Homework Solutions

1) Not orthogonal: $\begin{bmatrix}-1\\4\\-3\end{bmatrix} \cdot \begin{bmatrix}3\\-4\\-7\end{bmatrix} = -3-16+21 \neq 0.$

2) Is orthogonal:

$$\begin{bmatrix}1\\-2\\1\end{bmatrix}\begin{bmatrix}0\\1\\2\end{bmatrix} = -2+2=0, \quad \begin{bmatrix}1\\-2\\1\end{bmatrix}\begin{bmatrix}-5\\-2\\1\end{bmatrix} = -5+4+1=0,$$

$$\begin{bmatrix}0 & -5\\1 & -2\\2 & 1\end{bmatrix} = -2+2=0.$$

7) $\begin{bmatrix}2\\-3\end{bmatrix} \cdot \begin{bmatrix}6\\4\end{bmatrix} = 0$, so linear indep., so span R^2 since dimension $(R^2)=2$.

$$\underline{x} = \left(\frac{\underline{x}\cdot\underline{u}_1}{\underline{u}_1\cdot\underline{u}_1}\right)\underline{u}_1 + \left(\frac{\underline{x}\cdot\underline{u}_2}{\underline{u}_2\cdot\underline{u}_2}\right)\underline{u}_2$$

$$= \frac{39}{13}\,\underline{u}_1 + \frac{26}{52}\,\underline{u}_2 = 3 u_1 + 0.5 u_2$$

12) $\text{proj}_{\text{span}\left(\begin{bmatrix}-1\\3\end{bmatrix}\right)} \begin{bmatrix}1\\-1\end{bmatrix} = \left(\dfrac{\begin{bmatrix}1\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\3\end{bmatrix}}{\begin{bmatrix}-1\\3\end{bmatrix}\begin{bmatrix}-1\\3\end{bmatrix}}\right) \cdot \begin{bmatrix}-1\\3\end{bmatrix} = \dfrac{-4}{10}\begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}2/5\\-6/5\end{bmatrix}$

17) Is orthogonal, normalized set is

$$\left\{ \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}, \begin{bmatrix}-\sqrt{2}\\0\\\sqrt{2}\end{bmatrix} \right\}$$

18) Not orthogonal.

2b) Non-zero orthogonal vectors are lin. indep, so we have n lin. indep vectors in $\mathbb{R}^4$, so they span $\mathbb{R}^4$.

3) $\underline{u}_1 \cdot \underline{u}_2 = -1+1=0$, so orthogonal.

$$\text{proj}_{\langle \underline{u}_1, \underline{u}_2 \rangle} \underline{y} = \left(\frac{\underline{y}\cdot\underline{u}_1}{\underline{u}_1 \underline{u}_1}\right)\underline{u}_1 + \left(\frac{\underline{y}\cdot\underline{u}_2}{\underline{u}_2 \underline{u}_2}\right)\underline{u}_2$$

$$= \left(\frac{3}{2}\right)\begin{bmatrix}1\\1\\0\end{bmatrix} + \left(\frac{5}{2}\right)\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\4\\0\end{bmatrix}$$

2(a) True. If $\underline{w} \in W$ then $\underline{w} = a_1\underline{u}_1 + a_2\underline{u}_2$,

& $\underline{z}\cdot\underline{w} = a_1 \underline{z}\cdot\underline{u}_1 + a_2 \underline{z}\cdot\underline{u}_2 = 0$, so $\underline{z} \in W^\perp$.

b) True. ~~Let $\underline{w} \in W$. Then~~ This is the definition of $\text{proj}_W \underline{y}$.

~~$\underline{w}\cdot(\underline{y} - \text{proj}_W \underline{y})$~~

c) False. It is the unique $\hat{\underline{y}} \in W$ s.t. $\exists\ \underline{z} \in W^\perp$ with $\underline{y} = \hat{\underline{y}} + \underline{z}$.

d) True. Take $\hat{\underline{y}} = \underline{y}$ & $\underline{z} = \underline{0}$ $\in W^\perp$, then $\underline{y} = \hat{\underline{y}} + \underline{z}$, done by uniqueness.

9) We apply Gram-Schmidt.

$$\underline{v}_1 = \underline{w}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\underline{v}_2 = \underline{w}_2 - \operatorname{proj}_{\langle \underline{v}_1 \rangle} \underline{w}_2 = \underline{w}_2 - \left(\frac{\underline{v}_1 \underline{w}_2}{\underline{v}_1 . \underline{v}_1} \right) \underline{v}_1$$

$$= \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \left(\frac{-40}{20} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\underline{v}_3 = \underline{w}_3 - \operatorname{proj}_{\langle \underline{v}_1, \underline{v}_2 \rangle} \underline{w}_3 = \underline{w}_3 - \left(\frac{\underline{v}_1 \underline{w}_3}{\underline{v}_1 \underline{v}_1} \right) \underline{v}_1 - \left(\frac{\underline{v}_2 \underline{w}_3}{\underline{v}_2 \underline{v}_2} \right) \underline{v}_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \left(\frac{30}{20} \right) \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \left(\frac{-10}{20} \right) \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

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