Lecture 9

Complex varieties and complex manifolds; analytification

9.1 Holomorphic functions in several variables

There is a very rich theory of holomorphic functions in several complex variables. We will only touch on a tiny part of it.

In this section, we work with the standard Euclidean topology on $\mathbb{C}^n$, which is not the same as the Zariski topology unless $n = 0$.

For further reading on the topics of this and the next lecture, we suggest to browse through [Hart, Appendix B].

Definition 9.1.1 Let $U \subset \mathbb{C}^n$ be an open subset, and $f \colon U \to \mathbb{C}$. Let $u = (u_1, \ldots, u_n) \in U$. We say f *is holomorphic at* u if there exist $\epsilon \in \mathbb{R}_{>0}$ and complex numbers $c_{\underline{i}} : \underline{i} \in \mathbb{N}^n$ such that on the ball $B_\epsilon(u)$ we have an equality of functions

$$f(z_1, \ldots, z_n) = \sum_{\underline{i} \in \mathbb{N}^n} c_{\underline{i}} \prod_{j=1}^{n} (z_j - u_j)^{i_j} \, .$$

Implicitly we mean that the right hand side converges absolutely at every point in $B_\epsilon(u)$.

We say f *is holomorphic on* U if f is holomorphic at u for every $u \in U$.

Lemma 9.1.2 *Holomorphic functions are continuous, even C^∞ (smooth).*

Proof Easy, omitted. □

Examples:

i. Any polynomial function, or power series which converges on U gives a holomorphic function.

ii. If f and g are polynomials and g has no zeros on U then the rational function f/g is holomorpic on U. For example, if $U = \mathbb{C} \setminus \{0\}$, $f = 1$ and $g = z$ then we see that not every holomorphic function can be globally defined by a power series.

iii. Not every holomorphic function can be written as a ratio of polynomials, even locally. For example, the exponential function.

Lemma 9.1.3 *i. Let $f: U \to \mathbb{C}$ be a holomorphic function which does not vanish anywhere. Then $1/f$ is also holomorphic.*

ii. Let $f: U \to V \subset \mathbb{C}^n$ and $g: V \to \mathbb{C}^m$ be holomorphic. Then $g \circ f$ is holomorphic.

Proof Omitted. □

Lemma 9.1.4 *Let $f: U \to \mathbb{C}^n$ be holomorphic. Then $\{u \in f : f(u) = 0\}$ is a closed subset (in the euclidean topology).*

Proof Immediate since f is continuous. □

9.2 Complex manifolds

Definition 9.2.1 Let $U \subset \mathbb{C}^n$ be Euclidean open. Define a $\mathbb{C}$-space $(U, \mathrm{hol}(U, \mathbb{C}))$ where U has the Euclidean topology, and $\mathrm{hol}(U, \mathbb{C})$ is the subsheaf of complex valued functions which are holomorphic.

These $\mathbb{C}$-spaces will play the role of 'affine varieties' in defining complex manifolds. Note that they are always *open* in $\mathbb{C}^n$, in contrast to affine varieties.

Definition 9.2.2 A *complex manifold* is a $\mathbb{C}$-space which is everywhere locally isomorphic to $(U, \mathrm{hol}(U, \mathbb{C}))$ for some n and some open subset $U \subset \mathbb{C}^n$.

A morphism of complex manifolds is just a morphism as $\mathbb{C}$-spaces (so the complex manifolds form a full subcategory of $\mathbb{C}$-spaces, just like $\mathbb{C}$-varieties).

There is an obvious notion of the dimension of a complex manifold. If you have seen real manifolds, note that the underlying topological space of a complex manifold of dimension n is a real manifold of dimension $2n$ - we will come back to this in the next lecture.

Example 9.2.3 i. Any union of open subsets of $\mathbb{C}^n$ gives a complex manifold, these are never compact unless empty or $n = 0$.

ii. Glueing complex manifolds works in exactly the same way as glueing varieties, cf. section 6.2. Let $X_1 = X_2 = \mathbb{C}$ with its sheaf of holomorphic functions. Let $X_{12} = \{z \in X_1 : z \neq 0\}$ and similarly $X_{21} = \{z \in X_2 : z \neq 0\}$, these are open submanifolds. Define $\varphi_{1,2}: X_{12} \to X_{21}$ by $\varphi(z) = 1/z$ (with the obvious map on sheaves, cf 9.1.3). Then the complex manifold obtained from this glueing data is called $\mathbb{CP}^1$, 'complex projective space'. As a ringed space, this is not isomorphic to the variety $\mathbb{P}^1_{\mathbb{C}}$. For example, on the level of topological spaces, $\mathbb{CP}^1$ is Hausdorff but $\mathbb{P}^1_{\mathbb{C}}$ is not! Note that the constructions of $\mathbb{P}^1_{\mathbb{C}}$ and $\mathbb{CP}^1$ look rather similar, though they are carried out in different categories. This will be generalised when we talk about 'analytification' of complex smooth varieties - it will turn out that $\mathbb{CP}^1$ is the analytification of $\mathbb{P}^1_{\mathbb{C}}$.

Though they are both special kinds of $\mathbb{C}$-spaces, $\mathbb{C}$-varieties and complex manifolds are very different - this is illustrated a bit in the exercises.

9.3 Sheaves on a base for a topology

For a moment we work in somewhat greater generality than usual, to develop an important tool that we will use to define the analytification.

Let T be a topological space. Recall that a *base* for T is a set B of open subsets of T such that every open $U \subset T$ can be written as a union of elements of B. For example, if T is $\mathbb{R}^n$ with the Euclidean topology then ϵ-balls around points give a base for the topology.

Let k be a field. Let T be a topological space and B a base for T.

Definition 9.3.1 Suppose for every $U \in B$ we are given a subset $\mathcal{F}(U) \subset \{f : U \to k\}$. We say the assignment $\mathcal{F}$ is a *sheaf on* B if

i. for all $V \subset U$ with $U, V \in B$ and for all f in $\mathcal{F}(U)$, $f|_V$ is in $\mathcal{F}(V)$;

ii. for all U in B and for all $f : U \to k$, f is in $\mathcal{F}(U)$ if and only if for all $P \in U$ there is a $U_P \subset U$ such that $U_P \in B$ and $P \in U_P$ and $f|_{U_P}$ is in $\mathcal{F}(U_P)$.

If we take B to be the set of all opens in T, then to give a sheaf on B is trivially the same as to give a k-space structure on the topological space T.

Theorem 9.3.2 *Let B, B' be two bases for the topological space T with $B' \subset B$.*

i. If $\mathcal{F}$ is a sheaf on B then restricting to opens in B' gives a sheaf on B';

ii. The above restriction map induces a bijection between sheaves on B and sheaves on B',

Proof Exercise. □

If B_1 and B_2 are bases and their intersection is also a base, and if $\mathcal{F}_1$ and $\mathcal{F}_2$ are sheaves on B_1 and B_2 respectively then we can see if $\mathcal{F}_1$ and $\mathcal{F}_2$ come from the same k-space structure by seeing if their restrictions to $B_1 \cap B_2$ are equal. This is a key thing we will need in defining analytifications. The most important examples will be of the following form: let X be a topological space, and $U = \{U_i\}_{i \in I}$ an open cover of X. Define a base B for the topology on X to consist of those opens which are contained in at least one U_i. Suppose U' is another cover, and define B' analogously. Then $B \cap B'$ is also a base, and so we can compare sheaves on B and B' by looking at their restrictions to $B \cap B'$.

9.4 Analytification

Let $\mathrm{SmVar}_{\mathbb{C}}$ be the full subcategory of $\mathrm{Var}_{\mathbb{C}}$ consisting of varieties that are smooth. The *analytification functor* takes as input a smooth complex variety (or map of such) and outputs a complex manifold (or map of such). From now until the end of this section, fix a smooth complex variety X. We will define a complex manifold X^{an}, called the 'analytification of X'.

9.4.1 The underlying set

This is easy: we define the underlying set of X^{an} to be the same as the underlying set of X.

9.4.2 The topology

First we treat the case where X is affine. Then there exist $n \geq 0$ and an ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$ and an isomorphism of varieties from X to $Z(I) \subset \mathbb{A}^n$. We give $\mathbb{A}^n = \mathbb{C}^n$ the Euclidean topology, and then we define the topology on X to be (the pullback of) the subspace topology. A priori this depends on the choice of the ideal I and the isomorphism, but in fact this is not the case, as can be easily deduced from the following lemma:

Lemma 9.4.3 *Let $I, J \subset \mathbb{C}[x_1, \dots, x_n]$ be ideals. Let $f\colon Z(I) \to Z(J)$ be an isomorphism of varieties. Then f is a homeomorphism between the sets $Z(I)$ and $Z(J)$ with the subspace topologies from the Euclidean topology.*

Proof Rational functions without poles are continuous in the Euclidean topology. □

Now we treat the general case: by definition, there is an open cover of X by affine $\mathbb{C}$-varieties. Choose such a cover $X = \cup_j X_j$. Then (applying again the above lemma) we find that on overlaps $X_i \cap X_j$ the subspace topologies from X_i and X_j coincide. We then define the topology on X to be the one induced by the X_i. Again, this depends a priori on the choice of cover, but applying the above lemma again we find this is not the case.

For interest and future use, we note:

Lemma 9.4.4 *Let X be a complex variety, let X_{Zar} be the underlying Zariski topological space, and X_{Eu} be the topology we have just defined. Let $id\colon X_{Eu} \to X_{Zar}$ be the identity map on sets. Then id is continuous.*

Proof Exercise. □

9.4.5 The $\mathbb{C}$-space structure

Up to now we have not used the smoothness of X, but at this point it will be crucial. We repeat definition 6.4.1 for the convenience of the reader:

Definition 9.4.6 Let X be a variety and d in $\mathbb{N}$. For P in X, X is *smooth of dimension d at P* if there is an open subvariety U of X containing P and an isomorphism $\varphi\colon U \xrightarrow{\sim} Z(f_1, \dots, f_{n-d}) \subset \mathbb{A}^n$ for some n and $f_1, \dots, f_{n-d}$, such that the rank of the $n-d$ by n matrix over k:

$$\left(\frac{\partial f_i}{\partial x_j}(\varphi P)\right)_{i,j}$$

equals $n-d$. The variety X is *smooth of dimension d* if it is smooth of dimension d at all its points. The variety X is *smooth at P* if it is smooth of dimension d at P for some d. Finally, X is *smooth* if at every point P it is smooth of some dimension d_P.

The key to the construction is the *implicit function theorem*, which we recall here without proof:

Theorem 9.4.7 (Holomorphic implicit function theorem) *Let $U \subset \mathbb{C}^n$ be Euclidean open and $f_1, \dots, f_{n-d}$ be holomorphic functions on U. Let $p \in U$ be such that the $n-d$ by $n-d$ matrix over $\mathbb{C}$*

$$\left(\frac{\partial f_i}{\partial x_j}(p)\right)_{1 \le i \le n-d, 1 \le j \le n-d}$$

is invertible. Then there exist

- *an open neighbourhood U' of p contained in U*
- *a open subset $W \subset \mathbb{C}^d$;*
- *holomorphic functions $w_1, \dots, w_{n-d}\colon W \to \mathbb{C}$;*

such that for all $(z_1, \cdots, z_n) \in U'$ we have that

$$(f_i(z_1, \cdots, z_n) = 0 \text{ for all } 1 \le i \le n-d) \iff (w_i(z_{n-d+1}, \cdots, z_n) = z_i \text{ for all } 1 \le i \le n-d)\,.$$

Proof Omitted, see for example [KK, Section 0.8] or Wikipedia. □

If you have never seen the classical (eg. differentiable) version of this theorem and some applications, it may help to look at the Wikipedia page on the implicit function theorem.

To define a $\mathbb{C}$-space structure, it suffices to define it on a base for the topology as discussed above. To check it is independent of choices, we only need to check that two sheaves obtained by different choices agree on a small enough base for the topology (by theorem 9.3.2).

Let $p \in X$ be a point. Because X is smooth at p (say of dimension d) there exist:

- an open subvariety U of X containing P;
- an isomorphism $\varphi\colon U \xrightarrow{\sim} Z(f_1, \ldots, f_{n-d}) \subset \mathbb{A}^n$ for some n and $f_1, \ldots, f_{n-d}$;

such that the rank of the $n-d$ by n matrix over $\mathbb{C}$:

$$\left(\frac{\partial f_i}{\partial x_j}(\varphi(p))\right)_{i,j}$$

equals $n-d$. Without loss of generality we assume that the left $n-d$ by $n-d$ block is invertible.

By the implicit function theorem, there exist

- a Euclidean open neighbourhood $V_{\varphi(p)}$ of $\varphi(p)$ in $\mathbb{A}^n$;
- a Euclidean open $W \subset \mathbb{C}^d$;
- holomorphic functions $w_1, \ldots, w_{n-d}\colon W \to \mathbb{C}$;

such that for all $(z_1, \ldots, z_n) \in V_{\varphi(p)}$ we have that

$$f(z_1, \ldots, z_n) = 0 \iff (w_i(z_{n-d+1}, \cdots, z_n) = z_i \text{ for all } 1 \le i \le n-d)\,.$$

In other words, we get a homeomorphism $\psi\colon W \to V_{\varphi(p)} \cap Z(f_1, \ldots, f_{n-d})$ by sending $z = (z_{n-d+1}, \ldots, z_n)$ to

$$(w_1(z), \ldots, w_{n-d}(z), z_{n-d+1}, \ldots, z_n),$$

where the inverse is given by just forgetting the first $n-d$ coordinates.

We will now define a sheaf of holomorphic functions on small open neighbourhoods of p.

Let V' be any open neighbourhood of p contained in $\varphi^{-1}V_{\varphi(p)}$. As p varies, it is clear that such V' give a base of the (analytic) topology on X that we defined above. So by theorem 9.3.2 it is enough to tell you what the holomorphic functions on V' are. Well, given $f\colon V' \to \mathbb{C}$, we say f is holomorphic if and only if the composite

$$\psi^{-1}V' \to V' \xrightarrow{f} \mathbb{C}$$

is holomorphic, which is defined because $\psi^{-1}V'$ is an open subset of $\mathbb{C}^d$.

It is not clear at this point that these holomorphic functions are well-defined (even after making the various choices that we have), because the same V' could have its holomorphic functions 'defined' with respect to several different points p. But using that composites of holomorphic functions are holomorphic, this can be checked.

We should check that we have defined a sheaf on the base. If $V'' \subset V'$ it is clear that the restriction of a holomorphic function on V' is again holomorphic on V''. The second condition follows from the local nature of the definition of a holomorphic function.

During the definition of the sheaf, we made several choices, and we must check that the definition is independent of the choices. This is largely analogous to checking that the definition of 'holomorphic functions on V'' does not depend on the p with respect to which it is taken, the key extra input is our theorem that sheaves on two different bases induce the same $\mathbb{C}$-space structure if they agree on a sub-base of the intersection of the bases.

9.4.8 Analytification of morphisms

If $f\colon X \to Y$ is a morphism of $\mathbb{C}$-varieties, we want to get a morphism of complex manifolds from X^{an} to Y^{an}. This is straightforward because rational functions without poles are holomorphic; we omit the details.

This sends the inclusion of open sub varieties to the inclusion of open submanifolds. For example, for smooth quasi-projective $\mathbb{C}$-varieties $X \subset \mathbb{P}^n_{\mathbb{C}}$ we can obtain X^{an} by restriction of the structure sheaf from the analytification $\mathbb{CP}^n$ of projective space $\mathbb{P}^n_{\mathbb{C}}$. We will study the latter a bit further in the next lecture.

9.5 Examples

We can now give a huge number of examples of complex manifolds - any smooth complex variety gives one after analytification!

9.5.1 Projective line

We have already seen $\mathbb{CP}^1$, but now you can check that $\mathbb{CP}^1 = \left(\mathbb{P}^1_{\mathbb{C}}\right)^{an}$. Note that the latter is compact and Hausdorff (it is a sphere). We will come back to this next week.

9.5.2 Affine space

The analytification of $\mathbb{A}^n$ is just $\mathbb{C}^n$ with the usual sheaf of holomorphic functions. It works similarly for any open subvariety of $\mathbb{A}^n$.

Note that (with the Zariski topology) any open subset X of $\mathbb{A}^n$ is compact. On the other hand, the analytification X^{an} of such a subset is never compact unless $n = 0$ or it is empty.

Again with the Zariski topology, an open subset X of $\mathbb{A}^n$ is Hausdorff if and only if it is empty or $n = 0$. On the other hand, the analytification X^{an} of such a subset is always Hausdorff, since $\mathbb{C}^n$ is.

This suggests that studying X^{an} may not be a good way to gain information on X, but in fact this is far from true, and in the next lecture we will begin to develop a bit of a dictionary between them.

9.6 Exercises

Exercise 9.6.1 Show that an open subset U of $\mathbb{C}^n$ in the Zariski topology is Hausdorff if and only if $n = 0$ or U is empty.

Exercise 9.6.2 Give an example of a holomorphic function on $\mathbb{C}$ whose zero set is not closed in the Zariski topology.

Exercise 9.6.3 Here we check some basic facts about rational and holomorphic functions, in the 1-variable case for simplicity. Let U be an open neighbourhood of $0 \in \mathbb{C}$. Let $f \in \mathbb{C}[x]$ be a polynomial which does not vanish at 0.

i. Show that the image of f in the ring $\mathbb{C}[[x]]$ of formal power series is a unit.

ii. Show that the formal inverse of f that you found above has a positive radius of convergence.

iii. If you are following the commutative algebra course, show that $\mathbb{C}[[x]]$ is a *local ring*.

Exercise 9.6.4 Show that the underlying topological space of $\mathbb{CP}^1$ is a sphere. If you get stuck, google 'stereographic projection'.

Exercise 9.6.5 Let X be a complex variety.

i. Assume X is separated. Show that X^{an} is Hausdorff.

ii. Assume X^{an} is connected. Show that X is connected.

In fact the converses also hold, but this is harder and is omitted.

Exercise 9.6.6 Prove theorem 9.3.2.

Exercise 9.6.7 Prove lemma 9.4.4.

Exercise 9.6.8 If you do not know what the fundamental group of a pointed topological space is, ignore this exercise (it is just for fun). Let X be the complement of the origin in $\mathbb{A}^1_{\mathbb{C}}$. Pick any basepoint in X.

i. Compute the fundamental group of X^{an} with the Euclidean topology.

ii. Compute the fundamental group of X with the Zariski topology.

It turns out that there is a good notion of the fundamental group of an algebraic variety, even for varieties not over $\mathbb{C}$ (the 'étale fundamental group'), but its definition takes more work.