Solution to Exercise 3.6.5

November 17, 2016

Let $\Psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the Segre embedding defined by

$$((x_0 : x_1), (y_0 : y_1)) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1)$$

and let $\{w_0, w_1, w_2, w_3\}$ be the coordinates on $\mathbb{P}^3$.

(i) Observe that all points $p = (a_0 : a_1 : a_2 : a_3) \in \operatorname{Im}\Psi$ satisfies $a_0a_3 - a_1a_2 = 0$, so

$$Q = \operatorname{Im}\Psi \subseteq Z_{\text{proj}}(w_0w_3 - w_1w_2) =: Z.$$

It remains to show that equality holds. There are two ways to show this:

(a) Check directly. Suppose $p = (a_0 : a_1 : a_2 : a_3) \in Z$, then $a_0a_3 = a_1a_2$. Consider the sets $S_1 = \{(a_0, a_2), (a_1, a_3)\}$ and $S_2 = \{(a_0, a_1), (a_2, a_3)\}$. By the definition of a point in $\mathbb{P}^3$, one of $\{a_0, \dots, a_3\}$ is non-zero, so there will be at least one pair in each S_i that is non-zero.

Suppose $a_0 \neq 0$, then $(a_0 : a_2)$ and $(a_0 : a_1)$ define points in $\mathbb{P}^1$. We obtain

$$\Psi((a_0 : a_2), (a_0 : a_1)) = (a_0^2 : a_0a_1 : a_0a_2 : a_0a_3) = (a_0 : a_1 : a_2 : a_3) = p.$$

The other cases are similar. So we have shown that $p \in Q$.

(b) Use the fact that Z is irreducible (check!). By the definition of dimension (Definition 1.4.1), any proper irreducible closed subset of Z must have dimension $< \dim Z$. We know that $\dim Z = \dim \mathbb{P}^3 - 1 = 2$.

From Exercise 3.6.4(c), Q is closed in $\mathbb{P}^3$, so in particular, it is a closed subset of Z. If it is a proper subset, then all irreducible components of Q must have dimension ≤ 1. However, $\dim Q = \dim \mathbb{P}^1 + \dim \mathbb{P}^1 = 2$, so we obtain a contradiction. Hence, $Q = Z$.

Hence, Q is defined by the homogeneous polynomial $w_0w_3 - w_1w_2 \in \mathbb{C}[w_0, \dots, w_3]$.

(ii) Let $P = (a_0 : a_1) \in \mathbb{P}^1$. Then

$$\begin{aligned}\Psi(P \times \mathbb{P}^1) &= \{(a_0y_0 : a_0y_1 : a_1y_0 : a_1y_1) \mid (y_0 : y_1) \in \mathbb{P}^1\} \\ &\subseteq Z_{\text{proj}}(a_0w_2 - a_1w_0, a_0w_3 - a_1w_1).\end{aligned}$$

As in part (i), we can either check directly or use the irreducibility of $Z_{\text{proj}}(a_0w_2 - a_1w_0, a_0w_3 - a_1w_1)$ to check that equality holds (I'll leave that as an exercise).

Hint: to prove that $Z_{\text{proj}}(a_0w_2 - a_1w_0, a_0w_3 - a_1w_1)$ is irreducible is equivalent to showing that the ideal $I = (a_0w_2 - a_1w_0, a_0w_3 - a_1w_1) \subset \mathbb{C}[w_0, \dots, w_3] = S$ is prime, or that S/I is an integral domain. If $a_0 \neq 0$, we can substitute $w_2 = a_1w_0/a_0$ and $w_3 = a_1w_1/a_0$ to show that $S/I \cong \mathbb{C}[w_0, w_1]$.

Therefore,

$$\Psi(P \times \mathbb{P}^1) = Z_{\text{proj}}(a_0w_2 - a_1w_0, a_0w_3 - a_1w_1) \quad \text{and} \quad \Psi(\mathbb{P}^1 \times P) = Z_{\text{proj}}(a_0w_1 - a_1w_0, a_0w_3 - a_1w_2)$$

are lines on $\mathbb{P}^3$.

(iii) Let $A = (A_1, A_2) = ((a_0 : a_1), (a_2 : a_3))$ and $B = (B_1, B_2) = ((b_0 : b_1), (b_2 : b_3))$ be two points on $\mathbb{P}^1 \times \mathbb{P}^1$.

What is a line L through two points $P = (p_0 : \dots : p_3)$ and $Q = (q_0 : \dots : q_3)$ in $\mathbb{P}^3$?

Consider the projection $q : \mathbb{A}^4 - \{0\} \to \mathbb{P}^3$. Then the closure of $q^{-1}P$ and $q^{-1}Q$ in $\mathbb{A}^4$ are lines passing through the origin. Let $L \subset \mathbb{P}^3$ be the line through P and Q, so the closure of $q^{-1}L$ in $\mathbb{A}^4$ is the plane containing the two lines $\overline{q^{-1}P}$ and $\overline{q^{-1}Q}$. Any point on this plane is parametrized by $\lambda(p_0, \ldots, p_3) + \mu(q_0, \ldots, q_3)$ with $\lambda, \mu \in k$. Hence, L is parametrized by $\lambda P + \mu Q$ with $(\lambda : \mu) \in \mathbb{P}^1$.

The line L through $\Psi(A)$ and $\Psi(B)$ lies in Q if and only if $(p_0 : \ldots : p_3) := \lambda\Psi(A) + \mu\Psi(B) \in Q$ for all $(\lambda : \mu) \in \mathbb{P}^1$ if and only if

$$\begin{aligned} 0 &= p_0p_3 - p_1p_2 \\ &= (\lambda a_0a_2 + \mu b_0b_2)(\lambda a_1a_3 + \mu b_1b_3) - (\lambda a_0a_3 + \mu b_0b_3)(\lambda a_1a_2 + \mu b_1b_2) \\ &= \lambda\mu(a_0b_1 - a_1b_0)(a_2b_3 - a_3b_2) \end{aligned}$$

for all $(\lambda : \mu) \in \mathbb{P}^1$.

If $\lambda\mu \neq 0$, then we require either $a_0b_1 = a_1b_0$ or $a_2b_3 = a_3b_2$, that is to say, $A_1 = B_1$ or $A_2 = B_2$. Suppose $A_1 = B_1$. By (ii), we see that $\Psi(\{A_1\} \times \mathbb{P}^1)$ is a line passing through $\Psi(A)$ and $\Psi(B)$, so it is equal to L. Since Ψ is injective by Exercise 3.6.4(b), we obtain $\Psi^{-1}(L) = A_1 \times \mathbb{P}^1$. Similarly, if $A_2 = B_2$, we get $\Psi^{-1}(L) = \mathbb{P}^1 \times A_2$.

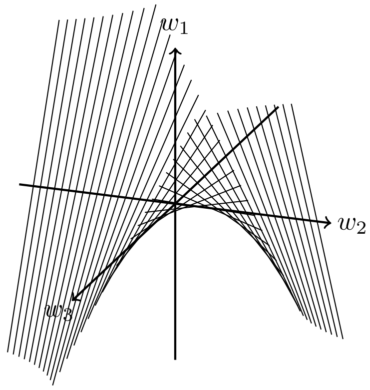
(iv) Let L_1 and L_2 be two lines on Q, and $l_i = \Psi^{-1}(L_i) \subset \mathbb{P}^1 \times \mathbb{P}^1$. Since L_i lie in the image of Ψ, we have $L_1 \cap L_2 = \Psi(l_1 \cap l_2)$.

If $L_1 = L_2$, then $L_1 \cap L_2 = L_1$ is a line. Suppose $L_1 \neq L_2$. Consider the cases described in (ii) and (iii).

If $l_1 = P_1 \times \mathbb{P}^1$ and $l_2 = P_2 \times \mathbb{P}^1$ with $P_1 \neq P_2$, then $l_1 \cap l_2 = \emptyset$, so $L_1 \cap L_2 = \emptyset$. Similarly for $l_1 = \mathbb{P}^1 \times P_1$ and $l_2 = \mathbb{P}^1 \times P_2$.

If $l_1 = P_1 \times \mathbb{P}^1$ and $l_2 = \mathbb{P}^1 \times P_2$, then $l_1 \cap l_2 = (P_1, P_2)$ and $L_1 \cap L_2 = \Psi(P_1, P_2)$ is a point.

(v) It is difficult to visualize Q directly, so first we take an affine slice, eg. $w_0 = 1$ and we draw $Q_0 \subset \mathbb{A}^3$. We shall take $k = \mathbb{R}$ (which is not algebraically closed!). Then, Q_0 can be seen as a spiral around the axis w_2 (and w_3, respectively) in $\mathbb{R}^3$ parametrized by lines in the w_1-w_3 (and w_1-w_3, respectively) plane.



(vi) The closed subset of $\mathbb{P}^1$ are

$$\{F \subset \mathbb{P}^1 \mid \#F < \infty\} \cup \{\emptyset, \mathbb{P}^1\}.$$

Hence, closed sets of $\mathbb{P}^1 \times \mathbb{P}^1$ in the product topology are finite unions of

$$F_1 \times F_2 \subset \mathbb{P}^1 \times \mathbb{P}^1 \quad \text{where} \quad \#F_i < \infty \text{ or } F_i = \mathbb{P}^1.$$

More explicitly, the closed sets of $\mathbb{P}^1 \times \mathbb{P}^1$ are of the form

$$F_1 \times \mathbb{P}^1 \cup \mathbb{P}^1 \times F_2 \cup S \tag{1}$$

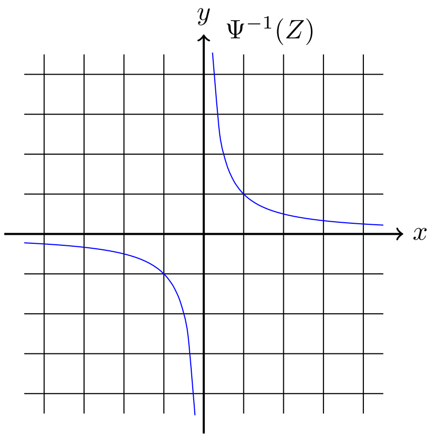
where $F_1, F_2 \subset \mathbb{P}^1$ and $S \subset \mathbb{P}^1 \times \mathbb{P}^1$ are finite subsets.

(vii) Consider the closed subset $Z := Z_{\text{proj}}(w_0 - w_3, w_0w_3 - w_1w_2) \subset Q$. Then,

$$\Psi^{-1}(Z) = \{((a_0 : a_1), (b_0 : b_1)) \mid a_0b_0 = a_1b_1\} = \{(\lambda : \mu), (\mu : \lambda) \mid (\lambda : \mu) \in \mathbb{P}^1\} \subsetneq \mathbb{P}^1 \times \mathbb{P}^1.$$

Clearly, $\Psi^{-1}(Z)$ is an infinite set. If $\Psi^{-1}(Z)$ is a closed set, it admits a decomposition in the form of (1). Then, either $F_1 \neq \emptyset$ or $F_2 \neq \emptyset$. However, note that $\Psi^{-1}(Z)$ intersects any line of the form $\{x\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{x\}$ at precisely one point, namely $(x, 1/x)$ or $(1/x, x)$. Hence, $\Psi^{-1}(Z)$ is not closed in the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, Ψ is not continuous if we take the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ and the topology induced from $\mathbb{P}^3$ on Q.

On an affine slice, we can visualize $\Psi^{-1}(Z)$ as the curve $y = \frac{1}{x}$ while the closed sets of the product topology are the vertical and horizontal lines:



In fact, one can show that if $Z = Z_{\text{proj}}(\lambda_0 w_0 + \cdots + \lambda_3 w_3, w_0 w_3 - w_1 w_2)$ with $\lambda_0 \lambda_3 \neq \lambda_1 \lambda 2$, then $\Psi^{-1}(Z)$ is not closed in the product topology.