Eigenvalues and Eigenvectors

So far:

- Definitions:

M an $n \times n$ matrix over field $k = \mathbb{R}$

Eigenvector: **nonzero** $v \in k^n$ such that there exists $\lambda \in k$ with $Mv = \lambda v$

Eigenvalue: $\lambda \in k$ such that there exists **nonzero** $v \in k^n$ with $Mv = \lambda v$

Eigenspace: $W_\lambda \subseteq k^n$, the set of eigenvectors with a given eigenvalue λ (linear subspace of k^n)

- Computation:

Eigenvalues are roots in k of *characteristic polynomial:* $\det(M - \lambda I_n)$

To find eigenvectors for eigenvalue λ, solve the matrix equation $Mv = \lambda v$ (Gaussian elimination/row reduction)

Why? Fish in a pond

- In year n we have a adult and b baby fish
- In year $n+1$ we have
  $0.9a + 0.2b$ adults
  $0.5a + 0.3b$ babies
- Let $M = \begin{bmatrix} 0.9 & 0.2 \\ 0.5 & 0.3 \end{bmatrix}$
- Population modelled by linear function
  $T\colon \mathbb{R}^2 \to \mathbb{R}^2$
  $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto M \begin{bmatrix} a \\ b \end{bmatrix}$

- Eigenvalues and vectors of M (approximate):

$$\lambda_1 = 1.04, \;\; v_1 = \begin{bmatrix} 83 \\ 56 \end{bmatrix}$$

$$\lambda_2 = 0.16, \;\; v_2 = \begin{bmatrix} -26 \\ 96 \end{bmatrix}$$

What happens in long term?

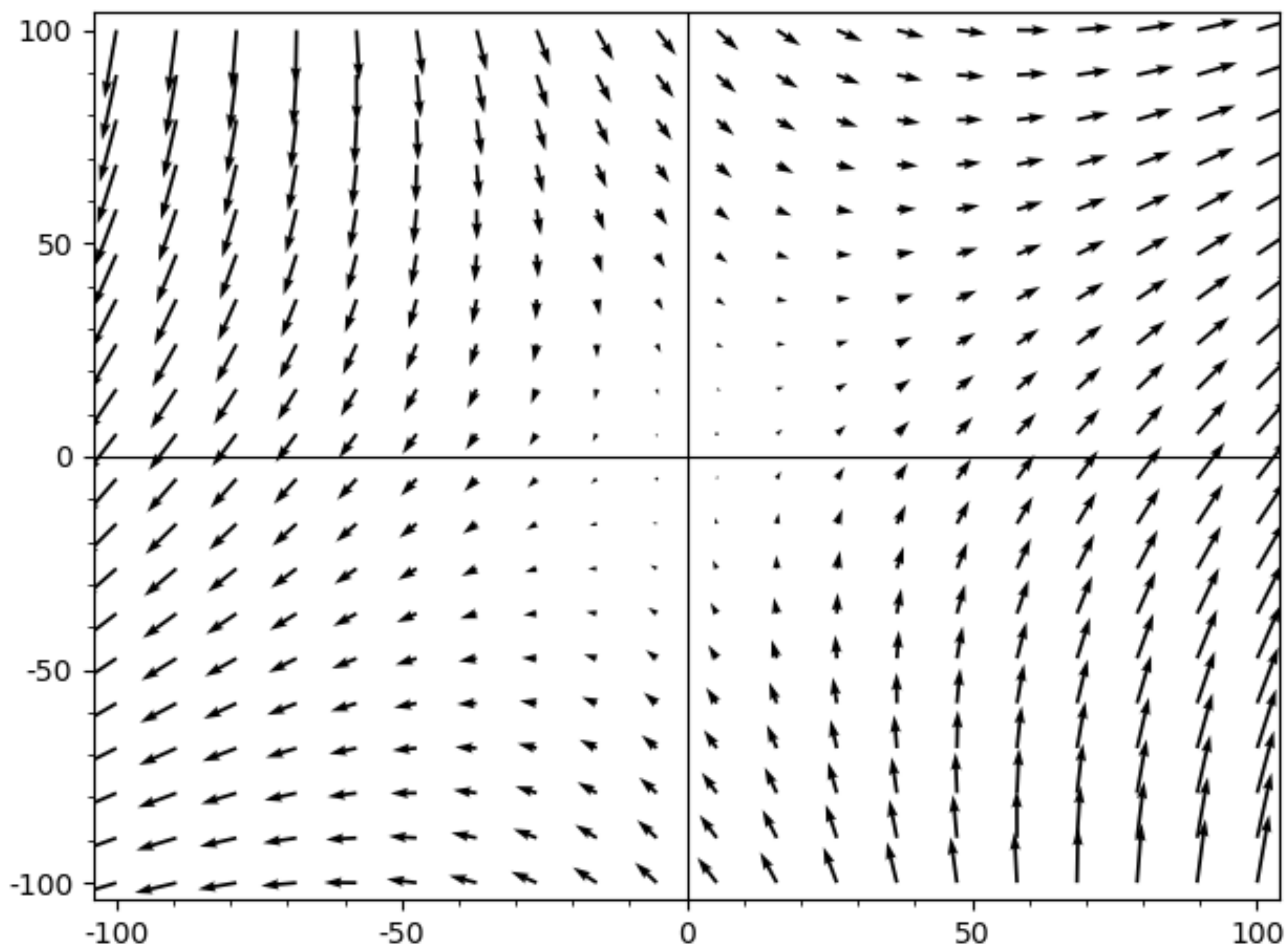
- Suppose the population is
  $a = 83$
  $b = 56$

- The population vector is (roughly) an eigenvector with eigenvalue $\lambda_1 = 1.04$

- So after r years, the population will be
  $$M^r \begin{bmatrix} 83 \\ 56 \end{bmatrix} = M^{r-1} \lambda_1 \begin{bmatrix} 83 \\ 56 \end{bmatrix} = \cdots = \lambda_1^r \begin{bmatrix} 83 \\ 56 \end{bmatrix} = 1.04^r \begin{bmatrix} 83 \\ 56 \end{bmatrix}$$

- Grows slowly over time.

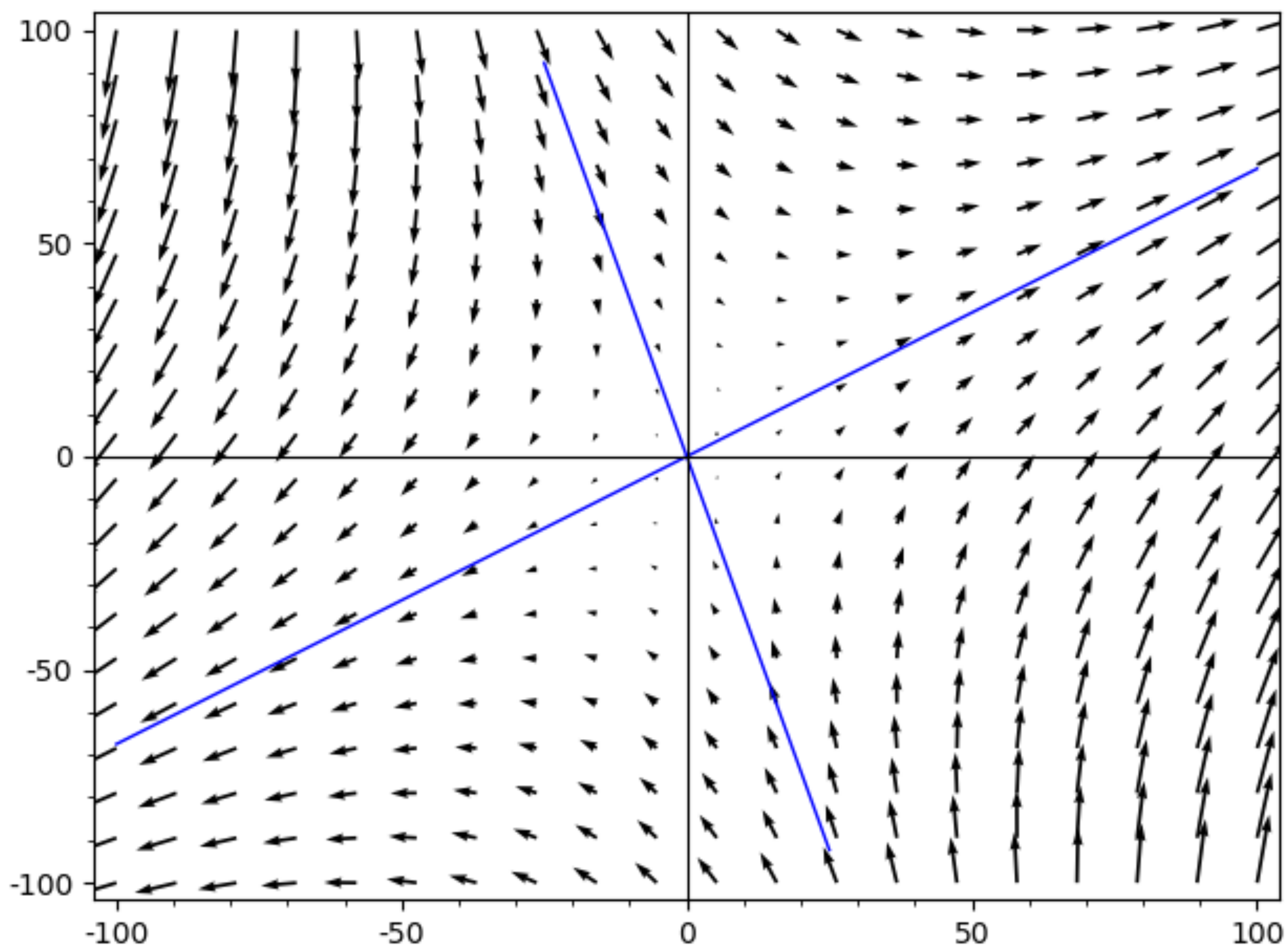
Graphical representation:



Stable population, slowly growing

Graphical representation:

$\lambda = 0.16$



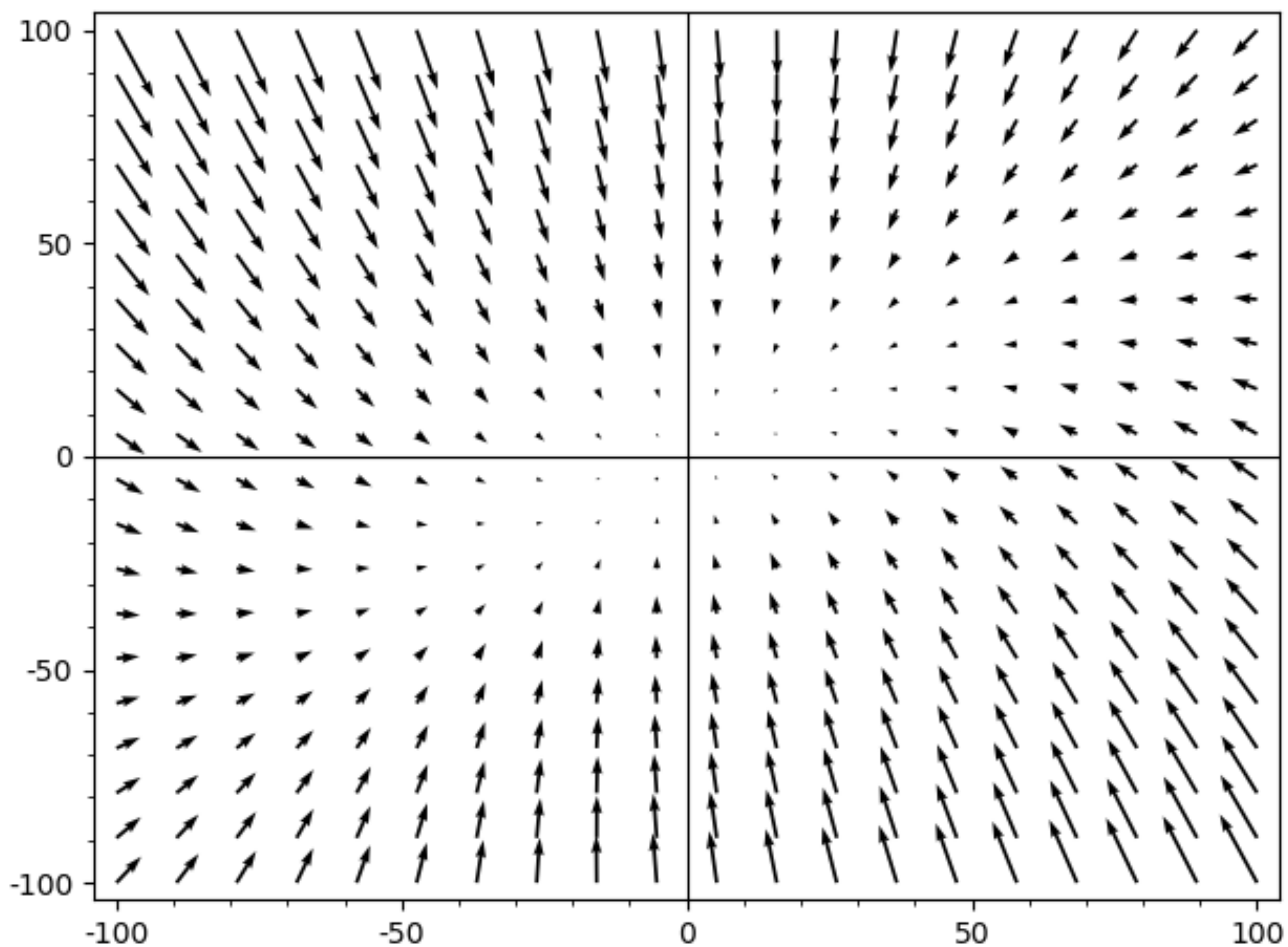
$\lambda = 1.04$

Converges to ratio $a/b = 83/56$

Effect of pollution

- Suppose some pollution is introduced to our pond, making fewer babies grow to adulthood
- New matrix
$$M' = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.3 \end{bmatrix}$$
- Eigenvectors *roughly* the same
- $\lambda_2 \approx 0.16$, *roughly* unchanged
- $\lambda_1 = 0.97$, slightly smaller (was 1.04)
- $(M')^r \begin{bmatrix} 83 \\ 56 \end{bmatrix} = \cdots = 0.97^r \begin{bmatrix} 83 \\ 56 \end{bmatrix}$, slowly goes to zero

Graphical representation (polluted):

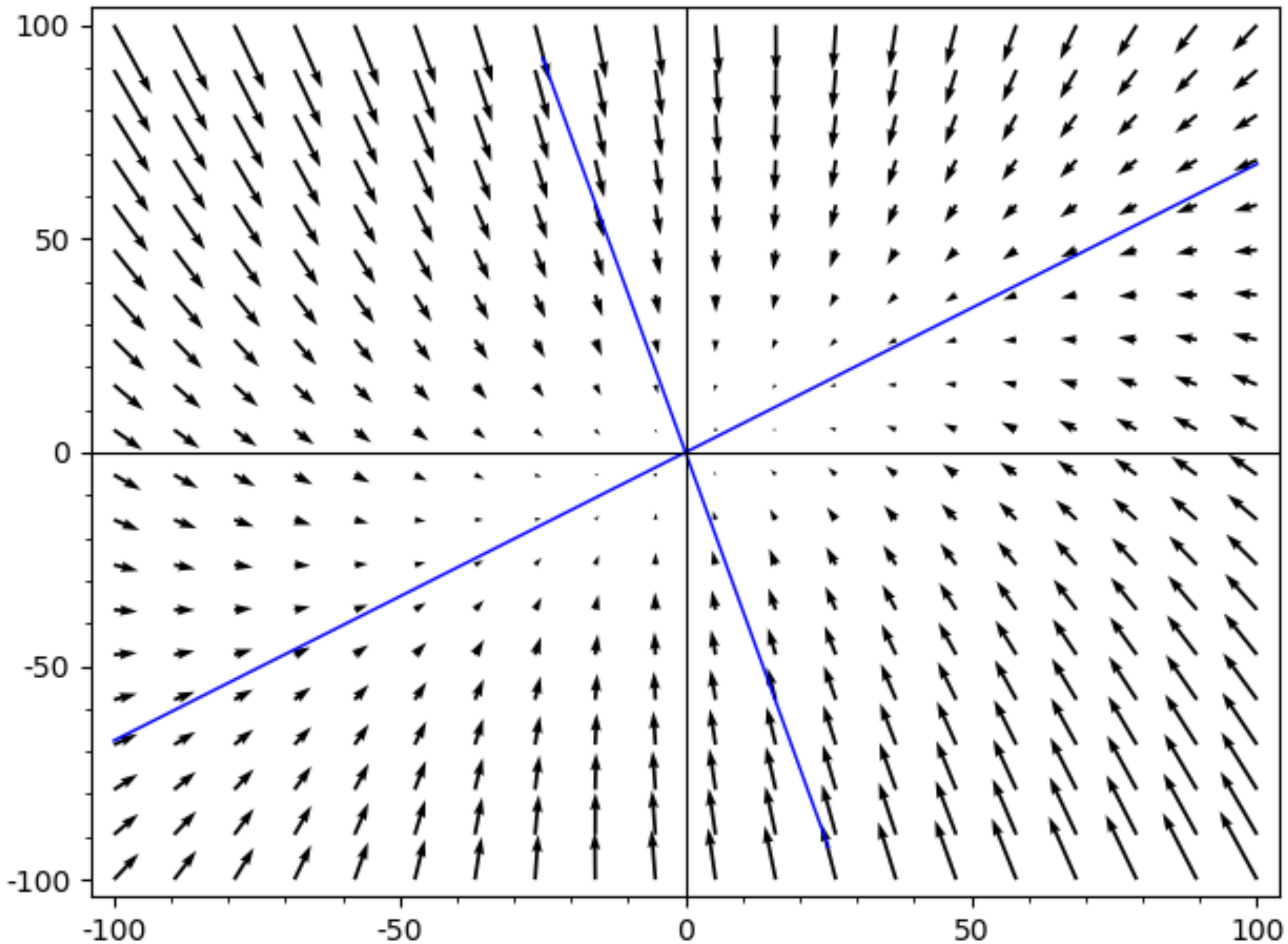


Population always collapses to zero!

Graphical representation (polluted):

$\lambda = 0.16$

$\lambda = 0.97$



Population always collapses to zero!

Conclusion

Take-away:

- small changes in parameters can have a big effect on long-term behaviour of linear models
- eigenvalues and vectors are a powerful tool to detect this
- especially important if we have more variables (male and female fish?)

Next:

- diagonalisation using eigenvalues and vectors
- using diagonalisation to compute large powers of a matrix
- computing the population of our pond in 1000 years, fast