Retake Exam – Lineaire Algebra 2

10 July 2018

Time: 3 hours.

Fill in your name and student number on all papers you hand in.

In total there are 6 questions, and each question is worth the same number of points.

In all questions, justify your answer fully and show all your work.

In this examination you are only allowed to use a pen and examination paper.

1. Define a matrix A and a vector $\mathbf{b}$ by

$$A = \begin{bmatrix} 3 & 5 & -1 & 2 \\ -1 & -2 & 4 & -4 \\ 1 & 2 & -4 & 4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}.$$

   a) Write down the augmented coefficient matrix of the linear system $A\mathbf{x} = \mathbf{b}$.

   b) Perform the row reduction algorithm on the augmented coefficient matrix, putting it in reduced row echelon form. You answer will be one of

$$M = \begin{bmatrix} 1 & 0 & 18 & -16 & 20 \\ 0 & 1 & -11 & 10 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 & 17 & -17 & 20 \\ 0 & 1 & -12 & 11 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   Which is it? *You should answer M or N.*

   c) Write the general solution of the matrix equation $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

2. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_3 - 4x_4 \\ x_2 - 2x_3 + 2x_4 \\ x_2 - 2x_3 + x_4 \end{bmatrix}.$$

   a) Write down the standard matrix for T.

   b) Write down a basis for the kernel of T. Is T injective (one-to-one)? Why?

   c) Write down a basis for the image of T. Is T surjective (onto)? Why?

3. (a) Let $M = \begin{bmatrix} 3 & 8 & 1 \\ 0 & 2 & 0 \\ 3 & 2 & 2 \end{bmatrix}$. What is $\det M$?

   A certain 4×4 matrix N has determinant 16.

   (b) Let A be the matrix obtained from N by swapping the first two rows. What is $\det A$?

   (c) Let B be the matrix obtained from N by adding 7 times the first row to the second row. What is $\det B$?

   (d) Let C be the matrix obtained by dividing every row in N by 2. What is $\det C$?

4. For each of the following 5 statements, say whether the statement is true or false. Justify your answer (either by an example if the statement is false, or a brief proof if it is true). If you are unsure, try some small examples.

   a) If $T\colon U \to V$ is a linear transformation and $\underline{u}_1, \underline{u}_2 \in U$, then $T(\underline{u}_1 - \underline{u}_2) = T(\underline{u}_1) - T(\underline{u}_2)$.

   b) Every 2×2 matrix is diagonalisable.

   c) If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are vectors in $\mathbb{R}^3$ with $\underline{a}$ orthogonal to $\underline{b}$ and $\underline{b}$ orthogonal to $\underline{c}$, then it must hold that $\underline{a}$ is orthogonal to $\underline{c}$.

   d) If B is a basis of $\mathbb{R}^3$ and S is a spanning set for $\mathbb{R}^3$, then S must contain some element of B.

   e) If $\underline{u}$ and $\underline{v}$ are non-zero vectors in $\mathbb{R}^n$ and $\underline{u}$ is a scalar multiple of $\underline{v}$, then the inner product $\underline{u} \cdot \underline{v}$ is greater than zero.

5. Define a matrix A by
$$A = \begin{bmatrix} 4 & 3 & 4 \\ -4 & -3 & 4 \\ 2 & -3 & 1 \end{bmatrix}.$$

   a) Use the Gram-Schmidt algorithm to find an orthogonal set of vectors with the same span as the columns of A. *Hint: in your final set of vectors, all the entries should be integers.*

   b) Are the columns of A linearly independent?

   c) What is the rank of A?

   d) Are the rows of A linearly independent?

6. Let A be the matrix
$$A = \begin{bmatrix} 5 & -6 \\ 4 & -5 \end{bmatrix}.$$

   a) Show that A has eigenvalues 1 and -1.

   b) For each eigenvalue in (a) give a basis for the corresponding eigenspace.

   c) Give an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

   d) Calculate A^{100}.