# Teleportation into quantum information

Or: elements of quantum information

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**Lecture hour 1** my short course
on Quantum Information and Statistical Science.
Lecture/hour 0 was the "warm up" on the Delft Bell experiment
Now we do a crash course on the Hilbert space stuff…
In **Lecture hour 2** we'll do examples.

Don't worry, we stick to finite dimensions and finite number of outcomes.
The Hilbert space is $\mathbb{C}^d$. Often, $d = 2$.
Or $d = 2^N$ ($N$ qubits)

# Baby quantum information
(more precisely: Kindergarten)

- Pure states, state vectors
- Projective (simple/projector-valued) measurements
- Entanglement
- Unitary evolution

# Toy quantum information

(more precisely: primary school)

- Mixed states and density matrices
- POVM’s (generalised measurements)
- Quantum instruments
- Kraus representation and the Kraus theorems

[After kindergarten and primary school, comes high school
I will call it: “QI for young adults, or if you prefer “grown-ups”.
After that there are higher levels still…]

# Special case: the qubit

- Two-dimensional Hilbert space, and tensor products of many copies! All (nearly all) of quantum computing, quantum cryptography, quantum information …

# Pure state

- A *d*-dimensional quantum state is **represented by** a unit length complex vector, thought of as a column vector (i.e., a $d \times 1$ matrix)
- We may write $\psi$, or $|\psi>$
- Denote complex conjugate and transpose with a star (physicists use a dagger)
- We may write $\psi^*$ or $<\psi|$
- $<\psi|\psi> = 1$
- $|\psi><\psi|$ is a $d \times d$ matrix, and it's the orthogonal projector onto the one-dimensional space spanned by $|\psi>$
- As we'll see, part of the representation of the state is redundant. It's enough to know $|\psi><\psi|$

# Observables

- Suppose $A$ is a self-adjoint matrix, i.e., $A^* = A$
- $A$ has real eigenvalues and one can find an orthonormal basis of eigenvectors.
- One can write $A = \Sigma_i\, a_i\, |\phi_i\rangle\langle\phi_i|$ where the $a_i$ are the eigenvalues, real, (the labelling is not unique), and the $\phi_i$ are the eigenvectors (may not be unique)
- I like to write $A = \Sigma_a\, a\, \mathrm{Proj}[A = a]$ where the $a$ are the distinct eigenvalues, $[A = a]$ is the eigenspace belonging to eigenvalue $a$, and $\mathrm{Proj}[A = a]$ projects onto that eigenspace.

# Measurement: **Born**'s law

- When the system is in state $\psi$ and we measure the observable *A*, we observe one of the eigenvalues *a*. The state "collapses" to the projection of $\psi$ onto the eigenspace corresponding to that eigenvalue, Proj[A = a] $\psi$ (divided by its length). The probability of seeing value *a* is the squared length: $||\text{Proj}[A = a]\ \psi||^2$

- By Pythagoras, $\text{sum}_a\ ||\text{Proj}[A = a]\ \psi||^2 = 1$

- This generalisation of the Born law is called the von Neumann-Lüders projection postulate

- One can call the measurement itself a "simple measurement" or a "projector-valued measurement"

# Unitary evolution

- Undisturbed, the state evolves according to Schrödinger's equation

- i $\hbar$ d/d$t$ $\psi$ = $H$ $\psi$ for some "Hamiltonian" $H$

  Take units s.t. "reduced Planck's constant" $\hbar = h/2\pi = 1$

- The Hamiltonian is a self-adjoint operator

- The solution of Schrödinger's equation is $\psi$(t) = exp(- i $H$ $t$) $\psi$(0)

- $U$ = exp(- i $H$ $t$) is a unitary operator, i.e. $U U^* = U^* U = \mathrm{Id}$

# Interaction between several systems

- If two systems of dimension *d* and *d* are interacting then they form a joint system of dimension $d \times d'$

- The Hilbert space of the joint system is the tensor product of the Hilbert spaces of the component systems

- This means that if $\phi_i$ and $\psi_i$ are state vectors of the two subsystems, and $c_i$ are complex numbers, not all zero, then $\Sigma_i \, c_i \, \phi_i \otimes \psi_i$, normalised to have length one, is a (possible) state vector of the joint system

# Entanglement

- Initially completely separate component systems can evolve into entangled systems of the joint state through time evolution with a Hamiltonian (or equivalently, a unitary) which is not itself of tensor product form.

# Randomisation

- We already saw that quantum measurement generates randomness
- We can think of *classical* randomness as a pure ingredient of quantum mechanics in itself – toss a coin, toss dice, shuffle cards… and then do something dependent on the outcome of the chance experiment

# Building blocks for a general picture

- *Measurement* according to projection postulate, bringing the system of interest into *interaction* with an "*ancillary*" (auxiliary) system in fixed state, *unitary evolution*, and classical *randomisation*, generate a vast range of ways in which a quantum system can be transformed while in the process generating classical information ("measurement results") which aren't necessarily "observed" at all.

# Mixed states, density matrices

- A **density matrix** is a non-negative self-adjoint matrix $\rho$ of trace 1
- Such a matrix can be written (not necessarily uniquely) as $\rho = \Sigma_i\, p_i\, |\psi_i\rangle\langle\psi_i|$
- The $p_i$ are nonnegative and add to 1

- Suppose we **prepare** a quantum system by creating it in pure state $|\psi_i\rangle$ with **probability** $p_i$
- We call this a **"system in a mixed state"**

# Theorem: density matrix $\rho = \Sigma_i \, p_i \, |\psi_i\rangle\langle\psi_i|$ is "the state" of the mixture

- The "state" of a physical system is the catalogue of all joint probability distributions of measurement results, given all collections of "generalised" measurements which can be performed on it

- In our case, a generalised measurement is the operation of combining any number of times: entanglement with ancillas, unitary evolution, randomization, projective measurements …

# Partial trace, subsystems

- Theorem: the state of a component of a larger system in a general entangled, mixed, state, is the partial trace of the density matrix of the joint system

# Generalised measurements

- A generalised measurement is determined by a collection of self-adjoint non-negative matrices $M_i$ which add to the identity; and an associated distinct outcome value $x_i$ for each component

- The probability of getting outcome $x_i$ is trace($\rho\ M_i$)

# Kraus matrices; instrument

- Suppose we are given matrices $A_{ij}$ and distinct outcome values $x_i$, satisfying $\Sigma_{ij}\ A_{ij}^*\ A_{ij}$ = Id.

- Consider a *transformation with observation* of a quantum system initially in state $\rho$: the system is transformed into the state $\Sigma_j\ A_{ij}\ \rho\ A_{ij}^*$ / trace($\Sigma_j\ A_{ij}\ \rho\ A_{ij}^*$) (depending on $i$) and one observes outcome $x_i$, with probability $\Sigma_\mathrm{j}$ trace($A_{ij}\ \rho\ A_{ij}^*$)

- The associated <u>instrument</u> is the mapping from density matrices $\rho$ to the combined quantum-classical state ($\Sigma_\mathrm{j}\ A_{ij}\ \rho\ A_{ij}^*$ : i ∈ $\mathscr{I}$), with classical outcome space ($x_i$ : $i \in \mathscr{I}$)

# Theorem: Kraus representation

- Every *totally* *positive*, *normalised, linear* transformation ($\rho \mapsto (\tau_i : i \in \mathscr{I})$) along with an outcome space ($x_i : i \in \mathscr{I}$) defines an instrument with a Kraus representation

- Every combination of entanglement with ancillary systems, unitary evolution, measurement by simple measurements on component sub-systems, classical randomisation using random measurement outcomes of earlier measurement … results in a *totally* *positive*, *normalised, linear* transformation of the density matrix

# Church of the larger Hilbert space

- Every instrument, every measurement, every state transformation can be represented by entanglement with a system in a fixed state in an ancillary Hilbert space, a unitary evolution of the joint system, measurement of the ancilla, and then discarding the ancilla.

Mathematically: this is called “dilation”, and the proof follows from the Naimark theorem

# Some mysteries

- If you create a mixed state and “lose” the information of how you did it, it can never be determined again, *how* you created the state.

- For instance, the completely mixed state Identity/dimension, can be created by picking *any* orthonormal basis and then picking one of the elements of the basis as state vector completely at random. Yet there is no way to detect, how it was created