EXERCISES FOR LECTURES ON THE DOUBLE RAMIFICATION CYCLE

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Abstract. Exercises for one day of a summer school for GQT in den Dolder, 2019

Contents

- There are lots of exercises — more than you can or should try to do in one day. Moreover, it is likely that some of them will be too easy for you, and some too hard. So try to pick ones that look non-obvious but doable. Doing parts of questions is of course fine - the goal is to learn things.
- The exercises are not in order of increasing difficulty (the latter depends somewhat on your background), instead I have tried somewhat to organise them thematically.
- In the lectures I will suggest many exercises; some of these will be reproduced here, others not.
- If you get stuck, talk to people, and feel free to ask for help!
- Unlike homework exercises for a lecture course, I have not written careful solutions for these. Some need knowledge not covered in the lectures. There may be mistakes... So if in doubt, or something seems harder than you expect, please ask.

1. Lecture 1

1.1 Prove Yoneda's lemma. First you should work out what the functor should do on morphisms (I only defined it on objects). If you get stuck take a look at `https://en.wikipedia.org/wiki/Yoneda_lemma`.

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1.2 Let C be a compact connected Riemann surface (or a smooth proper geometrically connected connected curve over a field k) of genus 1 with marked points $p_1, \dots, p_n$. Show that

$$\sum_{i=1}^{n} m_i p_i \sim qK_C \iff \sum_{i=1}^{n} m_i[p_i] = 0. \tag{1.0.1}$$

Here:

- $\sim$ denotes linear equivalence;
- K_C is the canonical divisor (AKA zero, since the genus is 1);
- On the right hand side, the sim is taken in the group law on the elliptic curve (C, p_1).

1.3 Let $S = \operatorname{Spec} \mathbb{Q}$, and define a curve C by the equation $x^2+y^2+z^2 = 0 \subseteq \mathbb{P}^2_{\mathbb{Q}}$.

(a) Show that C is smooth, projective, geometrically connected and of genus 0;

Consider the functor

$$\operatorname{Pic}_2 : \mathbf{Sch}^{op}_{\mathbb{Q}} \to \mathbf{Set}; T \mapsto \{\text{line bundles on } C \times_{\mathbb{Q}} T\}/\cong .$$

(b) Show that $\operatorname{Pic}_2(\mathbb{Q}(i))$ contains a Galois invariant element of degree 1;

(c) Show that $\operatorname{Pic}_2(\mathbb{Q})$ contains NO element of degree 1;

(d) Show that Pic_2 is not representable (by a scheme over $\mathbb{Q}$).

2. Lecture 2

In these exercises, you can use without proof that the 'naive definition 2' given of $\overline{\mathcal{M}}_{g,n}$ actually works fine when $g = 0$; more precisely, for $n \geq 3$ (stability condition) the functor

$$\overline{\mathcal{M}}_{0,n} : \mathbf{Sch}^{op} \to \mathbf{Set}; T \mapsto \{\text{stable (genus 0 curves with } n \text{ disjoint marked sections)}\}/\cong$$

is representable by a quasi-projective scheme. If you like you could even try to prove this, starting from small n.

2.1 Show that $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3}$ is a point. What is $C_{0,3}$?

2.2 construct an isomorphism $\mathcal{M}_{0,4} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

2.3 construct an open immersion (or just a map of functor, for now) $\mathcal{M}_{g,n+1} \to \mathcal{C}_{g,n}$.

2.4 Let $(C, p_1, \dots, p_n)$ be a proper nodal connected curve over an algebraically closed field, with n distinct marked points. Write $\tilde{C}$ for the normalisation of C, and call a point in $\tilde{C}$ *special* if it maps to a singular point or a marked point. Show the following are equivalent:

(a) $\# \operatorname{Aut}(C, p_1, \dots, p_n) < \infty$;

(b) Every genus 0 component of $\tilde{C}$ has at least 3 special points, and every genus 1 component of $\tilde{C}$ has at least 1 special point.

2.5 Construct an isomorphism $\overline{\mathcal{C}}_{g,n} \xrightarrow{\sim} \overline{\mathcal{M}}_{g,n+1}$. If you get stuck, take a look at [Knudsen, The projectivity of the moduli space of stable curves, II].

2.6 Is $\mathcal{C}_{g,n} \xrightarrow{\sim} \mathcal{M}_{g,n+1}$?

2.7 In the example at the end of lecture 2, for which (m_1, m_2) is the 'naive DRL' closed?

2.8 This is a longer exercise, intended for those students familiar with some algebraic geometry, to understand the structure of the boundary of the moduli space.

(a) Look up the definition of the Fitting ideal, in Wikipedia or the Stacks project.

Let $\pi\colon C \to S$ be a proper nodal curve. Write $\Omega_{C/S}$ for the sheaf of relative Kahler differentials, and $\mathcal{I}$ for the first Fitting ideal of $\Omega_{C/S}$ (so $\mathcal{I}$ is a sheaf of ideals on C). Write $\mathcal{J}$ for the 0-th Fitting ideal of $\mathcal{O}_C/\mathcal{I}$, viewed as an $\mathcal{O}_S$-module. We define the closed subscheme of S cut out by $\mathcal{J}$ to be the *boundary* of the family C/S.

(b) Let $S = \operatorname{Spec} R$, and suppose C is locally given by $\operatorname{Spec} R[x,y]/(xy-r)$ for some $r \in R$ (OK, this is not proper, but it won't matter in this case). What is $\mathcal{I}$? What is $\mathcal{J}$?

(c) Show that C/S is smooth if and only if the boundary is empty (maybe it will help to do the next part first...).

(d) Show formation of the boundary commutes with arbitrary base-change in S.

3. Lecture 3

3.1 Let G be a finite (undirected) graph with vertex set V. The *intersection matrix* I_G is a $V \times V$ integer matrix. If $v \neq v'$ then the corresponding entry is the number of edges joining v to v'. Then the diagonal is filled in so as to make the row (equivalently column) sums equal zero.

(a) Assume G is connected. For a vertex $v \in V$, write $\delta_v \in \mathbb{Z}^V$ for the function taking the value 1 at v and 0 elsewhere. Let $D \in \mathbb{Z}^V$(i.e. a divisor on G) be such that $D^T I_G v = 0$ for every vertex v. Show that D is constant.

(b) Is the assumption that G be connected necessary in the above?

(c) Let $S = \operatorname{Spec} k[[t]]$, and let C/S be proper, generically smooth and regular, with the central fibre having graph G. Let $\mathcal{L}$ a line bundle on C with multi degree D. Show the following are equivalent:
   - There exists a line bundle $\mathcal{L}'$ on C which coincides with $\mathcal{L}$ on the generic fibre, and has multidegree $\underline{0}$ on the closed fibre;
   - there exists a divisor D on G such that ID is equal to the multidegree of $\mathcal{L}$.

4. Lecture 4

4.1 We defined group schemes via functors of points. A more classical definition can be found in 3.1 of `https://www.math.ru.nl/~bmoonen/BookAV/BasGrSch.pdf`. Can you show they are equivalent? Checking all the axioms can be tedious, so stop whenever you feel you have the idea.

4.2 Show that $\mathrm{Pic}_{\mathbb{P}^1/k} = \mathbb{Z}_k$ (constant group scheme);

4.3 Recall that a morphism $X \to S$ is *separated* if the diagonal $X \to X \times_S X$ is a closed immersion. Show that a group scheme G/S is separated if and only if the unit section is a closed immersion.

4.4 Let $S = \operatorname{Spec} k[[t]]$ or a small disc around the origin in $\mathbb{C}$, with coordinate t. Define a curve over S by the equation $xy = t$ (more formally, $xy = tz^2$ in $\mathbb{P}^2_S$). Show that $\mathrm{Pic}^{\underline{0}}_{C/S} = S$. Show that $\mathrm{Pic}^{tot-0}_{C/S}$ is not separated over S.

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