Forensic Statistics and Graphical Models (4)

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Graphical Model

- Directed acyclic graph (V, G)
- Vertex $v \in V$: random variable X_v
- Arrows $v \rightsquigarrow w \in G$: (direct) statistical dependence
- For each vertex v, probability distributions law($X_v \mid X_{\mathrm{pa}(v)}$), often represented by conditional probability tables $p(x_v \mid x_{\mathrm{pa}(v)})$
    - pa(v) : graph parents of vertex v

Graphical Model

- Order vertices v such that $G \ni v_i \rightsquigarrow v_j \Rightarrow i <$ j

- w.l.o.g.: V = {1, … n },
  ordering of vertices respecting partial order of graph

- Joint distribution: $p(x_1, \ldots, x_n) = \prod_v p(x_v \mid x_{\mathrm{pa}(v)})$

- $X_v \perp X_{\{1,\ldots,v\text{-}1\} \setminus \mathrm{pa}(v)} \mid X_{\mathrm{pa}(v)}$

Graphical Model

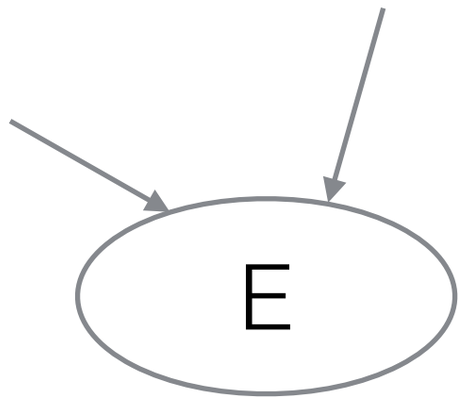
- Ancestral graph: include any ancestors

- Moralized graph: marry unmarried parents, forget direction of arrows

- **Theorem:** For any A, B, $C \subseteq V$,
  C separates A from B in $\mathrm{mor}(\mathrm{An}(A \cup B \cup C))$
  $\Rightarrow X_A \perp X_B \mid X_C$

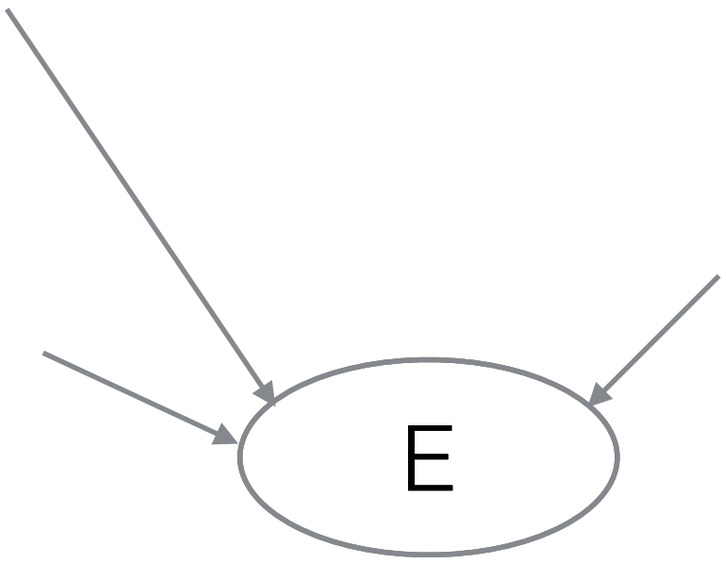
- **Theorem:** Pearl's d-separation criterium

Prosecution hypothesis



This DAG can compute Pr(E | Prosecution)

Defence hypothesis



This DAG can compute Pr(E | Defence)

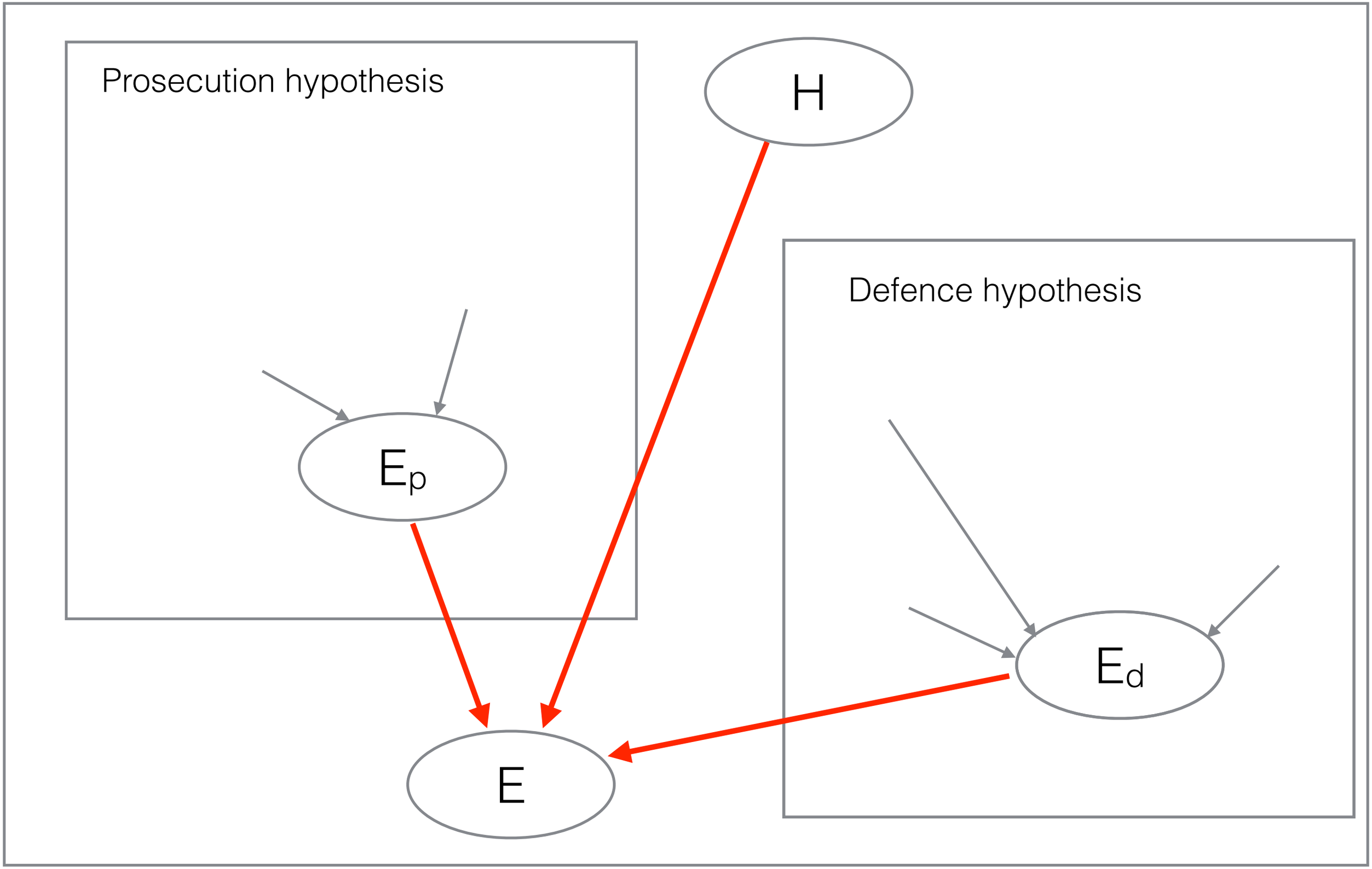
Bayes' rule

Posterior odds = prior odds **times** likelihood ratio

$P(H_p \mid E) : P(H_d \mid E) =$

$P(H_p) : P(H_d) \quad \times \quad P(E \mid H_p) : P(E \mid H_d)$

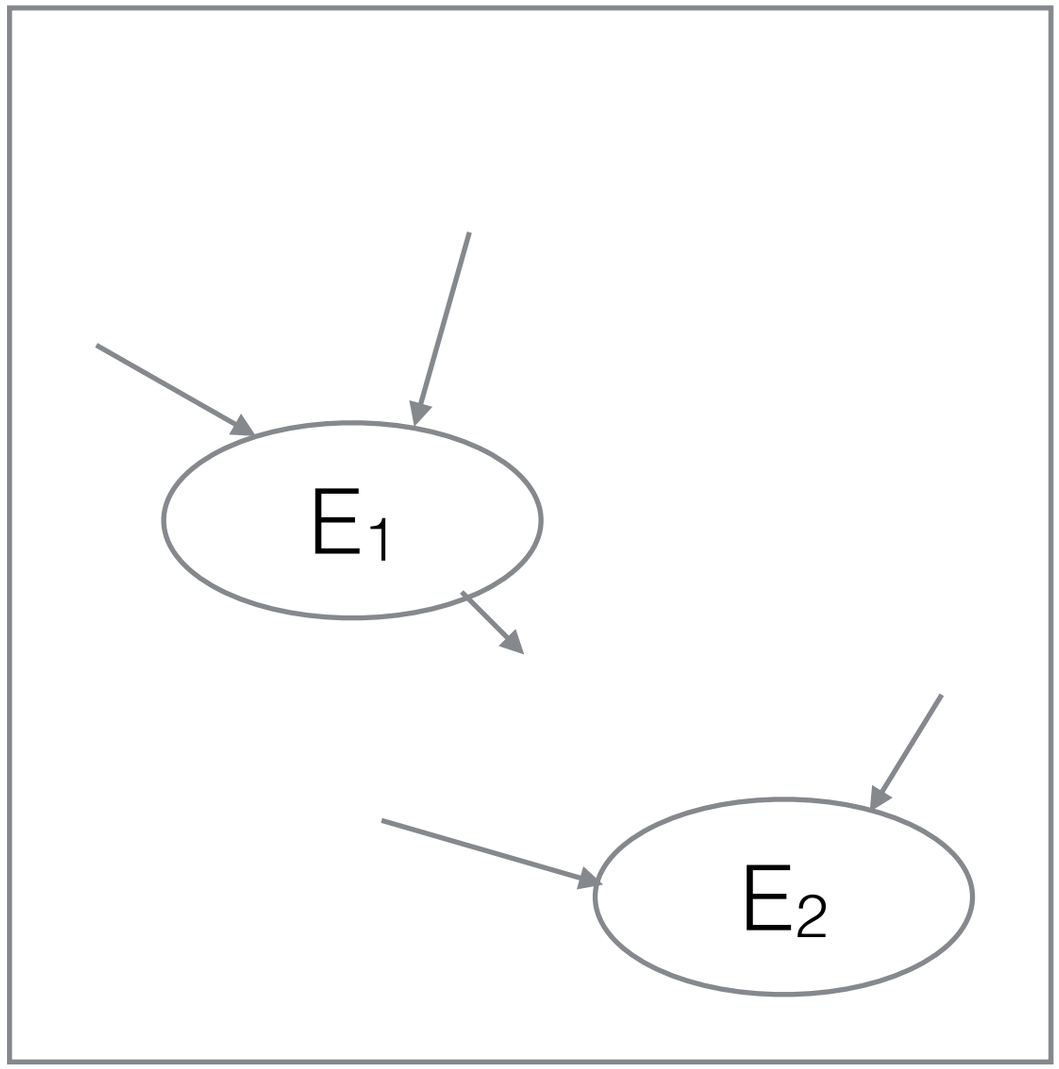
<u>If</u> (for convenience) we set prior odds = 1 : 1, then
<u>likelihood ratio = posterior odds</u>



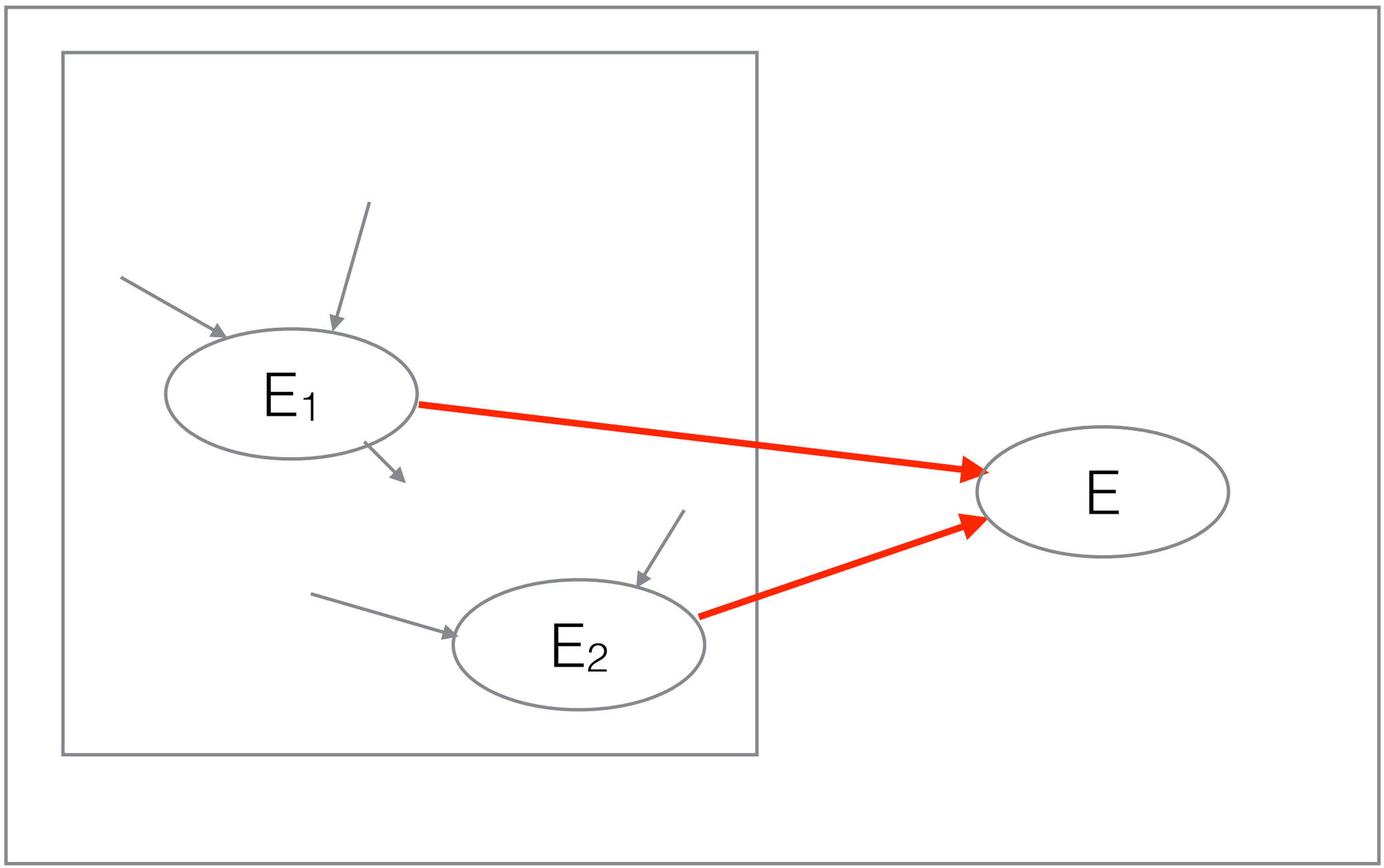
This DAG can compute
Pr(E | Prosecution) : Pr(E | Defence)

H = “prosecution” or “defence”
E ≡ E_p if H = “prosecution”
E ≡ E_d if H = “defence”

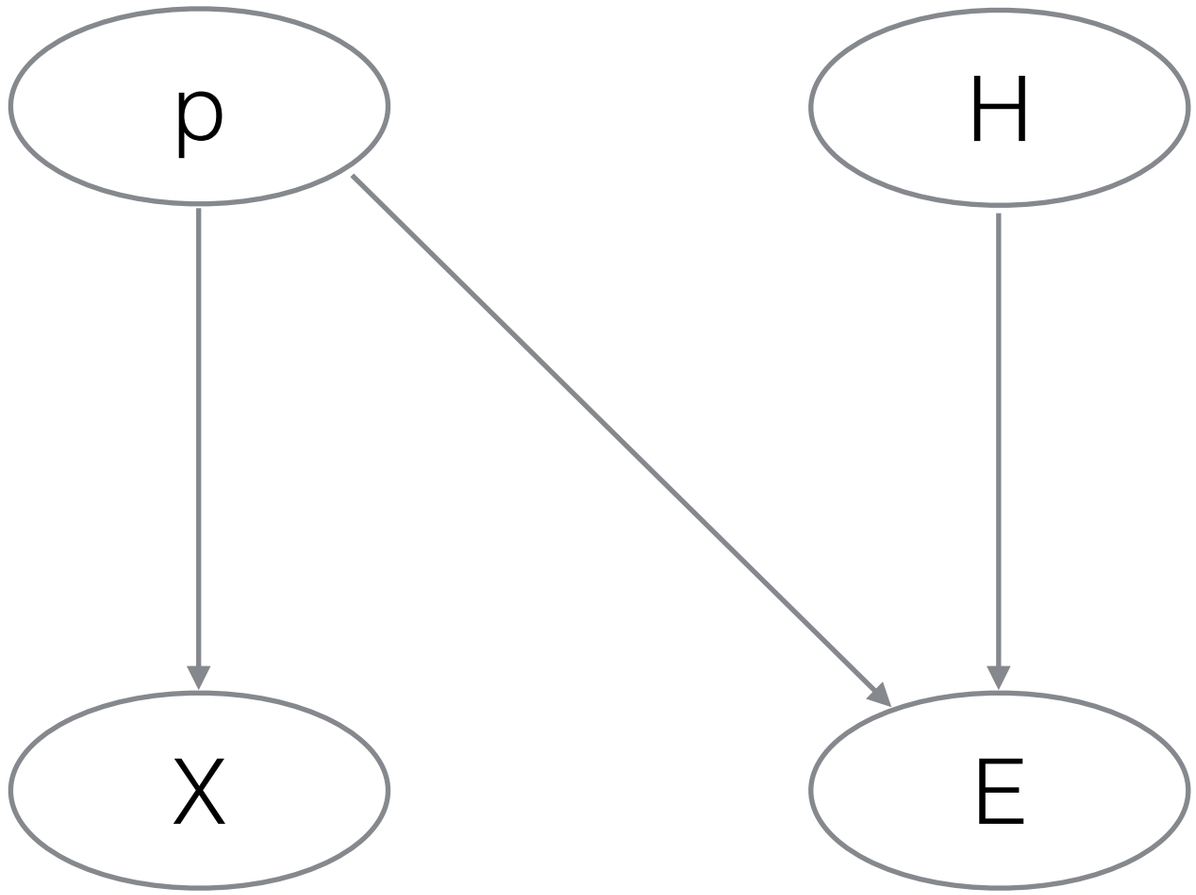
This DAG can compute Pr(E_1, E_2)



So can this one, if $E \equiv (E_1, E_2)$



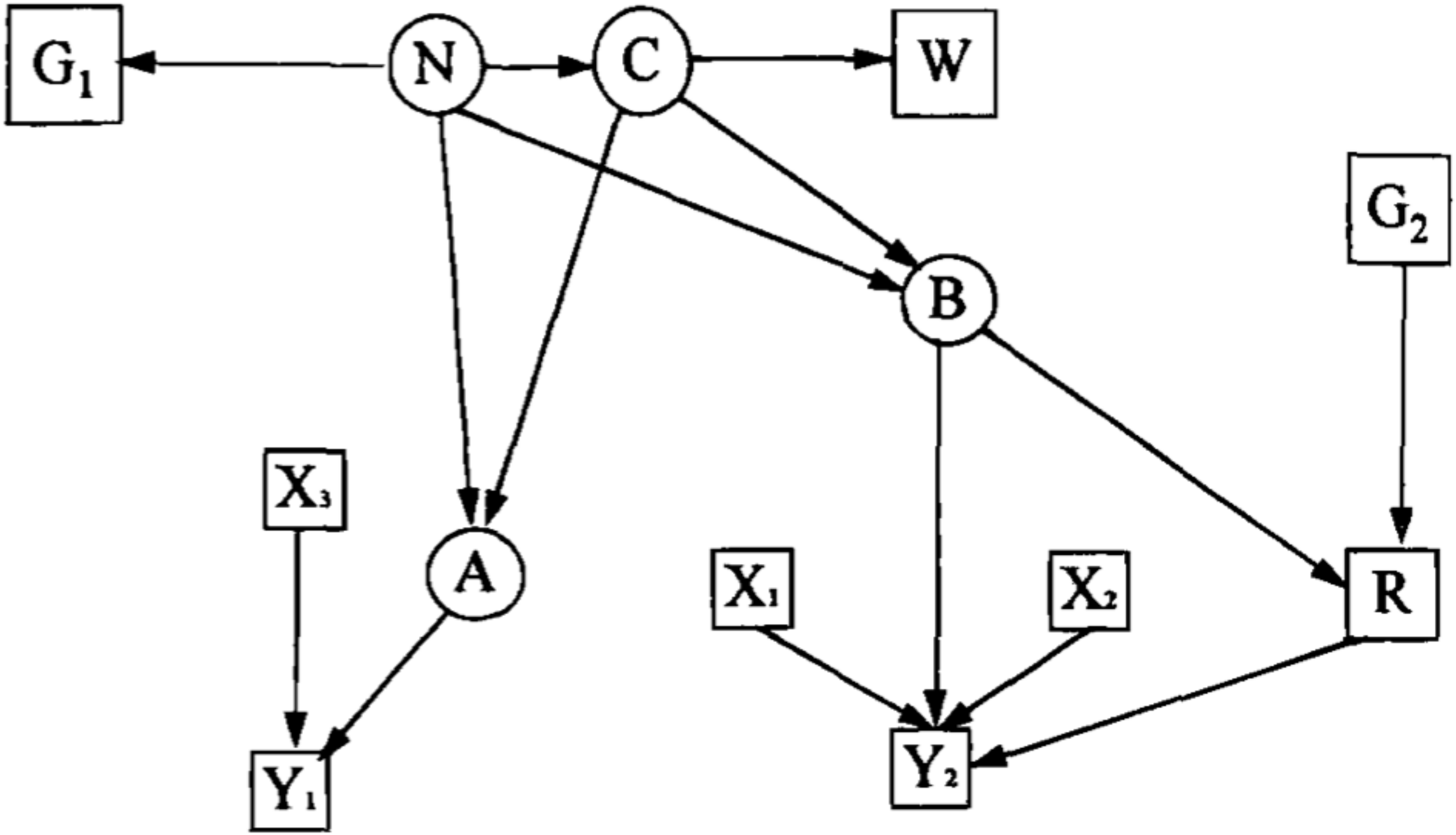
The rare haplotype problem



p ~ Beta(a,b)
H = “Prosecution” or “Defence”
X | p ~ Bin(n, p)
E = “match or “no match”
Pr(E = match | p, “Prosecution”) = p
Pr(E = match | p, “Defence”) = p^2

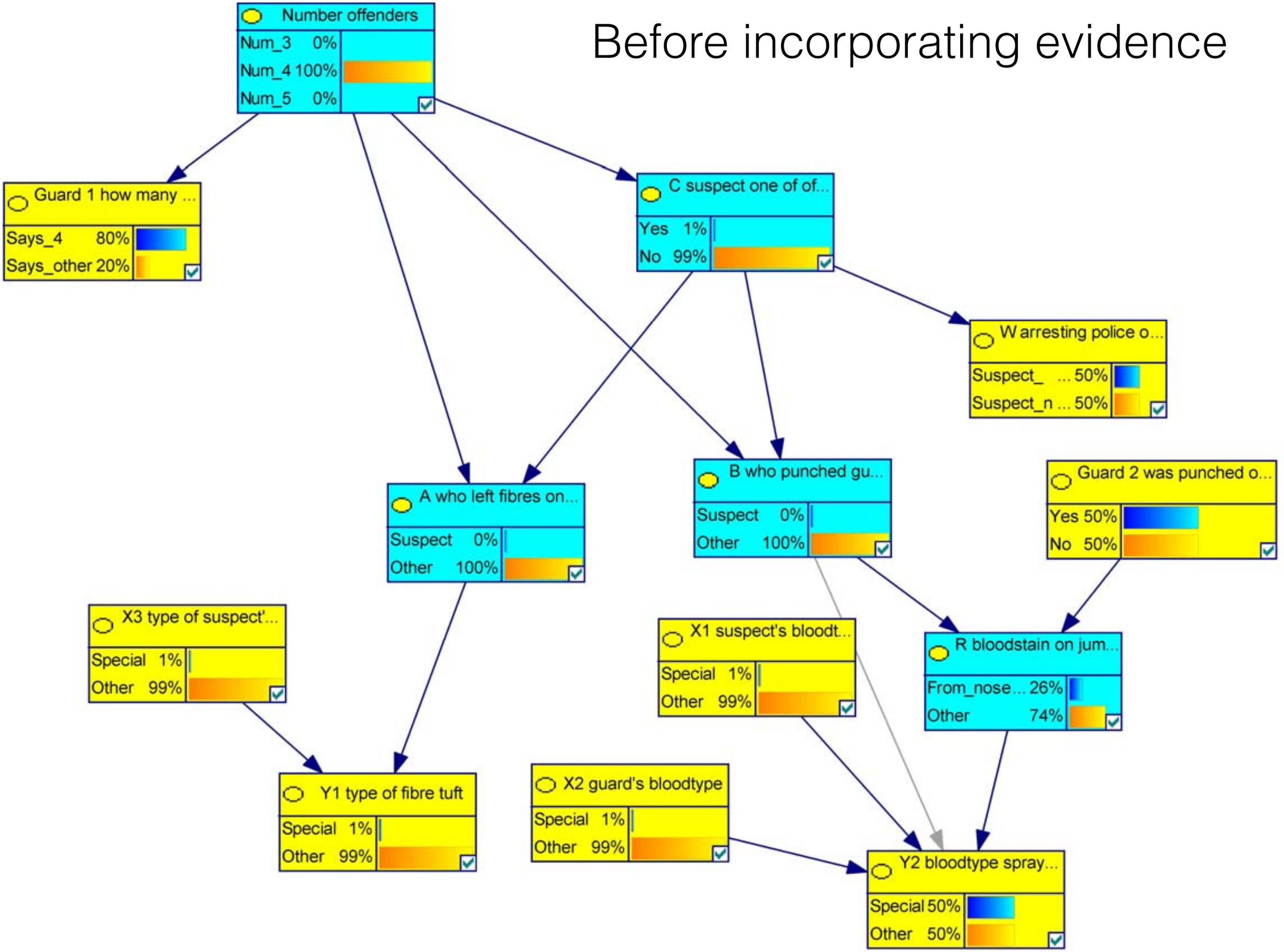
Problem: compute Pr(E, X | “Prosecution”) : Pr(E, X | “Defence”)

Example: Dawid and Evett (1997)



Squares = observed = evidence; circles = not observed; C = hypothesis of interest

Before incorporating evidence



After incorporating evidence