Solution to Exercise 5.4.6

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Exercise 5.4.6. Let $n > m$. Show that any morphism $f : \mathbb{P}^n \to \mathbb{P}^m$ is constant.

Proof. The proof consists of three main steps.

1. Parametrize the map f by rational sections s_i (to be defined).
2. Show that each section s_i can be determined by a homogeneous polynomial.
3. If $n > m$, show that these sections have a common zero unless they are non-zero constant.

1. Let $\{x_0, \dots, x_n\}$ and $\{y_0, \dots, y_m\}$ be coordinate functions on $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively.

   Let $\mathbb{A}_i^m = \mathbb{P}^m|_{y_i=1}$ be an affine slice of $\mathbb{P}^m$ and $f_i : X_i = f^{-1}\mathbb{A}_i^m \to \mathbb{A}_i^m$ be the restriction of f. The morphism f_i induces a ring homomorphism

   $$f_i^* : \mathcal{O}_{\mathbb{A}_i^m}(\mathbb{A}_i^m) \cong k[t_{0,i}, \dots, t_{i-1,i}, t_{i+1,i}, \dots, t_{m,i}] \to \mathcal{O}_{X_i}(X_i) = \mathcal{O}_{\mathbb{P}^n}(X_i).$$

   where the embedding of $\mathbb{A}_i^m \hookrightarrow \mathbb{P}^m$ is given by the identification $t_{ji} = \frac{y_j}{y_i}$. Let $s_{ji} = f_i^*(t_{ji})$, these are called *sections* of $\mathcal{O}_{\mathbb{P}^n}$. Note that the morphism f_i is parametrized by

   $$f_i : X_i \to \mathbb{A}_i^m, \quad x \mapsto (s_{0,i}(x), \dots, s_{i-1,i}(x), s_{i+1,i}(x), \dots, s_{m,i}(x)).$$

   On the intersection $\mathbb{A}_i^m \cap \mathbb{A}_{i'}^m$, there are equalities $t_{ii'} = t_{i'i}^{-1}$ and $t_{ji} = t_{ji'}t_{i'i}$. These induce equalities under f^*

   $$s_{ii'} = s_{i'i}^{-1}, \qquad s_{ji} = s_{ji'}s_{i'i}. \tag{1}$$

   Define $s_{ii} = 1$, so the first equality becomes a special case of the second. There exists i such that $\operatorname{Im} f \cap \mathbb{A}_i^m \neq \emptyset$. Without loss of generality, assume $i = 0$. Then X_0 is open and dense in $\mathbb{P}^n$.

   For any i where $X_i \neq \emptyset$, X_i is also dense, so $X_i \cap X_0 \neq \emptyset$. By (1), we get $s_{0,i} \neq 0$, so we can localize $\mathcal{O}_{Proj^n}(X_i)$ with respect to $s_{0,i}$ and note that

   $$s_{j,0} = \frac{s_{j,i}}{s_{0,i}} \in \mathcal{O}_{Proj^n}(X_i)_{s_{0,i}}|_{X_i \cap X_0}.$$

   Hence, there exists a rational section s_j on $\mathbb{P}^n$ defined by gluing $s_{j,0}$ with the sections $s_{j,i}/s_{0,i}$ whenever $X_i \neq \emptyset$. The map f is then defined by

   $$f : \mathbb{P}^n \to \mathbb{P}^m, \quad (x_0 : \dots : x_n) \mapsto (1 : s_1(x_0, \dots, x_n) : \dots : s_m(x_0, \dots, x_n)).$$

   Explicitly, on an open set X_i, we have

   $$(1 : s_1 : \dots : s_m) = (1 : s_{1,i}/s_{0,i} : \dots : s_{n,i}/s_{0,i}) = (s_{0,i} : \dots : s_{n,i}).$$

   Remark. In the language of rational functions (see Section 6.5 of the notes), s_i is an element of the function field $K(\mathbb{P}^n)$.

2. Consider a section $s_{ji} \in \mathcal{O}_{\mathbb{P}^n}(X_i)$. X_i can be covered by open subvarieties $X_{ik} = X_i \cap \mathbb{A}^n_k$. Each $X_{ik} \subset \mathbb{A}^n_k$ can in turn be covered by open affine subsets of the form $D(g)$ for some polynomials g. By Theorem 5.1.5,

$$\mathcal{O}_{\mathbb{A}^n_k}(D(g)) = k[u_0, \dots, \hat{u}_k, \dots, u_n, u_{n+1}]/(u_{n+1}g - 1).$$

From commutative algebra, it is known that the latter is precisely the localization of the polynomial ring $k[u_0, \dots, \hat{u}_k, \dots, u_n]$ with respect to g. We can thus write $s_{ji}|_{D(g)}$ in the form h/g^r for some polynomial $h \in k[u_0, \dots, \hat{u}_k, \dots, u_n]$ and $r \in \mathbb{N}$.

For any $D(g)$ and $D(g')$ in the open cover, the intersection $D(g) \cap D(g')$ is dense and open since k is algebraically closed, so $s_{ji}|_{D(g)}$ and $s_{ji}|_{D(g')}$ coincide on a dense open subset, and they must be identical as rational functions in $k(u_0, \dots, \hat{u}_k, \dots, u_n)$. Hence, $s_{ji}|_{X_{ik}}$ can be identified with a unique rational function in $k(u_0, \dots, \hat{u}_k, \dots, u_n)$.

The embedding $\mathbb{A}^n_k \to \mathbb{P}^n$ identifies $u_j = \frac{x_j}{x_k}$, thus $s_{ji}|_{X_{ik}}$ is a rational function, homogeneous of degree 0 in $k(x_0, \dots, x_n)$, i.e. there exist $g, h \in k[x_0, \dots, x_n]$, homogeneous with $\deg g = \deg h$ such that $s_{ji}|_{X_{ik}} = g/h$. A similar gluing argument shows that s_{ji} is defined globally by a rational function $g_{ji}/h_{ji} \in k(x_0, \dots, x_n)$.

Repeating the gluing over all i for a fixed j shows that $s_j = g_{j0}/h_{j0}$. Hence, clearing denominators, we get

$$(1 : s_1(x_0, \dots, x_n) : \dots : s_m(x_0, \dots, x_n)) = (g_0 : g_1 : \dots : g_m)$$

for some homogeneous polynomials $g_i \in k[x_0, \dots, x_n]$ with $\deg g_0 = \dots = \deg g_m$, and such that $\gcd(g_0, \dots, g_m) = 1$.

Remark. Proposition 6.5.3(i) and (ii) gives $K(\mathbb{P}^n) = K(\mathbb{A}^n_0) = k(u_1, \dots, u_n)$ where $u_i = \frac{x_i}{x_0}$. Thus, any element $s \in K(\mathbb{P}^n)$ is a rational function homogeneous of degree 0 in $k(x_0, \dots, x_n)$.

3. We need two lemmas from commutative algebra.

Definition. Let R be a noetherian ring. The height of a prime ideal $\mathfrak{p}$ is the maximal length m of any chain of prime ideals $0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_m = \mathfrak{p}$. The height of an ideal I is the minimum of the height of any prime ideal containing I.

Lemma 1 (Krull's hauptidealsatz)**.** *Let R be a Noetherian ring and I be a proper ideal generated by m elements in R, then the height of I is at most m.*

Proof. See [Mat80, 12.I (Theorem 18)]. □

Lemma 2. *Let $S = k[x_0, \dots, x_n]$ be the polynomial ring and $0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_m \subsetneq S$ be a maximal chain of prime ideals in S. Then, $m = n + 1$.*

Proof. This is an easy consequence of Noether's normalization lemma [Eis95, 8.2.1, Theorem A1]. □

Suppose $n > m$ and suppose that $\deg g_0 = \dots = \deg g_m = d > 0$. Let $I = (g_0, \dots, g_m) \lhd k[x_0, \dots, x_n] =: S$. Since $I \subset (x_0, \dots, x_n) =: \mathfrak{m}$, it is a proper ideal, so by Krull's Hauptidealsatz (Lemma 1), the height of I is at most m. Hence, there exists a prime ideal $\mathfrak{p}$ of height m containing I.

By Lemma 2, there exists a maximal chain of prime ideals of the form

$$0 = \mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_m = \mathfrak{p} \subsetneq \dots \subsetneq \mathfrak{p}_{n+1} = \mathfrak{m} \subsetneq S.$$

In particular, $Z_{\text{proj}}(I) \supset Z_{\text{proj}}(\mathfrak{p}) \neq \emptyset$. Hence, there exists a point $p \in \mathbb{P}^n$ such that $f(p) = (g_0(p) : \dots : g_m(p)) = (0 : \dots : 0)$ is not well-defined.

We obtained a contradiction, hence we must have $d = 0$. In this case, g_i are constant functions, so the image of f is constant in $\mathbb{P}^m$.

□

Remark. Note that the second assertion in the proof is only true when the domain is $\mathbb{P}^n$. For a morphism $f : X \to \mathbb{P}^m$ where $X \subset \mathbb{P}^n$ is a projective variety, the sections s_i may not be generated by polynomials in $\mathbb{C}[x_0, \dots, x_n]/I(X)$.

Remark. The first and second assertions of the proof can also be rephrased in terms of twisted sheaves on $\mathbb{P}^n$, see [Har77, Theorem II.7.1]

References

[Eis95] D. Eisenbud. "Commutative Algebra with a View Toward Algebraic Geometry". In: Graduate Texts in Mathematics Vol. 150 (1995).

[Har77] R. Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics Vol. 52. Springer, 1977.

[Mat80] H. Matsumura. *Commutative Algebra, Second Edition.* Mathematics Lecture Note Series 56. Benjamin/Cummings Publishing Company, 1980.