Lecture 3: $g, n, m_i \in \mathbb{Z}, \sum_i m_i = q(2g-2)$

~~C/S stable curve, p_i,~~ $(C/S, p_1 \ldots p_n)$ stable,

$$\mathcal{L} := \omega_{C/S}^{\otimes q}\left(-\sum_i m_i p_i\right)$$

(if not familiar w. rel. dual. sheaf ω, just take $q=0$).

Could näively try defining

$$DRL = \{ s \in S \mid \underbrace{\mathcal{L}_s \simeq \mathcal{O}_s}_{} \}.$$
$DRL \subseteq S$

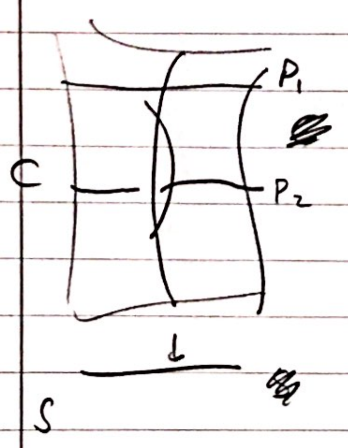
translate into divisors (when $q=0$):

$\exists$ a $\underbrace{\text{rat. fctn}}_{}$ f on C_s with $\operatorname{div} f = \sum_i m_i p_i(s)$

[a rat. fctn f_v for each conn. comp. V of the normalisation $\tilde{C}$ of C_s, s.t. f_v has no zeros or poles at preimages of nodes, & s.t. values at preimages of node agree]

But we saw last time that this set DRL need not be closed in S. Our example was

$S = \Delta_t$ or Spec $k[[t]]$, $\quad C: y^2 = (x-1)(x^2-t^2)$ blown up at (x,y,t),



p_1 at ∞, $\quad p_2$: $y=0, x=t$.

Then for $m_1 = 2, m_2 = -2$, we find all $t \neq 0$ lie in DRL, but $(t=0) \notin DRL$ for degree reasons.

To look more closely at this 'degree problem' we define the **multidegree** of a divisor / line bdle on a stable curve $(C_{\bar{k}}, p_1 \ldots p_n)$ over $k = k^{alg}$ to be the fctn

$$\mathrm{multideg}(\mathcal{L}) : \left\{ \begin{matrix} \text{conn}^{d}\text{. comps} \\ \text{of } \tilde{C} \end{matrix} \right\} \longrightarrow \mathbb{Z}$$

($\{\text{irred compts of } C\} = \{\text{conn. comps of } \tilde{C}\}$)

$$\left(V \rightarrowtail \tilde{C} \rightarrow C, \ V \xrightarrow{\iota_v} C \right) \longmapsto \deg(\iota_v^* \mathcal{L})$$

divisor/line bdle on smooth proper curve.

Recalling that dualgraph Γ has ~~vertex for~~ $\mathrm{vert}(\Gamma) = \{\text{con. comps of } \tilde{C}\}$, $\mathrm{edges} = \{\text{sing pts}\}$,

we can see the multidegree as a fctn

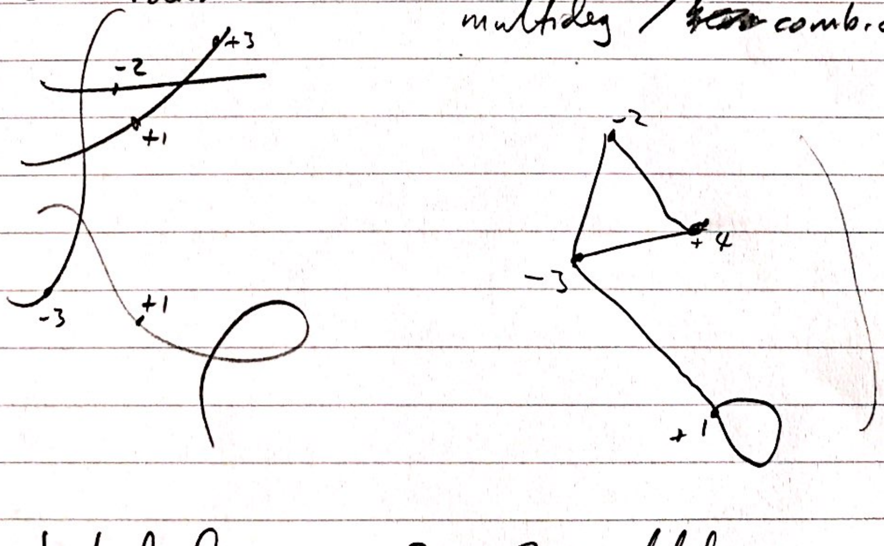
$$\mathrm{vert}(\Gamma) \longrightarrow \mathbb{Z},$$

AKA a combinatorial (/tropical) divisor.

<u>eg</u>

curve/divisor

multideg / ~~tro~~ comb. divisor



tot deg = 0.

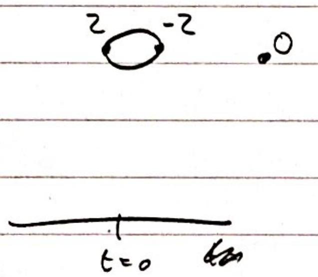
<u>Def</u>: total degree = sum of multideg.

Lem/ex: If $\mathcal{L} \cong \mathcal{O}$ then multideg $\mathcal{L} = \underline{0}$ (i.e. multdeg is isomorphism invariant!)

(If $D \sim_{lin} 0$ — " — $D = \underline{0}$.)

(converse implication fails, e.g. on smooth curves of $g>0$).

In our example family:



so over $t=0$ there is a 'combinatorial obstruction' to being in DRL.

Observation: in this example, we have natural divisors p_1, p_2 on C.

~~But~~ ~~& ... surface~~ But there are more 'natural divisors'.
Namely, fix any value of t, & look at the fibre C_t. This is a divisor on C for dimension reasons. But it's not v. interesting as it's principal: $\operatorname{div}(t) = C_t$, so $\mathcal{O}_C(C_t) \cong \mathcal{O}_C$.

But the fibre over $t=0$ breaks into two irred comps, Y_1 & Y_2 (s.t. p_i goes through Y_i).
Then these are not principal, so yield interesting line bdles on C.

Let's compute the multidegrees of $\mathcal{O}_C(Y_1)$, $\mathcal{O}_C(Y_2)$.

First, Y_1 & Y_2 meet at 2 pts, transversally. so

$\deg_{Y_1} \mathcal{O}(Y_2) = 2 = \deg_{Y_2} \mathcal{O}(Y_1)$.

What is $\deg_{Y_i}(Y_i)$? Note $Y_1 + Y_2 = C_0$ is principal,

so $\deg_{Y_1} \mathcal{O}(Y_1 + Y_2) = 0$, so $\deg_{Y_1} \mathcal{O}(Y_1) = -2$

($= \deg_{Y_2} \mathcal{O}(Y_2)$, similar argument).

~~So~~ Consider the line bundle

$$\mathcal{L}' = \mathcal{O}_C(2P_1 - 2P_2 + Y_1).$$

Then (ex), $\mathcal{L}'$ has multideg 0 on all fibres.

ex: $DRL' := \{ s \in S \mid \mathcal{L}'_s \simeq \mathcal{O}_{C_s} \}$ is closed in S.

Ok, so we've 'fixed' one example. May look a bit weird/unmotivated, but from jacobian perspective is more natural...

~~This motivates~~ Let's try to make a general def'n to capture the essence of this:

Def Let C/S with C & S regular, C smooth over a dense open $U \subseteq S$.
Suppose $\exists\ \mathcal{L}$ on C s.t.

- on all $s \in U$, have $\mathcal{L}_s \simeq \omega^{\otimes q}(-\sum m_i P_i)$
- $\forall s \in S$, $\mathcal{L}_s$ has multidegree 0.

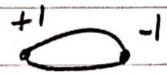
Then define $DRL = \{ s \in S \mid \mathcal{L}_s \simeq \mathcal{O}_{C_s} \}$.

Does this work? Well, a-priori it depends on choice of $\mathcal{L}$, but (Chaudish ex) such $\mathcal{L}$ is unique (up to ~~to~~ iso) if it exists.

Bigger problem: often, such an $\mathcal{L}$ will not exist.

Two types of eg., one harmless, one not.

1st eg: Same $\mathcal{C}_S$ as before, but now $m_1 = 1, m_2 = -1$.

Then multdeg$(\mathcal{O}(p_1 - p_2))$ = 

multdeg $\mathcal{O}(\gamma_1)$ = 

multdeg $\mathcal{O}(\gamma_2)$ = 

~~Obs~~ So no way to use γ_1, γ_2 to 'correct' multdeg of $p_1 - p_2$ to 0.

$\therefore$ ex: $\nexists$ $\mathcal{L}$ as in ~~statement~~ prev. def.

But this is a fairly harmless eg., as DRL $= \emptyset$ in this case, in particular it is closed.

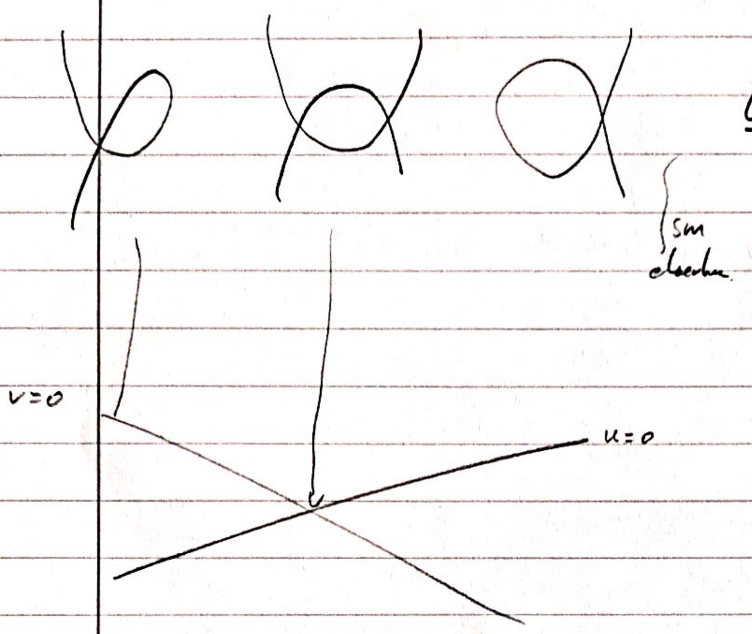
2^{nd} eg. Over $\Delta_u + \Delta_v$ on $\operatorname{Spec} k[[u,v]]$.

$$C : y^2 = ((x-1)^2 - u)((x+1)^2 - v)$$

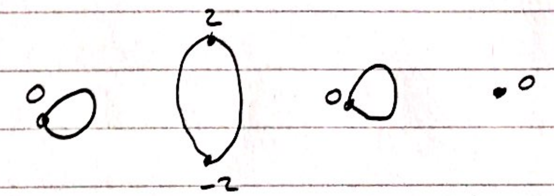
Mark the two pts at ∞ P_1, P_2.

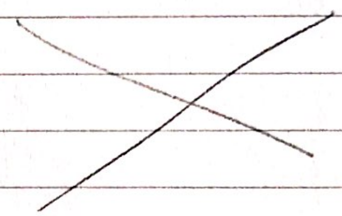
$m_1 = 2, m_2 = -2$.

Curve:



Graph:



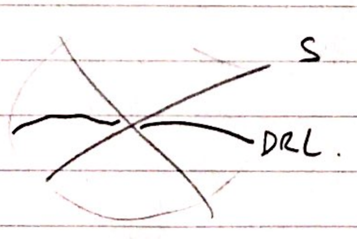


So there is a combinatorial obstruction; multideg $\neq 0$ on central fibre.

But there are no irreducible vertical divisors Y_1, Y_2 to correct the multidegree with, as only central fibre is reducible.

So $\nexists$ $\mathcal{L}^{\#}$ as in the def'n.

But ~~DRL~~ "naive DRL" defined with $\mathcal{O}(2P_1 - 2P_2)$ is not closed:



What to do? Intuitively, want to 'blow up' $u=v=0$ to stretch the pt to a divisor, so have vertical divisors to cure. But then problems... And anyway, what to do w. eg 1?

Final def:

Fix $g, n, q, m_1 \dots m_n$, $\sum m_i = q(2g-2)$ as always.

Def: $T \xrightarrow{t} \overline{M}_{gn}$ is non degen if T normal & $t^{-1} M_{gn}$ is dense in T.

• $T \xrightarrow{t} \overline{M}_{gn}$ is σ-extending if non-degen + ∃ line bdle ~~(locally on T)~~ $\mathcal{L}$ on C_T s.t.

• $\mathcal{L}_s$ has multideg 0 on C_s $\forall s \in T$

• $\mathcal{L}|_{t^{-1}M_{g,n}} \simeq \omega^{\otimes q}(-\sum m_i P_i)$

We've seen several egs of σ-ext & non-σ-ext.

Thm: The cat of σ-ext (alg spaces) over $\overline{M}_{gn}$ has a terminal object

$$\overline{M}_{gn}^{m,q} \to \overline{M}_{gn}.$$

DON'T ERASE.

On this $\overline{M}_{gn}^{m,q}$ have the l. bdle $\mathcal{L}$ ~~(maybe only exists locally, but ok, or see ... ex on Pic ...)~~

Then
$$DRL = \{ s \in \overline{M}_{gn}^{mq} : \mathcal{L}_s \simeq \mathcal{O}_{C_s} \}.$$

Again, this is just a set.

- How to make it a scheme (stack)?
- How to define cycle class?
- This lives on $\overline{M}_{gn}^{mq}$ not $\overline{M}_{gn}$; how to fix?