Extending DR, continued

We have this DRL from last time. Problems:

~~1 - L exists locally on T, messy!~~ (It's ok by rigidification)

1 ~~2~~ - How to show DRL $\subset \overline{M}^{mq}_{gn}$ algebraic / closed / ?

2 ~~3~~ - Upgrade to cycle?

3 ~~4~~ - push to $\overline{M}_{gn}$?

For first ~~3~~ 2, will proceed as in smooth case, by using jacobians.

§ Picard schemes of proper nodal curves.

markings & aut. play no role for now, so ignore.

In lecture 1, we defined the jacobian as a functor. In this more general setting the degree is less clear, but otherwise def goes through ok:

Def. C/S proper nodal curve, $p \in C(S)$ section.

$$\mathrm{Pic}_{C/S} : \underline{Sch}_S^{op} \longrightarrow \underline{Ab}$$

$$T \longmapsto \left\{ (\mathcal{L}, \varphi) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ line bdle on } C \times_S T \\ \varphi : p^*\mathcal{L} \xrightarrow{\sim} \mathcal{O}_T \end{array} \right\} \Big/ \sim$$

where $(\mathcal{L}, \varphi) \sim (\mathcal{L}', \varphi') \iff \exists\, \psi : \mathcal{L} \to \mathcal{L}'$ s.t.

$$\begin{tikzcd} p^*\mathcal{L} \arrow[rr, "p^*\psi"] \arrow[dr, "\varphi"'] & & p^*\mathcal{L}' \arrow[dl, "\varphi'"] \\ & \mathcal{O}_T & \end{tikzcd}$$

commutes.

Do 1st thm of 4·3 now!

Group Schemes.

~~Def. $G/_S$ a scheme. To give $G/_S$ the struct of a gp scheme is to give a factorisa~~

Def. A *group scheme* over S is a scheme $G/_S$ together w. a factorisation

$$
\begin{tikzcd}
\underline{Sch}_S^{op} \arrow[rr, "h_G(-) = Hom_S(-, G)"] \arrow[dr] & & \underline{Set} \\
& \underline{Gp} \arrow[ur, "forget"'] &
\end{tikzcd}
$$

Eg if $S = \operatorname{Spec} \mathbb{C}$, then have

$$
\begin{tikzcd}
S \in \underline{Sch}_S^{op} \arrow[rr] \arrow[dr] \arrow[ddr] & & \underline{Set} \ni G(\mathbb{C}) \\
& \underline{Gp} \arrow[ur] & \\
& \text{some gp} \arrow[uur] &
\end{tikzcd}
$$

So it equips $G(\mathbb{C})$ with a gp structure. But it does more;

eg

$$
\begin{tikzcd}
S \in \underline{Sch}_S^{op} \arrow[rr] \arrow[dr] & & \underline{Set} \ni G(S) \\
& \underline{Gp} \arrow[ur] &
\end{tikzcd}
$$

gives a distinguished unit $e \in G(S)$.

Similarly (ex), can get $m: G \times G \to G$,

$i: G \to G$

sat. all expected axioms.

examples:

$\mathbb{G}_m : \underline{Sch}_S^{gp}$

$G = \mathbb{A}^1_S = \operatorname{Spec}_S \mathcal{O}_S[x]$,

then $\operatorname{Hom}_S(T, G) = \mathcal{O}_T(T)$, which we equip with additive gp str.

$G = \mathbb{A}^1_S \setminus 0 = \operatorname{Spec}_S \mathcal{O}_S[x, x^{-1}] = \operatorname{Spec}_S \dfrac{\mathcal{O}_S[x,y]}{(xy-1)}$.

Then

$\operatorname{Hom}_S(T, G) = \mathcal{O}_T(T)^{*}$, equip w. mult-gp str.

As before, there's a non-trivial existence / representability thm

Thm [Groth. / Raynaud / ~~…~~ Mumford]:

$Pic_{C/S}$ is representable by a smooth gp. algspace over S.

(scheme, eg. if assume ired compnt fibres geom. ired; will not worry abt this.)

With a little more work, can easily use to show DRL algebraic, closed, & to give cycle class. But first let's work to understand this object better, with various examples etc.

(4.2) gp schemes.

§ Missing adjectives

Note we did not say Pic is projective. It's not! In general, neither q-cpct, sep, or ~~…~~ univ closed!

§ Not quasi-cpt:

This happens even for smooth curves, because we did not restrict the degree. ~~eg~~

Eg $S = \operatorname{Spec} k$, $C = \mathbb{P}^1$, then ~~…~~ (ex) $Pic_{C/S} = \mathbb{Z}_S$

constant gp scheme over ~~…~~ S.

To fix, we constrain degree. For smooth curves, we just said 'degree 0 on each fibre'. For nodal, have choice of multideg or total deg 0.

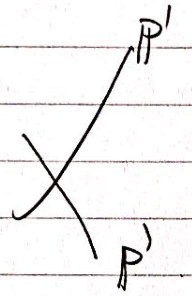
Def. $\mathrm{Pic}^{\mathrm{tot}\,0}_{C/S} : \underline{Sch}_S^{op} \longrightarrow \underline{Ab}$

$T \longmapsto \left\{ (\mathcal{L}, \varphi) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ on } C\times_S T, \\ \text{tot. deg } 0 \text{ on each geom. fib,} \\ \varphi : p^*\mathcal{L} \simeq \mathcal{O}_T \end{array} \right\} \Big/ \sim$

Def. $\mathrm{Pic}^0_{C/S}$: — " —

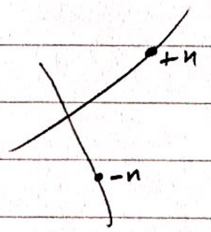
replace 'tot deg 0' by 'multideg 0'.

eg $S = \mathrm{Spec}\, k$, C:



Then $\mathrm{Pic}^0_{C/k} = pt$, but $\mathrm{Pic}^{\mathrm{tot}\,0}_{C/S} = \mathbb{Z}_S$,

when $n \in \mathbb{Z}$ corresp. to



Ingen:

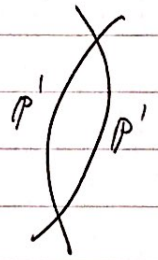
Thm: $\mathrm{Pic}^0_{C/S}$ is separated & Q-cpct (C/S still not proper, in gen)

~~Rk~~ ($\mathrm{Pic}^{\mathrm{tot}\,0}$ need not be sep. or Q-cpct)

see later. see above

The fact that $\mathrm{Pic}^0_{C/S}$ is in gen still not proper is not important for us, but nice to see an example.

Eg $k = \bar{k}$, $C =$



glue 0s together, glue ∞s together.

Then $\mathrm{Pic}^0_{C/k} = \mathbb{G}_m$ (so k-pts are k^*).

Why? More generally, recall $\pi : \tilde{C} \to C$ normalisation.

If $\mathcal{L}$ on C then $\pi^*\mathcal{L}$ is a line bdle on $\tilde{C}$, proper sm.

$\tilde{C} = \bigsqcup_v C_v$, & $\mathrm{Pic}_{\tilde{C}/k} = \prod_v \mathrm{Pic}_{C_v/k}$.

(take care w. markings; works out ok).

We get

$$\mathrm{Pic}_{C/k} \xrightarrow{\pi^*} \prod_v \mathrm{Pic}_{C_v/k}$$

map of gp-schm. proper.

ker/coker?

ϕ

$= \mathbb{Z} \times \mathbb{Z}$ in our above eg.

(gives $\mathrm{Pic}^0_{C/k} \longrightarrow \prod_v \mathrm{Pic}^0_{C_v/k}$, same ker & coker).

~~Thm [Ferrand, Logan H~~

Lem: π^* is surj.

Pf. Ex / see [Ferrand, pincement]. □

$\ker(\pi^*)$: - On each C_v have trivial bundle,

- at each node, specify how to glue together (elt of k^*)

- For each C_v, can scale whole bundle ($\rightsquigarrow$ elt of k^*)

~~Sofar~~ In our eg Γ, get $\dfrac{k^\times \times k^\times}{\underbrace{k^\times \times k^\times}_{\text{both factors acting diagonally}}} \simeq k^\times$.

In gen. have SES

$$1 \to k^\times \to k^{\times E} \to k^{\times V} \to \ker(\pi^*) \to 1$$

(ex: make maps precise),

So $\ker(\pi^*) \simeq k^{\times b_1(\Gamma)}$ "#holes".

So in gen, fibres of Pic^0 are ext. of ab. var by $\mathbb{G}_m^{b_1(\Gamma)}$.

~~§~~ § Back to DR.

Have $\mathrm{Pic}^0_{C/\overline{M}^{m,q}_{g,n}}$, & section σ given by L',

& — " — e — " — 0.

~~Then~~ Both cl. imm as Pic^0 separated, so ~~σ^*e is~~

$DRL = \sigma^* e$ is closed in $\overline{M}^{m,q}_{gn}$,

& can define cycle class $\sigma^{*!}[e]$ on DRL.

Thm: $DRL \xrightarrow{f} \overline{M}_{gn}$ is proper, ~~as~~ so $f_* \sigma^{*!}[e]$ makes sense, & gives codim g cycle on $\overline{M}_{gn}$ extending DR on M_{gn}.