Determine whether the sets in Exercises 1–8 are bases for $\mathbb{R}^3$. Of the sets that are *not* bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

1. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

4. $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 4 \end{bmatrix}$

11. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$. [*Hint:* Think of the equation as a "system" of homogeneous equations.]

In Exercises 13 and 14, assume that A is row equivalent to B. Find bases for Nul A and Col A.

13. $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 15–18, find a basis for the space spanned by the given vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_5$.

15. $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 10 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -6 \\ 9 \end{bmatrix}$

True or false:

22. a. A linearly independent set in a subspace H is a basis for H.

b. If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

c. A basis is a linearly independent set that is as large as possible.

e. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.

23. Suppose $\mathbb{R}^4 = \text{Span}\,\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$. Explain why $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

26. In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.

Exercises 31 and 32 reveal an important connection between linear independence and linear transformations and provide practice using the definition of linear dependence. Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a subset of V.

31. Show that if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent in V, then the set of images, $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$, is linearly dependent in W. This fact shows that if a linear transformation maps a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ onto a linearly *independent* set $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$, then the original set is linearly independent, too (because it cannot be linearly dependent).

32. Suppose that T is a one-to-one transformation, so that an equation $T(\mathbf{u}) = T(\mathbf{v})$ always implies $\mathbf{u} = \mathbf{v}$. Show that if the set of images $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly dependent, then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent. This fact shows that *a one-to-one linear transformation maps a linearly independent set onto a linearly independent set* (because in this case the set of images cannot be linearly dependent).

In Exercises 1–4, find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ and the given basis $\mathcal{B}$.

1. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

2. $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$

3. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \right\}, [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

In Exercises 5–8, find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to the given basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$.

5. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

6. $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} -1 \\ -6 \end{bmatrix}$

In Exercises 9 and 10, find the change-of-coordinates matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$.

9. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$

In Exercises 11 and 12, use an inverse matrix to find $[\mathbf{x}]_{\mathcal{B}}$ for the given $\mathbf{x}$ and $\mathcal{B}$.

11. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}, \mathbf{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

14. The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 1 + 3t - 6t^2$ relative to $\mathcal{B}$.

In Exercises 15 and 16, mark each statement True or False. Justify each answer. Unless stated otherwise, $\mathcal{B}$ is a basis for a vector space V.

15. a. If $\mathbf{x}$ is in V and if $\mathcal{B}$ contains n vectors, then the $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is in $\mathbb{R}^n$.

b. If $P_{\mathcal{B}}$ is the change-of-coordinates matrix, then $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}\mathbf{x}$, for $\mathbf{x}$ in V.

c. The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.

17. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 7 \end{bmatrix}$ span $\mathbb{R}^2$ but do not form a basis. Find two different ways to express $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Exercises 23–26 concern a vector space V, a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, and the coordinate mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$.

23. Show that the coordinate mapping is one-to-one. (*Hint:* Suppose $[\,\mathbf{u}\,]_{\mathcal{B}} = [\,\mathbf{w}\,]_{\mathcal{B}}$ for some $\mathbf{u}$ and $\mathbf{w}$ in V, and show that $\mathbf{u} = \mathbf{w}$.)

24. Show that the coordinate mapping is *onto* $\mathbb{R}^n$. That is, given any $\mathbf{y}$ in $\mathbb{R}^n$, with entries $y_1, \dots, y_n$, produce $\mathbf{u}$ in V such that $[\,\mathbf{u}\,]_{\mathcal{B}} = \mathbf{y}$.

25. Show that a subset $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ in V is linearly independent if and only if the set of coordinate vectors $\{[\,\mathbf{u}_1\,]_{\mathcal{B}}, \dots, [\,\mathbf{u}_p\,]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$. *Hint:* Since the coordinate mapping is one-to-one, the following equations have the same solutions, $c_1, \dots, c_p$.

$$c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0} \qquad \text{The zero vector in } V$$
$$[\,c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p\,]_{\mathcal{B}} = [\,\mathbf{0}\,]_{\mathcal{B}} \qquad \text{The zero vector in } \mathbb{R}^n$$

26. Given vectors $\mathbf{u}_1, \dots, \mathbf{u}_p$, and $\mathbf{w}$ in V, show that $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \dots, \mathbf{u}_p$ if and only if $[\,\mathbf{w}\,]_{\mathcal{B}}$ is a linear combination of the coordinate vectors $[\,\mathbf{u}_1\,]_{\mathcal{B}}, \dots, [\,\mathbf{u}_p\,]_{\mathcal{B}}$.

In Exercises 27–30, use coordinate vectors to test the linear independence of the sets of polynomials. Explain your work.

27. $1 + 2t^3$, $2 + t - 3t^2$, $-t + 2t^2 - t^3$