Lecture 7 (22 Oct 2013)

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October 22, 2013

Abstract

We will prove that the right derived functors of the push forward along a projective morphism sends coherent sheaves to coherent sheaves.

Theorem 0.1. *Let S be a locally noetherian scheme, and $f : X \to Y$ a proper morphism of H-quasi-projective S-schemes. Let $\mathscr{F} \in \mathrm{Mod}(X)$ be a coherent sheaf. Then for all $i \in \mathbb{Z}$, we have that $\mathrm{R}^i f_* \mathscr{F}$ is coherent.*

Recall that the right derived functors are computed for $f_* : \mathrm{Mod}(X) \to \mathrm{Mod}(Y)$. This is the standard definition of the cohomology of a sheaf on X/Y. We could also try to compute the derived functor for the push forward on the category of (quasi)-coherent modules. Does this exist (are there enough injectives)? If so, does it give the same answer? I don't know.

1 First reduction

Lemma 1.1. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. The sheaves $R^i f_* \mathcal{F}$ are the sheaves associated to the presheaves*

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F})$$

with obvious restriction mappings.

Proof. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $R^i f_* \mathcal{F}$ is by definition the ith cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \to f_*\mathcal{I}^1 \to f_*\mathcal{I}^2 \to \dots$$

By definition of the abelian category structure on $\mathcal{O}_Y$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \longmapsto \frac{\mathrm{Ker}(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\mathrm{Im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\mathrm{Ker}(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V)))}{\mathrm{Im}(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V)))}$$

which is equal to $H^i(f^{-1}(V), \mathcal{F})$ and we win. □

Lemma 1.2. *Let $f : X \to S$ be a q-compact separated morphism of schemes, and $\mathscr{F}$ a quasi-coherent sheaf on X. Then the sheaves $\mathrm{R}^p f_*\mathscr{F}$ are quasi-coherent for all p.*

Proof. Case 1: $q = 0$. Now $R^0 f_* = f_*$, so we need to prove that f_* of a quasi coherent sheaf is quasi coherent.

The question is local on S and hence we may assume that S is affine. Because X is quasi-compact we may write $X = \bigcup_{i=1}^n U_i$ with each U_i open affine. Because f is separated we may write $U_i \cap U_j = U_{ij}$ for some affine open U_{ij}, by exercise from previous lecture. Denote $f_i : U_i \to S$ and $f_{ij} : U_{ij} \to S$ the restrictions of f. For any open V of S and any sheaf $\mathcal{F}$ on X we have

$$\begin{aligned} f_*\mathcal{F}(V) &= \mathcal{F}(f^{-1}V) \\ &= \mathrm{Ker}\left(\bigoplus_i \mathcal{F}(f^{-1}V \cap U_i) \to \bigoplus_{i,j} \mathcal{F}(f^{-1}V \cap U_{ij})\right) \\ &= \mathrm{Ker}\left(\bigoplus_i f_{i,*}(\mathcal{F}|_{U_i})(V) \to \bigoplus_{i,j} f_{ij,*}(\mathcal{F}|_{U_{ij}})\right)(V) \\ &= \mathrm{Ker}\left(\bigoplus_i f_{i,*}(\mathcal{F}|_{U_i}) \to \bigoplus_{i,j} f_{ij,*}(\mathcal{F}|_{U_{ij}})\right)(V) \end{aligned}$$

In other words there is a short exact sequence of sheaves

$$0 \to f_*\mathcal{F} \to \bigoplus f_{i,*}\mathcal{F}_i \to \bigoplus f_{ij,*}\mathcal{F}_{ij}$$

where $\mathcal{F}_i, \mathcal{F}_{ij}$ denotes the restriction of $\mathcal{F}$ to the corresponding open. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-modules then $\mathcal{F}_i$, $\mathcal{F}_{ij}$ is a quasi-coherent $\mathcal{O}_{U_i}$, $\mathcal{O}_{U_{ij}}$-module. Hence by the characterisation of quasi coherent modules on affines, we see that the second and third terms of the exact sequence are quasi-coherent $\mathcal{O}_S$-modules. Thus we conclude that $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_S$-module.

General case: The idea is similar to the special case above. I have tried to spell things out more, but maybe I have just made it less clear...

We retain the notation from the first case. Fix $p \in \mathbb{Z}$. From lemma 1.1 we know that $R^p f_*\mathscr{F}$ is the sheaf associated to the presheaf $V \mapsto \mathrm{H}^p(f^{-1}V, \mathscr{F})$, and from Lecture 6 we can compute the $\mathrm{H}^p(f^{-1}V, \mathscr{F})$ using Cech cohomology.

Given an open subset $Z \subset S$, the open sets $U_i \cap f^{-1}Z$ form a cover of $f^{-1}Z$. We use this to construct a complex of presheaves on S, by

$$Z \mapsto \left[\prod_{0 \leq i_0 < \cdots < i_{p-1} \leq n} \mathscr{F}(Z \cap U_{i_0 \cdots i_{p-1}}) \to \prod_{0 \leq i_0 < \cdots < i_p \leq n} \mathscr{F}(Z \cap U_{i_0 \cdots i_p}) \to \prod_{0 \leq i_0 < \cdots < i_{p+1} \leq n} \mathscr{F}(Z \cap U_{i_0 \cdots i_{p+1}}) \right]$$

First, note that this is in fact a complex of quasi-coherent sheaves, since a finite product of quasi-coherent sheaves is again a sheaf. Secondly, note that the homology of this complex computed in the category of presheaves (kernel modulo image - all defined in an abelian category, see lecture 4) is nothing less than the presheaf

$$V \longmapsto H^i(f^{-1}(V), \mathcal{F}).$$

Now the homology of the complex computed in the category of *sheaves* is the sheafification of the above presheaf - this again follows from considerations in lecture 4, since forming the homology is done by taking a combination of kernels and cokernels, and for each of these we know that the sheafification of the object computed in PAb is the same as the object computed in Ab.

Finally, note that (since all $U_{\ldots}$ are affine by separateness), the homology in the complex can also be computed on the level of modules, after applying the global section functor. Since all the terms in the sequence are quasi-coherent, and forming kernels and cokernels of quasi-choerent sheaves commutes with global sections, we find that the 'sheaf associated to the module-homology' is exactly the sheaf $R^p f_* \mathscr{F}$ that we wanted, and it is quasi-coherent since it is the sheaf associated to a module on the (ring of functions on) the affine scheme S.

□

Lemma 1.3. *Let $f : X \to S$ be a separated quasi-compact morphism of schemes with S affine, and $\mathscr{F}$ a quasi-coherent sheaf on X. Then for all i we have*

$$\mathrm{H}^i(X, \mathscr{F}) = \Gamma(S, \mathrm{R}^i f_* \mathscr{F}).$$

Proof. Since f is q-cpct and separated, we know the $\mathrm{R}^i f_* \mathscr{F}$ are q-coherent. Since S is affine, we know $H^j(S, -) = 0$ for all $j > 0$. The result the follows from STA, tag 01F4 (it uses more homological algebra than we have set up). □

Lemma 1.4. *Let*

$$X \overset{f}{\rightarrowtail} Y \overset{g}{\to} Z$$

be morphisms of schemes with f a closed immersion. Then for all i there is a canonical isomorphism of sheaves

$$\mathrm{R}^i\, g_*(f_*\mathscr{F}) = \mathrm{R}^i(f\circ g)_*\mathscr{F}.$$

Proof. Very similar to the exercise at the top of page 5 of lecture 6. □

We will now show that the main result follows from the following lemma:

Lemma 1.5. *Let A a noetherian ring, $m \geq 0$, and $X = \mathbb{P}^m_A$. Let $\mathcal{F} \in \mathrm{Coh}(X)$. Then for all i, the A-module $\mathrm{H}^i(X, \mathscr{F})$ is finitely generated.*

Proof of theorem 0.1, assuming lemma 1.5. . We have a scheme S, a proper morphism $f : X \to Y$ of H-quasi-projective schemes over S, and a coherent sheaf $\mathscr{F}$ on X. From Lecture 3, we know the map f is in fact H-projective, so it factors as

$$X \overset{\iota}{\rightarrowtail} \mathbb{P}^n_Y \overset{\pi}{\to} Y$$

for some $n \geq 0$, with ι a closed immersion. Moreover, by exercise 4.1, the push forward $\iota_*\mathscr{F}$ is coherent. Hence by lemma 1.4 we are reduced to the case where $X = \mathbb{P}^n_Y$.

Coherence is a local property, and formation of $R^i f_*$ commutes with pullback along open immersions [exercise], hence we may assume $Y = \operatorname{Spec} A$ for some Noetherian ring A.

From lemma 1.2 we know that $\mathrm{R}^i\, f_*\mathscr{F}$ is quasi-coherent, and so (as Y is affine), it is the sheaf associated to its module of global sections $\Gamma(Y, \mathrm{R}^i\, f_*\mathscr{F})$. By Lemma 1.3 this coincides with $H^i(X, \mathscr{F})$, and we are done by lemma 1.5. □

2 Second reduction

In this section, we will prove lemma 1.5, assuming lemma 2.1:

Lemma 2.1. *Let $S = \operatorname{Spec} A$ be an affine scheme, $n, m, i \in \mathbb{Z}$ with $n, i \geq 0$. Then $H^i(\mathbb{P}^n_S, \mathcal{O}(m))$ is finitely generated as an A-module.*

Definition 2.2. Let S be a scheme, $m \in \mathbb{Z}_{\geq 0}$, $X = \mathbb{P}^m_S$. Let $\mathscr{F} \in \mathrm{Mod}(X, \mathcal{O}_X)$. Given $n \in \mathbb{Z}$, we write $\mathscr{F}(n) \overset{\mathrm{def}}{=} \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

Definition 2.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is *generated by global sections* if there exist a set I, and global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that the map

$$\bigoplus_{i\in I} \mathcal{O}_X \longrightarrow \mathcal{F}$$

which is the map associated to s_i on the summand corresponding to i, is surjective. In this case we say that the sections s_i *generate* $\mathcal{F}$.

Lemma 2.4. *Let A be a Noetherian ring, $m \in \mathbb{Z}_{\geq 0}$, and $X = \mathbb{P}^m_A$. Let $\mathscr{F}$ be a quasicoherent sheaf on X. Let $f \in \mathscr{F}(D^+(x_0))$. Then there exists $N > 0$ such that for all $n > N$, the section $x_0^n f$ lies in the image of the restriction map $\mathscr{F}(n)(X) \to \mathscr{F}(n)(D^+(x_0))$.*

Proof. Let x_i be coordinates on X. Let $i > 0$, and write f_i for the restriction of f to $\mathscr{F}(D^+(x_0 x_i)) = \mathscr{F}(D^+(x_i))[1/x_0]$. By definition of localisation, there exists $n_i > 0$ such that $x_0^{n_i} f_i$ lies in the image of the restriction from $\mathscr{F}(n)(D^+(x_i))$ to $\mathscr{F}(n)(D^+(x_0 x_i))$. Set $N = \max_i n_i$. □

Lemma 2.5. *Let A be a Noetherian ring, $m \in \mathbb{Z}_{\geq 0}$, and $X = \mathbb{P}^m_A$. Let $\mathscr{F}$ be a coherent sheaf on X. Then there exists an integer N such that for all $n \geq N$, the sheaf $\mathscr{F}(n)$ is generated by global sections.*

Proof. Let x_i be coordinates on X, $B_i := A[x_0/x_i, \cdots, x_m/x_i]$, and write M_i for the b_i-module $\mathscr{F}(D^+(x_i))$. By coherence, each M_i is finitely generated, say by $s_{i,j}$. By previous lemma, for each s_{ij} there exists $N_{i,j}$ such that for all $n > N_{ij}$, the element $x_i^n s_{ij} \in M_i \otimes_{B_i} \mathcal{O}(n)(D^+(x_i)) =: M_i'$ lies in the image of restriction from $\mathscr{F}(n)(X)$.

As such, there exists N such that for all $n > N$, every section $x_i^n s_{ij}$ extends to a global section of $\mathscr{F}(n)$. The 'multiplication by x_i^n map' $\mathscr{F} \to \mathscr{F}(n)$ induces an isomorphism $M_i \to M_i'$ (because $\mathcal{O}(n)$ is free of rank 1 and generated by x_i^n over $D^+(x_i)$). Hence the sections $x_i^n s_{ij}$ generate $\mathscr{F}(n)$. □

Lemma 2.6. *Let A be a Noetherian ring, $m \in \mathbb{Z}_{\geq 0}$, and $X = \mathbb{P}^m_A$. Let $\mathscr{F}$ be a coherent sheaf on X. The $\mathscr{F}$ can be written as a quotient of a finite direct sum of sheaves $\mathcal{O}(n)$ for various n.*

Proof. Choose N such that $\mathscr{F}(N)$ is generated by a finite number of global sections. From this we obtain a surjection

$$\oplus_i \mathcal{O}_X \to \mathscr{F}(N).$$

Tensor both sides with $\mathcal{O}(-N)$. □

Proof of lemma 1.5 assuming lemma 2.1. We have a noetherian ring A, and a coherent sheaf $\mathscr{F}$ on $X = \mathbb{P}^m_A$. We need to prove that $\mathrm{H}^i(X, \mathscr{F})$ is finitely generated as an A-module.

Note first that $\mathrm{H}^i(X, \mathscr{F}) = 0$ for all $i > m$, since X can be covered by $m+1$ affine opens (use cech cohomology from lecture 6[*1]). We now proceed by descending induction on i. Thus we assume that for all coherent sheaves

1: maybe an (easy) exercise?

$\mathscr{G}$ on X, the module $\mathrm{H}^{i+1}(X, \mathscr{G})$ is finitely generated. By lemma 2.6, we may write $\mathscr{F}$ in an exact sequence

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{F} \to 0$$

where $\mathscr{E}$ is a direct sum of copies of $\mathcal{O}(n)$ for various n. We have an exact sequence (from the long exact sequence in cohomology):

$$\cdots \to \mathrm{H}^i(X, \mathscr{E}) \to \mathrm{H}^i(X, \mathscr{F}) \to \mathrm{H}^{i+1}(X, \mathscr{K}) \to \cdots .$$

Now $\mathrm{H}^i(X, \mathscr{E})$ is finitely generated by lemma 2.1 and the fact that cohomology commutes with finite direct sums, and $\mathrm{H}^{i+1}(X, \mathscr{K})$ is finitely generated by the induction hypothesis and the fact that A is noetherian, so $\mathscr{K}$ is coherent. Using again that A is noetherian, this implies that $\mathrm{H}^i(X, \mathscr{F})$ is finitely generated, and we are done. □

3 Completing the proof: cohomology of $\mathcal{O}(n)$ on $\mathbb{P}^m$

In this section we will prove the following statement, which immediately implies (IS?!) lemma 2.1, and hence theorem 0.1

Lemma 3.1. *Let A be a ring. Let n, d, i be integers with $n, m \geq 0$. Then $\mathrm{H}^i(\mathbb{P}^n_A, \mathcal{O}(m))$ is finitely generated as an A-module.*

Proof. Let $\mathcal{U} = \{U_r\}_{r=0}^n$ denote the standard affine cover of X. We will use the Cech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{O}(d))$ to compute the cohomology. Set $B := A[x_0, \ldots, x_n]$. Note

$$\check{\mathcal{C}}^r(\mathcal{U}, \mathcal{O}(d)) = \prod_{0 \leq j_0 < \cdots j_r \leq n} B\left[\frac{1}{x_{j_0} \cdots x_{j_r}}\right].$$

We know the result if $i < 0$ or $i > n$; in these cases the cohomology is zero.

Case $i = 0$: It is not hard to see (exercise) that $\mathrm{H}^0(X, \mathcal{O}(d)) = B^d$, the space of degree-$d$ monomials in the graded ring B, which is finitely generated as an A-module.

Case $i = n$: The cohomology group $\mathrm{H}^n(X, \mathcal{O}(d))$ is the cokernel of the differential

$$\delta : \prod_{0 \leq j \leq n} B\left[\frac{1}{x_0 \cdots \check{x}_j \cdots x_n}\right]^d \to B\left[\frac{1}{x_0 \cdots x_n}\right]^d .$$

We can think of $B\left[\frac{1}{x_0\cdots x_n}\right]^d$ as the free A-module with basis degree-d monomials of the form $x_0^{l_0}\cdots x_n^{l_n}$, $l_* \in \underline{Z}$. The image of δ is the free submodule generated by those basis elements for which at least one of the l_i is non-negative. Thus we can think of $\mathrm{H}^n(X, \mathcal{O}(d))$ as the free A-module generated by degree-d monomials as above such that *all* the l_i are negative - the set of such monomials is finite, and we are done.

Case $0 < i < n$*:* Define

$$E = \{e = (e_0, \cdots, e_n) \in \mathbb{Z}^{n+1} \text{ such that } \sum_j e_j = d\}.$$

Given $e \in E$, define

$$\mathrm{NEG}(e) = \{a \in \{0, \cdots, n\} \text{ such that } e_i < 0\}.$$

Given $e \in E$, define a complex with objects:

$$C^p(e) = \bigoplus_{0 \leq j_0 < \cdots j_p \leq n, \mathrm{NEG}(e) \subset \{j_0, \cdots, j_p\}} A \cdot x_0^{e_0} \cdots x_n^{e_n},$$

and differentials induced by the differentials in the Cech complex (equivalently, given by the same formula as in the Cech complex).

We see that

$$\check{C}^p(\mathcal{U} < \mathcal{O}(d)) = \bigoplus_{e \in E} C^p(e)$$

so we have a direct sum decomposition of the objects in the Cech complex. Since the differentials respect this decomposition, we can relate the homology of the complexes $C^\bullet(e)$ to that of the Cech complex.

Fix $e \in E$. We will show that the homology of $C^p(e)$ is zero if $\mathrm{NEG}(e)$ is neither empty nor $\{0, \cdots, n\}$, which implies that the ith Cech cohomology *vanishes* if $0 < i < n$.

Suppose then that $\mathrm{NEG}(e)$is neither empty nor $\{0, \cdots, n\}$.Let $J \in \{0, \cdots, n\} \setminus \mathrm{NEG}(e)$. Define a map

$$h : C^{p+1}(e) \to C^p(e)$$

by $h(s)_{j_0, \cdots, j_p} = s_{j_0, \cdots J \cdots j_p}$. A calculation then shows that

$$(h\delta + \delta h)(s)_{j_0, \cdots, j_p} = s_{j_0, \cdots, j_p},$$

so h is homotopic to the identity, and hence the homology of the complex vanishes. □

4 Exercises

Exercise 4.1. Let $f : X \to Y$ be a closed immersion, with Y locally Noetherian. Let $\mathscr{F}$ be a coherent sheaf on X. Show that $f_*\mathscr{F}$ is a coherent sheaf on Y.

Hint: you might want to start by showing that f is quasi-compact and separated.

Exercise 4.2. We work over a fixed affine scheme $\operatorname{Spec} A$. Let X be $\mathbb{P}^1$ with the point $(0 : 1)$ doubled (c.f. the affine line with doubled origin). Let $f, g : \mathbb{P}^1 \to X$ be the two natural open immersions from copies of $\mathbb{P}^1$ (so their images cover X).

1) Show how to construct X by glueing 3 affine schemes together;

2) Compute $\mathrm{H}^0(X, f_*\mathcal{O}(n) \oplus g_*\mathcal{O}(m))$ for all m, $n \in \mathbb{Z}$.

References

[1] Robin Hartshorne *Algebraic Geometry*

[2] Qing Liu *Arithmetic Geometry and Algebraic Curves*

[3] de Jong et al. *Stacks Project*