\sqrt{n}}\right)|\uparrow\rangle$$

- Collective observables $L_{x,y,z} := \sum_{i=1}^{n} \sigma_{x,y,z}^{(i)}$

- Quantum Central Limit Theorem

$$u_x, u_y \neq 0 \Longrightarrow \begin{cases} \frac{L_x}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(2u_x, 1) \\ \frac{L_y}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(2u_y, 1) \end{cases}$$

$$\left[\frac{L_x}{\sqrt{n}}, \frac{L_y}{\sqrt{n}}\right] = \frac{2i}{n}L_z \xrightarrow{l.l.n.} 2i\mathbf{1}$$



Gaussian approximation for mixed states

- n identically prepared spins with local parameter $\mathbf{u} = (u_x, u_y, u_z)$

$$\rho_{\frac{\mathbf{u}}{\sqrt{n}}} := e^{i\frac{u_y\sigma_x - u_x\sigma_y}{\sqrt{n}}} \begin{pmatrix} \mu + \frac{u_z}{\sqrt{n}} & 0 \\ 0 & 1-\mu-\frac{u_z}{\sqrt{n}} \end{pmatrix} e^{-i\frac{u_y\sigma_x - u_x\sigma_y}{\sqrt{n}}}$$

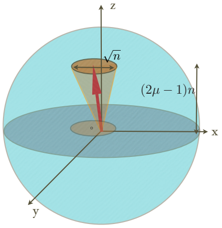
- Collective observables $L_{x,y,z} := \sum_{i=1}^{n} \sigma_{x,y,z}^{(i)}$

- Quantum Central Limit Theorem (mixed states)

$$\frac{L_{x,y}}{\sqrt{n}} \xrightarrow{\mathcal{D}} N\left(2(2\mu-1)u_{x,y}, 1\right)$$

$$\frac{L_z - n(2\mu-1)}{\sqrt{n}} \xrightarrow{\mathcal{D}} N\left(u_z, \mu(1-\mu)\right)$$

$$\left[\frac{L_x}{\sqrt{n}}, \frac{L_y}{\sqrt{n}}\right] = \frac{2i}{n} L_z \xrightarrow{l.l.n.} 2(2\mu-1)i\mathbf{1}$$

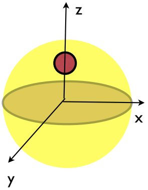


Local spin model and the Gaussian limit

- $\left\{\rho_{\mathbf{u}/\sqrt{n}} \;:\; \mathbf{u} = (u_x, u_y, u_z)\right\}$ neighbourhood of $\rho_0 := \mathrm{Diag}(\mu, 1-\mu)$

$$\rho_{\mathbf{u}/\sqrt{n}} := U_n\left(u_x, u_y\right) \begin{bmatrix} \mu + \frac{u_z}{\sqrt{n}} & 0 \\ 0 & 1 - \mu - \frac{u_z}{\sqrt{n}} \end{bmatrix} U_n\left(u_x, u_y\right)^*$$

$$U_n(u_x, u_y) := \exp(i(u_y\sigma_x - u_y\sigma_y)/\sqrt{n})$$



- Gaussian shift model: $N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}}$
  - Classical part: $N_{\mathbf{u}} := N(u_z, \mu(1-\mu))$
  - Quantum part: $\Phi_{\mathbf{u}} := \Phi\left(u_x\sqrt{2(2\mu-1)}\,,\, u_y\sqrt{2(2\mu-1)}\,;\, (2(2\mu-1))^{-1}\right)$

Local asymptotic normality for mixed spin states [5]

Theorem

Let $\rho_{\mathbf{u},n} := \left(\rho_{\mathbf{u}/\sqrt{n}}\right)^{\otimes n}$ be the state of n i.i.d. spins with $1/2 < \mu < 1$.

Then there exist quantum channels T_n, S_n such that for any $\eta < 1/4$

$$\lim_{n\to\infty} \sup_{\|\mathbf{u}\|<n^\eta} \|T_n(\rho_{\mathbf{u},n}) - N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}}\|_1 = 0,$$

and

$$\lim_{n\to\infty} \sup_{\|\mathbf{u}\|<n^\eta} \|\rho_{\mathbf{u},n} - S_n(N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}})\|_1 = 0.$$

- LAN + Optimal estimation of Gaussian shift $\Rightarrow$ asymptotically optimal state estimation

[5] M.G., B. Janssens and J. Kahn, *Commun. Math. Phys.* (2008)

Example: optimal qubit estimation with norm-one squared loss function

- Quadratic approximation for norm-one squared distance

$$\left\|\rho_{\hat{\mathbf{u}}/\sqrt{n}} - \rho_{\mathbf{u}/\sqrt{n}}\right\|_1^2 = \frac{4}{n}\left[(\hat{u}_z - u_z)^2 + (2\mu-1)^2((\hat{u}_x - u_x)^2 + (\hat{u}_y - u_y)^2)\right] + O(n^{-3/2})$$

- Gaussian limit model:

$$N(u_z, \mu(1-\mu)) \otimes \Phi\left(u_x\sqrt{2(2\mu-1)},\, u_y\sqrt{2(2\mu-1)};\ \frac{1}{2(2\mu-1)}\mathbf{1}\right)$$

- Probability distribution of heterodyne measurement on quantum part

$$N\left(u_x\sqrt{2(2\mu-1)},\, u_y\sqrt{2(2\mu-1)};\ \frac{1}{2(2\mu-1)}\mathbf{1} + \frac{1}{2}\mathbf{1}\right) \to N\left(u_x\,,\, u_y\,;\ \frac{\mu}{2(2\mu-1)^2}\mathbf{1}\right)$$

- Optimal risk

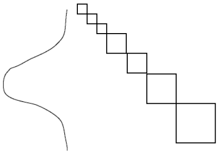
$$n\mathbb{E}\left\|\rho_{\hat{\mathbf{u}}/\sqrt{n}} - \rho_{\mathbf{u}/\sqrt{n}}\right\|_1^2 = 4\left(\frac{\mu}{2} + \frac{\mu}{2} + \mu(1-\mu)\right) = 8\mu - 4\mu^2$$

Idea of the proof

- Block diagonal form (Weyl Theorem)

$$\left(\mathbb{C}^2\right)^{\otimes n} = \bigoplus_{j=0,1/2}^{n/2} \mathbb{C}^{2j+1} \otimes \mathbb{C}^{d_j}$$

$$\rho_{\mathbf{u}/\sqrt{n}}^{\otimes n} = \bigoplus_{j=0,1/2}^{n/2} p_{\mathbf{u},n}(j)\,\rho_{\mathbf{u},n}(j) \otimes \frac{\mathbf{1}}{d_j}$$



- Classical part: $p_{\mathbf{u},n}(j) = \mathbb{P}[L = j]$ with L the total spin

$$L \approx L_z \sim \mathrm{Bin}(\mu + u_z/\sqrt{n}, n) \xrightarrow{s.} N_{\mathbf{u}}$$

- Quantum part: embed conditional state $\rho_{\mathbf{u},j}$ isometrically into $L^2(\mathbb{R})$

$$\begin{aligned} V_j &: \mathcal{H}_j \to L^2(\mathbb{R}) \\ T_j &: \rho_{\mathbf{u},j} \longmapsto V_j \rho_{\mathbf{u},j} V_j^* \end{aligned}$$

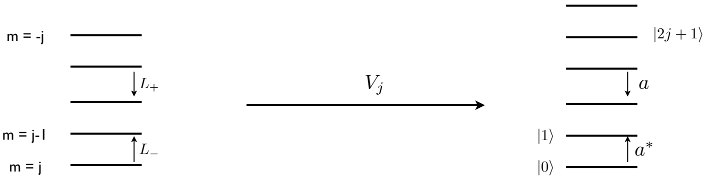
Isometric embedding

- Orthonormal bases

$$\begin{aligned} L_z|m,j\rangle &= m|m,j\rangle && (\ \mathbb{C}^{2j+1}\ ) \\ |k\rangle &= H_k(x)e^{-x^2/2} && (\ L^2(\mathbb{R})\ ) \end{aligned}$$

- Ladder operators

$$\begin{cases} L_+ & := L_x + iL_y \\ L_- & := L_x - iL_y \end{cases} \quad \text{and} \quad \begin{cases} a & := (Q+iP)/\sqrt{2} \\ a^* & := (Q-iP)/\sqrt{2} \end{cases}$$



Local asymptotic normality in d-dimensions

- Local model around $\rho_0 = \mathrm{Diag}(\mu_1, \dots, \mu_d)$ with $\mu_1 > \mu_2 > \cdots > \mu_d > 0$

$$\rho_{\mathbf{u}/\sqrt{n}} = \begin{bmatrix} \mu_1 + h_1/\sqrt{n} & \dots & z^*_{1,d}/\sqrt{n} \\ \vdots & \ddots & \vdots \\ z_{1,d}/\sqrt{n} & \dots & \mu_d - \sum_{i=1}^{d-1} h_i/\sqrt{n} \end{bmatrix} \qquad \mathbf{u} = (\mathbf{h}, \mathbf{z}) \in \mathbb{R}^{d-1} \times \mathbb{C}^{d(d-1)/2}$$

- Gaussian shift model: $N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}}$
  - Classical part: $N_{\mathbf{u}} := N(\mathbf{z}, I_{\mu}^{-1})$
  - Quantum part: $\Phi_{\mathbf{u}} := \bigotimes_{1 \leq j < k \leq d} \Phi\left(\frac{z_{j,k}}{2\sqrt{\mu_j - \mu_k}} \, ; \, \frac{\mu_j + \mu_k}{2(\mu_j - \mu_k)} \right)$

Local asymptotic normality in d-dimensions[14]

Theorem

Let $\rho_{\mathbf{u},n} := \left(\rho_{\mathbf{u}/\sqrt{n}}\right)^{\otimes n}$ be the state of n i.i.d systems with $\mu_1 > \cdots > \mu_d > 0$.

Then there exist quantum channels T_n, S_n such that

$$\begin{aligned} \lim_{n\to\infty} \sup_{\mathbf{u}\in\Theta_{n,\beta,\gamma}} \left\|T_n(\rho_{\mathbf{u},n}) - N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}}\right\|_1 &= 0 \\ \lim_{n\to\infty} \sup_{\mathbf{u}\in\Theta_{n,\beta,\gamma}} \left\|S_n(N_{\mathbf{u}} \otimes \Phi_{\mathbf{u}}) - \rho_{\mathbf{u},n}\right\|_1 &= 0 \end{aligned}$$

where

$$\Theta_{n,\beta,\gamma} = \left\{\mathbf{u} := (\mathbf{z}, \mathbf{d}) : \|\mathbf{z}\| \le n^{\beta}, \|\mathbf{d}\| \le n^{\gamma}\right\}, \text{ with } \beta < 1/9, \gamma < 1/4.$$

[14] M. G., J. Kahn, *Commun. Math. Phys. (2008)*

Blocks indexed by Young diagrams

- Block diagonal form

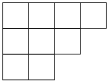
$$\left(\mathbb{C}^d\right)^{\otimes n} = \bigoplus_\lambda \mathcal{H}_\lambda \otimes \mathcal{K}_\lambda$$

$$\rho_{\mathbf{u}/\sqrt{n}}^{\otimes n} = \bigoplus_\lambda p_{\mathbf{u},n}(\lambda)\, \rho_{\mathbf{u},n}(\lambda) \otimes \mathrm{tr}_\lambda$$

- Young diagrams λ with d lines and n boxes

$\lambda_1 \approx n\mu_1$

$\lambda_d \approx n\mu_d$



- Classical part: $p_{\mathbf{u},n} \approx \mathrm{Mult}\left(\mu_1 + \frac{h_1}{\sqrt{n}}, \ldots, \mu_d - \sum_i \frac{h_i}{\sqrt{n}}; n\right) \Longrightarrow N_{\mathbf{u}}$

Bases and ladder operators in $\mathcal{H}_\lambda$

- Non-orthogonal basis $|t, \lambda\rangle = |\mathbf{m}, \lambda\rangle$

  $\mathbf{m} = (m_{i,j} = \sharp\text{j's in row i}\} : i < j)$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

semi-standard Young tableau t

- Typical vectors are $\approx$ orthogonal

  If $|\mathbf{m}|, |\mathbf{l}| = O(n^\eta)$ with $\eta < 2/9$ then

$$|\langle \mathbf{m}, \lambda \,|\, \mathbf{l}, \lambda\rangle| = O(n^{-c(\eta)})$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&2&2&3 \\ \hline 2&2&2&2&3&3 \\ \cline{1-6} 3&3&3 \\ \cline{1-3}\end{array}$$

typical Young tableau t

- Approximate ladder operators

$$L^*_{2,3} : \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&2&2&3 \\ \hline 2&2&2&2&3&3 \\ \cline{1-6} 3&3&3 \\ \cline{1-3}\end{array} \longrightarrow O(n^\eta) \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&\boxed{2}&3&3 \\ \hline 2&2&2&2&3&3 \\ \cline{1-6} 3&3&3 \\ \cline{1-3}\end{array} + O(n) \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1&1&1&1&1&1&1&2&2&3 \\ \hline 2&2&2&\boxed{3}&3&3 \\ \cline{1-6} 3&3&3 \\ \cline{1-3}\end{array}$$

- Approximate isometry

$$V_\lambda : |\mathbf{m}\rangle \longmapsto \bigotimes_{1 \le j < k \le d} |m_{j,k}\rangle$$

LAE for pure states on an infinite dimensional space [6]

- Sobolev class of 'nice' states $|\psi\rangle = \sum_j \psi_j |j\rangle \in \ell^2(\mathbb{N})$

$$S^{\alpha}(L) := \left\{ |\psi\rangle\langle\psi| : \sum_{j=0}^{\infty} |\psi_j|^2 j^{2\alpha} = \langle N^{2\alpha}\rangle \leq L, \text{ and } \|\psi\| = 1 \right\}, \qquad \alpha > 0, \quad L > 0.$$

- Unique local decomposition around fixed state $|\psi_0\rangle$

$$|\psi\rangle = |\psi_u\rangle := \sqrt{1 - \|u\|^2}|\psi_0\rangle + |u\rangle, \qquad |u\rangle \in \mathcal{H}_0$$

- Gaussian model: coherent states $|G(\sqrt{n}u)\rangle$ in the Fock space $\mathcal{F}(\mathcal{H}_0)$

- Local asymptotic equivalence

$$\{|\psi_u\rangle^{\otimes n} : \|u\| \leq \gamma_n\} \quad \approx \quad \{|\sqrt{n}u\rangle : \|u\| \leq \gamma_n\}$$

- Application: estimation rate for minimax optimal estimator for $|\psi\rangle \in S^{\alpha}(L)$

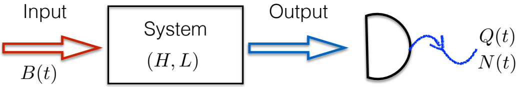
$$\sup_{|\psi\rangle \in S^{\alpha}(L)} \mathbb{E}_{\rho}\left[\|\hat{\rho}_n - \rho\|_1^2\right] \approx n^{-2\alpha/(2\alpha+1)}$$

[6] C. Butucea, M.G. , M. Nussbaum, *Ann. Statist.* (2018)

Outline

- Quantum Fisher information and quantum Cramér-Rao bound
- Local Asymptotic Normality for quantum IID ensembles
- Local Asymptotic Normality for quantum Markov processes

System identification and estimation with input-output open systems



- Unitary dynamics: singular coupling with incoming input fields (Q Stoch Diff Eq[7])

$$dU(t) = \left(-iHdt + LdA^*(t) - L^*dA(t) - \frac{1}{2}L^*Ldt \right) U(t)$$

- System identification: if $\theta \to (H_\theta, L_\theta)$, estimate θ by measuring the output[8]
  - which parameters can be identified ?
  - how does the output QFI scale with time t ?
  - how does this relate to dynamical properties, e.g. ergodicity, spectral gap...?
  - which measurements are informative ?
  - how to achieve high estimation accuracy ?

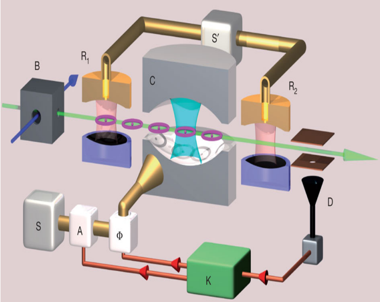
---

[7] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Springer Birkhäuser (1992)

[8] H. Mabuchi *Quant. Semiclass. Optics* (1996); J. Gambetta and H. M. Wiseman *Phys. Rev. A* (2001); S. Gammelmark and K. Molmer *Phys. Rev. A* (2013), S.Bonnabel, M.Mirrahimi, P.Rouchon, *Automatica* (2009)...

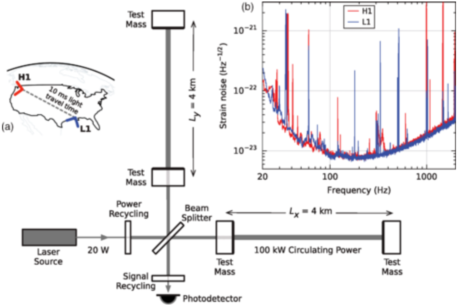
Quantum input-output systems[9]

- Input-output formalism describes controlled open system dynamics
- Quantum filtering, feedback control, quantum networks
- Control and system identification: two sides of the coin



Feedback control of cavity state in the atom maser

C. Sayrin *et al*, *Nature* (2011)

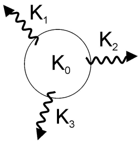


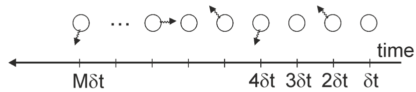
Advanced LIGO

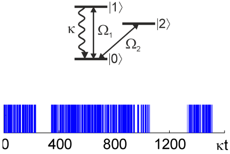
B. P. Abbott *et al. Phys. Rev. Lett.* (2016)

[9] C. W. Gardiner and P. Zoller, *Quantum Noise* (2004)
H. M. Wiseman and G. J. Milburn, *Quantum measurements and control* (2010)

Output state as superposition of quantum trajectories







- Monitoring the environment produces jump trajectories with infinitesimal Kraus operators
  - "no emission": $K^0_\theta = e^{-i\delta t H_\theta}\sqrt{1-\delta t \sum_j L^{j*}_\theta L^j_\theta}$
  - "emission" in channel j: $K^j_\theta = e^{-i\delta t H_\theta}\sqrt{\delta t} L^j_\theta$

- System-output state: coherent superposition of quantum trajectories, (continuous) MPS[10]

$$|\psi^{s+o}_\theta(t)\rangle = U_\theta(t)|\psi^{s+o}_{in}\rangle = \sum_{j_1,\ldots,j_n} K^{j_n}_\theta \ldots K^{j_1}_\theta |\psi\rangle \otimes |j_n \ldots j_1\rangle, \qquad n = t/\delta t$$

[10] M. Fannes, B. Nachtergale and R. Werner, *Commun. Math. Phys.*(1992);
D. Perez-Garcia, F. Verstraete, M. Wolf and I. Cirac, *Quantum Inf. Comput.* (2007)

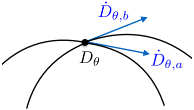
Generator of parameter change in system+output state

- Model dynamics with unknown parameter $\theta \in \mathbb{R}^m$

$$D_\theta = (H_\theta, L_\theta) \quad \longrightarrow \quad \left|\Psi_\theta^{\mathrm{s+o}}(t)\right\rangle = U_\theta(t)|\varphi \otimes \Omega\rangle$$

- Tangent vector at D_θ corresponding to changes in component θ_a

$$\dot{D}_{\theta,a} = (\dot{H}_{\theta,a}, \dot{L}_{\theta,a}) = \left(\frac{\partial H}{\partial \theta_a}, \frac{\partial L}{\partial \theta_a}\right)$$



- Generator of parameter change for component θ_a

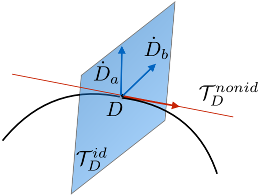
$$\frac{\partial}{\partial \theta_a}\left|\Psi_\theta^{\mathrm{s+o}}(t)\right\rangle = \dot{U}_{\theta,a}(t)|\varphi \otimes \Omega\rangle = U_\theta(t) G_{\theta,a}(t)|\varphi \otimes \Omega\rangle$$

- Generator is a quantum stochastic integral (fluctuation operator)

$$G_{\theta,a}(t) \quad := \quad \sqrt{t}\mathbb{F}_t(\dot{D}_{\theta,a}) = \int_0^t \dot{L}_{\theta,a}(s) dA^*(s) - i\mathcal{E}_D(\dot{D}_{\theta,a})(s) ds$$

$$\mathcal{E}_D(\dot{D}) \quad := \quad \dot{H} + \mathrm{Im}(\dot{L}^* L) - \mathrm{Tr}\left[\rho_{ss}^D (\dot{H} + \mathrm{Im}(\dot{L}^* L))\right] \mathbf{1}$$

Quantum information geometry of stationary output state[11]



Theorem (QFI of ergodic systems as Riemanian metric)

The quantum Fisher information matrix $F_{a,b}(t) = 4\mathrm{Re}\left\langle G^*_{\theta,a}(t)\cdot G_{\theta,b}(t)\right\rangle$ *grows linearly in* t *with rate* $F_{a,b}$ *given by the asymptotic Markov covariance of fluctuators*

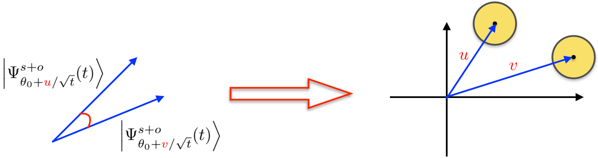
$$\begin{aligned} F_{a,b} &= 4\mathrm{Re}\left(\dot{D}_{\theta,a}, \dot{D}_{\theta,b}\right)_D \\ &:= 4\mathrm{Re}\,\mathrm{Tr}\left[\rho_{ss}\left(\dot{L}_{\theta,a} - i[L_\theta, \mathcal{L}^{-1}\circ\mathcal{E}_{\mathsf{D}}(\dot{\mathsf{D}}_{\theta,a})]\right)^*\cdot\left(\dot{L}_{\theta,b} - i[L_\theta, \mathcal{L}^{-1}\circ\mathcal{E}_{\mathsf{D}}(\dot{\mathsf{D}}_{\theta,b})]\right)\right]. \end{aligned}$$

The tangent space decomposes into identifiable and unidentifiable subspaces $\mathcal{T}_D = \mathcal{T}_D^{id}\oplus\mathcal{T}_D^{nonid}$

- $\mathcal{T}_D^{nonid} := \{\dot{D} : \dot{D} = i[K, D] + c(\mathbf{1}, 0)\} \quad\longrightarrow\quad (\dot{D}, \dot{D}')_D = 0$
- $\mathcal{T}_D^{id} = \{\dot{D} : \mathcal{E}_D(\dot{D}) = 0\} \quad\longrightarrow\quad (\dot{D}, \dot{D}')_D = \mathrm{Tr}(\rho^D_{ss}\dot{L}^*\dot{L}')$
- $F_{a,b}$ *defines a Riemannian metric on* $\mathcal{P} = \mathcal{D}/G$

[11] M.G., J. Kiukas, J. Math. Phys. (2017)

Gaussian approximation (LAN) for (system +) output state[12]



- Parameter uncertainty $\approx t^{-1/2} \Rightarrow$ interesting statistical features are local: $\theta = \theta_0 + u/\sqrt{t}$

$$D_{\theta_0+u/\sqrt{t}} = D_{\theta_0} + \frac{1}{\sqrt{t}}\dot{D}_u + O(t^{-1}) = D_{\theta_0} + \frac{1}{\sqrt{t}}\sum_a u_a \dot{D}_{\theta_0,a} + O(t^{-1})$$

Theorem (Local asymptotic normality)

Let $\mathcal{W}_D$ be the CCR algebra over $\mathcal{T}_D^{id}$ (continuous variable system) with Weyl unitaries $W(u)$ and "vacuum" state $|0\rangle$ satisfying

$$W(u)W(v) = e^{-i\mathrm{Im}(\dot{D}_u,\dot{D}_v)_D}\, W(u+v), \qquad \langle 0|W(u)|0\rangle := e^{-\frac{1}{2}\|\dot{D}_u\|_D^2}$$

System+output quantum model $|\Psi^{s+o}_{\theta_0+u/\sqrt{t}}(t)\rangle$ converges locally to coherent states (Gaussian) model $|u\rangle := W(u)|0\rangle$.

$$\lim_{t\to\infty}\left\langle \Psi^{s+o}_{\theta_0+u/\sqrt{t}}(t) \,\middle|\, \Psi^{s+o}_{\theta_0+v/\sqrt{t}}(t)\right\rangle = e^{-\frac{1}{2}\|\dot{D}_u-\dot{D}_v\|_D^2} = \langle u|v\rangle$$

[12] M.G., J. Kiukas, J. Math. Phys. (2017), Similar result for the reduced output state

The Holevo bound is achievable

- Holevo bound: quantum statistical model $\{\rho^\theta : \theta \in \Theta \subset \mathbb{R}^k\}$
  - $X_\theta := (X_{\theta,1}, \ldots, X_{\theta,k})$ s.t. $\mathrm{Tr}(\rho^\theta X_{\theta,i}) = 0, \quad \mathrm{Tr}(\frac{\partial \rho^\theta}{\partial \theta_i} X_{\theta,j}) = \delta_{i,j}$
  - $Z(X_\theta)_{i,j} := \mathrm{Tr}(\rho^\theta X_{\theta,j} X_{\theta,i})$

For any unbiased measurement with outcome $\hat{\theta} \in \mathbb{R}^k$

$$\mathbb{E}(\|\hat{\theta} - \theta\|^2) \geq C(\theta) := \inf_{X_\theta} \mathrm{Tr}\left(\mathrm{Re}(Z(X_\theta)) + |\mathrm{Im}(Z(X_\theta))|\right)$$