~~eg Harder eg. An eg where Def easier than prop:~~ -1 ~~5~~

Divisor of a rat. fctn.

eg. $\dim X = 1$, then prime divisor $=$ point.

- if p a pt then $\underset{\text{as pt}}{v_p} = \underset{\text{as divisor}}{v_p}$,

So $\operatorname{div} f$ is as in lecture 8.

eg $X = \mathbb{P}^2$. $f = \dfrac{x^2}{yz}$. $Z : x = 0$. ~~Choose the i~~

Cover $\mathcal{U} = \{ D^+(x), D^+(y), D^+(z) \}$.

Choose ~~$D^+(x)$~~ $D^+(y)$ to compute $v_Z(f)$. $f_i = \frac{x}{y}$

Write $f = \left(\frac{x^2}{y^2}\right) / \left(\frac{z}{y}\right)$

Then $\frac{x^2}{y} \in \mathcal{O}_X(D^+(y)) = k[\frac{x}{y}, \frac{z}{y}]$,

$v_Z\left(\frac{x^2}{y^2}\right) =$ largest n s.t. $\frac{x^2}{y} \in \left(\left(\frac{x}{y}\right)^n\right)$

$= 2$ (note y a unit).

$v_Z\left(\frac{z}{y}\right) = \dots = 0$,

$\rightsquigarrow v_Z(f) = 2$ (easier: decompose as $f = \frac{x^2}{yz}$!)

Similar: $v_{(y=0)}(f) = -1$

$v_{(z=0)}(f) = -1$. $v_Z(f) = 0 \ \forall$ other Z

~~Def~~ $\operatorname{div} f = 2.(x=0) - (y=0) - (z=0)$.

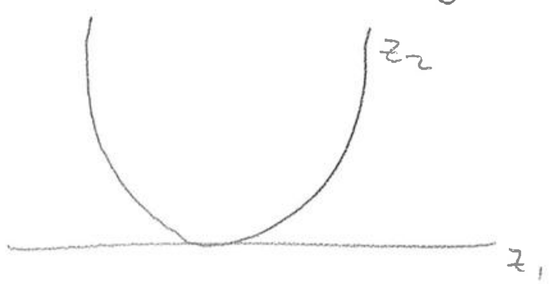
Intuition for intersection pairing on surfaces

Given two curves on a surface, can try to count # pts where they meet (over $\bar{k}$).

Idea: Should be invariant under deforming the curve a bit.

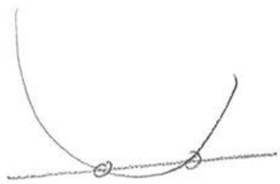
eg $Z_1 : y = 0$

$Z_2 : yz = x^2$



Only meet at 1 pt.

If move a bit, at 2 pts - eg $Z_1' : y = \varepsilon$.



Idea: define intersection 'with multiplicity' to take correct this.

Also want self-intersections.

We start w. def 11.2.1, 3 egs.

Then move on to Prop 11.2.7; doesn't always work, but often easier ~~in egs~~ when it does.

Examples w. def #11.2.1 (1)

Easy: Distinct lines in $\mathbb{P}^2$.

i) $Z_1 : x = 0$ $\quad U_1 = D^+(x) \leftrightarrow k[\frac{y}{x}, \frac{z}{x}]$ $\quad Z_1 \cap U_1 = \phi$, take $f_1 = 1$

$Z_2 : y = 0.$ $\quad U_2 = D^+(z) \leftrightarrow k[\frac{x}{z}, \frac{y}{z}]$ $\quad Z_1 \cap U_2 = \mathrm{div}_{U_2} \frac{x}{z}$, take $f_2 = \frac{x}{z}$.

ii) $U_{12} = k[\frac{x}{z}, \frac{y}{z}, \frac{z}{x}]$. f_1, f_2 both generate unit ideal.

$f_1 = f_{12} \cdot f_2 \qquad f_{12} = \frac{z}{x}$

$f_2 = f_{21} \cdot f_1 \qquad f_{21} = \frac{x}{z}$ (No tripple overlaps to check)

iii) $g_1 = f_{11} = 1$ ✓

$g_2 = f_{21} = \frac{x}{z}$.

$Z_2 - Z_2 \cap U_1 = \{\underbrace{(0:0:1)}_{P}\}$, $P \in U_2$

$\Rightarrow Z_1 \cdot Z_2 = \cancel{\ell_{(0:0:1)}\dim} \; v_{(0:0:1)} g_2$

So $Z_2 \simeq \mathbb{P}^1_{x,z}$, $g_2 = \frac{x}{z}$, $P = \cancel{\text{pt at infty}}$ origin of chart 1. Then claim $v_P\, g_2 = 1$.

Indeed: take ~~...~~ any open whd V of P in Z_2 not containing $(1:0)$,

then $\frac{x}{z}$ has no zeros on V, & ~~$v_P(\frac{x}{z}) = v_P(\frac{z}{x})$~~

$v_P(\frac{x}{z}) = \dim_k \frac{k[\frac{x}{z}]}{(\frac{x}{z})} = 1.$ ~~$= \dim_k \frac{k[\frac{z}{x}]}{(\frac{z}{x})} = 1$~~

So $Z_1 \cdot Z_2 = 1$.

Another way to calculate, using Thm 11.2.12. Postpone to end of (P3)

· Do eg(1) as before.

· eg(2): $Z_2 : yz - x^2 = 0$. $Z_2 - 2\cdot(x=0) = \mathrm{div}\left(\frac{yz - x^2}{x^2}\right)$, so $Z_1 \cdot Z_2 = 2 \cdot (\text{ans from eg}_1) = 2$.

(3): $Z_2 : y = 0$. $Z_2 - (x=0) = \mathrm{div}\left(\frac{y}{x}\right)$ so $Z_1 \cdot Z_2 = \left(\begin{smallmatrix}\text{ans from} \\ \text{eg } 1\end{smallmatrix}\right) = 1$.

Or Bézout: just use degrees (only works for $\mathbb{P}^2$!)

Slightly harder! Line & conic in $\mathbb{P}^2$.

$Z_1 : \cancel{xz}\ \ yz - x^2 = 0$

$Z_2 : y = 0.$

i) $\left.\begin{array}{l} U_1 = D^+(x), \\ U_2 = D^+(z) \end{array}\right\}$ as before, $\quad Z_1 \cap U_1 : \left(\frac{y}{x}\right)\left(\frac{z}{x}\right) - 1 = 0.$

$f_1 = \frac{y}{x} \cdot \frac{z}{x} - 1$

$Z_1 \cap U_2 : f_2 = \frac{y}{z} - \left(\frac{x}{z}\right)^2.$

ii) $U_{12} = k\left[\frac{x}{z}, \frac{y}{z}, \frac{z}{x}\right].$ $\quad f_1, f_2$ generate same ideal ✓.

$\cancel{f_1 = f_{12} \cdot f_2} \quad f_1 = f_{12} \cdot f_2 \qquad f_{21} = \left(\frac{x}{z}\right)^2$

$\frac{y}{x} \cdot \frac{z}{x} - 1 = \frac{z^2}{x^2} \cdot \left(\frac{y}{z} - \left(\frac{x}{z}\right)^2\right)$ ✓.

iii) $g_1 = 1$

$g_2 = f_{21} = \left(\frac{x}{z}\right)^2$. $\quad P = (0:0:1)$

$Z_1 \cdot Z_2 = \nu_P\left(\left(\frac{x}{z}\right)^2\right) = 2\nu_P\left(\frac{x}{z}\right) = 2. \quad \cancel{2\nu_P\left(\frac{z}{x}\right) = 2}$

from before.

(eg) Self intersection of a line in $\mathbb{P}^2$.

$\cancel{Z_1 =}\ Z_1 : y = 0 \qquad U_1 = D^+(x) \qquad Z_1 \cap U_1 = \operatorname{div}\left(\frac{y}{x}\right) \qquad f_1 = \frac{y}{x}$

$Z_2 : y = 0. \qquad U_2 = D^+(z) \qquad Z_1 \cap U_2 = \operatorname{div}\left(\frac{y}{z}\right)$ (empty). $\quad f_2 = \frac{y}{z}$

$U_1 \quad f_2 = f_{21} \cdot f_1 \Rightarrow f_{21} = \frac{x}{z}$ Again, $P = (0:0:1)$

$g_1 = 1$

$g_2 = \frac{y}{x} \qquad Z_1 \cdot Z_2 = \nu_P\left(\frac{x}{z}\right) = 1$ again!

Now do 1st two egs. w. Prop 11.2.7.

1 $Z_1 : x = 0$

$Z_2 : y = 0.$ $\quad Z_1 \cap Z_2 = \{\underbrace{(0:0:1)}_{P}\}.$

Take $U_P = D^+(z)$, then can take

$f_{1,P} = \frac{x}{z}$ $\quad$ in $\mathcal{O}_X(U_P) = k[\frac{x}{z}, \frac{y}{z}]$

$f_{2,P} = \frac{y}{z}.$

Then $Z_1 \cdot Z_2 = \dim_k \frac{\mathcal{O}_X(U_P)}{(f_{1,P}\ f_{2,P})} = \dim_k \frac{k[\frac{x}{z}\ \frac{y}{z}]}{(\frac{x}{z}, \frac{y}{z})} = \dim_k k = 1.$ ✓

~~$\mathcal{O}_X(U_P) =$~~

2 $Z_1 : yz - x^2 = 0$

$Z_2 : y = 0.$ $\quad Z_1 \cap Z_2 = \{(0:0:1)\}$ $\overset{\shortparallel}{P}$

Take $U_P = D^+(z)$, $\mathcal{O}_X(U_P) = k[\frac{x}{z}, \frac{y}{z}]$

$f_{1,P} = \frac{y}{z} - \frac{x^2}{z^2}$

$f_{2,P} = \frac{y}{z}.$

$Z_1 \cdot Z_2 = \dim_k \frac{k[\frac{x}{z}, \frac{y}{z}]}{(\frac{y}{z} - \frac{x^2}{z^2}, \frac{y}{z})} = \dim_k \frac{k[\frac{x}{z}]}{((\frac{x}{z})^2)}$ ~~$= \dim_k$~~

$= \dim_k k\langle 1, \frac{x}{z}\rangle = 2.$

Now go back to and do P1.