Algebraic Geometry–Final Exam 31 May 2016

- Time allowed: 3 hours.
- You should answer all 6 questions.
- You may quote results from the lectures without proof in the exam. If you wish to use results from the exercises then you are expected to re-prove them in the exam.
- Pen and paper only allowed - no books, notes, calculators etc.
- Throughout, k denotes an algebraically closed field, and all varieties we consider are varieties over the field k.

1. (a) State the Nullstellensatz.

   (b) Let m and n be positive integers. Let $f_1, \dots, f_n \in k[x_1, \dots, x_m]$ and consider the morphism $\varphi \colon \mathbb{A}^m \to \mathbb{A}^n$ given by $\varphi(P) := (f_1(P), \dots, f_n(P))$. Show that φ is surjective if and only if for all $a_1, \dots, a_n \in k$ the ideal $(f_1 - a_1, \dots, f_n - a_n)$ is not the whole ring $k[x_1, \dots, x_m]$.

2. (a) Show that $\mathbb{P}^1 \times \mathbb{P}^1$ is a projective variety. That is, give a projective variety $Z \subset \mathbb{P}^N$ for some N together with an isomorphism $\psi \colon \mathbb{P}^1 \times \mathbb{P}^1 \to Z$.

   (b) Show that $\mathbb{P}^1 \times \mathbb{P}^1$ is irreducible.

   (c) Show that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ have isomorphic function fields.

   (d) Show that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are not isomorphic.

3. Let X be a smooth irreducible projective curve of genus g. Let D be a divisor on X. Let K be a canonical divisor on X.

   (a) State the Riemann-Roch theorem applied to D.

   (b) State Serre Duality applied to D.

   (c) Let $f \in K(X)^*$. Show that $\deg(\mathrm{div}(f)) = 0$.

   (d) Show that $H^0(X, D) = \{0\}$ if $\deg D < 0$.

   (e) Show that $\dim H^0(X, D) = \deg D + 1 - g$ if $\deg D > \deg K$.

   (f) Let $P \in X$. Show that $\mathcal{O}_X(X \setminus \{P\})$ is infinite dimensional.

   (g) Show that $X \setminus \{P\}$ is not projective.

4. (a) Let X be a variety, $x \in X$ a point, and $d \geq 0$ an integer. State what it means for X to be *smooth of dimension d* at x.

   (b) Let Y be the union of the coordinate axes in $\mathbb{A}^4$. Write down a set of generators for $I(Y)$. Hint: it may help to observe that $I(V \cup W) = I(V) \cap I(W)$ for closed subvarieties V, $W \subset \mathbb{A}^n$.

   (c) Let $Z \subset \mathbb{A}^2$ be given by the vanishing of $xy(x-y)(x+y)$. Show that Y and Z are not isomorphic.

Please turn over for the last two questions.

5. (a) Assume that k is an algebraic closure of a finite field $\mathbb{F}_q$ with q elements. Let $X = \mathbb{P}^1$ and let $F_X : \mathbb{P}^1 \to \mathbb{P}^1$ be the q-Frobenius endomorphism of X. Show that F_X is bijective, but not an isomorphism.

   (b) State the definition of a prime divisor on a smooth irreducible projective curve over $\mathbb{F}_q$.

   (c) Describe all prime divisors on $\mathbb{P}^1_{\mathbb{F}_q}$. Briefly justify your answer (a detailed proof is not required).

6. Note that in this course complex manifolds were not required to be Hausdorff or second countable.

   (a) State the definition of a k-space.

   (b) Give an example of a $\mathbb{C}$-space which is a complex manifold but is not an algebraic variety over $\mathbb{C}$. Prove your answer.

   (c) Give an example of a $\mathbb{C}$-space which is an algebraic variety over $\mathbb{C}$ but is not a complex manifold. Prove your answer.

Algebraic Geometry–Retake Exam 21 June 2016

- Time allowed: 3 hours.
- There are 6 questions, of which you should answer 5. If you answer more than 5, then only your best 5 answers will count.
- You may quote results from the lectures without proof in the exam. If you wish to use results from the exercises then you are expected to re-prove them in the exam.
- Pen and paper only allowed - no books, notes, calculators etc
- Throughout, k denotes an algebraically closed field, and all varieties we consider are varieties over the field k.

1. (a) If one identifies $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

   (b) Let X be the variety $\mathbb{A}^1 \times \mathbb{P}^1$. Prove that X is not affine and not projective.

2. Let X be an affine variety. For f in $\mathcal{O}(X)$ define $D(f) = \{p \in X \,:\, f(p) \neq 0\}$ and $Z(f) = X \setminus D(f) = \{p \in X \,:\, f(p) = 0\}$. Prove:

   (a) $X = \cup_{i \in I} D(f_i) \Leftrightarrow \cap_{i \in I} Z(f_i) = \varnothing \Leftrightarrow$ the f_i generate $\mathcal{O}(X)$ as an ideal.

   (b) The $D(f)$ with f running through $\mathcal{O}(X)$ form a basis for the Zariski topology on X.

   (c) X is compact, i.e., every covering of X with open subsets has a finite subcovering.

3. Let X be a smooth projective irreducible curve and D a divisor on X.

   (a) State the definition of $H^0(X, \mathcal{O}_X(D))$ and $H^1(X, \mathcal{O}_X(D))$.

   (b) Define the genus of X.

   (c) Compute the genus of $\mathbb{P}^1$.

4. (a) Let $U \subset \mathbb{A}^1$ be non-empty open and let $f\colon U \to \mathbb{A}^1$ be a morphism. Show that f extends uniquely as a morphism $\bar{f}\colon \mathbb{P}^1 \to \mathbb{P}^1$.

   (b) Let $P_1, \ldots, P_r, Q_1, \ldots, Q_s$ be distinct points of $\mathbb{A}^1$. Assume that $\mathbb{A}^1 \setminus \{P_1, \ldots, P_r\}$ and $\mathbb{A}^1 \setminus \{Q_1, \ldots, Q_s\}$ are isomorphic. Show that $r = s$.

   (c) Describe explicitly $\mathrm{Aut}(\mathbb{P}^1)$ (you don't need to prove your answer).

   (d) Does the converse hold to (b) hold? More precisely, is it true that for every positive integer r and every set $P_1, \ldots, P_r, Q_1, \ldots, Q_r$ of distinct points in $\mathbb{P}^2$ we have

   $$\mathbb{A}^1 \setminus \{P_1, \ldots, P_r\} \cong \mathbb{A}^1 \setminus \{Q_1, \ldots, Q_s\}?$$

5. Let X be a smooth irreducible projective curve of genus $g = 1$.

   (a) State the Riemann-Roch theorem for X.

   (b) State Serre Duality for X.

   (c) Show that the degree of a canonical divisor K on X is zero.

   (d) Let $\varphi\colon \mathrm{Div}\, X \to \mathrm{Pic}\, X$ be the map sending a divisor to its class modulo principal divisors. Show that the class $\varphi(K)$ of K in $\mathrm{Pic}\, X$ is zero.

   (e) Let $O \in X$. Show that the map $\psi\colon X \to \mathrm{Pic}^0 X$ given by $P \mapsto \varphi(P - O)$ is bijective (here we write $\mathrm{Pic}^0 X$ for the image of the degree-zero divisors in $\mathrm{Pic}\, X$).

6. Assume that k is an algebraic closure of a finite field $\mathbb{F}_q$ with q elements. Let X be a smooth irreducible projective curve defined over $\mathbb{F}_q$ and of genus g. For a divisor D on X we write $h^0(D) = \dim H^0(X, \mathcal{O}_X(D))$. Let

$$G(t) = \sum_{\substack{[D]\in \mathrm{Pic}\, X_0 \\ 0\leq \deg D\leq 2g-2}} q^{h^0(D)} t^{\deg D} \,.$$

(a) Assume that $X(\mathbb{F}_q) \neq \varnothing$. Show that $G(t)$ is a polynomial.

(b) Show that $G(t)$ satisfies the functional equation

$$G(t) = q^{g-1} t^{2g-2} G(1/qt) \,.$$