Perfect Passion at a Distance

how to win at Polish poker : use quantum dice

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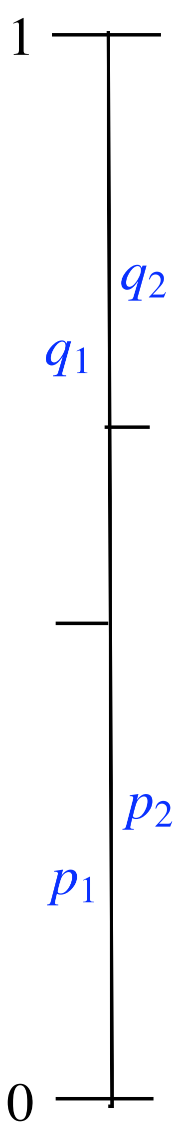
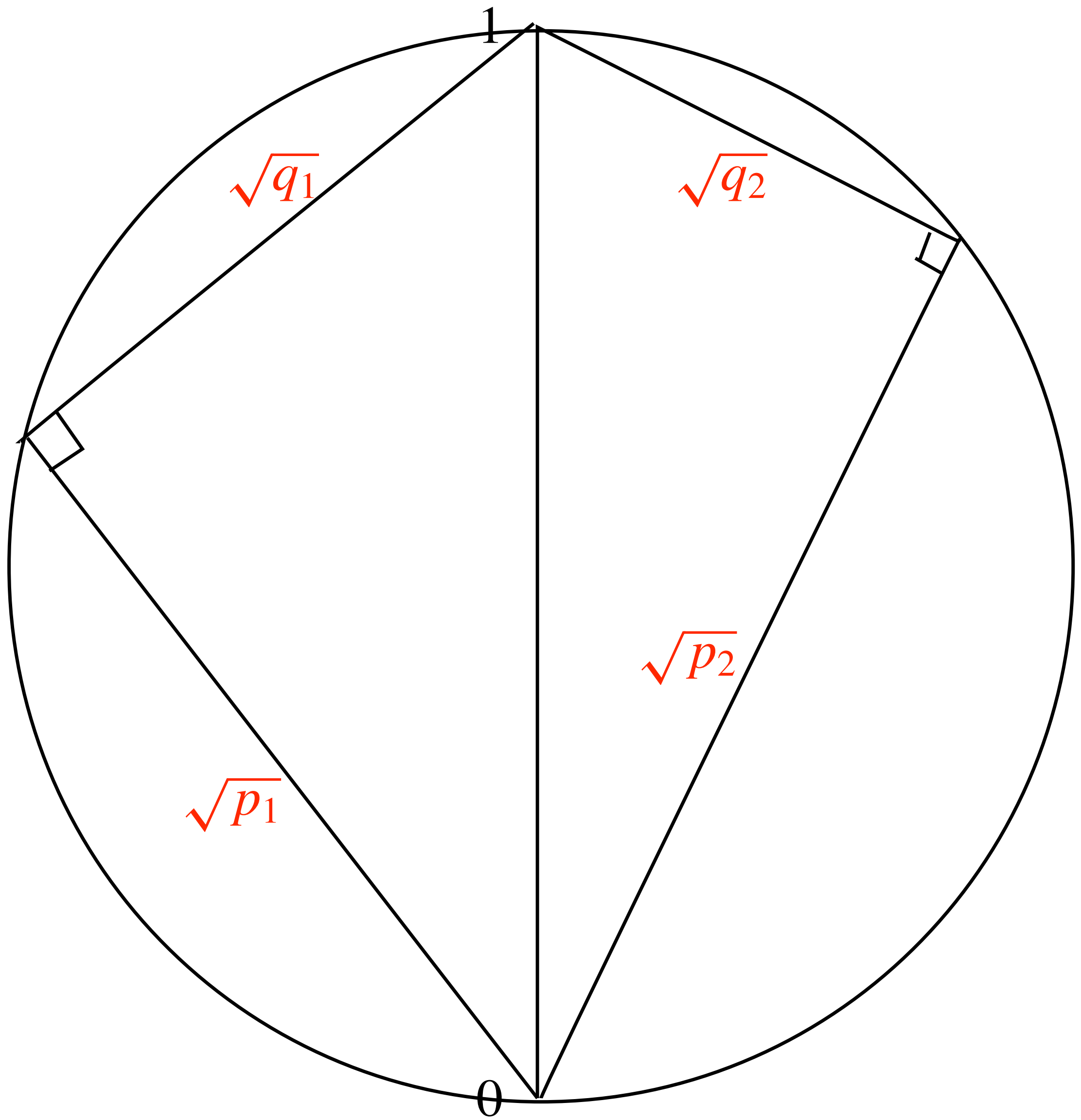
What it's all about :

Nature's two ways





$p_1 + q_1 = 1 = p_2 + q_2$

$p_1 + q_1 = 1 = p_2 + q_2$

Why it all works :

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Polish poker

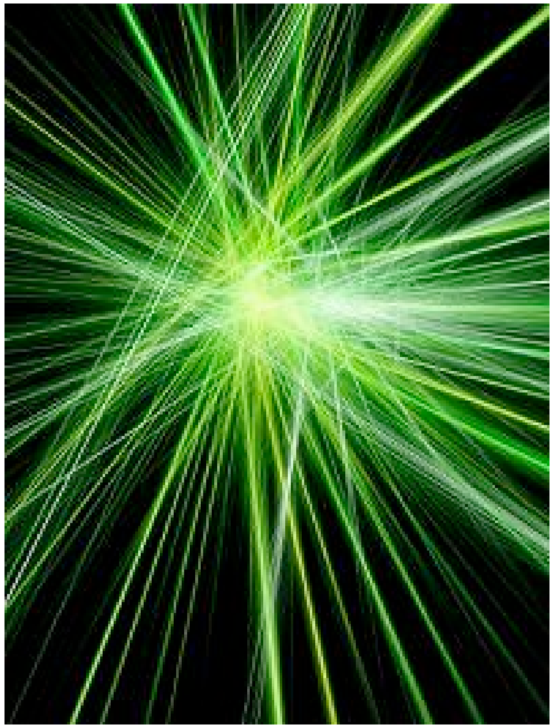
- I, Polish visitor to Moscow, meet you, Russian guy, in bar ; we get drinking
- I suggest you put ₽1 on the table & think of number between 0 , 20
- I put down ₽10 & ask your number
- You say (e.g.) "17"
- I say (e.g.) "3" and take all the money

thanks to Marek Żukowski, Gdansk

Distance no worries for spooky particles

(2x2x2)

Stephen Pincock
ABC Science Online
Friday, 8 December 2006



A message sent using entangled, or spooky, particles of light has been beamed across the ocean *(Image: iStockphoto)*

Scientists have used quantum physics to zap an encrypted message more than 140 kilometres between two Spanish islands. Professor Anton Zeilinger from the University of Vienna and an international team of scientists used 'spooky' pulses of light to send the message. They say this is an important step towards making international communications more secure. Zeilinger described the study this week at the Australian Institute of Physics meeting in Brisbane. The photons they sent were linked together through a process known as quantum entanglement. This means that their properties remained tightly entwined or entangled, even when separated by large distances, a property Einstein called spooky. The group's achievement is important for the emerging field of quantum cryptography, which aims to use properties such as entanglement to send encrypted messages. Research groups around the world are working in this field. But until now they have only been able to send messages relatively short distances, limiting their usefulness. Zeilinger's team wants to be able to beam the messages to satellites in space, so they could theoretically be relayed anywhere on the planet.

To test their system, the team went to Tenerife in the Canary Islands, where the European Space Agency operates a telescope specifically designed to communicate with satellites. Instead of pointing the telescope at the stars, Zeilinger says, the scientists turned it to the horizontal and aimed it towards a photon sending station 144 kilometres away on the neighbouring island of La Palma. "Very broadly speaking, we were able to establish a quantum communication connection," he says. "We worried a lot about whether atmospheric turbulence would destroy the quantum states. But it turned out to work much better than we feared." The results suggest it should be possible to send encrypted photons to a satellite orbiting 300 or 400 kilometres above the Earth, he says. "This is our hope. We believe that such a system is feasible." The next step is to try the system out with an actual satellite, a project which is likely to involve the European Space Agency and others. "This is about developing quantum communications on a grand scale," Zeilinger says. His team expects to publish its results soon.

Quantum Physics, abstract quant-ph/0607182

(2x2x2)

From: Rupert Ursin [view email]
Date (v1): Wed, 26 Jul 2006 14:29:14 GMT (201kb)
Date (revised v2): Thu, 27 Jul 2006 07:47:40 GMT (374kb)

Free-Space distribution of entanglement and single photons over 144 km

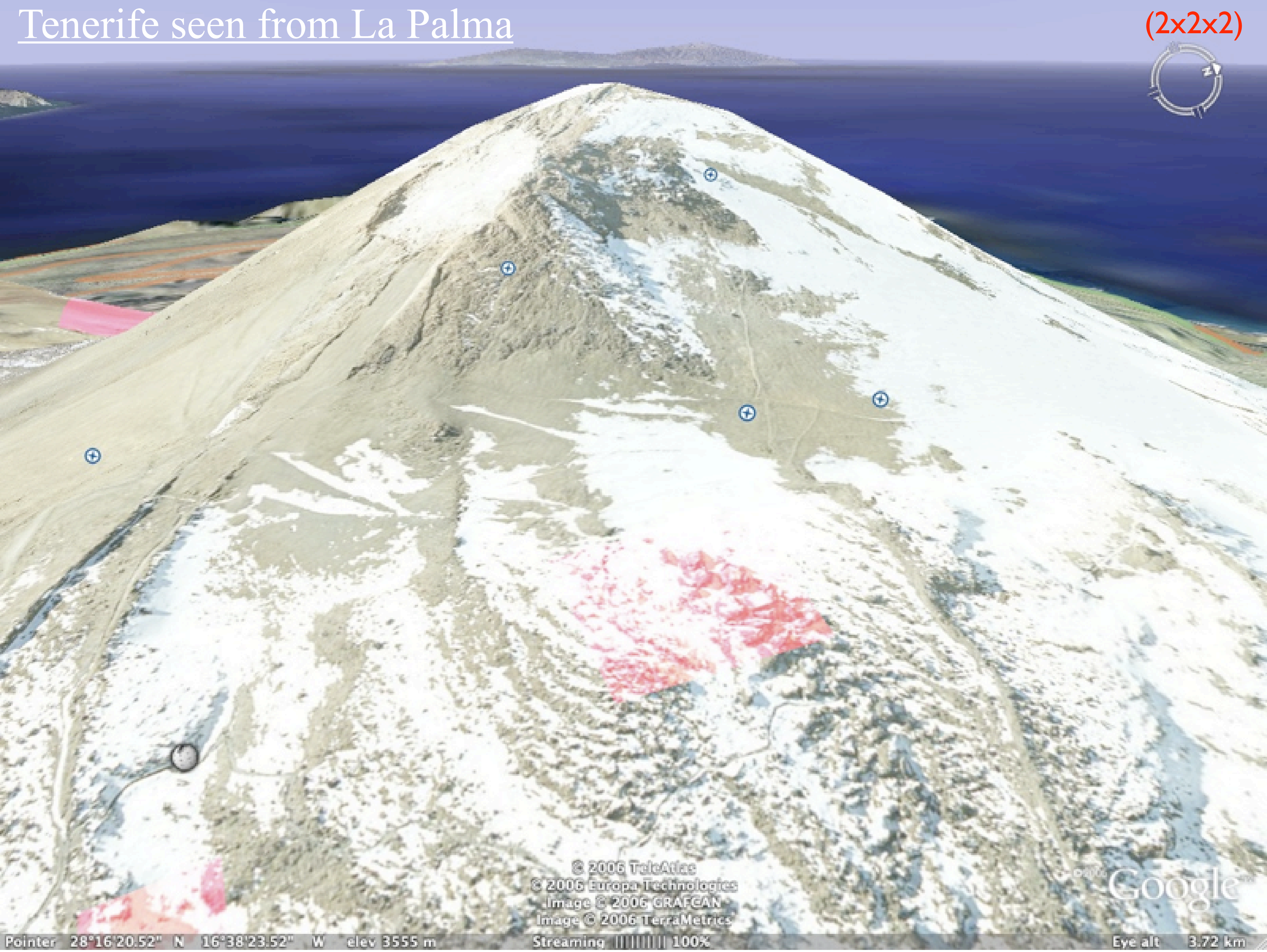
Authors: R. Ursin, F. Tiefenbacher, T. Schmitt-Manderbach, H. Weier, T. Scheidl, M. Lindenthal, B. Blauensteiner, T. Jennewein, J. Perdigues, P. Trojek, B. Oemer, M. Fuerst, M. Meyenburg, J. Rarity, Z. Sodnik, C. Barbieri, H. Weinfurter, A. Zeilinger

Comments: 10 pages including 2 figures and 1 table, Corrected typos.

Quantum Entanglement is the essence of quantum physics and inspires fundamental questions about the principles of nature. Moreover it is also the basis for emerging technologies of quantum information processing such as quantum cryptography, quantum teleportation and quantum computation. Bell's discovery, that correlations measured on entangled quantum systems are at variance with a local realistic picture led to a flurry of experiments confirming the quantum predictions. However, it is still experimentally undecided whether quantum entanglement can survive global distances, as predicted by quantum theory. Here we report the violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality measured by two observers separated by 144 km between the Canary Islands of La Palma and Tenerife via an optical free-space link using the Optical Ground Station (OGS) of the European Space Agency (ESA). Furthermore we used the entangled pairs to generate a quantum cryptographic key under experimental conditions and constraints characteristic for a Space-to-ground experiment. The distance in our experiment exceeds all previous free-space experiments by more than one order of magnitude and exploits the limit for ground-based free-space communication; significantly longer distances can only be reached using air- or space-based platforms. The range achieved thereby demonstrates the feasibility of quantum communication in space, involving satellites or the International Space Station

Tenerife seen from La Palma

(2x2x2)



Pan, Bouwmeester, Daniell, Weinfurter, Zeilinger (2000), *Nature*

Experimental test of quantum nonlocality in three-photon Greenberger–Horne–Zeilinger entanglement

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Bell's theorem[1] states that certain statistical correlations predicted by quantum physics for measurements on two-particle systems cannot be understood within a realistic picture based on local properties of each individual particle—even if the two particles are separated by large distances. Einstein, Podolsky and Rosen first recognized[2] the fundamental significance of these quantum correlations (termed 'entanglement' by Schrödinger[3]) and the

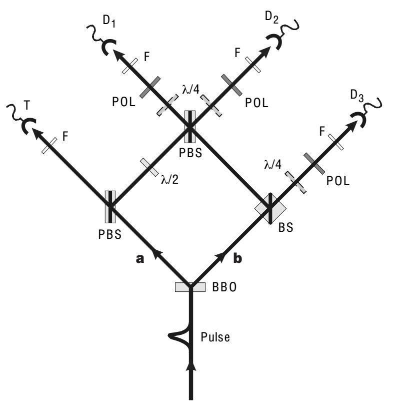


Figure 1 Experimental set-up for Greenberger–Horne–Zeilinger (GHZ) tests of quantum nonlocality. Pairs of polarization-entangled photons[28] (one photon *H* polarized and the other *V*) are generated by a short pulse of ultraviolet light (~ 200 fs, λ = 394 nm). Observation of the desired GHZ correlations requires fourfold coincidence and therefore two pairs[29]. The photon registered at T is always *H* and thus its partner in **b** must be *V*. The photon reflected at the polarizing beam-splitter (PBS) in arm **a** is always *V*, being turned into equal superposition of *V* and *H* by the λ/2 plate, and its partner in arm **b** must be *H*. Thus if all four detectors register at the same time, the two photons in D_1 and D_2 must either both have been *V V* and reflected by the last PBS or *HH* and transmitted. The photon at D_3 was therefore *H* or *V*, respectively. Both possibilities are made indistinguishable by having equal path lengths via **a** and **b** to D_1 (D_2) and by using narrow bandwidth filters (*F* ≈ 4 nm) to stretch the coherence time to about 500 fs, substantially larger than the pulse length[30]. This effectively erases the prior correlation information and, owing to indistinguishability, the three photons registered at D_1, D_2 and D_3 exhibit the desired GHZ correlations predicted by the state of equation (1), where for simplicity we assume the polarizations at D_3 to be defined at right angles relative to the others. Polarizers oriented at 45° and λ/4 plates in front of the detectors allow measurement of linear H'/V' (circular *R/L*) polarization.

(3x2x2)

ization. Then by the third term in equation (4), photon 3 will definitely be V' polarized.

By cyclic permutation, we can obtain analogous expressions for any experiment measuring circular polarization on two photons and H'/V' linear polarization on the remaining one. Thus, in every one of the three *yyx*, *yxy*, and *xyy* experiments, any individual measurement result—both for circular polarization and for linear H'/V' polarization—can be predicted with certainty for every photon given the corresponding measurement results of the other two.

Now we will analyse the implications for local realism. As these predictions are independent both of the spatial separation and of the relative time order of the three measurements, we consider them performed simultaneously in a given reference frame—say, for conceptual simplicity, in the reference frame of the source. Then, as Einstein locality implies that no information can travel faster than the speed of light, this requires any specific measurement result obtained for any photon never to depend on which specific measurements are performed simultaneously on the other two nor on their outcome. The only way then for local realism to

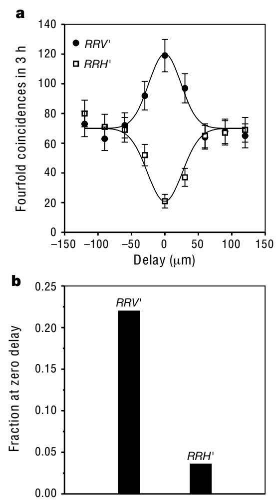


Figure 2 A typical experimental result used in the GHZ argument. This is the *yyx* experiment measuring circular polarization on photons 1 and 2 and linear polarization on the third. **a**, Fourfold coincidences between the trigger detector T, detectors D_1 and D_2 (both set to measure a right-handed polarized photon), and detector D_3 (set to measure a linearly polarized H' (lower curve) and V' (upper curve) photon as a function of the delay between photon 1 and 2 at the final polarizing beam-splitter). We could adjust the time delay between paths **a** and **b** in Fig. 1 by translating the final polarizing beam-splitter (PBS) and using additional mirrors (not shown in Fig. 1) to ensure overlap of both beams, independent of mirror displacement. At large delay, that is, outside the region of coherent superposition, the two possibilities *HHH* and *VVV* are distinguishable and no entanglement results. In agreement with this explanation, it was observed within the experimental accuracy that for large delay the eight possible outcomes in the *yyx* experiment (and also the other experiments) have the same coincidence rate, whose mean value was chosen as a normalization standard. **b**, At zero delay maximum GHZ entanglement results; the experimentally determined fractions of *RRV'* and *RRH'* triples (out of the eight possible outcomes in the *yyx* experiment) are deduced from the measurements at zero delay. The fractions were obtained by dividing the normalized fourfold coincidences of a specific outcome by the sum of all possible outcomes in each experiment—here, the *yyx* experiment.

explain the perfect correlations predicted by equation (4) is to assume that each photon carries elements of reality for both *x* and *y* measurements that determine the specific individual measurement result[5,6,8].

For photon *i* we call these elements of reality X_i with values +1(−1) for $H'(V')$ polarizations and Y_i with values +1(−1) for *R*(*L*); we thus obtain the relations[8] $Y_1Y_2X_3 = -1$, $Y_1X_2Y_3 = -1$ and $X_1Y_2Y_3 = -1$, in order to be able to reproduce the quantum predictions of equation (4) and its permutations.

We now consider a fourth experiment measuring linear H'/V' polarization on all three photons, that is, an *xxx* experiment. We investigate the possible outcomes that will be predicted by local realism based on the elements of reality introduced to explain the earlier *yyx*, *yxy* and *xyy* experiments.

Because of Einstein locality any specific measurement for *x* must be independent of whether an *x* or *y* measurement is performed on the other photon. As $Y_iY_i = +1$, we can write $X_1X_2X_3 = (X_1Y_2Y_3)(Y_1X_2Y_3)(Y_1Y_2X_3)$ and obtain $X_1X_2X_3 = -1$. Thus from a local realist point of view the only possible results for an *xxx* experiment are $V'V'V'$, $H'H'V'$, $H'V'H'$, and $V'H'H'$.

How do these predictions of local realism for an *xxx* experiment compare with those of quantum mechanics? If we express the state given in equation (1) in terms of H'/V' polarization using equation (2) we obtain:

$$|\Psi\rangle = \frac{1}{2}(|H'\rangle_1|H'\rangle_2|H'\rangle_3 + |H'\rangle_1|V'\rangle_2|V'\rangle_3 + |V'\rangle_1|H'\rangle_2|V'\rangle_3 + |V'\rangle_1|V'\rangle_2|H'\rangle_3) \quad (5)$$

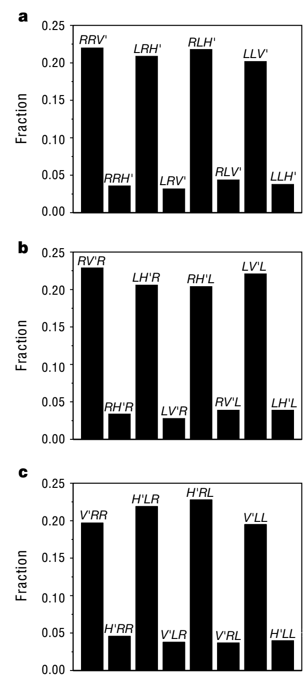


Figure 3 All outcomes observed in the *yyx*, *yxy* and *xyy* experiments, obtained as in Fig. 2. **a**, *yyx*; **b**, *yxy*; **c**, *xyy*. The experimental data show that we observe the GHZ terms predicted by quantum physics (tall bars) in a fraction of 0.85 ± 0.04 of all cases and 0.15 ± 0.02 of the spurious events (short bars) in a fraction of all cases. Within our experimental error we thus confirm the GHZ predictions for the experiments.

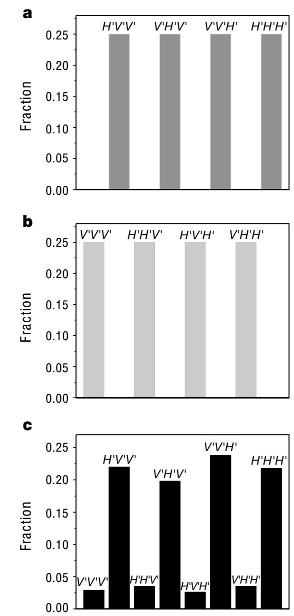


Figure 4 Predictions of quantum mechanics and of local realism, and observed results for the *xxx* experiment. **a, b**, The maximum possible conflict arises between the predictions for quantum mechanics (**a**) and local realism (**b**) because the predicted correlations are exactly opposite. **c**, The experimental results clearly confirm the quantum predictions within experimental error and are in conflict with local realism.

are shown in Fig. 2. The six remaining possible outcomes of a *yyx* experiment have also been measured in the same way and likewise in the *yxy* and *xyy* experiments. For all three experiments this results in 24 possible outcomes whose individual fractions thus obtained are shown in Fig. 3.

Adopting our first strategy, we assume that the spurious events are attributable to unavoidable experimental errors; within the experimental accuracy, we conclude that the desired correlations in these experiments confirm the quantum predictions for GHZ entanglement. Thus we compare the predictions of quantum mechanics and local realism with the results of an *xxx* experiment (Fig. 4) and we observe that, again within experimental error, the triple coincidences predicted by quantum mechanics occur and not those predicted by local realism. In this sense, we believe that we have experimentally realized the first three-particle test of local realism following the GHZ argument.

We then investigated whether local realism could reproduce the *xxx* experimental results shown in Fig. 4c, if we assume that the spurious non-GHZ events in the other three experiments (Fig. 3) actually indicate a deviation from quantum physics. To answer this we adopt our second strategy and consider the best prediction a local realistic theory could obtain using these spurious terms. How, for example, could a local realist obtain the quantum prediction $H'H'H'$? One possibility is to assume that triple events producing $H'H'H'$ would be described by a specific set of local hidden variables such that they would give events that are in agreement with quantum theory in both an *xyy* and a *yxy* experiment (for example, the results $H'LR$ and $LH'R$), but give a spurious event for a *yyx* experiment (in this case, LLH'). In this way any local realistic

The Scientific Story

(2x2x∞)

(2x2x∞)	EPR (1935)	Einstein Podolsky Rosen
	Bell (1964)	
(2x2x2)	CHSH (1969)	Clauser Horne Shimony Holt
	Aspect *et al.* (1982)	Aspect Dalibard Roger
(3x2x2)	GHZ (1988)	Greenberger Horne Zeilinger
	Pan *et al.* (2002)	Pan Bouwmeester Daniell Weinfurter Zeilinger

and so on

2 parties 2 settings 2 outcomes →

m parties *n* settings *d* outcomes

and beyond

this talk : 2x2x*d*

Bell poker

(2x2x2)

Eve versus Alice and Bob

Eve chooses RED and green, for both Alice and Bob

Alice chooses: x , X Bob chooses : y , Y

Alice and Bob want to achieve:

$$x \geq y$$

$$X \leq y \quad \text{i.e.} \quad y \geq X$$

$$x \leq Y \quad \text{i.e.} \quad Y \geq x$$

but also:

$$X > Y$$

In each round :

Alice & Bob bet ¥ 300
Eve bets ¥ 99
Eve gives *Alice & Bob* each a random setting (colour)
Alice doesn't know Bob's and vice-versa
Alice and Bob each declare a number

With only ordinary physics ("Local Realism") at their disposal Alice & Bob will lose all their money

With quantum entanglement they get very rich
even with arbitrarily large stake (¥ 300 $\rightarrow$ ¥ M, arbitrary $M < \infty$)

Suppose Alice has two dice, outcomes x, X
Bob has two dice, outcomes y, Y

$Y \geq x$ and $x \geq y$ and $y \geq X$ implies $Y \geq X$
Therefore: $Y < X$ implies $Y < x$ or $x < y$ or $y < X$
Therefore: $\Pr(Y < X) \leq \Pr(Y < x) + \Pr(x < y) + \Pr(y < X)$

Pr (Alice-Bob pay Eve) $\leq$ 3 Pr (Eve pays Alice-Bob)

If only values –1 and +1 are allowed
this is Bell's (1964) inequality ! (CHSH version; & assuming no-signalling)

If only values $-d/2, -d/2+1, \ldots, +d/2$ are allowed
this is the CGLMP (2002) inequality ! (assuming no-signalling)

Collins, Gisin, Linden, Massar and Popescu, *PRL*

Bell (1964)/CGLMP (2002); assuming Local Realism:

$$\Pr(Y < X) - \Pr(Y < x) - \Pr(x < y) - \Pr(y < X) \leq 0$$

Inequalities under QM :

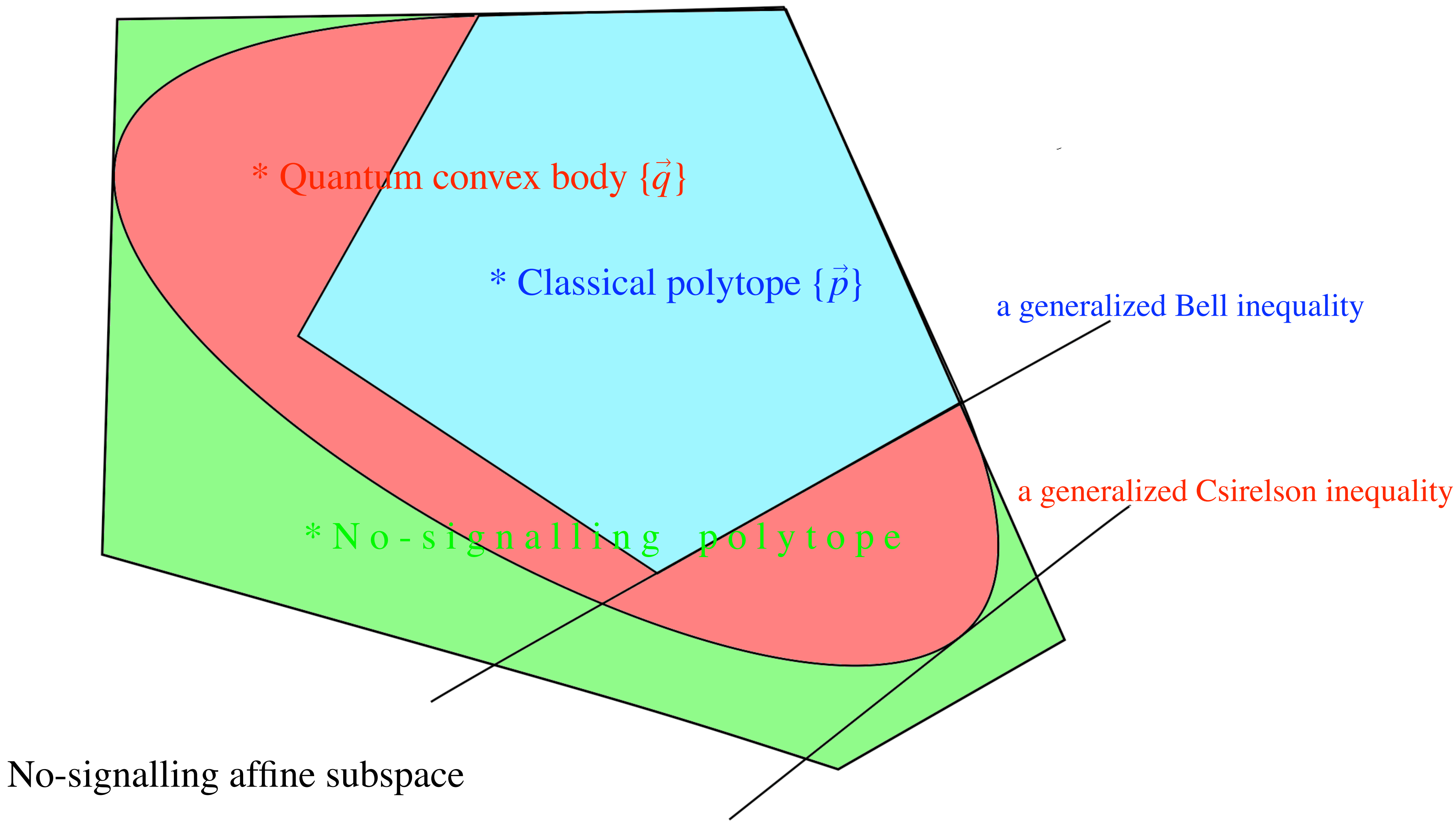
QM, d=2 Tsirelson (1980)

$$\Pr(Y < X) - \Pr(Y < x) - \Pr(x < y) - \Pr(y < X) \leq \sqrt{2} - 1$$

QM, $d=\infty$ Gill and Zohren (2006) `quant-ph/0612020`, to appear in *PRL*

$$\Pr(Y < X) - \Pr(Y < x) - \Pr(x < y) - \Pr(y < X) \leq 1$$

The no-signalling and local polytopes, and quantum convex body, for Bell-type experiment



Possible vectors of joint probabilities of settings and outcomes in the (*m* party, *n* settings, *d* outcomes) set-up

Conjectured best CGLMP state, measurements

$$|\Psi\rangle = \sum_{j=0}^{d-1} c_j \, |j\ j\rangle$$ for certain $c_j \geq 0$, U-shaped

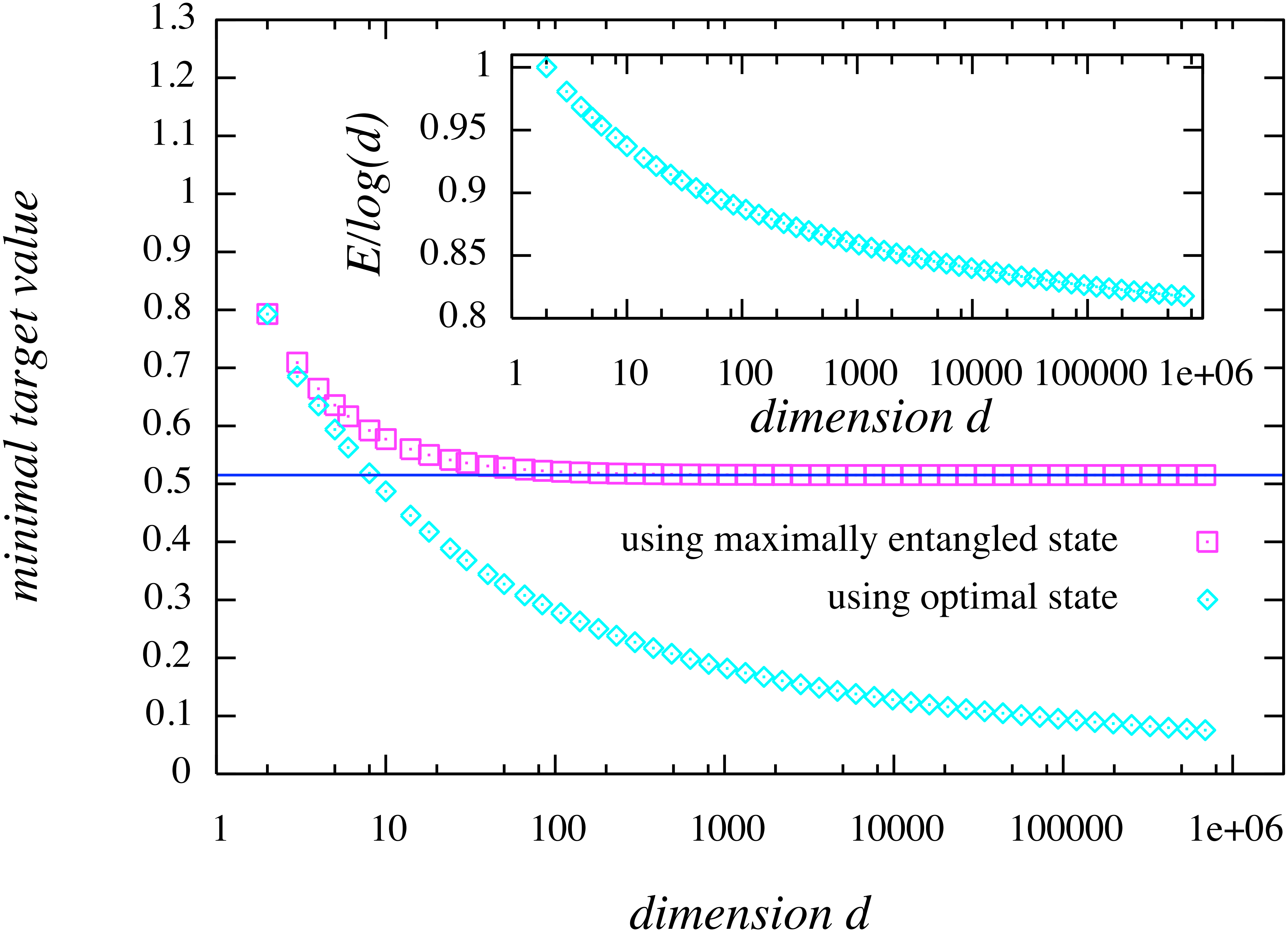
Alice chooses $\alpha = 0$ or $\pi/2$

Bob chooses $\beta = \pi/4$ or $-\pi/4$

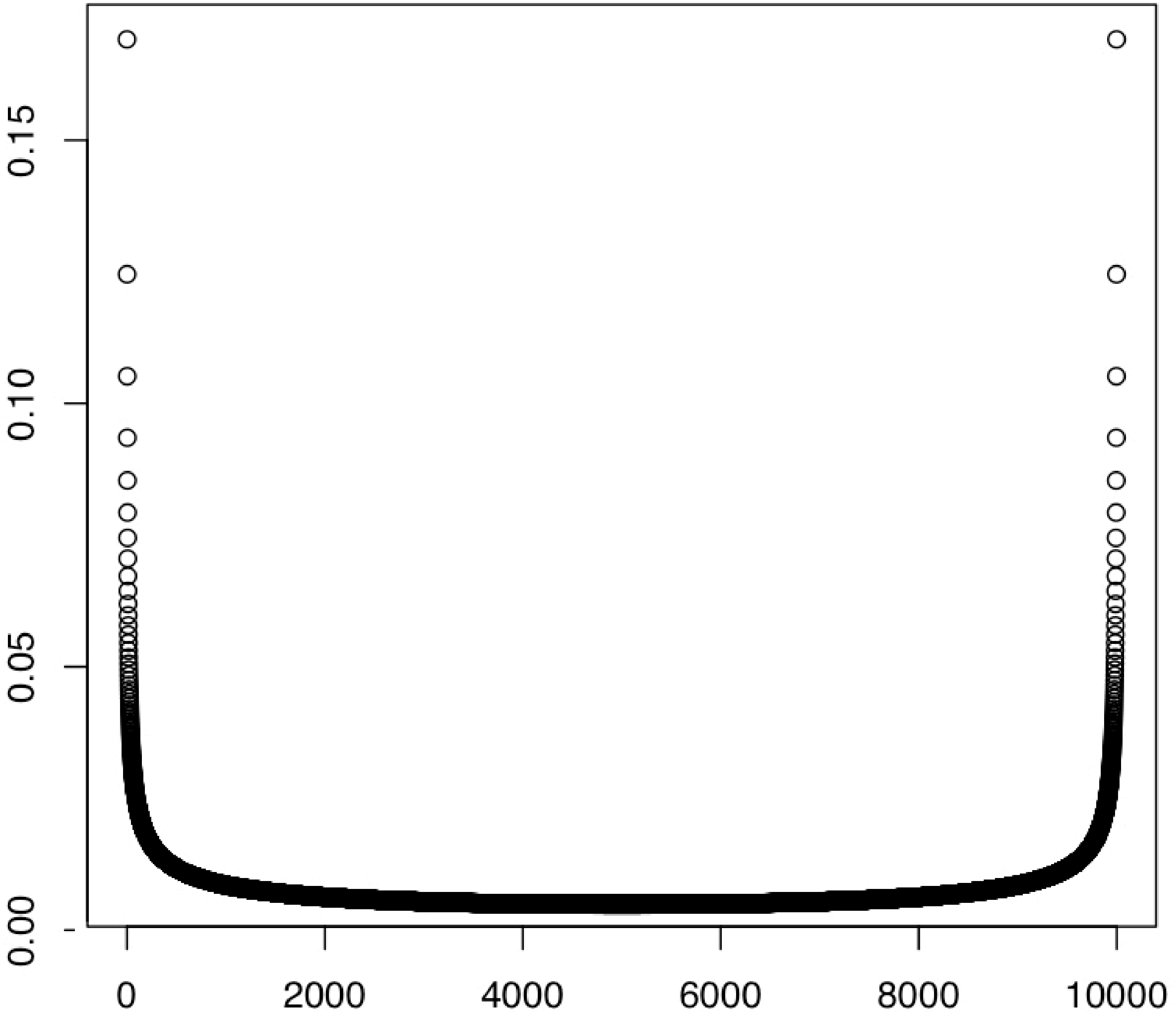
Alice, Bob apply diagonal unitaries $e^{i\frac{j\alpha}{d}}$, $e^{i\frac{j\beta}{d}}$

Alice does QFT, Bob $\mathrm{QFT}^{\dagger}$, they measure in computational basis

Are these state and (generalized CHSH) measurements, the best state and measurements, for CGLMP?



minimal target value
dimension d
using maximally entangled state
using optimal state
E/log(d)
dimension d
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
10
100
1000
10000
100000
1e+06
1
0.95
0.9
0.85
0.8



(instead of Euclidean distance)

$d = 10\,000$: Schmidt coefficients of best state (wrt relative entropy)

Notation

Fix # parties, # settings, # outcomes

* $p\,(x\ y\,..\ |\ a\ b\,..)$, $q\,(x\ y\,..\ |\ a\ b\,..)$

  probability of joint outcomes $x\ y\,..$ given joint settings $a\ b\,..$

  under classical, resp. quantum theory

* $\pi\ (a\ b\,..)$ probability of settings $a\ b\,..$; chosen by experimenter;

  mostly: kept fixed

* $p\,(a\ b\,..\ x\ y..) = \pi\ (a\ b\,..)\ p\,(x\ y\,..\ |\ a\ b\,..)$ and

  $q\,(a\ b\,..\ x\ y..) = \pi\ (a\ b\,..)\ q\,(x\ y\,..\ |\ a\ b\,..)$

  defines vectors $\vec{p}$ and $\vec{q}$; sets $\{\vec{p}\}$ and $\{\vec{q}\}$

Key facts

No-signalling affine subspace $\supset$ no-signalling polytope

$\supset$ quantum convex body $\{\vec{q}\}$ $\supset$ classical polytope $\{\vec{p}\}$

$\ni$ completely random point

Two sets of linear equality constraints, one set of linear inequalities:

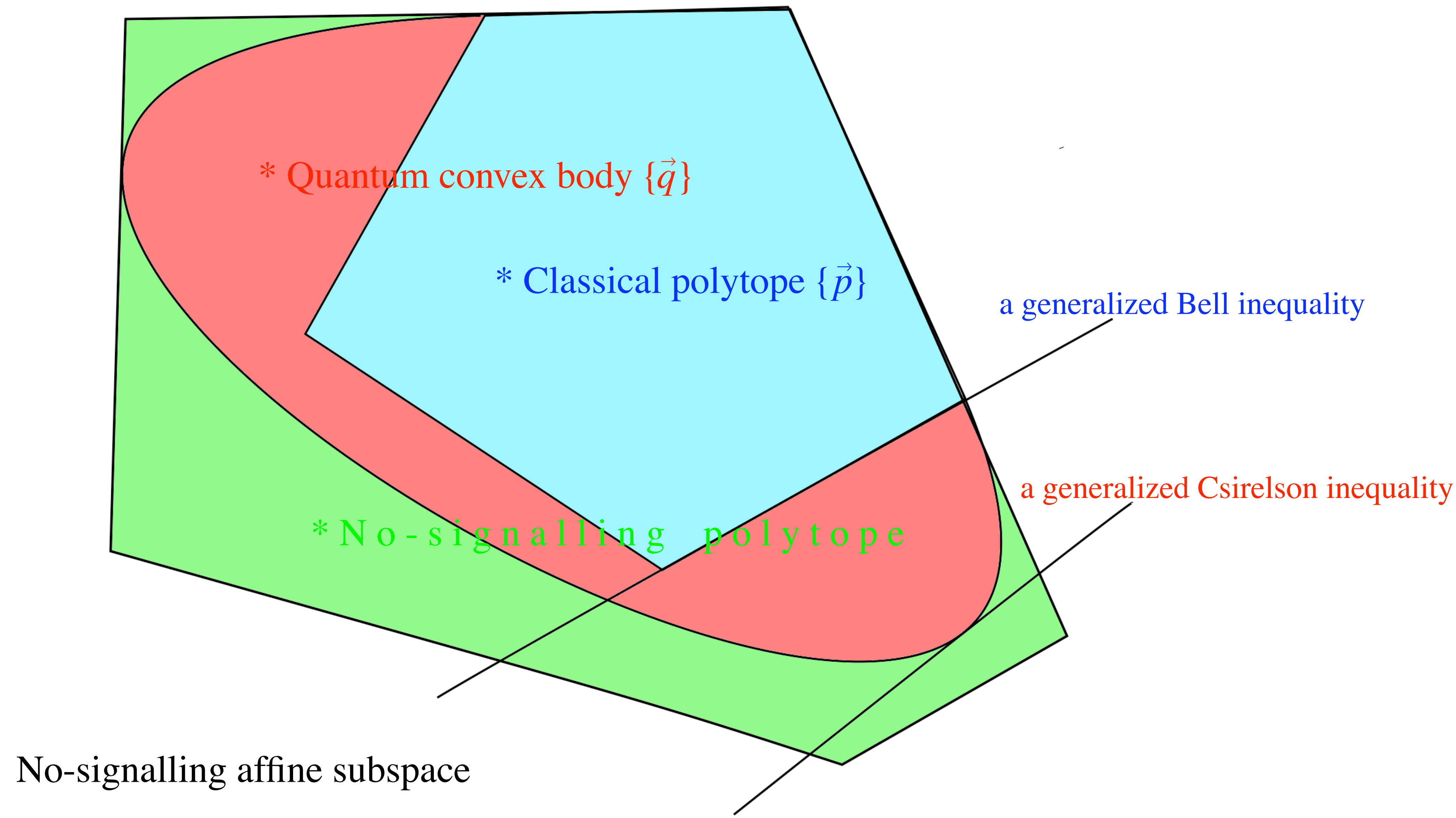
Normalization; No-signalling; Non-negativity

$$\forall_{ab\,..} \quad \sum_{xy\,..} p\,(xy\,..\mid ab\,..) \;=\; 1$$

$$\forall_{axbb'\,..} \quad \sum_{y\,..} p\,(xy\,..\mid ab\,..) \;=\; \sum_{y\,..} p\left(xy\,..\mid ab'\,..\right) \quad \text{etc.}$$

$$\forall_{ab\,..\,xy\,..} \quad p\,(xy\,..\mid ab\,..) \;\geq\; 0$$

The no-signalling and local polytopes, and quantum convex body, for Bell-type experiment



Possible vectors of joint probabilities of settings and outcomes in the (*m* party, *n* settings, *d* outcomes) set-up

Best experiment $\vec{q}$ for given # parties, # settings, # outcomes, solves

$$\sup_q \inf_p \sum_{a\,b\,..\,x\,y\,..} q(a\,b\,..\;x\;y\,..)\;\log_2 \frac{q(a\,b\,..\;x\;y\,..)}{p(a\,b\,..\;x\;y\,..)}$$

more precisely:

sup over experimental parameters: state, measurements, joint setting probabilities

inf over classical theories

computed by missing-data maximum-likelihood (Groeneboom; programs)

Changing the range of the optimization in various ways leads to non-locality measures for states, set-ups, . . .

The classical polytope $\{\vec{p}\}$

aka “local realism”, “the local polytope”, “local hidden variables”

$$\exists\ (X_a\, Y_b\, ..)_{ab..} \quad \text{such that} \quad \forall_{ab..}\ \big(p\,(xy..\mid ab..)\big)_{xy..} = \text{law}(X_a\, Y_b ..)$$

Results of unperformed experiments exist, too

Counterfactual outcomes of nonmeasured observables

The quantum body $\{\vec{q}\}$

$\exists$ closed subspaces L_x^a M_y^b .. of Hilbert spaces $\mathscr{H}$ $\mathscr{K}$ .. such that

* $\forall_a$ $(L_x^a)_x$ is an orthogonal decomposition of $\mathscr{H}$
* $\forall_b$ $(M_y^b)_y$ is an orthogonal decomposition of $\mathscr{K}$
* ..

$\exists\, \Psi \in \mathscr{H} \otimes \mathscr{K} \otimes ..$ $\quad \|\Psi\|^2 = 1$ such that

* $q\,(xy..\,|\,ab..) \;=\; \|\Pi_{L_x^a \otimes M_y^b \otimes ..}\Psi\|^2$

The quantum body $\{\vec{q}\}$

$\exists$ observables (self-adjoint operators) $X_a\ Y_b$ ..
such that each X_a commutes with each Y_b, each .. , and
$\exists$ a state Ψ such that

$$* \quad q\,(xy..\mid ab..) \;=\; \|\Pi_{\{X_a=x\}\,\cap\,\{Y_b=y\}\,\cap\,..}\,\Psi\|^2$$

The classical polytope $\{\vec{p}\}$

All $X_a\ Y_b$ .. commute

OR

There exist random variables and a probability measure …

GHZ paradox, Pan et al. experiment

Suppose $X_a^2 \equiv Y_b^2 \equiv Z_c^2 \equiv 1$

Classical :

If $Y_2 Y_1 = Y_1 Y_2$ then $(X_1 Y_2 Z_2)(X_2 Y_1 Z_2)(X_2 Y_2 Z_1) = (X_1 Y_1 Z_1)$

So $X_1 Y_2 Z_2 \equiv X_2 Y_1 Z_2 \equiv X_2 Y_2 Z_1 \equiv +1 \implies X_1 Y_1 Z_1 \equiv +1$

Quantum :

But if $Y_2 Y_1 = -Y_1 Y_2$ then $(X_1 Y_2 Z_2)(X_2 Y_1 Z_2)(X_2 Y_2 Z_1) = -(X_1 Y_1 Z_1)$

So $X_1 Y_2 Z_2 \equiv X_2 Y_1 Z_2 \equiv X_2 Y_2 Z_1 \equiv +1 \implies X_1 Y_1 Z_1 \equiv -1$

This can be arranged
  theoretically: GHZ
  experimentally: Pan, Bouwmeester, ...

Results :

- GHZ is potentially 9 times better than CHSH ... but actually only 9/8 (van Dam et al. 2005) [and actually ... !]
- New experiments, new inequalities ...
- ...

Conjectures (2x2x*d*) :

- All faces of the 2x2x*d* polytope are CGLMP faces
- The QFT measurements are optimal
- The no-signalling bound is attainable in the limit $d \rightarrow \infty$

van Dam, Gill & Grünwald (2005) *IEEE-IT* ; `quant-ph/0307125`
The statistical strength of nonlocality proofs

Acin, Gill & Gisin (2005) *Phys Rev Lett* ; `quant-ph/0506225`
Optimal Bell tests do not use maximally entangled states

Gill (2006) in: *IMS Monograph* **NN**; `math.ST/0610115`
Passion at a Distance: Better Bell Inequalities

Zohren & Gill (2006) `quant-ph/0612020` *Phys Rev Lett*
On the maximal violation of the CGLMP inequality for infinite dimensional states

ongoing work with Stefan Zohren (Utrecht), Marco Barbieri (Rome),
Jan-Åke Larsson (Linköping), Marek Żukowski (Gdansk), Philipp Pluch (Klagenfurt)

Peres (2000) *Fortsch Phys*; `quant-ph/9905084`
Bayesian analysis of Bell inequalities

Groeneboom, Jongbloed & Wellner (2005) `math.ST/040551`
Support reduction algorithm computing nonparametric function estimates mixture models

Gill (2003) *Växjö II proceedings*; `quant-ph/0301059`
Time, Finite Statistics and Bell's Fifth Position

Thank you !