Final Exam – Lineaire Algebra 1

12 January 2018

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 6 question, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

1. Consider the matrices M and A given by

$$M = \begin{bmatrix} -4 & -2 & 1 & -1 \\ 3 & 2 & 1 & -3 \\ 1 & 1 & -2 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -4 & -2 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & -2 \end{bmatrix}.$$

   a) Put the matrix M in reduced echelon form.

   b) Is the vector $\begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$ in the subspace of $\mathbb{R}^3$ spanned by the columns of A?

   c) Does the homogeneous linear system $A\underline{x} = \underline{0}$ have a non-trivial solution?

2. Define matrices A and B by

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 2 & -1 & -1 & 2 \\ 1 & 3 & -2 & 1 \\ -3 & 0 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 3 & 2 & 0 \\ -1 & 3 & 2 & -1 \\ -1 & 1 & 3 & 3 \\ -2 & -1 & -1 & 2 \end{bmatrix}$$

   a) Compute the determinant of A.

   b) Compute the determinant of B.

   c) What is the determinant of A^{-1}?

   d) What is the determinant of A^2B^3?

3. To answer this question, you will need the extra sheet. Please write your answers to this question on that sheet (you can use the back if you need it). Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

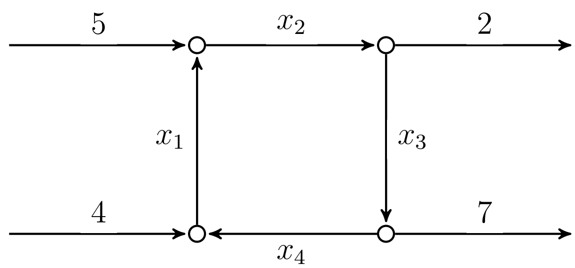
   a) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A (so that the standard matrix of T is A). Draw the image of the triangle in figure (a) on the extra sheet under the linear transformation T. You should draw your answer on the same figure.

b) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A^3. Draw the image under T of the triangle from figure (a) on figure (b).

c) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A^4. Draw the image under T of the triangle from figure (a) on figure (c).

d) Notice that $2018 = 2 + 4 \times 504$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by the matrix A^{2018}. Draw the image under T of the triangle from figure (a) on figure (d).

4. Consider the following network:



a) Write down a linear system describing the flow in the network.

b) Put the augmented matrix of the linear system from (a) in row reduced echelon form.

c) Can you find a solution where all the flows are positive?

5. Consider the set of vectors in $\mathbb{R}^2$ given by

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ a \end{bmatrix} \right\}$$

where a is a real number.

(a) Describe all the values of a for which the set S *linearly independent*.

(b) Describe all the values of a for which the span of the set S equal to the whole of $\mathbb{R}^2$.

[Hint: in parts (a) and (b), if you cannot determine all the values of a, you will still get some points for trying a few values of a].

Now consider the set of vectors in $\mathbb{R}^3$ given by

$$T = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(c) Does the set T span the whole of $\mathbb{R}^3$?

6. For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

    a) If $\underline{u}$ and $\underline{v}$ are both solutions to the homogeneous linear system $A\underline{x} = \underline{0}$ then $\underline{u} + \underline{v}$ is also a solution.

    b) If A and B are both 3×3 matrices and AB is the zero matrix then at least one of A and B is the zero matrix.

    c) If A is an invertible 4×4 matrix then $\det(2A) = 2\det(A)$.

    d) Every linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ is surjective.

    e) There exists a surjective linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$.