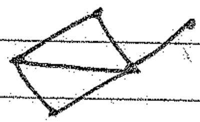
Graphs, spanning trees & determinants

A <u>graph</u> consists of a finite set of points (vertices), & edges between some of the vertices:



- at most one edge between 2 vertices - not
- edges must go between different vertices: not

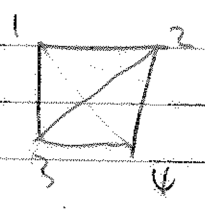
Label the vertices with integers 1, 2, ...
Then make the <u>laplacian matrix</u>:

Say n vertices. Then laplacian is a ~~(symmetric)~~ $n \times n$ matrix L

If $i \neq j$ then $l_{ij} = \begin{cases} -1 & \text{if } \exists \text{ edge from } i \text{ to } j \\ 0 & \text{else.} \end{cases}$

We choose the l_{ii} s.t. each row sums to zero.

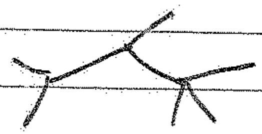
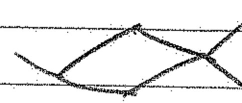
<u>eg</u>:



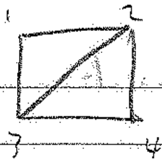
	1	2	3	4
1	2	-1	-1	0
2	-1	3	-1	-1
3	-1	-1	3	-1
4	0	-1	-1	2

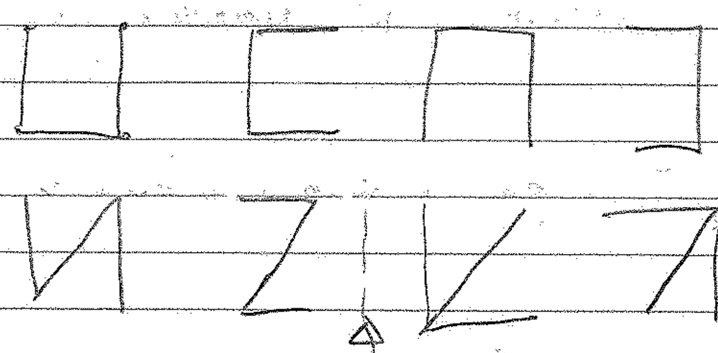
(note: symmetric)

A <u>tree</u> in a graph is a collection of edges which together contain no 'cycles', and which together form a connected subgraph.

eg.  <u>not</u> 

A <u>spanning tree</u> is a tree which visits every vertex (so every vertex is connected to every other vertex).

eg. Graph is    egs of spanning trees:



Are there any others? Ask here first!

In general, how to compute # spanning trees?
Can try all subsets of edges. Eg if graph has 1000 edges then must try 2^{1000} subsets - not practical!
$\approx 10^{300}$

<u>Thm</u>: let L be laplacian of connected graph G. Then # spanning trees = any cofactor of L.
Pf omitted, ~see Wikipedia

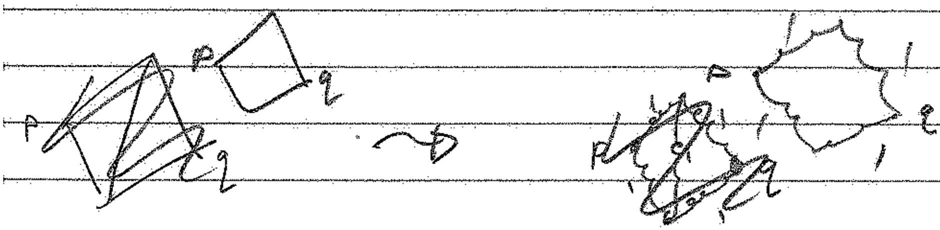
eg: $\text{cof}_{11} L$  again.

$$\text{cof}_{11} L = (-1)^{1+1} \det\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} = 3(6-1) + 1(-2-1) - 1(1+3)$$
$$= 15 - 3 - 4 = 8 \checkmark$$

$$\text{cof}_{14} L = (-1)^{1+4} \det\begin{pmatrix} -1 & 3 & -1 \\ -1 & -1 & 3 \\ 0 & -1 & -1 \end{pmatrix} = -\big(0(\text{stuff}) - (-1)(-3-1) + (-1)(1+3)\big)$$
$$= 8 \checkmark$$

Connection to ~~resista~~ electrical resistance.
(elektrische weerstand).

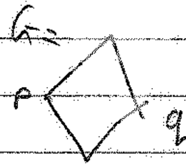
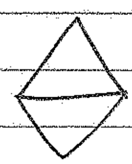
~~Given graph~~ Graph ⇝ electrical network, replace each edge with a 1-ohm resistor!



What is resistance between p & q? (Ans: $\frac{1}{\frac{1}{2}+\frac{1}{2}} = 1$)
In a big graph, not so easy.

Thm: ~~As~~ Let p, q be two vertices with no edge between them. Let G' be graph obtained by putting an edge between them.
Let T = # spanning trees in G
T' = — — — G'.
Then resistance between p & q $= \dfrac{\#T' - \#T}{\#T}$.

compute w/ dets.

eg. G =  G' =  T' = 8 (from before)

T = 4

⇒ ~~Res~~ res(p, q) $= \dfrac{8-4}{4} = 1$ ✓.

(∃ version allowing edge, or can adapt this).