CONTRACTIONS AND CANONICAL MODELS

ENİS KAYA

Abstract. Notes for the TAG seminar at University of Leiden, Spring 2018. See http://pub.math.leidenuniv.nl/ holmesdst/teaching/2017-2018/TAG/index.html for further details.

1. Desingularization

Before introducing the notion of canonical model, we need to recall some basic facts from the theory of desingularization.

Definition 1.1. Let X be a reduced locally Noetherian scheme. A proper birational morphism $\pi : Z \to X$ with Z regular is called a *desingularization of* X (or a *resolution of singularities of* X). If π is an isomorphism above every regular point of X, we say that it is a *desingularization in the strong sense.*

Example 1.2. Let X be a reduced curve over a field k; the normalization $X' \to X$ is a desingularization. More generally, let X be an excellent, reduced, Noetherian scheme of dimension 1; then the normalization $X' \to X$ is a desingularization. Hence the problem of the existence of desingularizations essentially concerns schemes in higher dimensions.

Let X be an excellent, reduced, Noetherian scheme of dimension 2. Consider the following sequence of proper birational morphisms:

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \to X, \qquad (*)$$

where $X_1 \to X$ is the normalization of X, and for every $i \geq 1$, $X_{i+1} \to X_i$ is the composition of the blowing up $X_i^{'} \to X_i$ of the singular locus $\mathrm{Sing}(X_i) = X_i \setminus \mathrm{Reg}(X_i)$ (which is closed because X_i excellent) endowed with the reduced scheme structure, and of the normalization $X_{i+1} \to X_i^{'}$. The sequence stops at n when X_n is regular.

Theorem 1.3. *Let $X \to S$ be a fibered surface. Let us suppose that* $\dim S = 1$ *and that X_η is smooth. Then the sequence (*) is defined and finite. In particular, X admits a desingularization $Z \to X$ in the strong sense. Moreover, $Z \to X$ is projective if S affine.*

Proof. See [Liu, Theorem 8.3.50] and [Liu, Corollary 8.3.51]. □

The following classical example will be used in the main example (Example 3.7) of the talk.

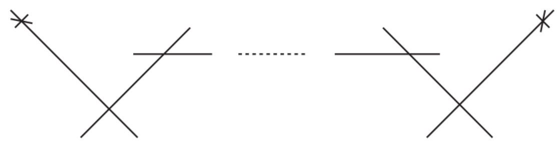
Date: April 19, 2018.

Example 1.4. Let $\mathcal{O}_K$ be a DVR with discrete valuation ν. Let us set $S = \text{Spec } \mathcal{O}_K$ and take $a \in \mathcal{O}_K$ such that $\nu(a) \geq 2$. Consider

$$X = \text{Spec } \mathcal{O}_K[x, y]/(xy - a).$$

Notice that X contains a unique singular point corresponding to the maximal ideal (t, x, y), say p.

It turns out that after $[e/2]$ successive blowing-ups of singular points, we end up with a desingularization $f : Z \to X$ of X. Moreover, the special fiber Z_s has $e + 1$ irreducible components and is of the form



where the components on the edges are affine lines over k (a cross means that we have removed a point from a projective curve) and the other components are projective lines over k. Furthermore, the fiber Z_p at the singular point p under f is the chain of the $e - 1$ projective lines given above. See [Liu, Example 8.3.53] for further details.

2. Contraction

Definition 2.1. Let $X \to S$ be a normal fibered surface. Let $\mathcal{E}$ be a finite set of integral vertical curves on X. A normal fibered surface $Y \to S$ together with a projective birational morphism $f : X \to Y$ such that for every integral vertical curve E on X, the set $f(E)$ is a point if and only if $E \in \mathcal{E}$ is called a *contraction of the* $E \in \mathcal{E}$.

The existence of a contraction morphism is a big deal, but it satisfies nice properties if it exists:

Proposition 2.2. *Let $X \to S$ be a normal fibered surface. Let $\mathcal{E}$ be a finite set of integral vertical curves on X and $\Delta = \cup_{E \in \mathcal{E}} E$. If a contraction $f : X \to Y$ of the $E \in \mathcal{E}$ exists, then it induces an isomorphism*

$$X \setminus \Delta \to Y \setminus f(\Delta)$$

and it is unique up to unique isomorphism.

Proof. By Zariski's connectedness principle ([Liu, Theorem 5.3.15]), for any $y \in f(\Delta)$, $f^{-1}(y)$ is connected of dimension 1. Therefore it is union of some $E \in \mathcal{E}$ which implies that $f^{-1}(f(\Delta)) = \Delta$. Hence

$$X \setminus \Delta = f^{-1}(Y \setminus f(\Delta)) \to Y \setminus f(\Delta)$$

is projective, quasi-finite, hence finite, and birational; so it is an isomorphism.

Let $f' : X \to Y'$ be another contraction. By above, we have isomorphisms

$$X \setminus \Delta \to Y \setminus f(\Delta), \quad X \setminus \Delta \to Y' \setminus f'(\Delta)$$

which give a unique birational map $g : Y \dashrightarrow Y'$ such that $f' = g \circ f$. It follows from [Liu, Corollary 8.3.23] that g is in fact an isomorphism. □

The proposition above says that Δ is the exceptional locus of f, see [Liu, Definition 7.2.21].

3. Canonical Models

Let $X \to S$ be a normal fibered surface and $\mathcal{E}$ be a finite set of integral vertical curves on X. By Proposition 2.2, if a contraction $f : X \to Y$ of the $E \in \mathcal{E}$ exists, then it is unique. This section is devoted to the existence of such morphism under additional conditions.

Proposition 3.1. *Let $X \to S$ be a regular fibered surface. Let $\Gamma_1, \dots, \Gamma_r$ be vertical prime divisors contained in a closed fiber X_s such that*

$$K_{X/S} \cdot \Gamma_i = 0, \quad i = 1, \dots, r,$$

and that the intersection matrix $(\Gamma_i \cdot \Gamma_j)_{1 \le i,j \le r}$ is negative definite. Then there exists a contraction morhism $f : X \to Y$ of the Γ_i.

Proof. This is [Liu, Corollary 9.4.7]. □

Proposition 3.2. *Let $X \to S$ be an arithmetic surface with $p_a(X_\eta) \ge 2$. Let $s \in S$ be a closed point, let Γ be an irreducible component of X_s, and $k' = H^0(\Gamma, \mathcal{O}_\Gamma)$. Then the following properties are equivalent.*

(1) $K_{X/S} \cdot \Gamma = 0$.
(2) $H^1(\Gamma, \mathcal{O}_\Gamma) = 0$ *and* $\Gamma^2 = -2[k' : k(s)]$.

Proof. The adjunction formula

$$p_a(\Gamma) = 1 + \frac{1}{2}(\Gamma^2 + K_{X/S} \cdot \Gamma)$$

(see [Liu, Theorem 9.1.37]) and the calculation

$$\begin{aligned} p_a(\Gamma) &= 1 - \chi_{k(s)}(\Gamma) \\ &= 1 - (\dim_{k(s)} H^0(\Gamma, \mathcal{O}_\Gamma) - \dim_{k(s)} H^1(\Gamma, \mathcal{O}_\Gamma)) \\ &= 1 - [k' : k(s)](\dim_{k'} H^0(\Gamma, \mathcal{O}_\Gamma) - \dim_{k'} H^1(\Gamma, \mathcal{O}_\Gamma)) \\ &= 1 + [k' : k(s)](-1 + \dim_{k'} H^1(\Gamma, \mathcal{O}_\Gamma)). \end{aligned}$$

give

$$\Gamma^2 + K_{X/S} \cdot \Gamma = 2[k' : k(s)](-1 + \dim_{k'} H^1(\Gamma, \mathcal{O}_\Gamma)).$$

Therefore (2) clearly implies (1).

Now assume that (1) holds. Then we have

$$\Gamma^2 = 2[k' : k(s)](-1 + \dim_{k'} H^1(\Gamma, \mathcal{O}_\Gamma)).$$

It is enough to show that $\Gamma^2 < 0$. Suppose $\Gamma^2 = 0$. Then Γ is a connected component of X_s ([Liu, Exercise 9.1.6]). By suitably replacing S, we may assume that X_s connected (see [Liu, Proposition 8.3.8]). Then

$$X_s = d\Gamma$$

for some $d \ge 1$. The equality

$$2p_a(X_\eta) - 2 = dK_{X/S} \cdot \Gamma$$

([Liu, Proposition 9.1.35]) implies that $p_a(X_\eta) = 1$, a contradiction. □

Definition 3.3. The vertical prime divisors Γ verifying condition (2) of the Proposition 3.2 and smooth over k' are called (-2)-*curves*.

We are ready to state the main result of the talk.

Proposition 3.4. *Let $X \to S$ be a minimal arithmetic surface with $p_a(X_\eta) \geq 2$. Set*

$$\mathcal{E} = \{\text{vertical prime divisors } \Gamma \text{ such that } K_{X/S} \cdot \Gamma = 0\}.$$

Then the following properties are true.

(1) *The set $\mathcal{E}$ is finite.*
(2) *There exists a contraction morphism $f : X \to Y$ of the $\Gamma \in \mathcal{E}$.*
(3) *If S is affine, then the sheaf $\omega_{Y/S}$ is ample.*

Proof. Set $K = K(S)$.

(1) By [Liu, Lemma 8.3.3] and [Liu, Corollary 8.3.6(d)], X_η is a l.c.i. projective curve over K. [Liu, Corollary 8.3.31] says that

$$\deg\, \omega_{X_\eta/K} = 2(p_a(X_\eta) - 1) > 0.$$

Again by [Liu, Corollary 8.3.6(d)], we have

$$\omega_{X/S}|_{X_\eta} = \omega_{X_\eta/K}.$$

Therefore $\omega_{X/S}$ is ample over X_η according to [Liu, Proposition 7.5.5]. Then by [Liu, Proposition 5.1.37(b)], there exists an open subscheme V of S such that $\omega_{X_V/V}$ is ample. Then

$$\mathcal{E} \subseteq \{\text{irreducible components of the } X_s,\ s \in S \setminus V\}.$$

But the latter is finite.

(2) As for Proposition 3.2, we see that the intersection matrix of the components of $\mathcal{E}$ is negative definite. Therefore there exists a contraction morphism $f : X \to Y$ of the $\Gamma \in \mathcal{E}$ by virtue of Proposition 3.1.

(3) Take $s \in S$ closed. Let Γ be an irreducible component of Y_s and Γ' its strict transform in X. Then the restriction

$$h : \Gamma' \to \Gamma$$

of f is a finite birational morphism. By [Liu, Corollary 9.4.18], we have

$$\omega_{X/S}|_{\Gamma'} \simeq (f^*\omega_{Y/S})|_{\Gamma'} = h^*(\omega_{Y/S}|_\Gamma)$$

which gives

$$\deg\, \omega_{Y/S}|_\Gamma = \deg\, h^*(\omega_{Y/S}|_\Gamma) = \deg\, \omega_{X/S}|_{\Gamma'}$$

thanks to [Liu, Proposition 7.3.8]. As $\Gamma' \notin \mathcal{E}$, $\deg\, \omega_{X/S}|_{\Gamma'} > 0$ so that $\deg\, \omega_{Y/S}|_\Gamma > 0$. Then $\omega_{Y/S}|_{Y_s}$ is ample by [Liu, Proposition 7.5.5]. By virtue of [Liu, Corollary 5.3.24] (this is the step that requires the assumption S is affine), the sheaf $\omega_{Y/S}$ is ample. □

Definition 3.5. Let $X \to S$ be a minimal arithmetic surface with $p_a(X_\eta) \geq 2$. Let $f : X \to Y$ be the contraction of the vertical prime divisors Γ such that $K_{X/S} \cdot \Gamma = 0$ (which exists by the proposition above). The surface $Y \to S$ is called the *canonical model* of X.

Remark 3.6. According to Proposition 3.4, the canonical sheaf $\omega_{Y/S}$ is ample if S is affine and that is great news! The prize we have to pay for this is that the canonical model is singular as soon as there exists at least one contracted component. Life is not that simple...

Example 3.7. Let $\mathcal{O}_K$ be a DVR with uniformizing parameter t, field of fractions K and residue field k of $\mathrm{char}(k) \neq 2, 3$. Let us set $S = \mathrm{Spec}\ \mathcal{O}_K$ and fix $n \geq 1$. Let us consider the scheme X_0 over S, normalization of

$$\mathbb{P}^1_{\mathcal{O}_K} = \mathrm{Spec}\ \mathcal{O}_K[x] \cup \mathrm{Spec}\ \mathcal{O}_K[1/x]$$

in

$$K(x)[y]/(y^2 - (x^2 + t^n)(x^3 + 1)).$$

Then X_0 is the union of the open subschemes

$$U = \mathrm{Spec}\ \mathcal{O}_K[x, y]/(y^2 - (x^2 + t^n)(x^3 + 1)),$$

$$V = \mathrm{Spec}\ \mathcal{O}_K[x_1, y_1]/(y_1^2 - x_1(1 + t^n x_1^2)(1 + x_1^3)),\ x_1 = 1/x,\ y_1 = y/x^3.$$

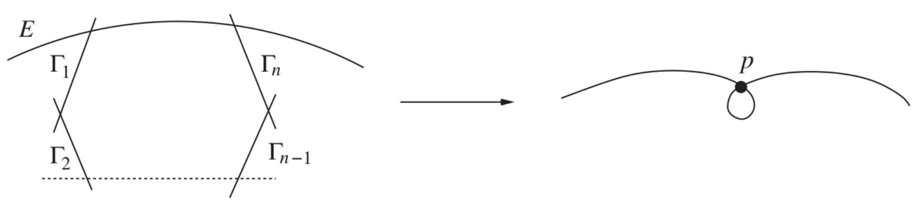
Notice that V is smooth over S and U contains a unique singular point corresponding to the maximal ideal (t, x, y), say p. By Theorem 1.3, X_0 admits a desingularization $X \to X_0$ in the strong sense and it is obtained by blowing-up and normalizing. Moreover it is projective since S is affine.

Let $z \in \widehat{\mathcal{O}}_{U,p}$ be a square root of $1 + x^3$ ([Liu, Exercise 1.3.9]). We see that

$$\widehat{\mathcal{O}}_{U,p} \simeq \widehat{\mathcal{O}}_K[[x, v]]/((v - x)(v + x) - t^n),\ v = y/z.$$

By Example 1.4, X_p is made up of a chain of n projective lines $\Gamma_1, \dots, \Gamma_n$ over k. With the help of [Liu, Proposition 9.1.21(b)], we have $\Gamma_i^2 = -2$.

The normalization of the irreducible singular curve U_s is an elliptic curve E ([Liu, Exercise 4.1.18]). After the blowing ups, what we get as the special fiber is a normal crossing divisor, so its irreducible components are regular ([Liu, Exercise 9.1.3]). Thus the strict transform of U_s is normal and therefore equal to its normalization E. Hence X_s is the union of the elliptic curve E and of the Γ_i.



By Castelnuovo's criterion ([Liu, Theorem 9.3.8]), X does not contain any exceptional divisor, i.e., it is relatively minimal. Moreover, by [Liu, Corollary 9.3.24], it is minimal. Thus, we can talk about the canonical model of X.

We will show that the canonical model of X is none other than X_0 itself. By adjunction formula ([Liu, Theorem 9.1.37]),

$$0 = p_a(\Gamma_i) = 1 + \frac{1}{2}(\Gamma_i^2 + K_{X/S} \cdot \Gamma_i),\ i = 1, \dots, n$$

so that

$$K_{X/S} \cdot \Gamma_i = 0,\ i = 1, \dots, n.$$

Combining these equalities with [Liu, Proposition 9.1.35] we have

$$2p_a(X_\eta) - 2 = K_{X/S} \cdot X_s = K_{X/S} \cdot E$$

so that $K_{X/S} \cdot E \neq 0$. Hence the set $\mathcal{E}$ of vertical prime divisors Γ such that $K_{X/S} \cdot \Gamma = 0$ is precisely $\{\Gamma_1, \dots, \Gamma_n\}$. By Proposition 2.2, the canonical model of X is X_0. Notice that the canonical model is singular.

Let $X \to S$ be a minimal arithmetic surface with $p_a(X_\eta) \geq 2$ and let $Y \to S$ be the canonical model of X. As we have seen in the Example 3.7, we may lose the regularity of the surface. On the other hand, we gain in the simplicity of the closed fibers. More precisely, for a closed point $s \in S$, the closed fiber X_s can have as many irreducible components as we want as we have seen in the example above; but this is not true for Y_s:

Proposition 3.8. *Let us keep the notation above and let n be the number of irreducible components of Y_s. Then*

$$n \leq 2p_a(X_\eta) - 2.$$

Proof. By [Liu, Corollary 9.4.18] Y is a l.c.i. over S. By [Liu, Corollary 6.3.24] this implies that Y_s is a l.c.i. over $k(s)$. Moreover, by [Liu, Theorem 6.4.9(b)], we have

$$\omega_{Y/S}|_{Y_s} = \omega_{Y_s/k(s)}.$$

Let us calculate the degrees. [Liu, Corollary 7.3.31] and [Liu, Proposition 5.3.28] give that

$$\deg_{k(s)}\omega_{Y_s/k(s)} = -2\chi_{k(s)}(\mathcal{O}_{Y_s}) = -2\chi_{k(\eta)}(\mathcal{O}_{Y_\eta}) = 2p_a(X_\eta) - 2.$$

Let $F_1, \ldots, F_n$ be the irreducible components of Y_s with respective multiplicities $d_1, \ldots, d_n$. By Proposition 3.4(b) $\omega_{Y/S}$ is ample. Since the restriction of an ample divisor to a closed subscheme remains ample, $\omega_{Y/S}|_{F_i}$ is ample for all i. This implies, by [Liu, Proposition 7.5.5], that $\deg_{k(s)}\omega_{Y/S}|_{F_i} > 0$. We then have

$$\deg_{k(s)}\omega_{Y/S}|_{Y_s} = \sum_{1\leq i\leq n} d_i \deg_{k(s)}(\omega_{Y/S}|_{F_i}) \geq \sum_{1\leq i\leq n} d_i \geq n$$

by virtue of [Liu, Proposition 7.5.7]. □

References

[Liu] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics **6**, Oxford University Press, 2002.

Enis Kaya, Johann Bernoulli Institute, University of Groningen Nijenborgh 9, 9747 AG Groningen, The Netherlands

E-mail address: e.kaya@rug.nl