Lecture 2: $g, n, 2, m_i, \sum m_i = q(2g-2)$

Last hour, saw how, ~~to define ZDRL ⊆ S~~
given C/S (smooth curve), $p_1 \ldots p_n \in C(S)$, how to
define DRL ⊆ S. And, (using some extra tools) ~~next time~~, how to
give algebraic structure, & show closed.

We want to 'describe' DRL ⊆ S, but what would such a description look like (for all S)? Better approach:
take C/S universal family.

§ moduli of smooth curves

$g \geq 0, \; n \geq 0, \; 3g-3+n > 0.$

Want to define moduli space of (sm. proper) curves / cpct connected R-surfaces of genus g w. n distinct marked pts (ordered)

1st approx. of def:

$$M_{g,n} = \left\{ (C, p_1 \ldots p_n) \;\middle|\; \begin{array}{l} C \text{ (cpct-conn) R-surface of} \\ \text{genus } g \text{ w. } n \text{ markings} \end{array} \right\} \Big/ \cong .$$

Problem: it's just a set... Want a $\mathbb{C}$-mfd/scheme.
cf. ~~Prior~~ Jacobians. Motivated by that;

2nd approx. of def:

$$M_{g,n} : \underline{Sch}^{op} \longrightarrow \underline{Set}$$

$$S \longmapsto \left\{ (C/S, p_1 \ldots p_n) \;\middle|\; \begin{array}{l} C/S \text{ family of} \\ \text{R-S of genus } g, \\ p_i, p_n \in C(S) \\ \text{disjt} \end{array} \right\} \Big/ \cong$$

Problem: this is enough data to specify a scheme, but it just doesn't exist. Problem is basically that automorphisms mean families don't glue nicely.

Actual Def:

$$\mathcal{M}_{g,n} : \underline{Sch}^{op} \longrightarrow \underline{Grpoid}$$ — cat where only morphisms are isos.

$$S \longmapsto \{(C/_S, p_1 \dots p_n) \dots\}$$ Don't divide out by isos!

Q: What exactly is a 'functor to groupoids'? Omitted..

Thm This $\mathcal{M}_{g,n}$ is 'representable' by a smooth, fin. type (DM stack) of dim $3g-3+n$.

for purpose of the today, will ignore this technical detail, & treat it as a smooth variety / $\mathbb{C}$-mfd.

~~We have id: $\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$, so we can evaluate~~

In gen, by def, $\mathrm{Hom}(S, \mathcal{M}_{g,n}) = \{(C/_S, p_1, \dots, p_n)\}$.

In particular, $\mathrm{Hom}(\mathcal{M}_{g,n}, \mathcal{M}_{g,n})$

$$id \longmapsto (C^{un}/_{\mathcal{M}_{g,n}}, p_1 \dots p_n)$$

'universal curve'.

Then define

$$DRL \hookrightarrow \mathcal{M}_{g,n} \text{ as locus where } \sum_i m_i p_i \sim q k_C$$

(define using jacobians as before, to get subscheme / subfunctor).

On $\mathcal{M}_{g,n}$ we basically 'know' what DRL is.

- There is a formula due to Hain for the cycle class. ~~And we know (Schm~~
- If $g>0$ & $\exists i$: $m_i<0$ or $g \nmid m_i$, then DRL is smooth & pure codim g (Schmitt)
- :

End of story?

Not really. Problem: $\mathcal{M}_{g,n}$ is generally not cpct/proper (eg. $g=0, n=4$; see exercises).

So eg. if want to integrate/intersect things w. DRL it will generally not work/give 0 for stupid reasons.

§ Stable curves

Idea is to compactify $\mathcal{M}_{g,n}$ by allowing curves w. mild singularities, & then try to extend DRL to this compactification.

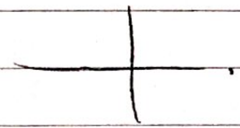
Of course, $\mathcal{M}_{g,n}$ doesn't have a ! cpct, but there is a particularly nice one, due to Deligne-Mumford-Knudsen. It allows stable curves, which we now define.

To do moduli we need families, but for now treat 'one at a time':

Idea: take some smooth curves / R-surfaces, & 'glue together' at some pts, keeping tangent directions distinct.
(Formalize w. fibred coproducts, see [Ferrand, Pincement.-])

More formal:

$C/\bar{k}$ is a nodal curve if it is analytically / étale locally

isomorphic to $\frac{k[x,y]}{xy}$ .

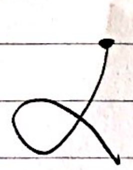
sections of sm locus

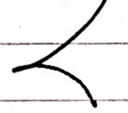
$(C/\bar{k}, p_1 \dots p_n)$ is stable if it is proper, nodal, connected,

& has $< \infty$ automorphisms. (see ex. for comb. version).

Families: 'nice map whose fibres are stable curves':

$(C/S, p_1 \dots p_n)$: C/S proper, flat, fin. pres, geom fibres stable.

Eg: in $\mathbb{P}^2$, $y^2 = x^2(x-1)$, mark pt at ∞:  stable ✓

$y^2 = x^3$, — " —  not nodal, so not stable

Then we define moduli of stable curves:

$$\overline{M}_{g,n} : Sch^{op} \longrightarrow Gpoid$$

g: fibres of genus g

$$S \longmapsto \{(C/S, p_1 \dots p_n) \text{ stable}\}$$

p_i disj't sections of sm. locus.

Thm [DM, K3]: $\overline{M}_{g,n}$ is a smooth, proper (DM stack) of dim $3g-3+n$,

$M_{gn} \hookrightarrow \overline{M}_{gn}$ dense open,

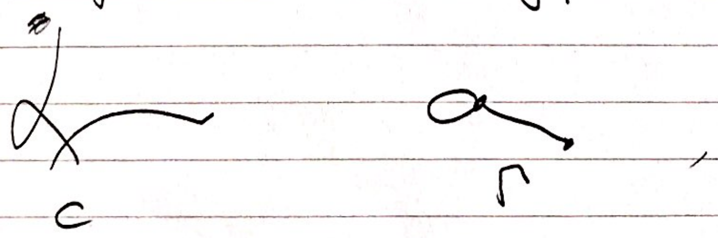
boundary is a NCD.

Have universal stable $\overline{C}_{gn}/\overline{M}_{gn}$ as before (see ex for more)

genus of a stable curve. normalisation $\tilde{C} \to C$: separate out all nodes

Dual graph of stable $(C/k, p_1 \dots p_n)$:

- vertex for each irred. comp
- edge for each sing pt.



$$\text{genus}(C) = b_1(\Gamma) + \sum_{v \in \pi_0(\tilde{C})} g(v).$$

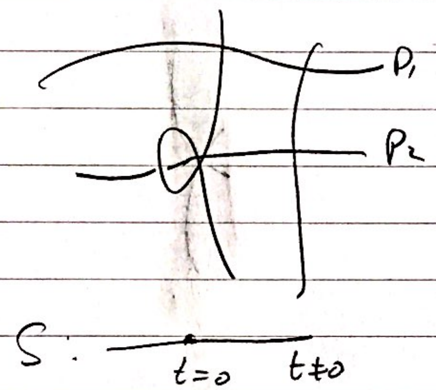
Thm: genus constant in flat families.

Or define as $h^1(\mathcal{O})$!

§ Extending DRL to ~~...~~ $\overline{M}_{g,n}$?!

~~Eg~~ We will study various families of stable curves as a warmup.

eg $\quad y^2 = (x-1)(x^2-t^2)$ over $k[[t]]$ / or over a small disc $\Delta \subseteq \mathbb{C}$ with parameter t.



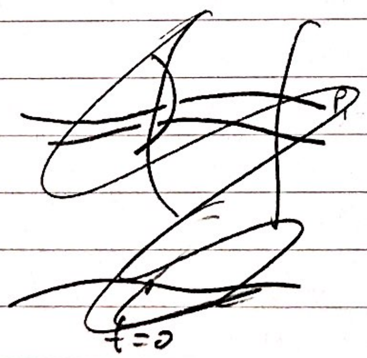
marked pts: one at ∞ (P_1), one given by $y=0$, $x=t$ (P_2).

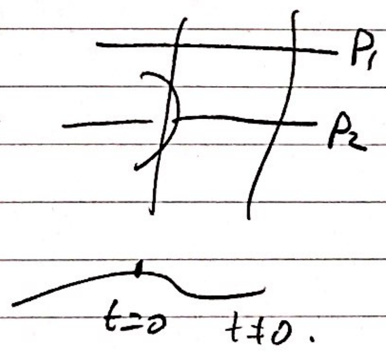
So generic fibre gives pt in $M_{1,2}$,

~~so closed pt lies in~~ ...

but section goes through singular pt (x,y,t).

Ans: blow up this singular pt;





This gives a pt in $\overline{M}_{1,2}$.

~~Xx~~ $g=1$, so $k \simeq 0$, so q doesn't matter.

So specifying m_1, m_2 will define a DRL $\subseteq S$.

$(m_1+m_2=0)$

eg $m_1 = m_2 = 0$: then $DRL = S$, as $\mathcal{O}_C(0\cdot P_1 - 0\cdot P_2) \simeq \mathcal{O}_C$.

$m_1 = -m_2 = 1$: then (ex). $DRL = \phi$: $\mathcal{O}_{C_t}(P_1 - P_2) \not\simeq \mathcal{O}_{C_t}$ for any t.

eg $m_1 = -m_2 = 2$.

Q: for which t is

$$\mathcal{O}_{C_t}(2P_1 - 2P_2) \simeq \mathcal{O}_{C_t} \quad ?$$

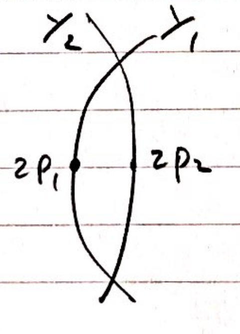
$t \neq 0$: then on a smooth curve, &

$$\operatorname{div}\left(\frac{z}{x-tz}\right) = 2P_1 - 2P_2.$$

So $\mathcal{O}_{C_t}(2P_1 - 2P_2) \simeq \mathcal{O}_{C_t}$.

But if $t=0$, then $\mathcal{O}_{C_0}(2P_1 - 2P_2) \not\simeq \mathcal{O}_{C_t}$,

since the degrees of the line bdles on the two irred comps are not equal:



$\mathcal{L} := \mathcal{O}_{C_0}(2P_1 - 2P_2)$

$\deg_{Y_1}(\mathcal{L}) = 2 \qquad \deg_{Y_1}(\mathcal{O}) = 0$

$\deg_{Y_2}(\mathcal{L}) = -2 \qquad \deg_{Y_2}(\mathcal{O}) = 0.$

So this naive version of DRL is not closed!

This is bad, eg if want to integrate/attach cycle class/chow class/do $\cap$ theory.