# Graphs, spanning trees & determinants

A graph consists of a finite set of points (vertices), & edges between some of the vertices.

- at most one edge between 2 vertices - not ⟲
- edges must go between different vertices: not ⟲

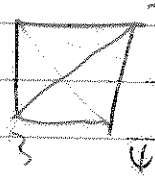
Label the vertices with integers 1, 2, ...
Then make the Laplacian matrix:

Say n vertices. Then Laplacian is a symmetric $n \times n$ matrix $L$.

If $i \neq j$ then $L_{ij} = \begin{cases} -1 & \text{if } \exists \text{ edge from } i \text{ to } j \\ 0 & \text{else} \end{cases}$

We choose the $L_{ii}$ s.t. each row sums to zero.

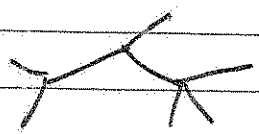
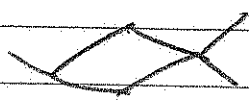
eg:




$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix} \end{array}$$

(note: symmetric)

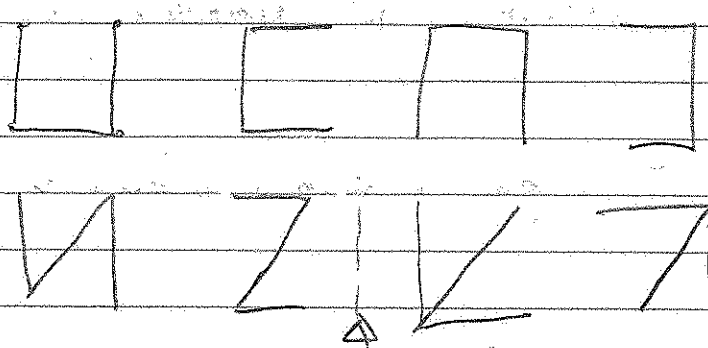
A tree in a graph is a collection of edges which together contain no 'cycles'

eg.  not 

A spanning tree is a tree which visits every vertex

eg. Graph is 

eg's of spanning trees:



Are there any others? Ask here first!

In general, how to compute # spanning trees?
Can try all subsets of edges. Eg if graph has 1000 edges then must try $2^{1000} \approx 10^{300}$ subsets - not practical!

Thm: Let $L$ be Laplacian of connected graph $G$. Then # spanning trees = any cofactor of $L$.

Pf omitted, ~ see Wikipedia.

eg: Let's find $\text{cof}_{11}$ of  again.

$$\text{cof}_{11} L = (-1)^{1+1} \det \begin{pmatrix} 3 & -1 & +1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} = 3(6-1) + 1(-2-1) + 1(1+3)$$

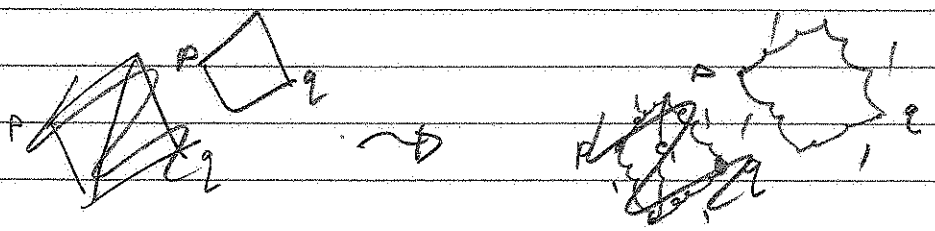
$$= 15 - 3 + 4 = 8 \quad \checkmark$$

$$\text{cof}_{14} L = (-1)^{1+4} \det \begin{pmatrix} -1 & 3 & -1 \\ -1 & -1 & 3 \\ 0 & -1 & -1 \end{pmatrix} = -\left(0 \text{ (stuff)} + -(-1)(-3-1) + (-1)(1+3)\right)$$

$$= 8 \quad \checkmark$$

Connection to ~~resista~~ electrical resistance.
(elektrische weerstand).

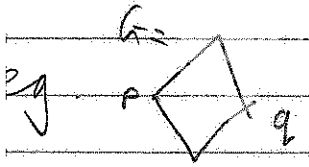
Graph $\leadsto$ electrical network, replace each edge with a 1-ohm resistor!

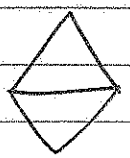



What is resistance between p & q? (Ans: $\frac{1}{\frac{1}{2}+\frac{1}{2}} = 1$)
In a big graph, not so easy.

hm Let p, q be two vertices with no edge between them. Let $G'$ be graph obtained by putting an edge between them.
Let $T$ = # spanning trees in $G$
$T'$ = — — — — $G'$.
Then resistance between p & q = $\frac{\#T' - \#T}{\#T}$.

compute w. dets.

eg. 


$T' = 8$ (Frombetre)

$T = 4$

$\leadsto$ res $(p, q) = \frac{8-4}{4} = 1$ ✓

(∃ version allowing edge, or can adapt this).