When are spans equal?

~~Wh~~ Write $\mathcal{U} = \text{Span}\{\underline{u}_1, \ldots, \underline{u}_r\}$, $V = \text{Span}\{\underline{v}_1, \ldots, \underline{v}_p\}$,

with all $\underline{u}_i$, $\underline{v}_i$ vectors in $\mathbb{R}^n$.

How can we decide whether $\mathcal{U} = V$?
It is enough to be able to decide whether $\mathcal{U} \subseteq V$.

Theorem $\mathcal{U} \subseteq V$ if and only if every $\underline{u}_i$ is in V.

If you want, you can skip the proof below. Go to top of page ②

Proof: First, note every $\underline{u}_i$ is in $\mathcal{U}$, eg. $\underline{u}_1 = 1 \cdot \underline{u}_1 + 0\underline{u}_2 + \cdots + 0\underline{u}_r$.

So if ~~...~~ $\mathcal{U} \subseteq V$ then every $\underline{u}_i$ is in V.

Suppose conversely that every $\underline{u}_i$ is in V. We want to show $\mathcal{U} \subseteq V$.

Step 1: If $\underline{a} \in V$ & $\underline{b} \in V$ then $\underline{a} + \underline{b} \in V$.
If $\underline{a} \in V$ & ~~...~~ $c \in \mathbb{R}$ then $c\underline{a} \in V$.

To see this, write $\underline{a} = a_1 \underline{v}_1 + \cdots + a_p \underline{v}_p$
$\underline{b} = b_1 \underline{v}_1 + \cdots + b_p \underline{v}_p$.

Then $\underline{a} + \underline{b} = (a_1 + b_1)\underline{v}_1 + \cdots + (a_p + b_p)\underline{v}_p$ so $\underline{a} + \underline{b}$ is in V,
and $r\underline{a} = (ra_1)\underline{v}_1 + \cdots + (ra_p)\underline{v}_p$ so $r\underline{a}$ is in V.

Step 2: Let y be any element of $\mathcal{U}$. Assume every $\underline{u}_i$ is in V.
We want to show $y \in V$.

Well, since $y \in \mathcal{U}$ we can write $y = x_1 \underline{u}_1 + \cdots + x_r \underline{u}_r$.

But every $\underline{u}_i$ is in V, so by step 1 every $x_i \underline{u}_i$ is in V,
so by step 1 again ~~every~~ the sum $x_1 \underline{u}_1 + \cdots + x_r \underline{u}_r$ is in V.

This concludes the proof.

② <u>Example</u> with vectors in $\mathbb{R}^3$.

Let $\underline{u}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\underline{u}_2 = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\underline{v}_1 = \begin{bmatrix}2\\2\\2\end{bmatrix}$.

So $\mathcal{U} = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ & $V = \text{Span}\left\{\begin{bmatrix}2\\2\\2\end{bmatrix}\right\}$.

Is $\mathcal{U} = V$?

First, is $V \subseteq \mathcal{U}$? By our theorem, we need to check whether $\begin{bmatrix}2\\2\\2\end{bmatrix}$ is in $\mathcal{U}$. But this is clear: $\begin{bmatrix}2\\2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} + 0\begin{bmatrix}1\\2\\3\end{bmatrix}$

Alternately, you could solve the vector equation

$$x_1\begin{bmatrix}1\\1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}$$ by row reduction.

You will find the solution $x_1 = 2$, $x_2 = 0$.

So $V \subseteq \mathcal{U}$.

Next, is $\mathcal{U} \subseteq V$? Need to check ① is $\underline{u}_1 \in V$?
② is $\underline{u}_2 \in V$?

① <u>Yes</u>: $\underline{u}_1 = \frac{1}{2}\begin{bmatrix}2\\2\\2\end{bmatrix} \in V$. Or solve the vector equation

$$x_1\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

② <u>No</u>: The vector equation $x_1\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ is inconsistent - there is no solution.

Conclusion: We have $V \subseteq \mathcal{U}$ but $\mathcal{U} \not\subseteq V$.

So $\mathcal{U} \neq V$.